EUDOXUS' AXIOM AND ARCHIMEDES' LEMMA

BY

JOHANNES HJELMSLEV*

1. It is a well-known fact that modern students of the writings of the old mathematicians have great difficulty in resisting the distracting impact of the modern mechanism of science on their thoughts. I believe I have found a striking example of this kind in the prevailing interpretation of ARCHIMEDES' theory of magnitudes, as for this reason nobody—as far as I can see—has yet succeeded in penetrating sufficiently into Archimedes' thoughts. Further demonstration of this is attempted in what follows.

2. With Archimedes, and particularly with his treatise *On the Sphere and Cylinder,* the Greek theory of magnitudes enters on a new epoch. Quite new magnitudes, such as the length of curved lines and the area of curved surfaces, are drawn into the field of investigation, and for these magnitudes definite rules are set up, formulated in the following postulates of magnitudes (here given in a somewhat abbreviated form after T. L. HEATH, *The Works of Archimedes*):

1°. Of all lines which have the same extremities the straight line is the least.

2°. Of two convex lines which have the same extremities, and of which one encloses the other, the outer is the greater.

3°. A plane area is smaller than a curved area with the same circumference.

4°. Of two convex surfaces covering the same plane area, and one of which encloses the other, the outer is the greater.

* Late professor of mathematics, University of Copenhagen.

Centaurus 1950: 1: 2-11

5°. Further, of unequal lines, unequal surfaces, and unequal solids, the greater exceeds the less by such a magnitude as, when added to itself, can be made to exceed any assigned magnitude among those which are comparable with [it and with] one another.

3. The last supposition (5°) (*Archimedes' lemma*) will receive speciel treatment in what follows.

In it lies the instrument that makes possible an extension of the main theorems in the theory of magnitudes founded by EUDOXUS—as we know it from EUCLID, Book V—in such a way that it can be applied to the new magnitudes.

But first of all, of course, the introductory definitions in the Eudoxean theory of proportions have been taken over, especially

A. *Eudoxus' axiom* (Eucl. V, def. 4), which gives the conditions under which two magnitudes, a, b, can be said to have any ratio to each other:

Magnitudes are said to have a ratio if any one of them by being added a sufficient number of times to itself can be made to exceed the other.

$$(a < b, a + a + \ldots\ldots + a > b).$$

Further

B. The definition of equal ratios a/b and c/d: If $ma \gtreqless nb$, then respectively $mc \gtreqless nd$ for each set of integers m, n.

C. The definition of unequal ratios: $a/b > c/d$, if there exists a set of integers m, n so that $ma > nb$, but $mc \leqq nd$.

From the theory of proportions proper it is in Archimedes' investigations the main rule that is applied, stating that

$$a \gtreqless b \text{ yields respectively}$$

$$a/c \gtreqless b/c \text{ (or } c/a \lesseqgtr c/b).$$

But the proof of this proposition presupposes (Eucl. V, 8)—regard is here had only to the uppermost sign $(a > b)$—the existence of an integral number n so that

$$n\,(a—b) > c.$$

In the Eudoxean theory of proportions this was a direct consequence of Eudoxus' axiom, as for the fields of magnitudes considered there a—b always existed as a magnitude of the same kind as those given. But for the new field of magnitudes with which Archimedes was now concerned this could by no means be presupposed. What, for instance, was to be understood by a—b when a was the arc of a circle, and b a straight line? Or a the surface of a sphere, and b a plane area?

As his way out of the difficulty Archimedes chose to put his lemma as the last step in the fundamental assumptions concerning the new magnitudes, or, if you prefer it: as the last step in the definition of these:

When a $>$ b, the difference a—b is an *ideal magnitude,* to which the same procedure is applicable as in the Eudoxean theory of magnitudes: that to any case that may arise there always exists such an integer n that n (a—b) is greater than any given magnitude c of the same kind as a and b.

4. That this was Archimedes' view can be seen clearly enough already from the care—one might almost say awe—with which in his introductory letters he mentions his lemma, already in the treatise *On the Quadrature of the Parabola,* but especially in the treatise *On the Sphere and Cylinder,* where the basic principles are stated in detail for the first time; later on also in his treatise *On Spirals.* But it appears with absolute certainty from the whole systematic structure of Archimedes' system, as will be further illustrated below.

Hitherto nobody seems to have realized the special significance of Archimedes' lemma. It has simply been interpreted as equivalent to Eudoxus' axiom, and it is a well-known fact that in generally accepted modern mathematical usage Eudoxus' axiom is referred to as Archimedes' axiom[1].

Moreover the view had been advanced that Archimedes himself took it for granted that a ratio can always be represented as a ratio between two lines. That this is an evident misunderstanding can be seen already from the fact that if it were so Archimedes' lemma would be quite superfluous.

[1] The mistake goes back to O. Stolz, *Zur Geometrie der Alten, insbesondere über ein Axiom des Archimedes* ("Eine Grösse kann so oft vervielfältigt werden, dass sie jede andere ihr gleichwertig übertrifft"); Innsbr. Ber. XII, 1882; Math. Ann., XXII.

Later adopted by G. Veronese in *Fondamenti di geometria* ... 1891 (German edition: *Grundzüge der Geometrie* ... 1894; footnote p. 95): "Stolz ist, so viel wir wissen, der erste gewesen, der die Aufmerksamkeit der Mathematiker auf diesen von ihm mit Recht *Axiom des Archimedes* genannten Satz gelenkt hat ..."

Later the same designation was adopted by Hilbert in *Grundlagen der Geometrie,* 1899.

H. G. Zeuthen has repeatedly protested against the misnomer (thus at the Heidelberg congress, 1904).

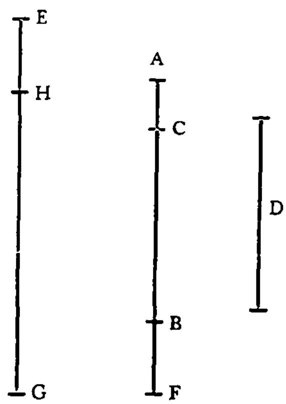
5. But we shall now proceed to a more detailed documentation taking a concrete example of Archimedes' demonstration, viz. Proposition 2 in his treatise *On the Sphere and Cylinder I.* We shall begin by quoting T. L. Heath, *The Works of Archimedes,* p. 5:

Given two unequal magnitudes, it is possible to find two unequal straight lines such that the greater straight line has to the less a ratio less than the greater magnitude has to the less.

Let AB, D represent the two unequal magnitudes, AB being the greater.

Suppose BC measured along BA equal to D, and let GH be any straight line.

Then, if CA be added to itself a sufficient number of times, the sum will exceed D. Let AF be this sum, and take E on GH produced such that GH is the same multiple of HE that AF is of AC.



Thus $EH/HG = AC/AF$.

But since $AF > D$ (or CB),

$$AC/AF < AC/CB.$$

Therefore, *componendo,*

$$EG/GH < AB/D.$$

Hence EG, GH are two lines satisfying the given condition.

6. On the proof reproduced above we shall now make the following comments:

In the figure the given magnitudes are symbolized by the straight lines AB and D, $AB > D$, so that the difference can easily be symbolized (by AC), together with its multiple, which according to Archimedes' lemma should exceed D. But all this is only a symbolic representation to enable us to retain the magnitudes in question in our minds. Beside is represented a real, straight line, on which the required segments should be measured.

In the study of Archimedes it is very important to insist on this distinction between ordinary magnitudes and segments of lines.

Proposition 2, with which we are here concerned, finds one of its most important applications in the proof that the surface O of the sphere = 4 times the great circle c (Prop. 33); the proof is given indirectly by showing that O can be neither greater nor less than 4c, in both cases by making use of Prop. 2 applied to the case where the given magnitudes are O and 4c, which of course Archimedes does not think of replacing by "proportional lines".

7. It is true that a contrast to the above remarks is found in a curious fact, viz. the following:

As the Greek text has come down to us (HEIBERG's Archimedes edition I, pp. 14-15) it contains a reference to Euclid's *Elements* I,2 occasioned by the introduction of the difference shown in the figure (on the above figure: AC) between the given magnitudes AB and D. As if this difference could have anything to do with the construction mentioned in Euclid I,2 (which, by the way, at any rate is an error for I,3)! In a footnote Heiberg quotes a passage from PROCLUS, from which it should appear that the Archimedes texts already at that time contained the same reference to Euclid. But the reference is at any rate quite naïve and must have been inserted by an inexpert copyist .

When in a treatise from 1909[2] ZEUTHEN by the mentioned reference is led to give the following general statement of the basis of Archimedes' ideas: "ein Verhältniss kann immer als das Verhältniss zweier Strecken dargestellt werden," then this is quite incomprehensible. *Archimedes can have meant no such thing.* Nowhere has he made any proposition to this effect, and by means of the propositions he has made he cannot prove it.

If this was the case he might have saved himself the trouble of his considerations concerning the proof of Prop. 2: He might have been content with a reference to the effect that the two magnitudes were proportional to two segments of a line, and then simply make the greater of these a little less.

8. But we proceed with our discussion of the proof of Prop. 2. The next point to be mentioned is the following:

From $AF > BC$ we conclude that $AC/AF < AC/BC$.

The rule here applied we express by simpler notation thus:

From $a > b$ follows $c/a < c/b$.

It has been taken over from Eudoxus' theory of proportions, and as mentioned above (3) had there been proved by the fact that we could

[2] *Über einige archimedische Postulate* (Archiv f. d. Geschichte der Naturwissenschaften und der Technik, I, 1909).

find an integer n so that $n(a-b) > c$, which follows from Eudoxus' axiom so long as a and b are such magnitudes that their difference proves to be a magnitude of the same kind as those given; but when a and b are magnitudes of a more general kind, as here in Archimedes' theory of magnitudes, the mentioned condition necessarily leads to the formulation of Archimedes' lemma.

9. The last question we shall mention in connection with the proof of Prop. 2 is the transition from the inequality

$$EH/HG < AC/BC,$$

to the inequality

$$EG/GH < AB/D,$$

or with simpler notation

$$\text{from } a/b < c/d$$
$$\text{to } (a+b)/b < (c+d)/d.$$

Of this conclusion Archimedes gives no proof, and it is not mentioned in the Elements (in Euclid is only mentioned the corresponding proposition with the sign of equation, Euclid V, 18). In EUTOCIUS' commentary is given a proof which we shall reproduce here:

We first find a magnitude x so that

$$b/a = d/x, \text{ or } a/b = x/d,$$

from which follows according to Eucl. V,18

$$(a+b)/b = (x+d)/d.$$

But $a/b < c/d$, consequently $x/d < c/d$, $x < c$,

from which again follows

$$x+d < c+d, \qquad (x+d)/d < (c+d)/d,$$

and as

$$(x+d)/a = (a+b)/b$$

it finally follows that

$$(a+b)/b < (c+d)/d, \text{ q.e.d.}$$

A similar proof is found in PAPPUS.

To these proofs can be objected that they do not answer Archimedes' purpose, as they are based on the existence of a fourth proportional x corresponding to three magnitudes b, a, d:

$$b/a = d/x,$$

in the case when the magnitudes on one side (b and a) are segments of lines, while the magnitudes on the other side (d and x) are magnitudes in general, (or vice versa). The existence of such a fourth proportional is, however, outside Archimedes' assumptions and cannot be proved by these.

This consideration leads to such questions as the above: whether we can presuppose the existence of segments of lines proportional to two given general magnitudes, a presupposition which Archimedes has not formulated and in the whole of his exposition of the general theory of magnitudes (in his treatise *On the Sphere and Cylinder* and in later works) clearly enough endeavoured to avoid, in which he also succeeded completely.

Thus Eutocius' and Pappus' proofs serve as commentaries on Archimedes' last step in the proof of Prop. 2. On closer examination, however, we see that no commentary is needed. The proposition required follows directly from the definition of unequal ratios:

From $a/b < c/d$, i.e. $ma < nb$, $mc \gtreqless nd$,

follows $m(a+b) < (m+n)b$, $\qquad m(c+d) \gtreqless (m+n)d$,

consequently $(a+b)/b < (c+d)/d$,

which proves the proposition.

Apparently Archimedes has thought it superfluous to give this proof.

10. To throw more light on the fundamental questions concerning the understanding of Archimedes' theory of magnitudes to which I have called attention in the preceding pages we shall with modern aids construct the following algebraic example: We build up a coordinate-geometry with Pythagorean metric where the coordinates do not include all real numbers, but only all real, algebraic numbers. In this geometry it is possible to establish a theory of magnitudes in which all Archimedes' propositions are valid. But in this geometry no segment of a straight line exists = the circumference of a circle. Hence follows further that given a segment of a line l and the circumference of a circle c, two segments of lines, a and b, proportional to l and c do not exist, as in that case the fourth proportional to the segments of lines a, b, l would define a segment of line = c.

But this proves that Archimedes' theory of magnitudes neither explicitly nor implicitly contains any proposition to the effect that there always exist two lines proportional to two given general magnitudes, or that there always exists a fourth proportional to three given magnitudes.

11. That by his theory of magnitudes Archimedes is unable to prove the existence of a line equal to a given circumference of a circle of course also follows from the above. But that he—on purpose—planned his theory of magnitudes in such a way that he did not beforehand incorporate any assumption concerning the existence of such a segment of line is manifest from his treatise *On Spirals,* Prop. 4, which runs as follows (after Heath):

"Given two unequal lines, viz. a straight line and the circumference of a circle, it is possible to find a straight line less than the greater and greater than the less:

For, by the Lemma, the excess can, by being added a sufficient number of times to itself, be made to exceed the lesser line.

Thus e.g. if $c > l$ (where c is the circumference of the circle and l the length of the straight line), we can find a number n such that.

$$n\ (c-l) > l.$$

$$\text{Therefore } c-l > \frac{l}{n},$$

$$\text{and } c > l + \frac{l}{n} > l.$$

Hence we have only to divide l into n equal parts and add one of them to l. The resulting line will satisfy the condition."

If here Archimedes had taken c to be equal to a definite straight line, there would not be any reason to set up the above proposition as in that case the statement would be obvious.

12. It is, however, interesting to see where he finds a later use for the proposition. He does so when writing on the tangent of the spiral, Prop. 18, where it is stated that (in modern usage) the polar subtangent OB at the point A at the end of the first complete turn of the spiral OA is equal to the circumference of a circle with the radius OA. The proof (not repeated here) it to the effect that the polar subtangent can be neither, greater nor less than the said circumference.

Thus the existence of the spiral tangent, which Archimedes evidently does not doubt, implies the existence of a straight line = the circumference of the said circle. It seems that from the moment this fact was before him Archimedes here saw a certain supplementary basis for further investigations on the rectification and quadrature of the circle, such as he made later in his treatise *On the Measurement of the Circle.*

It is true that after the discovery and deciphering of Archimedes' *Method* in 1906 Heiberg and Zeuthen, induced by a certain passage in it, were inclined—in contrast to their former views—to come round to the view that the treatise *On the Measurement of a Circle* must be older than the treatise *On the Sphere and Cylinder.* The passage runs as follows in Heiberg's translation[3]:

"Durch diesen Lehrsatz, dass eine Kugel viermal so gross ist als der Kegel, dessen Grundfläche der grösste Kreis, die Höhe aber gleich dem Radius der Kugel, ist mir der Gedanke gekommen, dass die Oberfläche einer Kugel viermal so gross ist als ihr grösster Kreis, indem ich von der Vorstellung ausging, dass, wie ein Kreis einem Dreieck gleich ist, dessen Grundlinie die Kreisperipherie, die Höhe aber dem Radius des Kreises gleich, ebenso ist die Kugel einem Kegel gleich, dessen Grundfläche die Oberfläche der Kugel, die Höhe aber dem Radius der Kugel gleich."

But the starting-point of the conception mentioned here, i.e. the relation between the circumference and the area of the circle, was surely of old so familiar at Archimedes' time that it can at any rate be traced as far back as the studies of the quadratrix, so that the fact that it is mentioned in the connection referred to above should not be any strong motive for us to date it together with or before the Method.

It seems to carry greater weight—besides the above consideration in the treatise *On Spirals*—that not only have some introductory passages on inscribed and circumscribed polygons from the theory of proportion in the treatise *On the Sphere and Cylinder* (under Prop. 1 and Prop. 6) been used in the treatise *On the Measurement of a Circle,* but one of these passages (the last under Prop. 6): "that it is possible to circumscribe such a polygon about the circle that its area exceeds that of the circle by a magnitude less than any given area", has been proved in both treatises, but in a simpler way in the treatise *On the Measurement of a Circle* than in the other treatise.

[3] Heiberg-Zeuthen, *Eine neue Schrift des Archimedes,* Bibliotheca Mathematica, Dritte Folge, VII, 1907, S. 328.

13. As mentioned above Archimedes has no possibility of proving the existence of a straight line = a given circumference of a circle by means of his propositions of magnitude. But surely he has realized the fact that while the ratio of the surface of the sphere to the great circle could be expressed by the simple integer 4, the ratio of the circumference of the circle to the diameter was of a much more complicated nature, so that it could only be described by the use of less numbers and greater numbers.

But his investigations in his theory of magnitudes border so closely on modern existence proofs that the instrument that might lead direct to these is of a purely formal nature: the extension of the Eudoxean theory of the proportions of segments of straight lines so that it also includes "non-terminated segments of lines". The latter conception is arrived at in the following way: If on a segment AB of a straight line are given a series of segments AA_1, AA_2, ..., each of them forming part of the following one and there exists no segment of a line that is exhausted by this series, then the set of points they comprise defines a "non-terminated segment of line."

All terminated and non-terminated segments of lines form a field of magnitude for which the whole of the Eudoxean theory of proportions is easily proved to be valid as soon as we introduce the definitions of Sum, Difference, Greater, and Less that naturally suggest themselves. And this takes us direct from the ancient to the modern theory of magnitudes.