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Euler's Troublesome Series: An Early Example of the Use of Trigonometric Series

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This paper discusses the role of the series expansion of $(1 - g \cos \omega)^{-\mu}$ in the works of Leonhard Euler. Two of his papers are considered in detail, his 1748 prize-winning essay on Saturn and Jupiter to the Paris Academy, and his 1756 prize-winning essay, also to the Paris Academy, on planetary perturbations. A close examination of these works indicates that Euler was more concerned with convergence issues than he traditionally has been credited with being. © 1993 Academic Press, Inc.

Cet article discute le rôle des développements en série de $(1 - g \cos \omega)^{-\mu}$ dans les oeuvres de Leonhard Euler. Deux de ses articles sont analysés en détail, la pièce au sujet de Saturne et Jupiter qui a remporté le prix de l'académie de Paris en 1748 et celle au sujet des perturbations planétaires qui a remporté le prix de l'académie de Paris en 1756. Un examen attentif de ses oeuvres montre que Euler était plus interessé aux questions concernant la convergence qu'on a cru. © 1993 Academic Press, Inc.

Der Aufsatz behandelt die Rolle der Reihenentwicklung $(1 - g \cos \omega)^{-\mu}$ in den Arbeiten Leonhard Eulers. Es werden zwei seiner Werke genau erörtert, sein von der Pariser Akademie preisgekrönter Aufsatz von 1748 über Saturn und Jupiter und sein ebenfalls von der Pariser Akademie preisgekrönter Aufsatz von 1756 über Planetenstörungen. Eine sorgfältige Untersuchung dieser Werke zeigt, daß sich Euler mehr mit Konvergenzfragen befasst hat als man von ihm gewöhnlich geglaubt hat. © 1993 Academic Press, Inc.

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I. INTRODUCTION

Although it took until the 19th century for a rigorous theory of convergence to become established, it is well known that infinite series were used by 17th and 18th century mathematicians. Many believed that the earlier mathematicians did not bother with questions of convergence but simply manipulated them formally. This last point was presented by Knopp in his famous work on infinite series: "Practically the whole of the 19th century was required to establish the convergence tests set forth in the preceding sections and to elucidate their meanings. . . . How great a distance had to be traversed before this point could be reached is clear if we reflect that Euler never troubled himself at all about questions of

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convergence; when a series occurred, he would attribute to it, without any hesitation, the value of the expression which gave rise to the series'' [Knopp 1928, 298].

The same view is echoed by Carl Boyer in his work on the history of calculus as: ''If $(x + o)^n$ is to be expanded by the binomial theorem, the number of terms will be infinite for values of n which are not positive integers. No conclusion can in general be drawn from an application of the theorem unless the series is convergent, but neither Newton nor his successors for a century later fully appreciated the need for investigations into the question of convergence'' [Boyer 1949, 207].

Contrary to Knopp's statement Leonhard Euler (1707–1783) did indeed ''trouble himself with questions of convergence,'' though he may not have ''fully appreciated'' them according to our present understanding. If we use the distance analogy of Knopp, Euler stood near the beginning of a long road which led to our present understanding of convergence, but he was very aware of and did anticipate many questions of convergence associated with trigonometric series expansions.

Euler made major contributions to the field we now call celestial mechanics [1], and was one of the first investigators into the three-body problem. In a prize-winning essay to the Paris Academy on the inequalities of Saturn and Jupiter [Euler 1749b], Euler needed to determine the integral of the term $(1 - g \cos \omega)^{-3/2}$, where g is a constant near $\frac{4}{5}$. He succeeded in determining this integral by expanding the term into a trigonometric series and integrating it term by term. This was well before Fourier and his analysis.

II. BACKGROUND—THE INEQUALITIES OF SATURN AND JUPITER

Euler addressed the Berlin Academy of Sciences in June 1747 concerning Newton's universal law of gravitation. He noted the differences between recent observations and those which were computed according to the planetary theory of Newton and Kepler. Quoting directly from his main argument:

> The theory of Astronomy is therefore still much more removed from the degree of perfection to which it has been thought to be already carried. Because if the forces, by which the Sun acts upon the Planets, and the latter upon each other, were exactly in the inverse ratio of the squares of the distances, they would be known, and consequently the perfection of the theory would depend on the solution of this problem: *That the forces by which a Planet is moved being known, the motion of this Planet is determined.* [Euler 1749a, 6; Waff 1975, 59]

This 1747 paper formed the basis for Euler's later work on the three-body problem. The problem of a body orbiting around a fixed center is treated in a very general manner, including several modifications to Newton's inverse square law of gravitational attraction. In problems 5 and 6 of the paper Euler derives the equations of motion of a body orbiting about a fixed center by means of an arbitrary attractive force.

There were two serious problems left unresolved by Newton. The first was the motion of the lunar apsides of about 3°3′ per revolution and the second the action of Jupiter on Saturn. The question was whether the Newtonian law of gravitation could successfully explain these two anomalies or should other forces, or other force laws be considered. In 1747 Euler definitely believed the latter; however, his further investigations into both of these three-body problems led him to consider only the inverse square law.

Euler wrote first on the inequalities of Saturn and Jupiter in response to the prize offered by the Royal Academy of Science of Paris for the year 1748. He states in his introduction to the work:

> The Royal Academy of Sciences of Paris, proposed as a subject for the prize of the year 1748, a theory of Saturn and Jupiter, by which one could explain the inequalities of the two planets which is provided by their mutual cause, principally about their conjunction.
>
> We know, first of all, that there is no doubt, that the Royal Academy is of the view that the theory of Newton, founded on universal gravitation, which is found to be quite admirably well in accord with all of the celestial motions, that those which are the inequalities which are discovered in the motions of the planets, one is boldly able to maintain, that the mutual attraction of the planets is the cause. Therefore as the Astronomers had perceived the various inequalities in the motion of Saturn, one concludes, very likely, that they are caused by the force with which this planet is attracted toward Jupiter which not only is closest to Saturn, but also exceeds it in mass, and by consequence in attractive force all of the other planets together, such that their effects are indefinitely small compared to that of Jupiter. For the same reason, the force of Saturn on Jupiter so exceeds that of all of the other planets, that to determine the disturbances to which the motion of Saturn and Jupiter are subjected, one can without error, neglect the forces of the other planets.
>
> Now following this theory, the cause of the inequalities which the Astronomers have observed in the motions of Saturn and Jupiter, is made known, and in order to answer the proposed question, one will have only to determine the motions of three bodies which are mutually attracted in ratios composed of their masses, and by the inverse square ratio of their distances, and then put in place of one of the three bodies the Sun, and the bodies Saturn and Jupiter in lieu of the other two. By this, one sees the question proposed is reduced to the solution of a problem purely of mechanics: but it is necessary to admit that this problem is one of the most difficult ones of mechanics and hence one must not seek a perfect solution, until much more progress is made in analysis. [Euler 1749b, 45]

The four differential equations which described the motion of Jupiter and Saturn were determined by Euler, by applying the results of his 1747 paper, to be

$$\text{I.}\quad ddz - zd\varphi^2 = -a^3d\zeta^2[(1+\nu)\cos\psi^3/z^2 + nz/v^3 + n\cos\omega/y^2 - ny\cos\omega/v^3]$$

$$\text{II.}\quad 2dzd\varphi + zdd\varphi = -na^3d\zeta^2\sin\omega(1/y^2 - y/v^3)$$

$$\text{III.}\quad d\pi = [(na^2d\zeta^2\sin(\varphi-\pi)\sin(\vartheta-\pi))/(zd\varphi)][1/y^2 - y/v^3]$$

$$\text{IV.}\quad d\log\tan G = [(na^3d\zeta^2\cos(\varphi-\pi)\sin(\vartheta-\pi))/(zd\varphi)][1/y^2 - y/v^3],$$

where z is the shortened distance from Saturn to the Sun, φ is the longitude of Saturn, π is the longitude of the ascending node, G is the inclination of the orbital planes, ϑ is the longitude of Jupiter, ψ is the latitude of Saturn, ν is the mass of Saturn divided by the mass of the Sun with a value of 1/3021, a is the mean distance of Jupiter to the Sun, n is the mass of Jupiter divided by the mass of the Sun and equal to 1/1067, ζ is the mean anomaly of Jupiter, y is the distance of

Jupiter to the Sun, ω is the elongation of Saturn and Jupiter, and v, which is the distance between Saturn and Jupiter, is equal to $(y^2 + z^2/\cos^2\psi - 2yz \cos \omega)^{1/2}$ [Euler 1749b, 58] [2].

In Section III of the essay Euler begins to solve these equations by reducing the problem with simplifying assumptions. First assume that the motion of both planets occur in the same plane. This reduces the number of equations from four to just the first two. It also implies that $\psi = 0$ and that $\cos \psi = 1$, which simplifies the first equation and reduces v to $(y^2 + z^2 - 2yz \cos \omega)^{1/2}$. Next assume that the orbit of Jupiter is a circle; then y can be replaced by its mean distance a. Now we have the following equations:

$$\text{I.}' \quad ddz - zd\varphi^2 = -a^3 d\zeta^2[(1 + \nu)/z^2 + nz/v^3 + (n \cos \omega)/a^2 - na \cos \omega/v^3].$$

$$\text{II.}' \quad 2dzd\varphi + zdd\varphi = -na^3 d\zeta^2 \sin \omega(1/a^2 - a/v^3).$$

Euler next assumes that if Jupiter were not present, then the (unperturbed) orbit of Saturn would be a circle; i.e., its eccentricity would be zero. He states that the rate of change of Saturn's longitude is approximately proportional to the rate of change of Jupiter's longitude. The difference is represented by a term, $n\ dx$, which accounts for the effect of Jupiter on Saturn; i.e., $d\varphi = m\ d\zeta + n\ dx$, where m represents the constant of proportionality and the term $n\ dx$ is small in comparison to the term $m\ d\zeta$. Next Euler considers z, the shortened distance of Saturn to the Sun. He notes that without the effect of Jupiter z would be equal to f, the mean Saturn–Sun distance, but with Jupiter's effect, $z = f(1 + nr)$, where the term nr is small. Euler further states that the terms nr and $n\ dx$ depend uniquely on the angle ω, the Saturn–Jupiter elongation. Since $n\ dx$ is small, he uses the relation $d\varphi = m\ d\zeta$, and it follows that the rate of change of the elongation is $d\omega = (1 - m)\ d\zeta$. He then defines $f = \lambda a$, $g = 2\lambda/(1 + \lambda^2)$, and $h = \lambda(1 + \lambda^2)^{3/2}$. The term g can be seen to be equal to $2af/(a^2 + f^2)$, v is now equal to $(a^2 + f^2 - 2af \cos \omega)^{1/2}$, and $(1 + nr)^{-1/2}$ is approximately equal to $(1 - 2nr)$. These relations permit Euler to rewrite the equations as

$$\text{I.}'' \quad m^2\ d\zeta + 2mn\ dx + m^2 nr\ d\zeta - n\ ddr/d\zeta = (1 + \nu)\ d\zeta/\lambda^3 - 2nrd\zeta/\lambda^3 + nd\zeta \cos \omega/\lambda + nd\zeta(\lambda - \cos \omega)/(h(1 - g \cos \omega)^{3/2})$$

$$\text{II.}'' \quad 2mdr + ddx/d\zeta = -\ d\zeta \sin \omega/\lambda + d\zeta \sin \omega/(h(1 - g \cos \omega)^{3/2}).$$

[Euler 1749b, 59–60]

In order to integrate the two equations Euler notes that one must deal with the integral of the term $(1 - g \cos \omega)^{-3/2}$. A closed form expression for this integral is not possible, and so he makes a significant mathematical aside on how to obtain this in a series expansion.

III. THE FIRST APPEARANCE OF THE TRIGONOMETRIC SERIES

The main impedement to finding the solution of these differential equations (I″, II″) of motion was given by Euler as:

> To take advantage of these equations, the greatest difficulty is found with the irrational formula $(1 - g \cos \omega)^{-\mu}$ which is not possible to resolve into a convergent series, seeing that g is near to $\frac{4}{5}$ [*laquelle ne se peut résoudre dans une suite convergente, vû que la valeur de g est environ* = $\frac{4}{5}$]. [Euler 1749b, 60]

The series he refers to arises by expanding the term $(1 - g \cos \omega)^{-\mu}$ using the binomial expansion. Since Euler does not give us a definition of the word "convergence," we will examine how he actually uses these series, rather than impose later meanings onto his words. The rate of convergence of the above series is dependent on the value g, which is defined in terms of the mean distances from the sun of the two planets. Indeed to obtain an approximate error of less than 0.001, without making any assumption about the value of $\cos \omega$ other than that it is less than 1 in absolute value, when g is set to 0.8 one would need to compute at least 30 terms of the series. Thus, the series is for practical purposes *not convergent*. Euler continues:

> This circumstance at first led me to believe that retaining this irrational formula in the calculations would render the solution almost impractical, seeing that one must discover the integral values by the measurement of the area of curved lines; which gives a very laborious approximation, and certainly many steps. [Euler 1749b, 60]

Euler has rejected the approximation method "area under curved lines," which is a numerical technique to obtain an approximate value for the integral, in favor of developing a mathematical approach of solving the problem.

> It is true that the last equation . . . could be integrated were it not that one has to resolve the irrational formula $(1 - g \cos \omega)^{-3/2}$; but this integration hardly helps in the first equation, unless one wishes to resort to calculating the area under curved lines, a method which, although it is practical in the present hypothesis, is not of any use, when one will have to consider the eccentricity of one or the other of the orbits. This circumstance obliges me to make a digression about the formula $(1 - g \cos \omega)^{-3/2}$, which I consider in a more general form as follows, $(1 - g \cos \omega)^{-\mu}$ . . . whose resolution, following ordinary rules is:
>
> $$\begin{aligned}(1 - g \cos \omega)^{-\mu} = 1 &+ \mu/1 \cdot g \cos \omega \\ &+ \mu(\mu + 1)/(1 \cdot 2) \cdot g^2 \cos^2 \omega + \mu(\mu + 1)\mu(\mu + 2)/(1 \cdot 2 \cdot 3) \\ &\cdot g^3 \cos^3 \omega + \ldots \text{etc.}\end{aligned}$$
>
> but this series is not suitable for my purpose, in as much as it is not sufficiently *convergent* (*tant parce qu'elle n'est pas assez convergente*), since it contains powers of $\cos \omega$. As for the last inconvenience, one can remedy it by reducing the powers of the cosine of the angle ω, to the cosines of multiples of the angle, by means of the following rules, founded on those of the Trigonometry:
>
> $$\cos \omega = \cos \omega$$
>
> $$2 \cos^2 \omega = \cos 2\omega + 1/2 \cdot 2/1$$
>
> $$4 \cos^3 \omega = \cos 3\omega + 3/1 \cos \omega$$
>
> $$8 \cos^4 \omega = \cos 4\omega + 4/1 \cos 2\omega + 1/2 \cdot 4/1 \cdot 3/2$$
>
> etc.
>
> where the law of the progression is evident, with the remark that the absolute or constant terms are all multiplied by 1/2.

Having made these substitutions, as much as the expression becomes very complicated, we can assume that it becomes:

$$(1 - g \cos \omega)^{-\mu} = A + B \cos \omega + C \cos 2\omega + D \cos 3\omega + E \cos 4\omega$$
$$+ F \cos 5\omega + G \cos 6\omega + H \cos 7\omega + \text{etc.}$$ [Euler 1749b, 61]

The coefficients of this cosine series are again infinite series themselves which must all be evaluated. To simplify this evaluation he proceeds to derive a method of determining all the coefficients once the values of A and B are known.

For let

$$s = A + B \cos \omega + C \cos 2\omega + \text{etc.}$$
$$= (1 - g \cos \omega)^{-\mu}.$$

He then derives the term ds/s by differentiating

$$\log s = -\mu \cdot \log(1 - g \cos \omega)$$

to get

$$ds/s = -\mu\, g \sin \omega\, d\omega/(1 - g \cos \omega),$$

which gives

$$(1 - g \cos \omega)\, ds/d\omega + \mu g\, s \sin \omega = 0. \qquad \text{(A)}$$

Noting that

$$ds/d\omega = -B \sin \omega - 2C \sin 2\omega - 3D \sin 3\omega \ldots$$

and using the relations

$$\sin n\omega \cdot \cos \omega = (1/2) \sin(n + 1)\omega + (1/2) \sin(n - 1)\omega$$

and

$$\cos n\, \omega \cdot \sin \omega = (1/2) \sin(n + 1)\omega - (1/2) \sin(n - 1)\omega,$$

Eq. (A) reduces to a sine series as follows:

$$(-B + gC + \mu gA - \mu gC/2) \sin \omega + (-2C + gB/2 + 3gD/2 + \mu gB/2 - \mu gD/2) \sin 2\omega + \ldots = 0.$$

This sine series can be identically zero only if all of the coefficients vanish, hence

$$C = (2B - 2\mu gA)/[(2 - \mu)g], \qquad D = (4C - (\mu + 1)gB)/[(3 - \mu)g],$$

etc.

In order to determine the values of A and B Euler states: ". . . I have discovered a special method, to determine these sums sequentially: it is founded on the division of a right angle into as many parts as one wants because the sine of these parts provides a property which has a good agreement with the series, which then gives the values of A and (1/2) B. Let q designate a right angle, then I say that the following expressions approach more and more these values" [3].

$$\text{I.} \qquad A = +(1/2)(1 - g \sin q/2)^{-\mu} + (1/2)(1 + g \sin q/2)^{-\mu}.$$

$$(1/2)B = +(1/2) \sin (q/2)(1 - g \sin q/2)^{-\mu} - (1/2) \sin(q/2)(1 + g \sin q/2)^{-\mu}.$$

$$\text{II.} \qquad A = +(1/4)(1 - g \sin q/4)^{-\mu} + (1/4)(1 - g \sin 3q/4)^{-\mu} + (1/4)(1 + g \sin q/4)^{-\mu} + (1/4)(1 + g \sin 3q/4)^{-\mu}.$$

$$(1/2)B = +(1/4) \sin (q/4)(1 - g \sin q/4)^{-\mu} + (1/4) \sin(3q/4)(1 - g \sin 3q/4)^{-\mu} - (1/4) \sin(q/4)(1 + g \sin q/4)^{-\mu} - (1/4) \sin(3q/4)(1 + g \sin 3q/4)^{-\mu}.$$

$$\text{III.} \qquad A = +(1/6)[(1 - g \sin q/6)^{-\mu} + (1 - g \sin 3q/6)^{-\mu} + (1 - g \sin 5q/6)^{-\mu} + (1 + g \sin q/6)^{-\mu} + (1 + g \sin 3q/6)^{-\mu} + (1 + g \sin 5q/6)^{-\mu}]$$

$$(1/2)B = +(1/6)[\sin(q/6)(1 - g \sin q/6)^{-\mu} + \sin(3q/6)(1 - g \sin 3q/6)^{-\mu} + \sin(5q/6)(1 - g \sin 5q/6)^{-\mu} - \sin(q/6)(1 + g \sin q/6)^{-\mu} - \sin(3q/6)(1 + g \sin 3q/6)^{-\mu} - \sin 5q/6(1 + g \sin 5q/6)^{-\mu}] \text{ [Euler 1749b, 65].}$$

We can reconstruct the method by which Euler derived the sequence for approximating the coefficients A and B. We use radian measure for clarity in this reconstruction, although it should be understood that Euler used the older degree notation. We present the argument for the coefficient A; the case for B is similar. The argument is dependent upon the coefficients A, B, C, D, E, etc. eventually becoming small.

Euler begins with the equation

$$(1 - g \cos \omega)^{-\mu} = A + B \cos \omega + C \cos 2\omega + D \cos 3\omega + E \cos 4\omega + F \cos 5\omega + G \cos 6\omega + H \cos 7\omega + \text{etc.}$$

If we replace the argument ω with $\omega + \pi$ and use the identities

$$\cos(\omega + \pi) = \cos \omega \cos \pi - \sin \omega \sin \pi = -\cos \omega$$

and

$$\cos(n(\omega + \pi)) = \cos n\omega \cos(n\pi) - \sin n\omega \sin(n\pi) = (-1)^n \cos n\omega,$$

we then get the equation

$$(1 + g \cos \omega)^{-\mu} = A - B \cos \omega + C \cos 2\omega - D \cos 3\omega + E \cos 4\omega - F \cos 5\omega + G \cos 6\omega - H \cos 7\omega + \text{etc.}$$

If we now add the two equations and divide by 2 we get

$$(1/2)[(1 - g \cos \omega)^{-\mu} + (1 + g \cos \omega)^{-\mu}] = A + C \cos 2\omega + E \cos 4\omega + G \cos 6\omega + I \cos 8\omega. \ . \ . \ .$$

Now replacing ω with $\omega - \pi/2$ and using the identities

$$\cos(\omega - \pi/2) = \cos \omega \cos \pi/2 + \sin \omega \sin \pi/2 = \sin \omega,$$

$$\cos(2n(\omega - \pi/2) = \cos 2n\omega \cos 2n \cdot \pi/2 + \sin 2n\omega \sin 2n \cdot \pi/2 = (-1)^n \cos 2n\omega,$$

we get the following, which we will refer to as (*):

$$(1/2)[(1 - g \sin \omega)^{-\mu} + (1 + g \sin \omega)^{-\mu}] = A - C \cos 2\omega + E \cos 4\omega - G \cos 6\omega + I \cos 8\omega - \ . \ . \ . \ .$$

If we evaluate this equation at $\omega = \pi/4$ we get:

$$(1/2)[(1 - g \sin \pi/4)^{-\mu} + (1 + g \sin \pi/4)^{-\mu}] = A - C \cos \pi/2 + E \cos \pi - G \cos 3\pi/2 + I \cos 2\pi + \ . \ . \ . \ . = A - E + I \ . \ . \ .$$

Hence

$$A = (1/2)[(1 - g \sin \pi/4)^{-\mu} + (1 + g \sin \pi/4)^{-\mu}] + E - I + M \ . \ . \ . \text{ etc.}$$

This expression approximates A with an error approximately equal to E. This gives us Euler's approximation I to A.

The technique is now to divide $\pi/2$ into four parts, i.e., $\pi/8$, $\pi/4$, $3\pi/8$. First evaluate (*) at $\pi/8$,

$$(1/2)[(1 - g \sin \pi/8)^{-\mu} + (1 + g \sin \pi/8)^{-\mu}] = A - C \cos \pi/4 + E \cos \pi/2 - G \cos 3\pi/4 + I \cos \pi + \ . \ . \ . \ ,$$

and evaluate (*) at $3\pi/8$,

$$(1/2)[(1 - g \sin 3\pi/8)^{-\mu} + (1 + g \sin 3\pi/8)^{-\mu} = A - C \cos 3\pi/4 + E \cos 3\pi/2 - G \cos 9\pi/4 + \ . \ . \ . \ .$$

We add these terms together and divide by 2 to get

$$\begin{aligned}
&(1/4)[(1 - g \sin \pi/8)^{-\mu} + (1 + g \sin \pi/8)^{-\mu} \\
&\quad + (1 - g \sin 3\pi/8)^{-\mu} + (1 + g \sin 3\pi/8)^{-\mu}] \\
&= A - C(\cos \pi/4 + \cos 3\pi/4) + E(\cos \pi/2 + \cos 3\pi/2) \\
&\quad - G(\cos 3\pi/4 + \cos 9\pi/4) + I(\cos \pi + \cos \pi) \\
&\quad - K(\cos 5\pi/4 + \cos 7\pi/4) + M(\cos 3\pi/2 + \cos \pi/2) \ldots \\
&= A - C \cdot 0 + E \cdot 0 - G \cdot 0 + I \cdot 0 - K \cdot 0 + M \cdot 0 + O \cdot 0 - Q \ldots
\end{aligned}$$

Hence

$$\begin{aligned}
A = {} & (1/4)[(1 - g \sin \pi/8)^{-\mu} + (1 + g \sin \pi/8)^{-\mu} + (1 - g \sin 3\pi/8)^{-\mu} \\
& + (1 + g \sin 3\pi/8)^{-\mu}] - Q + \ldots \text{etc.}
\end{aligned}$$

Therefore we can approximate A with an error less than Q. This gives us Euler's approximation II to A. Continuing in this manner we can get a more and more accurate determination of A [4] [Golland 1991, 35–37].

Euler uses the value 0.8405 for g and he calculates a value of 3.21789 for A by dividing the right angle into 10 parts. For comparison the following table was calculated, using this same value for g, as in Eqs. I, II, III:

Division of right angle into n parts	A
2 parts	2.18346
4 parts	3.08396
6 parts	3.20336
8 parts	3.21616
10 parts	3.21746
12 parts	3.21758
14 parts	3.21759
16 parts	3.21759

The correct value for A is calculated as

$$\begin{aligned}
A &= (1/2)\pi \int_{-\pi}^{\pi} (1 - g \cos \omega)^{-3/2}\, d\omega \\
&= 3.217598.
\end{aligned}$$

The difference between our calculated value for $n = 10$ and that of Euler is most probably due to the number of significant digits carried in the calculations, and to the number of digits used in calculating the cosine function [Euler 1749b, 62–72].

Euler then proceeds to substitute the series expansion into Eqs. I′ and II′ and, by means of term by term integration, obtains the final results. He states that the integrations result in a series which converges faster than the original [Euler 1749b, 67].

IV. EULER'S LATER USE OF THE SERIES

Euler uses this same trigonometric series in a later paper on planetary perturbations which won the Paris prize in 1756 [Euler 1769, 50–51]. In this paper he has generalized the problem from Saturn and Jupiter to that of any two planets, one to be called the perturbed planet and the other the perturbing planet. He is aware that in order to consider all planetary perturbations he would be doing the n-body problem. But he argues there is no loss in generality if one considers the problem to be a sequence of three-body problems, one for each of the remaining planets. The true motion of a given planet would become a sequence of corrections to the results obtained for the previous three body problem.

He has two possible methods for obtaining the actual planetary orbits. Both begin by transforming the four fundamental differential equations (I–IV) into expressions for the differentials of the following six orbital elements; the perturbing planet's longitude; the perturbed planet's semiparameter; the perturbed planet's eccentricity; the perturbed planet's apsidal line; the longitude of the ascending node of the perturbed planet; and the perturbed planet's inclination. At this point either he can solve for the elements by integration as he did in the Saturn and Jupiter paper, or he can use an alternative method which he developed in a paper of 1752 [Euler 1758]. This latter method would require first establishing the elements which describe the initial orbit of the perturbed planet at some point in time and then after a small increment of time modifying the elements using the differential changes as given by the equations. Euler argues that this method will be very laborious since one must constantly apply the differential corrections as often as once an hour or once a day and more importantly, the accuracy of the method is dependent upon the accuracy with which one can determine the initial ellipse. He is concerned that any errors in determining this initial ellipse will tend to grow with each incremental correction. For these reasons, he decides to do the integrations, and he again employs the method of expanding the disturbing terms in the differentials into trigonometric series and integrating term by term [Euler 1769, 45–46].

The differentials of the orbital elements are given by Euler as

$$\begin{aligned}
d\theta &= ad\omega/x^2 \cdot (ap)^{1/2};\\
dp &= -2nMaxd\omega(ap)^{1/2};\\
dq &= na\,d\omega(2M\cos v + N\sin v - (Mq(\sin v)^2)/(1 - q\cos v)) \cdot (ap)^{1/2};\\
d\varphi - dv &= (nad\omega/q)(2M\sin v - N\cos v\\
&\quad + (Mq\sin v\cos v)/(1 - q\cos v))(ap)^{1/2}.\\
d\pi &= -naxyd\omega\sin(\varphi - \pi)\sin(\theta - \pi)(1/z^3 - 1/y^3)(a/p)^{1/2};\\
d\log\tan G &= -naxyd\omega\cos(\varphi - \pi)\sin(\theta - \pi)(1/z^3 - 1/y^3)(a/p)^{1/2}
\end{aligned}$$

[Euler 1769, 41–44],

where the perturbing planet moves along an ellipse defined by $y = c/(1 - e\cos u)$, with

c its semiparameter,
e its eccentricity,
u its true anomaly,
θ its longitude;

the projection of the orbit of the perturbed planet moves along an ellipse $x = p/(1 - q \cos v)$, with

p its semiparameter,
q its eccentricity,
v its true anomaly,
φ its longitude;

a is the mean distance from the sun to the perturbed planet;
ω is the mean anomaly of the perturbed planet;
n is the ratio of the mass of the perturbing planet to that of the sun;
z is the distance between the perturbing planet and the projection of the perturbed planet (the curtate) and is equal to $(x^2 + y^2 - 2xy \cos(\varphi - \theta))^{1/2}$;
M is an expression equal to $y(1/z^3 - 1/y^3) \sin(\varphi - \theta)$;
N is an expression equal to $x/z^3 - y(1/z^3 - 1/y^3) \cos(\varphi - \theta)$.

He notes that the expression for $1/z^3$ is very "troublesome," and wishes to transform it into a convergent series which can be integrated term by term [Euler 1769, 49–50]. He defines $\eta = \varphi - \theta$, $x^2 + y^2 = r^2$, and $s = 2xy/(x^2 + y^2)$, then $z = r(1 - s \cos \eta)^{1/2}$ and hence $1/z^3 = (1 - s \cos \eta)^{-3/2}/r^3$. Thus the expression $(1 - s \cos \eta)^{-3/2}$ expanded in an infinite series will converge since s is less than 1, although the convergence will be "excessively slow" unless $s \cos \eta$ is very small [Euler 1769, 49–50]. After making the same standard trigonometric substitution for the product of the cosines of two angles as he did in the earlier paper, he finds

$$\begin{aligned}(1 - s \cos\eta)^{-3/2} = {} & P + Qs \cos\eta + Rs^2 \cos 2\eta \\ & + Ss^3 \cos 3\eta + Ts^4 \cos 4\eta + \text{etc.}\end{aligned}$$

This allows him to give the following expression:

$$1/z^3 = 1/r^3(P + Qs \cos \eta + Rs^2 \cos 2\eta + Ss^3 \cos 3\eta + Ts^4 \cos 4\eta + \text{etc}).$$

It should be noted that this cosine series expansion for $(1 - s \cos \eta)^{-3/2}$ is different from that used in the earlier Saturn and Jupiter paper. By explicitly having the powers of s appear in the expansion, Euler is able to show that the coefficients P, $(1/2)Q$, $(1/2)R$, $(1/2)S$, . . . etc., which are series expansions in s, are dominated by the geometric series $1 + s^2 + s^4 + s^6 + \ldots$. Since this geometric series has sum equal to $1/(1 - s^2)$, he reasons that multiplying each term of the original series by $(1 - s^2)$ would result in an expansion which would have a sum near 1, and would be easier to compute. Thus he works with the series for $P(1 - s^2)$, $(1/2)Q(1 - s^2)$, $(1/2)R(1 - s^2)$, etc. Once the value for s is known for the particular three body problem, it becomes very easy to recover the values for $P, Q, R, \ldots$ etc. and hence the series expansion for $(1 - s \cos \eta)^{-3/2}$.

V. CONCLUSION

Euler expands various algebraic expressions resulting from transformations of the equations into infinite series, which can be integrated term by term to obtain approximate solutions. The series $(1 - s \cos \eta)^{-3/2}$ plays a key role in this method.

Euler's treatment of perturbation theory from 1747 onward is fundamentally concerned with the expansion of $(1 - s \cos \eta)^{-3/2}$ as a specific case of $(1 - s \cos \eta)^{-\mu}$, which is of interest mathematically as part of his contribution to the theory of convergence. He enters into a significant mathematical discussion of the series $(1 - g \cos \eta)^{-\mu}$ in the Saturn and Jupiter paper. The major attempts to cope with planetary perturbations during the remainder of the eighteenth century by Clairaut, D'Alembert, Lagrange, and Laplace all use this series [Wilson 1985, 18]. However, Euler was the first to investigate it mathematically and use it in perturbation theory.

In the case of Saturn and Jupiter the value of g is approximately $\frac{4}{5}$, which prompts Euler to comment on the convergence of the series: "it is not possible to resolve this into a convergent series, seeing that g is near to $\frac{4}{5}$" (*laquelle ne se peut résoudre dans une suite convergente, vû que la valeur de g est environ* $=$ $\frac{4}{5}$) [Euler 1749, 60]. He seems here to be saying that the series does not converge, but he proceeds as if it did. A more reasonable interpretation of the statement is that he is worried about the rate of convergence of the series.

In the span between these two papers, Euler continued the use of expanding the disturbing functions in terms of trigonometric series. Euler is vague on the question of convergence of the series expansion in his 1748 paper resulting in a very mechanical method, but the 1756 paper demonstrates a more sophisticated understanding of the use and convergence of the series expansion. In the latter paper he directly states that convergence of the series depends on the term $s \cos v$ being less than 1 in magnitude; he reformulates the calculations of the coefficients, $P, Q, R, \ldots$, etc. of the series noting that they are comparable with the geometric series $1 + s^2 + s^4 + \ldots$, thus enabling easier computations; and he states that the magnitude of the term $s \cos v$ determines the rate of convergence of the series approximation in actual calculations.

In neither of these two papers does Euler define what he means by convergence. This is not all that unusual since they were papers on astronomy, and not mathematics. However, Euler published a famous paper on divergent series in 1760, in which he defined convergence and divergence and clearly expressed his ideas on these issues [Euler 1760]. From it we can infer that a convergent series must exhibit three related characteristics. It must be possible to obtain a number which is the "sum"; the existence of a limit without knowing this value would be a useless concept to Euler, a view that remains valid for applications. We must also be able to find this value, which can be by the process of term by term addition. Last, of course, the terms must go to zero. This last condition is necessary but not sufficient for convergence by modern standards.

Euler abandoned his use of trigonometric series after the 1756 paper. Wilson has suggested that his age and loss of eyesight made alternative methods more attractive to him [Wilson 1980]. This cannot be denied, but even this is bound to the rate of convergence of the series. The series, however, became the standard tool of all of the theoreticians who followed, and the problems of convergence which were apparent to Euler remained problems into this century as we can see from Brown's 1896 comment that there was "lack of any certain knowledge on the subject [the convergence of the series]" [Brown 1896, v–vi].

It may have taken until the 19th century for a rigorous theory of convergence to become established, and until the early 20th century for a convergence theory of trigonometric series to be developed. But before such theories could be developed the need for them had to be appreciated. Euler worked in a period when these questions were just becoming apparent. Although he did have difficulties with convergence questions it is apparent that Knopp's criticism was overly harsh. Euler was the first to use the trigonometric series in celestial mechanics, and as his work matured he began to clarify his ideas on convergence. All necessary steps have to be taken before a rigorous theory can be developed. In this regard Euler and his "troublesome" series left a legacy both to celestial mechanics and to analysis.

NOTES

1. This term is due to Laplace; other terms are gravitational astronomy and physical astronomy.
2. More complete discussions of the astronomy and the associated models may be found in [Golland 1991] and [Wilson 1980 and 1985].
3. This translation is intentionally literal since the temptation is to say the expressions "converge" to the values, but Euler does not use the word "converge" in this passage.
4. Problem 34 of [Euler 1768] provided helpful information in this reconstruction.

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