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Leonhard Euler's use and understanding of mathematical transcendence

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Abstract

Leonhard Euler primarily applied the term "transcendental" to quantities which could be variable or determined. Analyzing Euler's use and understanding of mathematical transcendence as applied to operations, functions, progressions, and determined quantities as well as the eighteenth century practice of definition allows the author to evaluate claims that Euler provided the first modern definition of a transcendental number. The author argues that Euler's informal and pragmatic use of mathematical transcendence highlights the general nature of eighteenth century mathematics and proposes an alternate perspective on the issue at hand: transcendental numbers inherited their transcendental classification from functions.

Résumé

Leonhard Euler appliqua le terme «transcendantal» surtout aux quantités qui pouvaient être soit variables, soit fixes. En analysant comment Euler appliqua la transcendance mathématique aux opérations, aux fonctions, aux progressions, et aux quantités fixes tout en considérant la manière dont les définitions mathématiques furent élaborées au XVIIIe siècle, l'auteur évalue l'affirmation qu'Euler a fourni la première définition moderne d'un nombre transcendant. L'auteur soutient que le concept de transcendance mathématique employé par Euler, étant informel et pragmatique, met en évidence le caractère générale des mathématiques du XVIIIe siècle, et l'auteur propose ensuite une perspective différente sur la question : nombres transcendants ont hérité de leur classification des fonctions transcendantes.

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1. Introduction

Leonhard Euler was loosely credited by Paul Erdös and Underwood Dudley [1983, 297] as having provided today's definition of a transcendental number. While their article falls under Ivor Grattan-Guinness's [2004] heritage category (an approach to history concerned primarily with the question "how did we get here?" emphasizing past notions for their similarities to current mathematics), it is clear that Erdös and Dudley are not strongly committed to their claim. In addition to Euler, Johann Lambert and Joseph Liouville have also been credited with establishing the current definition of a transcendental number [Serfati, 2010; Karatsuba, 2009, 77]. This is unusual given there is a century of mathematical development between Euler and Liouville but also in light of Fel'dman and Shidlovskii's [1967, 2] claim that "Each new major achievement in the theory of transcendental numbers is linked with the emergence of a new method." The lack of a clear consensus among historians of mathematics concerning the genesis of the definition of a transcendental number necessitates a closer look at Euler's use and understanding of mathematical transcendence. Today, transcendental numbers are defined as non-algebraic. Algebraic numbers are defined as numbers which are roots of polynomials with integer coefficients. In other words, transcendental numbers are not roots of any algebraic equation $a_0x^n + a_1x^{n-1} + \cdots + a_n = 0$ where $a_i \in \mathbb{Z}$ and $a_0 \neq 0$. Joseph Liouville [1844a,b, 1851] is credited with proving the existence of transcendental numbers by constructing what are now called Liouville numbers. Even though Georg Cantor's [1874] work demonstrated that the set of transcendental numbers is uncountable, relatively few numbers have been proven to be transcendental and, in general, it is quite difficult to do so. Charles Hermite [1873] first proved e is transcendental and Ferdinand von Lindemann [1882] first proved π is transcendental. The transcendence of e^{π} and $2^{\sqrt{2}}$ were part of David Hilbert's problems in his 1900 Paris Address implying that transcendental number theory was part of the mathematical research frontier [Hilbert, 1902]. Euler, however, believed the transcendence of a number was not significant for mathematics. In fact, he used and understood mathematical transcendence in a different manner.

Today's confusion regarding Euler's mysterious application of the term "transcendental" to numbers is to be expected because his convention of doing so seemed to be consistent with implying that something was not algebraic and therefore deceivingly familiar. Today, the transcendental classification hinges on a mathematical property of a number but Euler only declared objects to be transcendental as a generic description. The phrase "transcendental number" was also not used directly by Euler. Instead, he used the phrase "transcendental quantity" because he did not consider transcendental numbers to be "true" numbers [Ferraro, 2004, 41–2]. Euler distinguished between quantity and number [Euler, 1770, 2]. This apparent semantic distinction is worthwhile because quantities were not necessarily interchangeable with numbers [Euler, 1770, 2]. For Euler, quantities were anything that could be increased or decreased and could be constant or variable [Euler, 1748, 1, 1770, 2]. In this sense, single numbers represented specific determinations (measures of magnitude) of variable quantities [Euler, 1770, 2; Ferraro, 2004, 44]. Moreover, Euler held that "... a function itself of a variable quantity is a variable quantity" [Euler, 1748, 3]. Euler's use of the word "quantity" to refer to function is primarily responsible for much of the abovementioned confusion regarding Euler's expression "transcendental quantity" because his most frequent use of it referred not to determined quantities but to functions. It is with regard to functions that Euler's application of "transcendental" could be considered synonymous with "non-algebraic"; albeit, Euler considered some functions algebraic that we would not consider so today such as z^{π} [Euler, 1748, 5]. However, Euler labeled constant quantities "transcendental" if the function describing their relationship to the unity was transcendental. For example, π and e were related to the unity through the transcendental functions of arc-length and the natural logarithm respectively. This paper will demonstrate that mathematical objects, including numbers, inherited their transcendental classification from what Euler called "transcendental operations."

2. Eighteenth century definitions

Understanding the role of definition in eighteenth century mathematics not only places Euler's use and understanding of mathematical transcendence in context but also allows us to evaluate Erdös and Dudley's claim that Euler provided today's definition of transcendental numbers. Strictly speaking, Euler could not have done so because the purpose of eighteenth century mathematical definitions was different than today's [Ferraro, 1999, 102]. Interpreting Euler's definitions as current is not only anachronistic but also misses the opportunity to accentuate implicit dissimilar aspects of eighteenth century mathematical definitions regarding description, ontology, and precision. The primary goal of Euler's definitions was to describe mathematical objects. While ostensibly obvious the ramifications of such an objective are substantial.

Consider the following definition Euler provided for function:

> A function of a variable quantity is an analytic expression composed in any way whatsoever of the variable quantity and numbers or constant quantities. [Euler, 1748, 3]

With Euler, the function concept occupied a central role in analysis. However, Euler's definition of function has more in common with today's concept of formula than today's concept of function. The purpose of this definition was to describe the relationship between quantities resulting in a particular emphasis on operations. As a result, Eulerian analysis is considered today to be algebraic and algorithmic in character [Fraser 1989; Jahnke 2003; Sørensen 2005].

Eulerian analysis is one facet of a larger paradigm of eighteenth century mathematics, algebraic analysis. Craig Fraser [2003, 327] eloquently describes why the eighteenth century style of mathematics is captivating to today's mathematicians and historians:

> … algebraic analysis provides an interesting example of a mature mathematical paradigm that would be replaced by a quite different paradigm in the later development of the subject. The transition from Euler and Lagrange to Cauchy and Weierstrass constituted a profound intellectual transformation in conceptual thought… The case of mathematics is even in some important aspects more striking, because the point of view embodied in the older paradigm retains a certain intellectual interest and validity not found in quite the same way in the discarded theories of older physics.

Fraser has written extensively on algebraic analysis and he [1989, 319] comments

> … [Euler's and Lagrange's] work taken broadly shares an explicit emphasis on the analytical or "algebraic" character of the differential and integral calculus, both as a foundational description and as a theme to unify the different branches of the subject; on the need to separate the calculus from geometry, while continuing to cultivate geometrical and mechanical applications; and on a belief in generality as a primary goal of mathematics.

It is important to call attention to the rationales of algebraic analysis. First, Fraser believes algebraic analysis serves the purpose of unifying different branches of mathematics. Algebraic analysis serves as a manner to investigate mathematics using tools such as infinite series and as such can infiltrate various mathematical topics. Second, given concern over the infinitesimal (such as Bishop Berkeley's polemic attack against it in The Analyst [Berkeley, 1734]) there were attempts to separate calculus from such geometric notions. Again, algebraic analysis served well here. Joseph Louis Lagrange took this to an extreme in his 1788 *Mécanique Analytique* where he declared "One will not find figures in this work. The methods that I expound require neither constructions, nor geometrical or mechanical arguments but only algebraic operations, subject to a regular and uniform course."[1] [Lagrange, 1811, i] The need to separate calculus from geometry did not necessitate an abandonment of geometric and mechanical applications, however. Much of algebraic analysis found its value in its algorithms and therefore its applications. Algorithms were used in algebraic analysis to simplify conceptually challenging notions such as differentials into accessible algebraic rules. As such the strength of an algorithm was found in its broad applicability not narrowing precision. This interpretation was reinforced by Fraser's third rationale that generality was a primary goal of mathematics. This paradigm of algorithmic and algebraic analysis was underscored by an emphasis on operations, particularly algebraic operations. Operations provided the intermediate processes of the algorithms and were done so because simple algebraic operations were fundamentally base to mathematics. Operations were emphasized in Euler's definition of function because operations were the key ingredients to an analytic expression. Operations served as descriptors of the relationship between the variable and constant quantities.

Euler's definition of a determined quantity is needed to understand how he used transcendence and also highlights the descriptive character of his definitions; in this case, the description of the relationship between a quantity and the unit.

> So that the determination, or the measure of magnitude of all kinds, is reduced to this: fix at pleasure upon any one known magnitude of the same species with that which is to be determined, and consider it as the *measure* or *unit*; then, determine the proportion of the proposed magnitude to this known measure. This proportion is always expressed by numbers; so that a number is nothing but the proportion of one magnitude to another arbitrarily assumed as the unit... From this it appears, that all magnitudes may be represented by numbers... [Euler, 1770, 2]

For eighteenth century mathematicians such as Euler, definitions described the nature of mathematical objects which existed *a priori* whereas current definitions constitute the exact nature mathematical objects to be studied [Jahnke, 2003; Ferraro, 1998, 1999; Fraser, 1989, 2003]. It is necessary to draw attention to the ontological implications of Euler's definition of a determined quantity. The mathematical object in this passage is the quantity not the number. Notice that the quantity to be measured was assumed to exist and the role of number here was to describe the proportion of this magnitude to a given unit; i.e. a measure of quantity. Whereas Euler's declaration that "This proportion is always expressed by numbers" might suggest the existence of numbers in some mind-independent realm via some sort of Platonism, his statement that "a number is nothing but the proportion..." indicates

[1] On ne trouvera point de Figures dans cet Ouvrage. Les méthodes que j'y expose ne demandent ni constructions, ni raisonnemens géométriques ou mécaniques, mais seulement des opérations algébriques, assujéties à une marche régulière et uniforme.

that within his ontological framework, a number did not exist without a proportion for it to represent. Therefore Euler understood number only as a relationship between a given magnitude and the unity. This relationship and the concepts of "true" versus "fictitious" numbers will be analyzed in the next section. For now, it is enough to understand that Euler's concepts of quantity and number were different.

Today's concept of number must account for current mathematics such as set theory, cardinality, and the continuum which, obviously, were not concerns of Euler's. It is important to consider Euler's mathematical definitions in their context and contrasting them with modern definitions can highlight the different outlook of eighteenth century mathematics. Euler had no reason to prove that certain numbers existed because they measured *a priori* quantities. Therefore, historical context provides resolution to queries such as the one raised by Erdös and Dudley [1983, 297]:

> It is perhaps curious that [Euler] never proved their [transcendental numbers] existence. The proof of Liouville was well within his reach. Maybe Euler considered the existence of transcendental numbers as self-evident, which by our standards, is certainly not the case.

Erdös and Dudley's speculation that the existence of transcendental numbers was self-evident to Euler accurately reflects the paradigm of algebraic analysis. Since Euler operated within the paradigm of algebraic analysis, Liouville's existence proof was not only unnecessary for Euler it was also not within his reach. Liouville's proof exhibits a century of mathematical development after Euler published his *Introductio*. Liouville was not operating within the paradigm of algebraic analysis and as a result Liouville's practice and methodology differed substantially from Euler's. Moreover, their claim is founded on the premise that Euler understood transcendental numbers the way Liouville did. It is the objective of this paper to demonstrate the opposite.

Noticeably absent from Euler's definition was a way to account for complex numbers. Complex numbers are not things which can be increased or decreased or measure a quantity to a given unit in a manner consistent with Euler's theory of number. Even though one of Euler's most celebrated results is the formula connecting complex numbers to exponential and trigonometric functions, $e^{i\theta} = \cos\theta + i\sin\theta$, his use of imaginary numbers was purely instrumental and the symbol $\sqrt{-1}$ had no meaning for him; the number was "imaginary" or "impossible" [Euler, 1770, 43]. The symbol was simply a tool that highlights how Euler believed in the general applicability and validity of algebra, which is characteristic of eighteenth century mathematics [Fraser, 1989]. Fraser [2003, 319] also comments that "Eighteenth-century confidence in formal mathematics was almost unlimited." Judith Grabiner [1981] preceded Fraser by declaring mherore boldly that "Sometimes it seems to have been assumed that if one could just write down something which was symbolically coherent, the truth of the statement was guaranteed." Even though Euler considered $\sqrt{-1}$ to be purely symbolic with no mathematical meaning at all, his confidence in his algebraic analysis gave validity to algebraically consistent statements containing these imaginary quantities.

The absence of complex numbers from Euler's definition of determined quantities presents another difference between Euler's definitions and today's. His definitions could also be said to lack the completeness that characterizes today's precise definitions. Since Euler's goal of definition was to describe the nature of a mathematical object not constitute the exact nature of it, it was not necessary to exhaust the entire concept being defined [Ferraro, 1999, 104]. Today, however, precise definitions are required due to concerns about existence. Since this was not a concern for Euler, there was no need to be so precise. For instance, certain characteristics of mathematical objects would be taken as obvious or implicit [Ferraro, 1999, 2000]. Niels Hans Jahnke [2003] declares that it would be considered "ridiculously formalistic" to append $x \neq 0$ to the function $\frac{1}{x}$. Today a function maps one set to another and restrictions such as $x \neq 0$ restrict the domain and the applicability of the function. Instead, Euler emphasized that the analytic expression implied the existence of a relation that remained generally valid as the variables changed except possibly at a few exceptional values. In this context we can understand why Euler's definition of function is inconsistent with today's function concept and the emphasis on operations not set mapping is something more akin to today's formula, not function, concept. Since restrictions were not declared in eighteenth century mathematics, good mathematicians "being masters of their discipline… knew how to apply their intuitions" [Sørensen, 2005, 477] and avoided applying their algorithms in improper situations. In addition to being intuitive or even pragmatic, the strength of eighteenth century mathematics was its generality. Euler's analysis has been considered algorithmic, analytic or algebraic [Fraser, 1989, 317], implicitly algebraic [Jahnke, 2003, 107], and formula-centered [Sørensen, 2005, 454]. The algebraic character of Euler's mathematics led to the strength of its generalizations. Fraser [1989, 330] comments that "the generality of the formulas of algebra… assures the generality of the associated method and hence the generality of the mathematics itself."

Today's mathematicians must incorporate all aspects of a concept into a definition because it constitutes the exact nature of that object. This was unnecessary for Euler because the mathematical objects were assumed to exist and his goal was to describe the object without being overly formalistic but rather pragmatic and general. The strength of Euler's mathematics is its broad applicability not narrowing precision. It is in this spirit that a historian must examine Euler's use and understanding of mathematical transcendence. It thus seems that one could dismiss Erdös and Dudley's claim since Euler's use of definitions was inconsistent with the role definition plays in nineteenth century and later mathematics. Yet, the investigation into the finer points proves to be more enlightening.

3. Number and quantity

Euler employed the expression "transcendental quantity" rather than "transcendental number." As we have seen, for Euler "quantity" meant something quite different from number. A proper understanding of Euler's notion of quantity will clarify his use of the phrase "transcendental quantity."

The relationship between variables in a function revealed the nature of that function. Recall that Euler understood quantities to be determined or variable and variable quantities included functions. Ferraro [2004, 44] tells us "a single number expressed a quantum [single determination of a variable quantity] rather than a quantity." Therefore, that functional relationship between variables is of great importance for understanding how Euler applied the term "transcendental" to constant quantities. Since a constant quantity does not involve variables, the nature of constant quantity is exposed by comparing it to another constant instead, the unity.

Beyond the quantity-number distinction, Ferraro [2004] also distinguishes between "true" and "fictitious" numbers in Euler's work. Recalling Euler's definition of a determined quantity, it is clear that the natural and non-zero rational numbers could be intuitively understood as measures of quantity. This was not the case for zero, negative, irrational, or complex numbers. Ferraro states:

> … even though all numbers were abstract and symbolic entities, only some adequately reflected the concept of number as the exact result of a process of measurement and were "true" numbers. Other types of numbers did not fit the notion of a number (although for different reasons). In the strict sense of the term they were not true numbers. I shall term them "fictitious numbers" or "fictions." [Ferraro, 2004, 41]

And he continues:

> Eulerian mathematics effectively presents *an ontological difference* between natural and rational numbers (true numbers) and the other species of numbers (which did not correspond to the idea of numbers and therefore were fictitious numbers). [Ferraro, 2004, 42]

The difference between the quantity and number in conjunction with the ontological distinction Ferraro posits allows the reader to understand why Euler did not use the phrase "transcendental number" and instead used "transcendental quantity." Transcendental determined quantities were not considered "true" numbers and did not share their privileged ontological status. Therefore these objects were considered more appropriately labeled "quantities"; likewise for irrational, negative, and imaginary numbers [Ferraro, 2004, 43]. This practice was also common among other eighteenth century mathematicians [Ferraro, 2004, 43]. However it is important to notice that Euler rarely referred to a determined quantity as transcendental, although Euler applied this term to both e and π. With Euler's broad use of the word "quantity" understood, it is in the following section that we turn to Euler's use and understanding of transcendence.

4. Transcendence

Euler used the word transcendental to describe various mathematical objects. From his broad application of the term we can capture his motivation and use it to understand more clearly what he meant by calling determined quantities, such as e and π, transcendental.

First, Euler described certain operations as transcendental [Euler, 1748, 4]. Even though this occurred in a section talking about functions, he called the following operations algebraic: addition, subtraction, multiplication, division, raising to a power, and the extraction of roots. He then continued that there were other operations "such as exponentials, logarithms, and others which integral calculus supplies in abundance" [Euler, 1748, 4] which he labeled transcendental. Clearly Euler was using the word transcendental here to distinguish between algebraic and non-algebraic/transcendental operations. Euler used the transcendental classification not only to distinguish between algebraic and transcendental operations but also functions and progressions.

Euler provided considerable detail of his function concept in the *Introductio*. Consider the following selection:

> 7. Functions are divided into algebraic and transcendental. The former are those made up from only algebraic operations, the latter are those which involve transcendental operations…
> Something else about transcendental functions should be noted, and this is the fact that the function will be transcendental only if the transcendental operation not only enters in, but actually affects the variable quantity. If the transcendental operations only pertain to the constants, the function is to be considered algebraic. [Euler, 1748, 4–5]

Algebraic functions were not restricted to polynomials with integer exponents but also included fractions of polynomials as well as rational and irrational exponents. For example, Euler classified $\frac{a^2 - z\sqrt{a^2+z^2}}{a+z}$ an algebraic function of z; more specifically, an irrational algebraic function [Euler, 1748, 6]. Euler informed us how he employed transcendental operations to distinguish between algebraic and transcendental functions. A function inherited the transcendental classification from the transcendental operation only if it affected the variable quantity. Transcendence expressed the nature of a function and it was the variable quantities alone that influenced the nature of the function. Euler was explicit about this.

> 9. . . . it should be noted that the constant quantities [and coefficients]. . . whether they be positive or negative, integers or fractions, rational or irrational, or even transcendental, do not change the nature of the functions. [Euler, 1748, 7]

For example, $\log(x) + 7$ is transcendental because the transcendental operation affects the variable quantity but $\log(7) + x$ is algebraic because the transcendental operation only affects the constant. Similarly, Euler gave us the function $x^2 + y \log z$ as an example of a transcendental function of both y and z [Euler, 1748, 65]. He considered circular arcs and trigonometric functions to be transcendental as well [Euler, 1748, 101].

To belabor the point, Euler distinguished between algebraic and transcendental progressions. They too inherited their transcendental nature from other mathematical objects. In this case, if the function describing the general term of a progression was transcendental then Euler called the progression transcendental as well [Euler, 1738, 2]. Therefore operations passed on their transcendental nature to functions and they in turn passed it on to progressions. This inheritance was multi-tiered and had generational characteristics to it with operations as an abstract patriarch. This concept of inheriting transcendence is key to understanding Euler's application of the transcendental classification.

It is within the passages about functions that we can find a rare occurrence where Euler labeled a constant quantity transcendental. Consider again section 7 from the *Introductio*:

> 7. . . . For instance, if c [$= 2\pi$] denotes the circumference of a circle with radius equal to 1, c will be a transcendental quantity, nevertheless, these expressions: $c + z$, cz^2, $4z^c$, etc. are all algebraic functions of z. [Euler, 1748, 5]

Notice here that Euler called the function $4z^c$ algebraic which is not consistent with today's use of the term transcendental function. The above quotation is one of a few occasions where we see that Euler classified a specific value transcendental. Euler called both π [Euler, 1748, 5] and e [Euler, 1744, 311] "transcendental" but he did not reveal what he meant by that.

Since Euler distinguished between algebraic and non-algebraic operations, functions, and progressions, it is reasonable to extend this notion as he understood it to determined quantities. In other words, Euler called a determined quantity transcendental to imply that it was not algebraic. Unfortunately, Euler did not define what he meant by an algebraic determined quantity either. Instead of looking for a definition of a transcendental number, which Euler did not provide, I will outline my interpretation of Euler's understanding of the nature of transcendental determined quantities.

The nature of a function was understood by looking at the relationship between the variables. That was why transcendental operations had to affect the variable for the function to be classified transcendental. It is precisely in this sense, I argue, that transcendental functions inherited their transcendental nature from transcendental operations. Recalling that functions were considered variable quantities, I also want to draw attention to the relationship these variable quantities had to the unity. If the variation of such a quantity in relation to the unity could be represented through transcendental operations then that variable quantity would be transcendental.

Using his explicit definition of transcendental functions we can access Euler's implicit understanding of transcendental determined quantities by considering the functional relationship variable quantities had to the unity. Therefore, we must return to Euler's notion of quantity and number. The nature of a determined quantity was understood by looking at the relationship the magnitude had to the unity because Euler held that "we cannot measure or determine any quantity, except by considering some other quantity of the same kind as known, and pointing out their mutual relation" [Euler, 1770, 1]. Bearing in mind that transcendental progressions inherited their transcendence from transcendental functions and transcendental functions inherited their transcendence from transcendental operations affecting the variables, I propose the following concept of transcendental inheritance regarding determined variables:

> Euler considered constant quantities to be transcendental if the function describing their relationship to the unity was transcendental.

In other words, transcendental determined quantities inherited their transcendence from transcendental functions (just like progressions) and the nature of the numbers could not be understood before this relationship had been established. The nature of a quantity was classified by looking at the functional relationship it had with the unity. If that function was transcendental then the quantity inherited its transcendental classification. Classifying variable quantities was much less ambiguous than classifying constant quantities since constant quantities can have several different functional relationships with the unity. The most appropriate function for classifying the nature of a constant quantity was the function from which it arises. For example, integers came from basic arithmetic and therefore the relationship they had to the unity was algebraic. Euler understood quantities such as π and e, however, to arise from arc-length and the natural logarithm and therefore those functions were the most appropriate to use in their classification.

Consider the following examples, π and e. According to Euler, π was transcendental not because it was not the root of an algebraic equation with integer coefficients but rather because one could not understand the nature of π without understanding that π arises in the study of circular arcs and arc length ($L = r\theta$) was considered by Euler to be a transcendental function; i.e. π is the length of the semi-circle with radius 1. The relationship π has with the unity, 1, could be described through a transcendental function and π therefore inherited its transcendence from the arc-length function. Likewise with e. This value was considered transcendental because it was intuitively understood in connection with the natural logarithm. The natural logarithm, like many of Euler's transcendental functions, is generated by an integral: $\int_1^x \frac{1}{t}dt$. The natural logarithm connects e and the unity, $\ln(e) = 1$. Even though Euler expressed e in the *Introductio* using infinite series, earlier works such as E853 written circa 1727–1728 indicate that he first used the value to represent the base of the natural logarithm. One can find Cajori's translation of E853 in Smith's 1929 source book which contains the following statement "For the number whose logarithm is unity, let e be written..." [Cajori, 1929, 95].

Not only is my proposal consistent with Eulerian mathematics but it also fits the paradigm of eighteenth century mathematics. That a determined quantity inherits its transcendence from a function is consistent with Euler's attempt to make the function concept central to analysis [Jahnke, 2003]. Generality was key to eighteenth century mathematics [Fraser, 1989] and the rules of algebra held "true" or valid in general [Ferraro, 2004, 43] and my proposal generalizes the consistent use of the transcendence of operations, functions, and progressions to determined quantities. In addition, Fraser [1989, 329] continues "the primary fact, the meaning of the theorem, derives always from the underlying algebra" and my proposal appeals to this as the meaning comes from the underlying functional relationship between quantities and not an inherent property of number. Moreover, the emphasis on operations draws upon the algorithmic quality of analysis during the eighteenth century. My approach resonates well within this style of analysis. Euler was not using difficult mathematics or relying on "questionable" infinities or infinitesimals when dealing with the transcendence of numbers. Instead, the whole process was quite intuitive.

5. Criticism

This proposal, however, is not immune from criticism. For instance Euler would have considered the function $y = \pi x$ to be algebraic but surely transcendental quantities such as π, 2π, 3π, etc. arise from this. My response to this line of reasoning is that one needed to understand or have access to arc-length before using π as a coefficient in another function. The value π did not make sense in this context alone. You still needed to understand π in terms of arc length first.

Another line of reasoning follows from transcendental functions which have specific determinations which were algebraic such as $y = \sin(x)$. In other words, not all specific determinations would be considered transcendental. Clearly $\sin(\pi) = 0$ must be a counterexample to my proposal. A defence against this style of criticism can be found in Fraser [1989] and Sørensen [2005]. In the eighteenth century, counterexamples were considered exceptions and exceptions were considered as special cases and not mathematically significant. For instance Ferraro [2004, 36] informs us that "Indeed a [variable] quantity possessed its own properties, which might be false for certain of its determinations."

Another criticism might appeal to the different relationships π and e have to the unity. When paired with the unity, π is a value the function attains but in the latter case, the unity is the value of the function, not e. Again, this situation can be resolved when one respects the intuitive nature of eighteenth century mathematics. The nature of these numbers cannot be described without using transcendental functions. Eighteenth century mathematics was pragmatic and we must appreciate this fact if we are to effectively understand Euler's use and understanding of mathematical transcendence. In the case of determining whether or not certain objects actually fall into this transcendental classification Euler approached the problem descriptively and in an informal way. For Euler, determining the transcendence of an object was descriptive and not some sort of inherent property we might identify with today. For instance, if we return to the function $4z^c$, Euler said the following: "The doubt raised by some as to whether such expressions as z^c [$c = 2\pi$] are correctly classified as algebraic is of little importance" [Euler, 1748, 5].

6. Conclusion

Leonhard Euler did not define transcendental numbers as numbers which are not roots of algebraic equations with integer coefficients and he did not use or understand them in that manner. Euler's purpose of definition was not to precisely and exhaustively define something as we might today to create an object of study but rather to provide a general broad practical description of an object that already existed. Euler understood numbers to represent the proportion of a given magnitude to a unit. If Euler labeled a number such as e or π transcendental that meant he understood the nature of these quantities to be transcendental. That is, their relationship with the unity could be intuitively described using transcendental functions, functions in which transcendental operations affected the variable. Mathematical objects inherited their transcendence from other mathematical objects such as the transcendental operations. This description was a useful classification but not mathematically important for Euler. Understanding his pragmatism towards transcendence in conjunction with distinctions between quantity and number as well as ontological implications discussed in Ferraro [2004], we can account for the confusion regarding Euler's mysterious application of the phrase "transcendental quantity."

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