HISTORIA MATHEMATICA **25** (1998), 290–317
ARTICLE NO. HM982195

Some Aspects of Euler's Theory of Series: *Inexplicable* Functions and the Euler–Maclaurin Summation Formula

Giovanni Ferraro

Via Nazionale, 38, 80021 Afragola (Naples), Italy

Wallis's method of interpolation attracted the attention of the young Euler, who obtained some important results. The problem of interpolation led Euler to formulate the problem of integration, i.e., to express the general term of a series by means of an integral. The latter problem was connected to the question of expressing the sum of a series using an integral. The outcome of this research was Euler's derivation of what would later become known as the Euler–Maclaurin formula. Euler subsequently returned to interpolation and formulated the theory of inexplicable functions including the gamma function.

The methods used by Euler illustrate well the principles of 18th-century analysis. Eulerian procedures are based upon the notion of geometric quantity. A function is actually conceived as the expression of a quantity and, for this reason, it intrinsically possesses properties we can term continuity, differentiability, Taylor expansion. These correspond to the usual properties of a curve which has ''regular'' characteristics (lack of jumps, presence of tangents, curvature radius, etc.). They have a ''figural'' clarity. Although Eulerian analysis remains rooted in geometry, it dispenses with figural representation: it is substantially nonfigural geometry. Reasoning with figures (which integrates the proof in classical geometry) is replaced by reasoning with analytic symbols. These are general because they do not represent a particular quantity and are not subjected to restrictions, but are an abstract representation of quantity. © 1998 Academic Press

Il problema dell'interpolazione wallisiana attrasse l'attenzione del giovane Euler. Egli ottenne rapidamente risultati di grande interesse e proseguì le sue ricerche formulando il problema dell'integrazione consistente nell'esprimere il termine generale di una serie mediante un integrale. Quest'ultimo problema ben presto si evolse nella questione di esprimere la somma di una serie mediante un integrale che guidò Euler nella derivazione della formula sommatoria oggi detta di Euler-Maclaurin. Euler ritornò in seguito sul problema dell'interpolazione sviluppando la teoria delle funzioni inesplicabili che comprendono come caso particolare la gamma.

I metodi usati da Euler sono di grande interesse per una comprensione dell'analisi settecentesca. Infatti alla base delle procedure euleriane vi è la nozione di quantità geometrica, la quale comporta che la funzione, intesa come espressione della quantità, abbia intrinsecamente proprietà che, con linguaggio moderno, possiamo chiamare continuità, differenziabilità, sviluppabilità in serie de Taylor, e che corrispondono alle proprietà tipiche di una curva dotata di ''regolarità'' (non fare salti, esistenza della tangente, del raggio di curvatura, ecc.). Queste proprietà hanno un'immediata evidenza nelle curve comunemente studiate, un'evidenza, per cosi dire, figurale. L'analisi euleriana conserva un forte contenuto geometrico, ma elimina la rappresentazione figurale: sostanzialmente essa appare come una geometria non figurale. Il ragionamento sulla figura (che nella geometria classica integrava la dimostrazione) viene ora sostitutito dal ragionamento sui simboli analitici, i quali sono generali, perché non rappresentano questa o quella particolare quantità e non sone soggetti a limitazioni di sorta, ma sono una rappresentazione astratta della quantità. © 1998 Academic Press

0315-0860/98 $25.00

Le problème d'interpolation de Wallis attira l'attention du jeune Euler, qui obtint rapidement des résultats de grand intérêt. Il amena Euler à formuler le problème d'intégration consistant à exprimer le terme général d'une série par une intégrale. Ce dernier problème se transforma rapidement en celui d'exprimer la somme d'une série par une intégrale, menant à la formule appelée de nos jours la formule d'Euler-Mac Laurin. Euler reprit ultérieurement le problème d'interpolation, formulant la théorie des fonctions "inexplicables" qui incluent en particulier la fonction Gamma.

Les méthodes utilisées par Euler ont un grand intérêt pour comprendre l'analyse du XVIIIe siècle. En effet, à la base des procédures eulériennes, on trouve la notion de quantité géométrique, qui implique que la fonction, comprise comme une expression de cette quantité, possède intrinsèquement certaines propriétés—en langage moderne la continuité, la différentiabilité, le développement en séries de Taylor—correspondant aux propriétés typiques d'une courbe dotée de régularité—ne pas faire de sauts, avoir une tangente, un rayon de courbure. Ces propriétés étaient évidentes pour les courbes communément étudiées, une évidence pour ainsi dire figurale. L'analyse eulérienne conserve un important contenu géométrique, mais élimine la représentation figurale: concrêtement, elle apparaît comme une géométrie non figurale. Le raisonnement sur la figure, qui, dans la géométrie classique, intégrait la démonstration, est ici remplacé par le raisonnement sur les symboles analytiques: ceux-ci sont généraux parce qu'ils ne représentent pas une quantité particulière et ne sont pas soumis à des restrictions, mais sont une représentation abstraite de la quantité.

AMS 1991 subject classification: 01A50.
Key Words: Euler; series; gamma function; Euler–Maclaurin sum formula; transcendental functions; analysis/synthesis.

1. INTRODUCTION

The theory of inexplicable functions includes the so-called Eulerian gamma function and the summation formula later named after Euler and Maclaurin, two of the more interesting results from Euler's work on series. Euler derived them by interpreting the notions of the *n*th term and partial sum[1] of a series in terms of his notion of function, namely, as continuous and differentiable (to use modern terms) functions of the index (viewed as a continuous variable). In this paper, I examine the development of Euler's thought on this topic from 1730 to 1755, i.e., from his early articles to those written in maturity, highlighting the concepts underlying Euler's techniques and their close connection with some crucial notions of the analysis of infinity in the 18th century. In so doing, I aim to qualify the well-known interpretation of Euler as rejecting earlier geometric methods as well as to illustrate an aspect of the change of mathematical analysis from a "geometric" to an "algebraic" stage during the middle of the 18th century. In a subsequent paper, I plan to examine the development of this subject after 1760, when some of the more remarkable difficulties intrinsic to an algebraic conception became observable.

Euler's starting point in the theory of series was the problem of interpolation,

[1] In what follows, I will often use the terms "the general term" and "the summation term," respectively, in order both to respect Euler's terminology and to point out the differences between the modern and Eulerian concepts. For the same reason, I also prefer to use the word "inexplicable" instead of the vague and generic term "transcendental." According to Euler [17, 2: Section 367], inexplicable functions are not algebraic, and it is largely unknown what genus of transcendental functions they belong to.

namely, extending a number sequence a_n defined for integral values of n to non-integral values of n. For instance, the interpolation formula for the factorial sequences $a_1 = 1, a_2 = 2, a_3 = 6, a_4 = 24, \ldots$ amounted to finding the values of terms like $a_{1/2}, a_{3/2}, \ldots$ corresponding to non-integral indices. From a modern point of view, this problem, which was introduced by Wallis, is meaningless. A modern mathematician attributes meaning to new operations or symbols using appropriate definitions. Operations and symbols do not have a "natural" meaning, only a meaning that has been assigned by definition. If $a_n = n!$ is defined only for an integer value of n, then any value can be assigned to a new symbol such as $\frac{1}{2}!$.

Euler viewed the matter differently. For him, interpolating $n!$ required reconstructing the "nature" of the sequences 1, 2, 6, 24, ... just as one reconstructed the nature of a physical phenomenon by interpolating physical data. Euler, like Leibniz, held that mathematical objects have their foundations in nature. In the 18th century, however, analytical objects were considered as quantities directly expressed by variables or by means of formulas, i.e., functions (cf. [21]). *To determine the nature of a sequence a_n (i.e., to interpolate) thus meant to determine the quantity (in the form of a function) that generated it.* According to Euler, "we have perfect knowledge of the nature of a series if we know its general term, i.e., a formula which exhibits the term of index x, whether integer or fractional or irrational" [14, 467].[2] Indeed, only in this way are the non-integral terms determined without ambiguity or uncertainty.[3]

Nor does knowledge of the integral terms of a series enable us to determine its nature with certainty. For instance, the series $1 + 2 + 3 + 4 + 5 + \cdots$, i.e., the series which assigns the value x to the term of index x, may be interpolated not only by means of the function $y = x$, but also by

$$y = x + x^2 \sin \pi x, \tag{1.1}$$

or, more generally, by

$$y = x + \sum_{i=0}^{\infty} P_i(x) \sin i\pi x, \tag{1.2}$$

where $P_i(x)$ is a function of x for each i [14, 465–467]. Consequently, interpolation varies according to the function chosen as the general term. As Euler explained, "[t]hen, even if all the terms of a series that correspond to integral indices are determined, one can define the intermediate ones, which have fractional indices, in an infinite variety of ways so that the interpolation of this series continues to be

[2] "perfecte ... natura seriei cognoscitur, si eius terminus generalis, seu formula, quae cuivis indici x, sive integro sive fracto sive surdo, terminum respondentem exhibeat, fuerit cognita."

[3] Euler [14, 467–468] subdivided series into three classes according to the law of formation: (1) the series whose nth term is known (i.e., the terms are given as a function of the index): (2) the recursive series whose terms are functions of the previous terms and among which the most important are the recurrent ones; and (3) the series whose terms are expressed as a function of both index and antecedent terms.

indeterminate'' [14, 466].[4] According to Euler, the problem of interpolation consisted substantially in determining the general term of a series for which one knows the first terms or the recurrence law, that is, in determining a function (in Euler's sense) $f(x)$ such that $a_n = f(n)$. This function *is not arbitrary* but is *necessarily* generated from the calculus.

It is worthwhile noting that Euler saw substantially no difference between the concept of a general term and a function. In modern mathematics, the general term of a series is a function that differs from other functions only in the specific domain. For Euler, general terms and functions were formulas or analytical expressions[5] (cf. [21; 32]), and he did not have the means to distinguish one from the other. A given formula, for instance $x^2 + 3x$, can represent both the function $y(x) = x^2 + 3x$ and the xth series term $a_x = x^2 + 3x$. Naturally, general terms have the same properties as functions (in Euler's sense, cf. [21]), and therefore the general term a_n is *continuous*[6] *and can be differentiated*, *integrated*, *and expanded in Taylor series with regard to the index n*. Although I shall often refer to continuous or differentiable functions in what follows, it is important to note that the term ''continuous (or ''differentiable,'' etc.) function'' distorts Euler's thought because the expression ''continuous function'' implies the existence of discontinuous (or non-differentiable, etc.) functions. It was inconceivable to Euler for functions to be anything but continuous, differentiable, etc., as these properties are intrinsic to the classical concept of quantity on which he based his mathematics.

2. THE SEARCH FOR THE GENERAL TERM AND INTEGRATION

The problem of finding the general term of a series had been tackled in Russian academic circles around 1720. Daniel Bernoulli [1] determined the general term of a recurrent series and applied this result to an approximate determination of the roots of equations. Christian Goldbach [28; 29] expressed the general terms and the partial sum (*summa generale*) of certain series by means of finite differences (Euler developed this idea later; see [22]) and tried to generalize the results by examining the series derived by means of a variable ''law of progression.''[7] Goldbach

[4] Hinc etsi omnes seriei termini, qui indicibus integris respondet, sunt determinati, intermedios tamen, qui indices habent fractos, infinitis variis modis definire licet, ita ut interpolatio istius seriei maneat indeterminata.''

[5] Euler offered this definition: ''A general term is a formula that consists of constant quantities or any other quantities like n, which gives the order of terms; thus, if one wishes the third term, 3 can be set in the place of n'' [3, 4]. These words anticipate the well-known definition of a function in the *Introductio:* ''A function of a variable quantity is an analytical expression composed in any way from the variable and numbers or constant quantities'' [12, 1:18].

[6] The conception of the continuous as an original notion is hidden behind the typical 18th-century formulation of the problem of interpolation. The discrete is only the particularization of the continuous. The continuous generates the discrete, in the sense that given a sequence defined for $n = 1, 2, 3, \ldots$, one must go back to its original continuous structure (expressed by an analytical formula).

[7] ''A *law of progression* is a formula by means of which, given one or more terms of the series we can find another term antecedent or successive;'' for instance, $u_{n+1} = mu_n$, $u_{n+2} = (ku_n + ru_{n+1}) \div (u_n + u_{n+1})$ are two constant laws of progression for m, k, and r constants, where $u_{n+1} = nu_n$ is variable [29, 164].

considered the series $\sum_{n=1}^{\infty} a_n$ given by a recurrence relation and used a formula that was already known to Newton,

$$a_n = a_1 + \frac{n-1}{1}\Delta a_1 + \frac{(n-1)(n-2)}{1\cdot 2}\Delta^2 a_1 + \frac{(n-1)(n-2)(n-3)}{1\cdot 2\cdot 3}\Delta^3 a_1 + \cdots, \tag{2.1}$$

where $\Delta a_n = a_{n+1} - a_n$ and $\Delta^n a_n = \Delta^{n-1} a_{n+1} - \Delta^{n-1} a_n$.

The righthand side of (2.1) is the finite sum $\sum_{r=0}^{n-1} \binom{n-1}{r} \Delta^r a_1$, if n is a positive integer, and the infinite series $\sum_{r=0}^{\infty} \binom{n-1}{r} \Delta^r a_1$, if n is not an integer. I retain the ambiguity of the original notation,[8] as it is of crucial importance in Goldbach's procedure, which consisted of giving non-integral values to n in (2.1) and so producing an infinite expression. Goldbach said that it approximated the general term a_n as desired and was highly suitable (*precipuus*) for interpolation. For instance, in order to interpolate $a_n = 1/n!$ for $n = \frac{3}{2}$, he expressed $a_n = 1/n!$ by means of (2.1) as

$$1 - \frac{1}{2}(n-1) + \frac{1}{6}\frac{(n-1)(n-2)}{2} + \frac{1}{24}\frac{(n-1)(n-2)(n-3)}{6} + \cdots,$$

where

$$a_1 = 1, \qquad \Delta a_1 = -\frac{1}{2}, \qquad \Delta^2 a_1 = \frac{1}{6}, \qquad \Delta^3 a_1 = \frac{1}{24}, \ldots.$$

For $n = \frac{3}{2}$, he had

$$1 - \frac{1}{2\cdot 2} - \frac{1}{6}\frac{1}{2\cdot 4} + \frac{1}{24}\frac{3}{2\cdot 4\cdot 6} + \cdots.$$ [9]

It thus comes as no surprise that, in a letter dated 13 October, 1729 [26, 1:3–7], the

[8] In [29], Goldbach expressed the xth term as

$$a + (b-a)(x-1) + (c-2b+a)(x-1)\frac{x-2}{2} + (d-3c+3b-a)(x-1)\frac{x-2}{2}\frac{x-3}{3} + \text{etc.},$$

where $a, b, c, d, \ldots$ are the 1st, 2nd, 3rd, 4th, ... terms of a series "$a + b + c + d +$ etc." Such ambiguous notation was usual in the 18th century. Generally speaking, "$a + b + c + d +$ etc." denoted both $a_1 + a_2 + \cdots + a_n$, i.e., a finite sum, and $a_1 + a_2 + \cdots + a_n + \cdots$, i.e., an infinite series. The ambiguity arose not only due to inappropriate symbolism but also because, for 18th-century mathematicians, there was no difference between finite and infinite sums (see [32, 282]). They indeed considered a series as known when one could explicitly exhibit its first terms and knew the law for deriving the following terms. Whether the process of derivation of terms was finite or infinite was of no importance. Sometimes, like in Goldbach's procedure or in the early derivations of the binomial theorem, such ambiguity was precisely what enabled the extension of a result proved in a finite case to the infinite case.

[9] If we interpolate the hypergeometric sequence $n!$ for $n = \frac{3}{2}$, (2.1) provides the series $1 + \frac{1}{2} - \frac{3}{8} + \frac{11}{6} - \frac{265}{128} + \cdots$, which is divergent. Goldbach thought, however, that one can assume

$$\frac{3}{2}! = \left(\frac{1}{\frac{3}{2}!}\right)^{-1} = \frac{1}{1 - \frac{1}{2\cdot 2} - \frac{1}{6}\frac{1}{2\cdot 4} + \frac{1}{24}\frac{3}{2\cdot 4\cdot 6} + \cdots}.$$

We also find this procedure in Euler [16].

first in a long-lasting correspondence, Euler asked Goldbach to give his opinion on the formula

$$n! = \prod_{k=1}^{\infty} \frac{k^{1-n}(k+1)^n}{k+n}, \tag{2.2}$$

which he was about to publish [3].[10] This letter explicitly connected Euler's result to Goldbach's research.

As Euler made clear, however, he did not consider satisfactory the expression of the general term of a series by means of another series. He wrote to Goldbach [26, 1:4], stating that if one expresses the general term of a series by means of other series, then the intermediate terms, i.e., those with a non-integral index, are determined only approximately. For this reason, he stopped treating the matter using series and became interested in a method that would enable him to determine the real (and not approximate) intermediate terms. Furthermore, in [3, 3], he stated that he chose not to dwell upon (2.2) because he already had more suitable ways to express the nth term of the hypergeometric series. Thus, Euler posed the so-called problem of integration: to express the general term of a series by an integral, namely, by a formula of the kind $\int_0^b p(x, n)\, dx$. Euler explained that the function p depends on x and certain constants (of which one is n) and that the integration of $p(x, n)$ from 0 to the real number b yields a "function of the index n and constant quantities," namely, the general term [26, 1:12]. In [3], in order to clarify that the result of integration is precisely the nth term, Euler gave the quoted definition of the general term (see footnote 5). His specifications given both in the letter to Goldbach and in [3] seem superfluous to modern eyes, but they are due to the fact that Euler identified as formulas the notions of the general term and the function.

It is of interest that Euler considered infinite expressions unsuitable for providing general terms, which, as functions, are finite formulas. According to Craig Fraser, "infinite series are not themselves regarded as functions" [24, 322]. This concept is crucial in Euler's thought. One merely has to think of the Eulerian definition of the sum of the series, which is based on the idea that a series is only a transformed function. (I discuss in greater detail elsewhere the relation between series and functions and Fraser's views; see [21; 22].) However, in his earlier paper, Euler had not yet entirely developed his formal conception of function; he used a rather restrictive notion. Indeed, while he would later think that infinite formulas served to investigate functions exactly (in the sense that manipulating one is the same as manipulating the other), in [3] and in his letter to Goldbach, Euler conceived (2.2) only as a tool for computing the terms of series approximately, even though he actually calculated the exact value of $\frac{1}{2}!$ $(= \sqrt{\pi}/2 = \Gamma(3/2))$. In his early papers, Euler suggested that integration provided the "true" result, since integration was interpreted geometrically as a quadrature, i.e., the result was exact insofar as it was

[10] In [3], Euler merely verified (2.2) for $n = 0, 1, 2, 3$. He did not justify the transition from integral to non-integral explicitly. The same thing occurs in a derivation of (2.2) in [18] (for a discussion of this derivation, see [2, 20–21]).

geometrically conceived. I should mention that, at this time, even the sine and cosine were considered geometrically. According to Victor Katz, until 1739 "there was no sense of sine and cosine being expressed, like algebraic functions, as formulas involving letters and numbers" [30, 312].

In [3], Euler asserted that he had initially thought that even if $n!$ could not be expressed algebraically, it could at least be expressed exponentially. He realized that this was impossible after he noticed (by applying (2.2)) that $(\frac{1}{2})!$ depended on π and that therefore $n!$ might be connected with both the algebraic and the transcendental quantities dependent upon the quadrature of the circle. He also observed that the same is true for many integrals $\int_0^b p(x, n)dx$ which, although not expressible algebraically for every n, could be expressed either by an algebraic quantity or by a quantity dependent on the quadrature of curves for some n. Euler thus thought that such integral formulas were essential in order to express the general term of certain series [3, 3–4; 26, 2:5–12].[11]

His method involved considering a certain integral and finding the series whose general term corresponds to it. He later [5] termed such a procedure a *synthetic* method, since *a priori* knowledge of the result was supposed, and one simply verified that a certain integral (which was already known) expressed the general term of a series (see Section 4). For example, Euler[12] [3, 7–14] considered $\int_0^1 x^e(1 - x)^n\, dx$. By expanding $(1 - x)^n$, he derived

$$\int_0^1 x^e(1-x)^n\, dx = \int_0^1 x^e \sum_{h=0}^{\infty} \binom{n}{h} (-x)^h\, dx$$
$$= \int_0^1 \sum_{h=0}^{\infty} \binom{n}{h} (-1)^h x^{h+e}\, dx = \sum_{h=0}^{\infty} \binom{n}{h} \int_0^1 (-1)^h x^{h+e}\, dx,$$

and, by an integration term by term, obtained

$$\int_0^1 x^e(1-x)^n\, dx = \sum_{h=0}^{\infty} \binom{n}{h} \frac{(-1)^h}{e+h+1}. \tag{2.3}$$

Euler verified that the righthand side of (2.3) equals $n!/(e + 1)(e + 2) \cdots (e + n + 1)$ for $n = 0, 1, 2, 3$ and concluded that $\int_0^1 x^e(1 - x)^n\, dx$ is the nth term of the series $a_n = n!/(e + 1)(e + 2) \cdots (e + n + 1)$.

In order to determine an integral expression of $n!$, Euler multiplied

$$\int_0^1 x^e(1-x)^n\, dx = \frac{n!}{(e+1)(e+2)\cdots(e+n+1)}$$

[11] The progressions whose general terms cannot be expressed algebraically were called transcendental by Euler: "In the same manner as in geometry one usually calls transcendental what goes further than common algebra" [3, 4].

[12] Similar cases may be found in Wallis. See [34].

by $(e + n + 1)$ to get

$$(e+n+1)\int_0^1 x^e(1-x)^n\,dx = \frac{n!}{(e+1)(e+2)\cdots(e+n)}.$$

If $e = f/g$, then

$$a_n = \frac{f+(n+1)g}{g^{n+1}}\int_0^1 x^{f/g}(1-x)^n\,dx = \frac{n!}{(f+g)(f+2g)\cdots(f+ng)}, \qquad (2.4)$$

which becomes

$$\frac{1}{0^{n+1}}\int_0^1 x^{1/0}(1-x)^n\,dx = n!, \qquad \text{for } f = 1 \text{ and } g = 0. \qquad (2.5)$$

Today, this would be handled by first giving a meaning to $1/0^{n-1}\int_0^1 x^{1/0}(1-x)^n\,dx$ and, then, manipulating it; ensure existence first, calculate second. According to Euler, the problem lay in finding the value of such an expression; its existence was coincident with the determination of the value.[13]

This also comes through clearly in other cases, for instance in problems like "determining the value" of fractions $f(x)/g(x)$ when they assume the form 0/0 for a certain x or of expressions of the kind

$$\frac{\pi^2}{6n(n-1)} + \frac{1}{n(n-1)^2} - \frac{(2n-1)\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{n}\right)}{n^2(n-1)^2}$$

for $n = 1$ ([26, 2:229–231; 17, 2: Section 385]). The problem was to discover the value that the "nature" of sequences or functions dictated for these forms. In 18th-century mathematics, there were no sets of previously defined objects which were represented by symbols; rather symbols of mathematical objects had their *foundation in nature*. A combination of symbols was a statement concerning the "nature" of mathematical objects, and the fact that symbols like 0 in combinations such as 0/0 went further than their original meaning is a sign of a new and unexplored aspect of mathematical objects. By investigating the nature of these mathematical objects a combination of symbols would have a natural meaning.[14]

With respect to (2.5), Euler observed that the previous result also holds if we replace x by any function $f(x)$ (provided $f(0) = 0$ and $f(1) = 1$) in the above

[13] In 18th-century analysis, there were no existence theorems (except the fundamental theorem of algebra).

[14] In the special case of the ratios 0/0, there was a natural value since numbers were not the original objects of the calculus. The calculus primarily operated upon forms that expressed quantities. In analysis, numbers were conceived of as having originated from variable quantities, whose memory they, in some sense, preserved. Therefore, for c a root of $f(x) = 0$ and $g(x) = 0$, 0/0 has a meaning as $0/0 = (f(x))_{x=c} \div (g(x))_{x=c}$, namely, as the ratio of two quantities expressed by the function $f(x)$ and $g(x)$.

integrals (and obviously with df in place of dx). For $f(x) = x^{g/(f+g)}$, the lefthand side of (2.4) becomes

$$\frac{f+(n+1)g}{g^{n+1}}\int_0^1 x^{g/(f+g)}\,(1-x^{g/(f+g)})^n\,dx,$$

from which

$$\int_0^1 \frac{(1-x^0)^n}{0^n}\,dx = n!, \qquad \text{for } f = 1 \text{ and } g = 0.$$

Euler interpreted the integrand as $((1 - x^z)/z)_{z=0}$ and found that it is equal to $-\log x$, by applying the so-called "L'Hôpital rule." Consequently, we have the formula

$$n! = \int_0^1 (-\log x)^n\,dx, \tag{2.6}$$

which enables us to attribute a value to $n!$ for n non-integral. Finally, Euler verified (2.6) in certain particular cases and noted that (2.6) did not allow for an easy calculation of the value of $n!$. It is possible, however, to reduce the calculation of $\int_0^1 (-\log x)^n\,dx$ to the quadrature of certain algebraic curves by formulas of this type

$$\int_0^1 (-\log x)^n\,dx = \frac{(f+g)(f+2g)\cdots(f+(n+1)g)}{g^{n+1}}\int_0^1 x^{f/g}(1-x)^n\,dx$$

(i.e., in more modern notation

$$\Gamma(n+1) = \frac{(f+g)(f+2g)\cdots(f+(n+1)g)}{g^{n+1}}\,B\left(\frac{f}{g}+1, n+1\right),$$

where $\Gamma(x)$ and $B(x, y)$ are the gamma and beta functions). The latter formulas enable us to compute some values of $n!$ for non-integer n, if we assume the quadrature of the given algebraic curve to be known. In this sense, (2.6) is a more convenient and exact method for interpolating $n!$.

Historically, (2.6) was the first integral expression of the function $\Gamma(x)$; it is unusual today. Euler later went on to provide a different, and now more usual, integral expression (for instance, see [18]). From the modern viewpoint, however, its derivation is problematic as is the formulation of the question (the interpolation) which leads to (2.6) or (2.2). Furthermore, the final result (2.6) was not viewed as an analytic function in its own right; it was seen merely as a tool for evaluating and representing $n!$.

3. FROM THE PROBLEM OF INTEGRATION TO THE SUMMATION FORMULA

The problem of integration, therefore, consisted in seeking the most appropriate way of applying integral calculus to the determination of the nth term of a series. Euler immediately realized that the method used in [3] could be applied to determine the summation term as well as to sum certain series which could not be handled algebraically but which could be given by means of differential formulas to be integrated. In the early 1730s, Euler therefore gave up the problem of interpolation, only to return to it in the 1750s. In so doing, he opened a new field of research based on the increasingly widespread use of infinitesimal calculus. Among other things, his new approach enabled him to derive the summation formula named for him and Maclaurin. In order to come to terms with these developments in Euler's thought, we must first understand his method.

In 1732 Euler observed:

> Last year I proposed a method for summing innumerable progressions [he refers to [4]], which not only covers the series having an algebraic sum but also provides the sum of the series dependent on the quadrature of curves, which cannot be summed algebraically. I then used a synthetic method; indeed taking any general formula I asked myself what the series could be whose sums are expressed by that formula. In this way I obtained several series, whose sums I had been able to assign. ... In order to make it easier and clearer to find the sum of any proposed series, provided this can be achieved, I communicated this analytical method, which allows the discovery of the summation term by the nature of the series. [5, 42][15]

The synthetic method of 1731, published in [4], started with a *determinate* function (specifically, Euler considered an integral representation of the function of the kind $\int_c^b p(n, x)\, dx$)[16] and derived a (finite or infinite) series whose sum term $S(n, x)$ is equal to $\int_c^b p(n, x)\, dx$ ($\int_c^b p(\infty, x)\, dx = S(\infty, x)$ is a special case). Euler started from the result $\int_c^b p(n, x)\, dx$, supposed to be entirely determined, and arrived at the summation term. In the analytic method of 1732 [5], to which the previous quotation refers, Euler supposed that an *unknown* function $f(x)$ is the sum of the series and, operating on $f(x)$, tried to derive $\sum_{n=0}^{\infty} f_n(x) = f(x)$. In both [4] and [5], Euler operated on the sum $f(x)$, deriving certain results concerning the series $\sum_{n=0}^{\infty} f_n(x)$, but in the synthetic method, the function sum is already known, guessed at in some way, while for the analytic method the sum $f(x)$ is unknown.

In [5], Euler adopted Pappus's classical terminology[17] to the analysis of the infinite. The synthetic method of 1731 yields a synthetic solution (in Pappus's sense) of the converse problem (i.e., given a function $f(x)$ find the series whose summation

[15] "Proposui anno praeterito methodum innumeras progressiones summandi, quae non solum ad series algebraicam summam habentes extendit, sed earum etiam, quae algebraice summari nequeunt, summas a quadraturis curvarum pendentes exhibet. Synthetice tum usus sum methodo; generalibus enim assumtis formulis quaesivi serie, quarum summae his formulis exprimentur. Hocque modo plurimas series generales adeptus sum, quarum summas poteram assignare. ... Quo ... facilius magisque in promtu sit seriei cuiuscunque propositae summam, si quidem fieri potest, invenire, comunicabo hic methodum analyticam, qua ex ipsius seriei natura terminum summatorium eruere licet."

[16] In his correspondence with Goldbach, Euler actually used the term "function" with regard to these integral formulas [26, 1:12]. He may have considered $\int_c^b p(n, x)\, dx$ as a function insofar as he conceived the integral as a functional relation (with explicit reference to its geometric meaning). In [3], Euler spoke simply of formulas.

[17] Such terminology already seems residual and relative to a phase in which Euler's formalism was not yet entirely developed. It no longer appears in his later papers about series, where the term *analytical* has a different meaning (see Section 4).

term is $f(x)$), which, read backwards, becomes the solution to the direct problem. With regard to the theory of series, the use of Pappus's terminology implies the following elaborate scheme:

	To sum			To develop		
Analytical procedure	AS			AD		
Synthetic procedure		S_1S	S_2S		S_1D	S_2D

In the above scheme:

- AS is the analytical method of the sum. One operates upon the indeterminate object $f(x)$,[18] satisfying the condition *of being the sum of the given series* $\sum_{n=0}^{\infty} f_n(x)$ (this is the method used in [5]).
- AD is the analytical method of development. Given $f(x)$, one operates upon the indeterminate series[19] $\sum_{n=0}^{\infty} f_n(x)$, which satisfies the condition *of being the development of* $f(x)$, and obtains a known series which is the development of $f(x)$.[20]
- S_1S is a synthetic method of the sum. The series $\sum_{n=0}^{\infty} f_n(x)$ is given. The sum $f(x)$ is initially guessed at in some way, and one shows that $f(x) = \sum_{n=0}^{\infty} f_n(x)$. As Euler put it in [5, 42], to sum a series, one needs to compare it with known formulas and to investigate whether the series is derived from one of them (this is the method used in [4]).
- S_1D is a synthetic method of development. One guesses at the expansion $\sum_{n=0}^{\infty} f_n(x)$ of the given function $f(x)$ in some way and proceeds to derive $f(x) = \sum_{n=0}^{\infty} f_n(x)$.
- S_2S is a synthetic method of the sum. The series is given, and one derives $f(x)$.
- S_2D is a synthetic method of development. The function $f(x)$ is given, and one derives $\sum_{n=0}^{\infty} f_n(x)$.

In [4, 26–30], Euler applied the synthetic method of the sum as follows. Since

$$\int_0^1 \frac{1-x^n}{1-x}\,dx = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n}$$

for n an integer, this integral can be used to interpolate the series $1 + \frac{1}{2} + \frac{1}{3} + \cdots + 1/n$ (since it makes up its nth term) and to sum a finite number of the terms

[18] Usually, the indeterminate object on which one operates in any application of the analytical method is denoted by a symbol. For instance, we can denote the unknown sum of $\sum_{n=0}^{\infty} f_n(x)$ simply by the symbol $f(x)$. This fact is irrelevant to my argument. Of course, we can directly operate by means of $\sum_{n=0}^{\infty} f_n(x)$, which is always an unknown object as long as the sum of the series is unknown.

[19] Even if the series is indeterminate, its form is assumed to be of a particular kind, usually a power series.

[20] The above scheme emerged from a conversation with Marco Panza.

of the harmonic series (since it expresses the summation term of $\Sigma_n\ 1/n$). More generally, Euler denoted a function of x by $P(x)$ and considered

$$\int_0^k \frac{1-P^n(x)}{1-P(x)}\,dx = k + \int_0^k P(x)\,dx + \int_0^k P^2(x)\,dx + \cdots + \int_0^k P^{n-1}(x)\,dx$$

to be the summation term of the series $a_n = \int_0^k P^n(x)\,dx$. In particular, he studied $P(x) = x^\alpha/a^\alpha$ and reduced $\Sigma_n\, b^{(n-1)i+1}/(c + (n-1)a)$ to this case. If $b = 1$, the latter becomes $\Sigma_n\ 1/(c + (n-1)a)$, whose summation term is expressed by

$$\int_0^1 \frac{1-x^{na/c}}{(1-x^{a/c})c}\,dx.$$

Euler then generalized this result [4, 34–36], stating, for example, that the sum term of the series $\Sigma_n\ c/(c + (n-1)a)(c(\alpha+2) + (n-1)a)$ is

$$\int_0^1 x^\alpha\,dx \int_0^1 \frac{1-x^{na/c}}{(1-x^{a/c})c}\,dx.$$

He noted that

$$\frac{1-x^{na/c}}{1-x^{a/c}} = \frac{1}{1-x^{a/c}},$$

if $n = \infty$ and $(0 <)\ x < 1$. This equality was also considered valid when $x = 1$ because of the continuity[21] of the quantity x, and therefore Euler derived the sums of the series whose summation terms had been determined earlier. For instance [4, 36–37],[22]

$$\sum_n^\infty \frac{1}{c+(n-1)a} = \int_0^1 \frac{1}{(1-x^{a/c})c}\,dx \qquad \text{and}$$

$$\sum_{n=1}^\infty \frac{c}{(c+(n-1)a)(c\alpha+(n-1)a)} = \int_0^1 x^{\alpha-2}\,dx \int_0^1 \frac{1}{(1-x^{a/c})c}\,dx.$$

In contrast to the synthetic method usually used in [3; 4], Euler [5] proposed two analytical techniques for summing (finite or infinite) series. One of these is the summation formula which will be examined in Section 4. The other involves seeking the summation term (expressed by an integral or algebraic expression) by manipulating the series through appropriate (algebraic or differential) operations on its terms in order to reduce the given series either to another series (which can be more

[21] This represents an application of the well-known Leibnizian principle of continuity: what is true up to the limit is true at the limit. This principle implies that one must not consider the interval (0, 1) as a set of numbers, in which one may or may not include the number 0 and 1. Instead, an interval is viewed as a quantity (similar to a physical segment) which necessarily has two extremes that share the same nature of that quantity (i.e., the extremity of the quantity is termed the geometrical point).

[22] Euler used such results to transform slow convergent series into fast convergent series in [4, 38–41]. For instance, $1 + \frac14 + \frac19 + \cdots = 1 + \frac18 + \frac{1}{36} + \frac{1}{128} + \frac{1}{400} + \cdots + (\log 2)^2 = 1.164481 + 0.480453 = 1.644934$.

easily summed) or again to itself (which yields an equation providing the sum) [5, 44]. Euler set $S(m, x) = \sum_{n=1}^{m} a_n x^n$ and determined $S(m, x)$ by means of appropriate manipulations.[23]

Two examples will clarify this technique. The first does not involve calculus. Euler set $S(m, x) = \sum_{n=1}^{m} x^{(n-1)b+a}$ (see footnote 18) and, by the elementary technique still used today, derived $S(m, x) = ((1 - x^{mb})/(1 - x^b))x^a$. From this, under the condition $0 < x < 1$, he obtained $S(\infty, x) = (1/(1 - x^b))x^a$. In the second example, he set $S(m, x) = \sum_{n=1}^{m} x^{(n-1)\alpha+\beta}/(an + b)$ and manipulated this equation as follows. He multiplied it by px^r (p and r appropriate constants), differentiated it, and rearranged and summed the finite geometric progression on the righthand side. He then integrated both sides of the final equation to obtain

$$S(m, x) = \frac{\beta}{a} x^{(\alpha a - a\beta - b\beta)/a} \int x^{(a\beta + b\beta - \alpha)/a} \frac{1 - x^{m\beta}}{1 - x^{\beta}} \, dx. \tag{3.1}$$

He imposed the condition $S(m, 0) = 0$. When $m = \infty$, (3.1) becomes

$$S(\infty, x) = \frac{\beta}{a} x^{(\alpha a - a\beta - b\beta)/a} \int x^{(a\beta + b\beta - \alpha)/a} \frac{1}{1 - x^{\beta}} \, dx$$

since Euler was concerned with calculating $S(\infty, 1)$ and could limit himself to $0 < x < 1$.

4. THE SUMMATION FORMULA

The summation formula was first mentioned in [5] and was proved, for the first time, in [9]. It was again called analytical, but this term had a meaning different from that given in [5]. In [9], Euler spoke of an analytical method in contrast to the geometric method used in [8]: "When I gave more precise consideration to the mode of summing which I had dealt with by using the geometrical method in the above dissertation [8] and investigated it analytically, I discovered that what I had derived geometrically could be deduced from a peculiar method for summing that I mentioned three years before in a paper [5] on the sum of series" [9, 108].[24]

In [8], Euler based the determination of an approximating evaluation of the sum of a "convergent"[25] series $\sum a_n$ on a geometric representation, i.e., by means of

[23] In [4; 5], Euler obtained results on infinite series by imposing limitations on the range of variables. He calculated the sum of (finite or infinite) numerical series by using power series and setting $x = 1$. In modern terms, he calculated $\lim_{x\to 1^-} S(m, x)$, if the series is finite, and $\lim_{x\to 1^-} S(\infty, x)$ (having previously determined $S(\infty, x)$ for $0 < x < 1$), if the series is infinite. Euler actually considered the series only for $0 < x < 1$. For this reason, the integrals are calculated in $(0, 1)$ where x^n is infinitesimal for $n = \infty$ and can be neglected. This procedure differs from the more marked, later formalism (cf. [22]).

[24] "Cum, quae superiore dissertatione se summatione serierum methodo geometrica exposui, diligentius considerassem eandemque summandi rationem analytice investigassem, perpexi id, quod geometrice elicui, deduci posse ex peculiari quandam summandi methodo, cuius iam ante triennium in dissertatione de summatione serierum mentionem feceram."

[25] According to Euler, $\sum a_n$ is convergent, if a_n goes to zero and $a_n > a_{n+1}$.

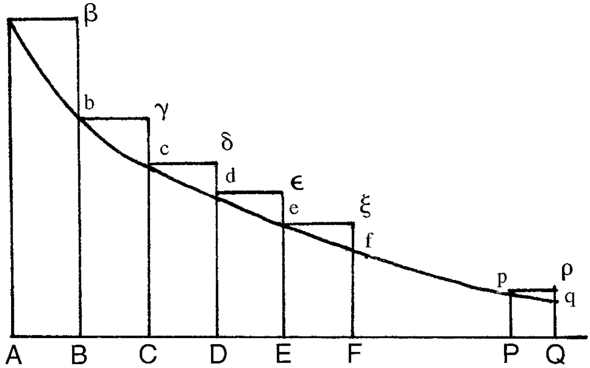

FIGURE 1

geometric figures. He wrote the series as $a + b + c + d + e + \cdots$ and denoted the nth and $(n + 1)$st terms by x and y, respectively. He considered the diagram shown in Fig. 1, where $aA = a_1$, $bB = a_2$, $cC = a_3$, $dD = a_4, \ldots, pP = a_n$, $qQ = a_{n+1}$ and $AB = BC = CD = DE = \cdots = PQ = 1$. Since "$x$ is a quantity composed of n and constants," $pP = x$ provides, according to Euler, the equation between AQ and qQ which expresses the nature of the curved line ap; i.e., the equation of curve is $x = a(n) = a_n$. Of course, $\sum_{i=1}^{n} a_i > \int_1^{n+1} a(n)\, dn$ or, as Euler said, $s_n = \sum_{i=1}^{n} a_i > \int a(n + 1)\, dn$ with the condition that the value of $\int a(t)\, dt = 0$ at s_0.[26]

In order to improve this approximation, he observed that the curvilinear triangles $ab\beta$, $bc\gamma$, $\ldots$, $pq\rho$ which have been neglected are greater than the rectilinear triangles $ab\beta$, $bc\gamma, \ldots, pq\rho$ (the curved line aq is convex, at least for large enough n). Since the sum of the areas of the rectilinear triangles $ab\beta$, $bc\gamma$, $\ldots$, $pq\rho$ is $(Aa - Qq)AB:2$, we have

$$\sum_{i=1}^{n} a_i > \int a(t+1)\, dt + \frac{a_1}{2} - \frac{a_{n+1}}{2}.^{27}$$

Finally, Euler considered the secant bc (Fig. 2) and approximated the arc ac by an appropriate arc of a parabola. Thus, the median bm is close to the tangent to the curve, and $S_{ab} = \frac{1}{3}T_1 = \frac{1}{6}T_2$, where the area between the curved line and the segment ab is denoted by S_{ab}, and the areas of triangles abm and abn are denoted

[26] Euler called this inequality an upper limit of series. Similarly, he derived $s_n = \sum_{i=1}^{n} a_i < \int_1^{n+1} a(t - 1)\, dt$ ($= \int a(t)\, dt$ under the condition that the value of $\int a(t)\, dt = 0$ at the origin), i.e., a lower limit.

[27] Analogously, he obtained

$$\sum_{i=1}^{n} a_i < \int_1^{n+1} a(t-1)\, dt - \frac{a_1}{2} + \frac{a_{n+1}}{2}.$$

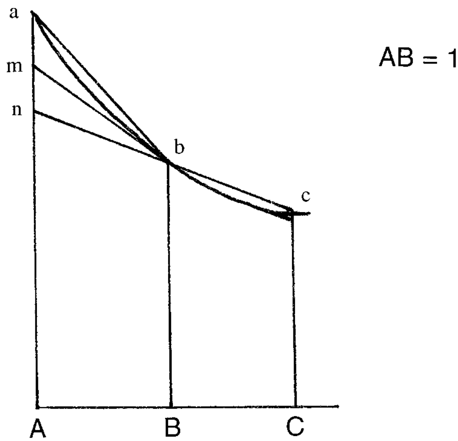

FIGURE 2

by T_1 and T_2. Since $na = Aa - 2Bb + Cc$ (indeed, in the trapezoid $ACcn$, if a segment Bb is drawn parallel to the sides An and Cc and bisecting AC, then $bB = (An + Cc)/2$), we have

$$T_2 = \frac{Aa - 2Bb + Cc}{2} = \frac{a_1 - 2a_2 + a_3}{2} \qquad \text{and} \qquad S_{ab} = \frac{a_1 - 2a_2 + a_3}{12}.$$

Of course, the sum of all the areas $S_{ab} + S_{bc} + S_{cd} + \cdots + S_{pq}$, which is neglected in the last inequality, is approximately equal to $(a_1 - a_2)/12 - (a_{n+1} - a_{n+2})/12$.

Therefore, Euler determined the approximating formula for "convergent" series

$$\sum_{i=1}^{n} a_i = \int a(t+1)\, dt + \frac{a_1}{2} - \frac{a_{n+1}}{2} + \frac{a_1 - a_2}{12} - \frac{a_{n+1} - a_{n+2}}{12}. \tag{4.1}$$

In proving (4.1), Euler implicitly assumed the convexity of the arc ab and hence of the curve. Furthermore, the errors are small when n is large and a_n is small. He actually applied (4.1) to sum the first 1,000,000 terms of the harmonic series.[28] For $n = 990{,}000$, (4.1) yields

$$\frac{1}{11} + \frac{1}{12} + \cdots + \frac{1}{1{,}000{,}000} = \log \frac{1{,}000{,}001}{11} + \frac{1}{22} + \frac{1}{132} - \frac{1}{144} - \frac{1}{2{,}000{,}002} - \frac{1}{12{,}000{,}012} + \frac{1}{12{,}000{,}024},$$

[28] This example clarifies the conditions of validity of (4.1) and, in a sense, is part of the proof of (4.1). On the peculiar use of examples in 18th-century demonstrations, see [23; 24].

and therefore

$$1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{1{,}000{,}000}=14.392669.$$

$$\left(\text{Note } 1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{10}=2.928968.\right)$$

When $n = \infty$, (4.1) becomes $\sum_{i=1}^{\infty} a_i = \int a(t+1)\,dt + 7a_1/12 - a_2/12$, and Euler illustrated this in the context of the example

$$1+\frac{1}{4}+\frac{1}{16}+\frac{1}{25}+\cdots=\left(1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{10^2}\right)+\left(\frac{1}{11^2}+\frac{1}{12^2}+\cdots\right)$$
$$=1.549768+\frac{1}{11}+\frac{7}{12\cdot 121}-\frac{1}{12\cdot 144}=1.644920.$$

The geometrical method of [8] hinged on using appropriate geometrical figures; some steps of the deduction were inferred by scrutinizing these figures. The analytical method of [9], i.e., the summation formula, dispensed with the geometrical representation. If, however, we look carefully at [8; 9], we note that both papers are based upon the same principles and, in effect, it would be easy to translate [8] into analytical symbols. Thus, in [8], the sequence a_n was viewed as a curved line whose equation is $y = a(n)$ and was assumed to be a continuous curve (both in the sense that it makes no jumps and in Euler's sense) possessing a tangent at each point. In [9], the sequence a_n was viewed as a continuous and infinitely differentiable function $a(n)$. Both articles substantially turned the study of the series $\sum a_n$ into the study of the function $a(n)$ and its integral $\int a_n\,dn$. As Euler put it, "I reduced summation to integration" [8, 101].[29]

Nowadays, the geometrical representation of [8] might seem a dispensable tool for facilitating the comprehension of proof; as in modern Euclidean geometry, figures improve the understanding of reasoning but are unessential. In reality, the modern proof is merely a linguistic deduction derived from explicit axioms and inference rules. This is not true for the classical conception of Euclidean geometry, where the reference to figures plays a crucial role (cf. [25; 33]). For instance, the geometric intuition connected to a geometric figure ensures that the circumference C and the segment s have a common point (Fig. 3). When Euler subsequently claimed the absence of geometric design in his analytic treatises, he asserted the absence of inference derived from the mere inspection of a figure (*inspectio figurae*) which was crucial in classical geometric proofs. For this reason, we can say that the analytical method is a *non-figural* method whereas the geometrical one can be described as *figural*.

[29] This integral expression of the summation differs from those of the type $s_n = \int f(x,n)\,dx$ examined above (see Section 3).

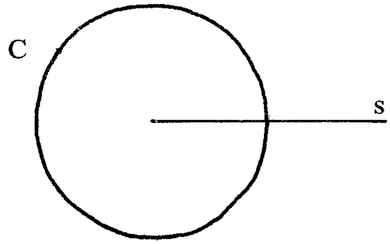

FIGURE 3

Furthermore, if it is true that Euler's analysis actually dispensed with figures, this does not mean that it also lacked geometrical references. Indeed, Euler based his analysis on the concept of geometric quantity which was translated into abstract terms like variable and function. The analytical procedures retained a remarkably geometric characterization. Thus, a function was actually conceived as the abstract and general expression of a geometric quantity. It therefore intrinsically possessed properties we can call continuity, differentiability, Taylor expansion, etc. These correspond to the usual properties of a "nice" curve, namely, lack of jumps, presence of the tangent, curvature radius, etc. Non-figural analysis was not a nongeometrical analysis. Thus, it may be more precise to compare 18th-century analysis to modern differential geometry than to modern analysis.

With respect to [8; 9], Euler stated that the analytical method was superior to the geometric one not only because it included the geometrically determined formula (4.1) but also because it led to an improvement of the sum so that the true result (*veram summam*) could be obtained through the addition of other terms (which were unlikely to be resolved by the geometric method). The power of the analytical method of [9] derives from the concept of the summation term $S_x = \sum_{n=1}^{x} a_n$ as a continuous and differentiable function of the index x.[30]

This concept appeared for the first time in [4, 29–30]. Euler took the summation term of the series $a_n = b^{(n-1)i+1}/(c + (n-1)e)$ to be A and said that we can consider n and A as flowing quantities (*quantitates fluentes*), when n is almost infinitely greater than 1, and therefore the differential dn is to dA as 1 is to $b^{ni+1}/(c + ne)$. He thus obtained the differential equation $dA = (b^{ni+1}/(c + ne))\, dn$, whose integral gives A as a function of n. For $i = 0$, Euler actually solved this equation and found $A = b/e \times \log(C(c + ne))$, C being an indeterminate constant.

This conception was later developed in [7], where, while deriving various results concerning the harmonic series $\sum c/(a + (i-1)b)$, Euler stated that, when i increases by one, the sum term s increases by $c/(a + ib)$ and therefore $\dfrac{di}{ds} = \dfrac{1}{c/(a + ib)}$. This equation furnishes the summation term $s = C +$

[30] The summation term recalls the integral $s(x) = \int_0^x a(n)\, dn$, in some aspects, but in others, it is really the sum of a finite number of terms.

$\left(\frac{c}{b}\right)\log(a+ib)$. In [8], Euler explicitly enunciated the general principle upon which the above results are based:

$$dn : ds_n = 1 : a_n, \qquad \text{i.e.,}$$
$$ds_n = a_n\,dn\ (s_n \text{ is the summation term of the series } \textstyle\sum a_n), \tag{4.2}$$

and he derived $s_n = \int a_n\,dn$ from this. He assumed the validity of (4.2) under the condition that n is large and the increment of s_n is very small. He was thus referring to "convergent" series. However, apart from the geometrical [8], where (4.2) is implicit, Euler only actually used (4.2) with respect to "convergent" series in [7]. Generally, he applied (4.2) to divergent series as well.

In [9], Euler proved the summation formula by using the Taylor series in a decisive way. He held that a function $y(x)$ could be expanded in Taylor series "if y is given in whatever way by means of x and constants" [9, 109]. The general term $X = X_i$ of a series and its summation term $S(x) = \sum_{i=1}^{x} X_i$ could also be expanded in Taylor series because "both S and X, in the case that the series is determined, are composed of x and constants" [9, 112]. As a consequence, he wrote

$$S(x-1) = \sum_{i=1}^{x-1} X_i = S(x) - \frac{dS}{1!\,dx} + \frac{d^2S}{2!\,dx^2} - \frac{d^3S}{3!\,dx^3} + \frac{d^4S}{4!\,dx^4} - \cdots$$
$$S(x) - S(x-1) = X = \frac{dS}{1!\,dx} - \frac{d^2S}{2!\,dx^2} + \frac{d^3S}{3!\,dx^3} - \frac{d^4S}{4!\,dx^4} + \cdots, \tag{4.3}$$

where (4.3) expresses the general term as a function of the summation term. Now, Euler wanted to derive S as a function of X. Setting

$$\frac{dS}{dx} = \sum_{n=0}^{\infty} a_n \frac{d^nX}{dx^n}$$

yielded

$$S = a_0 \int X\,dx + \sum_{n=1}^{\infty} a_n \frac{d^{n-1}X}{dx^{n-1}} \qquad \text{and} \qquad \frac{d^hS}{dx^h} = \sum_{n=0}^{\infty} a_n \frac{d^{n+h-1}X}{dx^{n+h-1}}.$$

Replacing the last formulas in (4.3), Euler inferred that

$$X = \sum_{h=1}^{\infty} (-1)^{h+1} \frac{d^hS}{h!\,dx^h} = \sum_{h=1}^{\infty} \frac{(-1)^{h+1}}{h!} \sum_{n=0}^{\infty} a_n \frac{d^{n+h-1}X}{dx^{n+h-1}}$$
$$= \sum_{n=0}^{\infty} \frac{d^nX}{dx^n} \sum_{h=1}^{n+1} \frac{(-1)^{h+1} a_{n+1-h}}{h!}. \tag{4.4}$$

By comparing the lefthand and the far righthand side of (4.4), he derived

$$a_0 = 1, \qquad a_n = \sum_{i=1}^{n} (-1)^{i+1} \frac{a_{n-i}}{(i+1)!},$$

and the summation formula

$$S(x) = \int X\,dx + \frac{X}{2!} + \frac{dX}{3!2\,dx} - \frac{d^3X}{5!6\,dx^3} + \frac{d^5X}{7!6\,dx^5} - \frac{3d^7X}{9!10\,dx^7} + \frac{5d^9X}{11!6\,dx^9} - \frac{691\,d^{11}X}{13!210\,dx^{11}} + \frac{35\,d^{13}X}{15!2\,dx^{13}} - \frac{3617\,d^{15}X}{17!30\,dx^{15}} + \cdots. \tag{4.5}$$

The indefinite integral yields a constant which is determined by the condition $S(0) = 0$.[31]

Immediately following the completion of [9], Euler returned to (4.5) in [10], where he modified the summation formula (now called the universal method) in order to make the calculation easier. He [10, 125] asserted that difficulties arise from the fact that the index of the general term increases by one unit at a time. He therefore considered a series $\sum_i X_{a+ib}$ and, setting $S(x) = \sum_{i=0}^{x} X_{a+ib}$, obtained

$$S(x-b) = \sum_{h=0}^{\infty} (-1)^h \frac{b^h d^h S}{h!\,dx^h} \qquad \text{and}$$

$$S(x) = \int \frac{X\,dx}{b} + \frac{X}{2!} + \frac{b\,dX}{3!2\,dx} - \frac{b^3d^3X}{5!6\,dx^3} + \frac{b^5d^5X}{7!6\,dx^5} - \frac{3b^7d^7X}{9!10\,dx^7} + \frac{5b^9d^9X}{11!6\,dx^9} - \frac{691b^{11}d^{11}X}{13!210\,dx^{11}} + \frac{35b^{13}d^{13}X}{15!2\,dx^{13}} - \frac{3617b^{15}d^{15}X}{17!30\,dx^{15}} + \cdots$$

with $S(a) = X_a$. Euler also added another term to this formula, but it is incorrect.

Euler's research into the sum formula continued throughout his life, and he applied it to numerous series. In [17], he explicitly linked the coefficients a_n with the Bernoulli numbers[32] (already studied in [11] and related to the sum of some remarkable series) and found

$$Sz = \int z\,dx + \frac{1}{2}z + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{B_{2n}}{(2n)!} \frac{d^{2n-1}z}{dx^{2n-1}} \qquad (\text{with } S(0) = 0). \tag{4.6}$$

[31] Stella Mills transforms (4.5) into

$$S(n) = \int_0^n X\,dn + \frac{X(n) - X(0)}{2} + \frac{1}{12}\left(\frac{dX(n)}{dn} - \frac{dX(0)}{dn}\right) - \frac{1}{720}\left(\frac{d^3X(n)}{dn^3} - \frac{d^3X(0)}{dn^3}\right) + \cdots$$

and points out a discrepancy in one of Euler's examples [31]. However, if we use the indefinite integral and apply the condition $S(0) = 0$ correctly (Euler explained how in [9, 115]), no discrepancy actually arises.

[32] Bernoulli numbers B_r are defined by the relation

$$\frac{t}{e^t - 1} = 1 + \sum_{r=1}^{\infty} (-1)^{[r/2]+1} \frac{B_r}{r!} t^r,$$

where $[x]$ is the integral part of x.

The demonstration of [17] is remarkable because it indicates an evolution in Euler's thought toward a more formal conception and procedures that seem to be a prelude to the calculus of operations. Euler [17, 2: Sections 167–168] denoted $\Sigma\, y_x$ by Sy (with $S(0) = 0$) and interpreted S as a symbolic operation that enjoys certain formal properties:

(1) finite and infinite additivity, which he explicitly formulated in these terms: If $y_x = p_x + q_x + r_x + \cdots$, then $Sy = Sp + Sq + Sr + \cdots$, that is, $\sum_{n=1}^{x} y_n = \sum_{n=1}^{x} p_n + \sum_{n=1}^{x} q_n + \sum_{n=1}^{x} r_n + \cdots$, and

(2) the commutativity of the operations S and d^n/d^nx: $(d^n/dx^n)\,(Sy) = S(dy^n/dx^n)$. (From a modern viewpoint, this formula has meaning only if we interpret S as an integral. In this case, it corresponds to $(d^n/dx^n) \int y\,dx = \int (dy^n/dx^n)\,dx$.)

Euler set $v = y(x-1) = y_{x-1} = v_x$ $(x = 2, 3, \ldots)$ and $v(1) = A$ to get $Sv = \sum_{n=1}^{x} y_n = A + \sum_{n=2}^{x} y_{n-1} = A + Sy - y$. By applying the additivity of the operation S to

$$v_x = y(x-1) = \sum_{n=0}^{\infty} \frac{(-1)^n d^n y}{n!\, dx^n}$$

and rearranging it, he derived

$$S\left(\frac{dy}{dx}\right) = y - A + \sum_{n=2}^{\infty} (-1)^n S\, \frac{d^n y}{n!\, dx^n} \qquad \text{and}$$

$$S(z) = \int z\, dx + \sum_{n=1}^{\infty} (-1)^{n+1} S\, \frac{d^n y}{(n+1)!\, dx^n}$$

(where $z = dy/dx$ and the condition $S(0) = 0$ holds). By differentiating and applying property (2), Euler found

$$S\frac{d^h z}{dx^h} = \frac{d^{h-1}z}{dx^{h-1}} + \sum_{n=2}^{\infty} \frac{(-1)^n}{n!} \cdot S\, \frac{d^{h+n-1}z}{dx^{h+n-1}}, \qquad h = 0, 1, 2, \ldots \left(\text{here } \frac{d^{-1}z}{dx^{-1}} = \int z\, dx\right).$$

He then expressed Sz as $Sz = \int z\,dx + \sum_{n=1}^{\infty} a_n(d^{n-1}z/dx^{n-1})$ and, proceeding as above, derived (4.6).

Later, he provided other proofs of the sum formula. The basic principles did not change; however, there are some significant differences, the analysis of which is beyond the scope of this paper.

5. THE SEARCH FOR THE GENERAL TERM AND INEXPLICABLE FUNCTIONS

In [14], Euler returned to the problem of the search for the general term,[33] which had earlier given rise to the problem of integration. He asserted that the innumerable

[33] Euler also dealt with the search for the general term in [12], where he provided his interpretation of Daniel Bernoulli's results [1] (on this subject, see [32]).

general terms of a series, given by a recursive rule, are contained in a general rule and "can be found by analysis without divination" (*sine divinatione per analysin invenire possunt*) [14, 470]. According to Euler, it is possible to determine, on the basis of the rules of calculus, the law, or rather the formula, that expresses the nth terms in a general and *necessary* way.

As mentioned above, in 1720 Euler believed the intermediate terms of the sequence $n!$ could only be found approximately by (2.2) and did not trust that expression of the general term. He thus sought a more convenient expression, i.e., (2.6), which provided the true and not merely the approximated values of the terms. By 1750, he considered the expression of the general term by series to be its most general expression. He expressed the nth term in the form of a series and seemed no longer reluctant to consider series as suitable for expressing the result of his research [14]. However, formulas like (5.1) are not actually the *final* result. Euler, in principle, imagined that such trigonometric series could be expressed as finite functions.

Euler considered the series $1 + 1 + 1 + \cdots$ and $a + (a + g) + (a + 2g) + \cdots$, namely, series whose general term can be represented in the form $a_n = a + (n - 1)g$, for a and g constants. He [14, 470–480] found that the most general expressions of their general term were, respectively,

$$a_n = 1 + \sum_{i=1}^{\infty} \alpha_i \sin 2i\pi n + \sum_{i=1}^{\infty} \beta_i(\cos 2i\pi n - 1)$$

and

$$a_n = a + g(n - 1) + \sum_{i=1}^{\infty} \alpha_i \sin 2i\pi n + \sum_{i=1}^{\infty} \beta_i (\cos 2i\pi n - 1). \tag{5.1}$$

Euler, however, seemed to conceive of the above trigonometric series only as expansions of rational functions of $\sin \pi n$ and $\cos \pi n$.

The latter example includes the series $1 + 2 + 3 + 4 + \cdots$, mentioned in Section 2. The "general" result that Euler arrived at, however, was less wide-ranging than (1.2). Indeed, (1.2) also comprises the case in which the coefficients $P_i(x)$ are not constant as in (1.1). Nevertheless, he considered formula (5.1) to be the most general expression of the nth term, as this was the formula derived through calculus procedures.

Euler reduced these two problems to the study of the functional equation $y(x + 1) = y(x)$ and $y(x + 1) = y(x) + g$, where $y(x)$ is an Eulerian *function*. They were then solved by Taylor expansion of $y(x)$, allowing Euler to apply the technique for solving differential equations of infinite degree that he had developed in [13]. In the same way, he solved (in [14]) eight other problems that reduce to the following equations: $y(x + 1) = ay(x)$ (problem 3); $y(x) = ay(x - 1) + b$ (problem 4); $y(x) = ay(x - 1) + by(x - 2)$ (problem 5); $y(x) = a_1y(x - 1) + a_2y(x - 2) + \cdots + a_ny(x - n)$ (problem 6); $y(x) = c + a_1y(x - 1) + a_2y(x - 2) + \cdots + a_ny(x - n)$ (problem 7); $y(x) = my(x) + a + bx$ (problem 8); $y(x) =$

$y(x-1)+F(x)$ (problem 9); $y(x+1)=xy(x)$ (problem 10) (where $a, b, c, g, m, a_1, a_2, \ldots, a_n$ are constants and $F(x)$ is an Eulerian function). Problems 5 and 6 concern the determination of the general term of a recurrent series with the relation scales (a, b) and $(a_1, \ldots, a_n)$, respectively. These generalize results due to Bernoulli in [1] and to Euler himself in [12].

Euler's research on the *n*th term gave rise to the theory of inexplicable functions. Euler termed inexplicable those functions having neither a determinate expression nor an expression by means of an equation (namely, an implicit algebraic expression). He actually considered the sums $S(x)=\sum_{n=1}^{x} a_n$ and the products $P(x)=\prod_{n=1}^{x} a_n$ and studied the functions obtained by giving nonintegral values to x. The old problem of interpolation now became autonomous; the interpolating functions of a series were studied as independent analytical objects.[34]

Euler aimed to determine the differentials of inexplicable functions. In [17, 2: Section 368], he noticed that this topic fit naturally in the first part of the treatise (where he studied the differentials of elementary functions), but since it required series, he was obliged to change his order of presentation. He effectively wanted to invent the calculus of these new objects and, consequently, to increase the number of functions. (He had already expanded the realm of functions with his interpretation of the sine, cosine, etc.; cf. [30].) Although Euler stated that "[g]enerally ... the notion of these inexplicable functions can be derived from series" [17, 2: Section 367] and, later, wrote that inexplicable functions are expressions "that cannot be expressed finitely as is usual" [17, 2: Section 389], these statements do not imply that an infinite formula on its own expresses a function. Indeed, inexplicable functions are defined not by series but by relations between quantities which are verbally expressed and denoted by a symbol of the kind $1+\frac{1}{2}+\frac{1}{3}+\cdots+1/x$ (see [21; 24; 32]). According to Euler, an inexplicable function is a *quantity* equal to a given finite sum $\sum_{n=1}^{x} a_n$ or product $\prod_{n=1}^{x} a_n$, for x an integer. The series only serves to derive the rules that enable the manipulation of that function.

In order to determine the differential of $S(x)=\sum_{n=1}^{x} a_n$, Euler [17, 2: Section 369–374] used ∞ as an infinitely large number, namely, ∞ is actually a number and therefore not only do $S(\infty)$ and a_∞ exist but so do $S(\infty+1), S(\infty+2), \ldots,$ and $a_{\infty+1}, a_{\infty+2}, a_{\infty+3}, \ldots$. (This was not the first time that Euler used the infinitely large as a number. For instance, see [7].) He put $T(x)=S(x+\omega)$ and considered the sequence $S(x), S(x+1), S(x+2), S(x+3), \ldots$. Since $S(x+n)-S(x+n-1)=a_{x+n}$, if one assumes that the numbers a_{x+n} converge to a number L, one has $S(\infty+1)-S(\infty)=a_{\infty+1}=L$, $S(\infty+2)-S(\infty+1)=a_{\infty+2}=L, \ldots$; i.e., $S(\infty)$, $S(\infty+1), S(\infty+2), S(\infty+3), \ldots$ are an arithmetic progression. Consequently,

$$\begin{aligned} T(\infty)=S(\infty+\omega)&=S(\infty)+(S(\infty+1)-S(\infty))\omega=S(\infty)+\omega(S(\infty)+a_{\infty+1}-S(\infty))\\ &=S(\infty)+\omega a_{\infty+1}, \end{aligned}$$

i.e., $T(\infty)=S(x)+\sum_{n=1}^{\infty} a_{x+n}+\omega a_{\infty+1}$.

[34] In fact, in [17], there are two separate chapters about this topic: Chapter XVI illustrates the theory of inexplicable functions and the following Chapter XVII applies the derived results to interpolation.

As $T(\infty) = T(x) + \sum_{n=1}^{\infty} a_{x+n+\omega}$, we have

$$S(x+\omega) = S(x) + \omega a_{\infty+1} + \sum_{n=1}^{\infty} a_{x+n} - \sum_{n=1}^{\infty} a_{x+n+\omega}. \tag{5.2}$$

If ω is infinitesimal, $S(x+\omega) - S(x)$ is the expression of the differential dS of the function. This aspect of (5.2) attracted Euler's attention, not the possibility that one could obtain a representation of the function $S(x)$ by means of an infinite series from (5.2). The infinite expression of $S(x)$ was only viewed as a tool for deriving the differential of $S(x)$ or for applying the function to the problem of interpolation. In no case did Euler conceive of the possibility of defining $S(x)$ by means of (5.2) or (5.4). For instance, in Section 371, he considered $H(x) = 1 + \frac{1}{2} + \frac{1}{3} + \cdots + 1/x$ and derived

$$\begin{aligned} H(x+\omega) &= H(x) + \sum_{n=1}^{\infty} \frac{1}{x+n} - \sum_{n=1}^{\infty} \frac{1}{x+n+\omega} = H(x) + \sum_{n=1}^{\infty} \frac{\omega}{(x+n)(x+n+\omega)} \\ &= H(x) + \sum_{n=1}^{\infty} \frac{\omega}{x+n} \sum_{h=1}^{\infty} (-1)^{h-1} \frac{\omega^{h-1}}{(x+n)^h} \\ &= H(x) + \sum_{h=1}^{\infty} (-1)^{h-1} \omega^h \sum_{n=1}^{\infty} \frac{1}{(x+n)^{h+1}}. \end{aligned}$$

For $\omega = dx$, he had

$$dH = \sum_{h=1}^{\infty} (-1)^{h-1}\, dx^h \sum_{n=1}^{\infty} \frac{1}{(x+n)^{h+1}}.$$

Only subsequently, in Section 372, did he derive the infinite expression of $H(x)$ in order to interpolate $1 + \frac{1}{2} + \frac{1}{3} + \cdots + 1/x$. Since $H(0) = 0$, he obtained the expansions

$$H(\omega) = \sum_{n=1}^{\infty} \frac{\omega}{n(n+\omega)} \qquad \text{and} \qquad H(\omega) = \sum_{h=1}^{\infty} (-1)^{h-1} \omega^h \sum_{n=1}^{\infty} \frac{1}{n^{h+1}}.$$

These provide the sum H, even if ω is not an integer, but they do not define $H(x)$.

For a generic inexplicable function $S(x)$, Euler considered Taylor's expansion of

$$a_{n+x+\omega} = a(n+\omega+x) = \sum_{h=0}^{\infty} \frac{\omega^h\, d^h a(x+n)}{h!\, dx^h}, \qquad \text{for } n = 0, 1, 2, 3, \ldots,$$

(he applied the methodology already used to seek the summation formula) and obtained

$$S(x+\omega) = S(x) + \omega a_{\infty+1} + \sum_{n=1}^{\infty} a(x+n) - a(x+n+\omega)$$

$$= S(x) + \omega a_{\infty+1} - \sum_{n=1}^{\infty}\sum_{h=1}^{\infty} \frac{\omega^h\, d^h a(x+n)}{h!\, dx^h}$$

$$= S(x) + \omega a_{\infty+1} - \sum_{h=1}^{\infty} \frac{\omega^h}{h!} \sum_{n=1}^{\infty} \frac{d^h a(x+n)}{dx^h}.$$

As $a_{\infty+1} = a(1) + \sum_{h=1}^{\infty} [a(h+1) - a(h)]$, he obtained

$$S(x+\omega) = S(x) + \omega a(1) + \omega \sum_{h=1}^{\infty} [a(h+1) - a(h)] - \sum_{h=1}^{\infty} \frac{\omega^h}{h!} \sum_{n=1}^{\infty} \frac{d^h a(x+n)}{dx^h}, \quad (5.3)$$

which provides the "complete differential"

$$dS = a(1)\, dx + dx \sum_{h=1}^{\infty} [a(h+1) - a(h)] - \sum_{h=1}^{\infty}\sum_{n=1}^{\infty} \frac{d^h a(x+n)}{h!}.$$

Euler later derived another expression of dS using the power series of $S(x)$. Setting $x = 0$ and

$$G_h = \sum_{n=1}^{\infty} \left[\frac{d^h a}{h!\, dx^h}\right]_n,$$

we have

$$S(\omega) = a(1)\omega + \sum_{h=1}^{\infty} [a(h+1) - a(h)]\omega - \sum_{h=1}^{\infty} G_h \omega^h$$

and, changing ω into x, we obtain

$$S(x) = a(1)x + \sum_{h=1}^{\infty} [a(h+1) - a(h)]x - \sum_{h=1}^{\infty} G_h x^h. \quad (5.4)$$

This expresses the value of the inexplicable function $S(x)$ "by means of an infinite series" but, above all, it allows us to derive the differential ratio dS/dx, d^2S/dx^2, $d^3S/dx^3, \ldots$, or, in Euler's words, the "complete differential" of S. In this manner, inexplicable functions submit to calculus.

Euler then generalized these achievements (derived subject to the condition that, for $n = \infty$, the sequence a_n becomes a constant L), considering the case in which the second or third differences of $S_\infty, S_{\infty+1}, S_{\infty+2}, \ldots$ equal 0. He reduced the products

$S(x) = \prod_{n=1}^{x} a_n$ to sums by means of logarithms. Under the conditions $\log a_\infty = 0$ and $\log a_{\infty+1} - \log a_\infty = 0$,[35] he derived

$$S(x+\omega) = S(x) \prod_{n=1}^{\infty} \frac{a_{n+x}}{a_{n+x+\omega}} \qquad \text{and} \qquad S(x+\omega) = S(x) a_{x+1}^{\omega} \prod_{n=1}^{\infty} \frac{a_{x+n+1}^{\omega} a_{x+n}^{1-\omega}}{a_{x+n+\omega}},$$

respectively. For $x = 0$, one has $S(0) = 1$ and changing ω into x, one obtains

$$S(x) = \prod_{n=1}^{\infty} \frac{a_n}{a_{n+x}} \qquad \text{and} \qquad S(x) = a_1^x \prod_{n=1}^{\infty} \frac{a_{n+1}^{x} a_n^{1-x}}{a_{n+x}}$$

[17, 2: Sections 381–382]. The latter may be applied to $G(x) = 1 \cdot 2 \cdot 3 \cdots x$ (in this case, the first difference of the logarithms of the terms whose index is infinite is 0) to obtain (2.2) [17, 2: Section 402]. For $\log S(x) = \log \prod_{n=1}^{x} a_n = \sum_{n=1}^{x} \log a_n$ and $\log a_{\infty+1} - \log a_\infty = 0$, (5.3) is transformed into

$$\begin{aligned} \log S(x+\omega) = \log S(x) + \omega \log a(x+1) + \omega \sum_{h=1}^{\infty} \log \frac{a(x+h+1)}{a(x+h)} \\ - \sum_{h=1}^{\infty} \frac{\omega^h}{h!} \sum_{n=1}^{\infty} \frac{d^h \log a(x+n)}{dx^h}, \end{aligned} \tag{5.5}$$

and then

$$\frac{dS}{S} = dx \log a(x+1) + dx \sum_{h=1}^{\infty} \log \frac{a(x+h+1)}{a(x+h)} - \sum_{h=1}^{\infty} \sum_{n=1}^{\infty} \frac{d^h \log a(x+n)}{h!}$$

[17, 2: Section 385]. Euler applied this formula to $G(x) = 1 \cdot 2 \cdot 3 \cdots x$ to obtain

$$\frac{dG}{G} = dx \log(x+1) + dx \sum_{h=1}^{\infty} \log \frac{x+h+1}{x+h} + \sum_{h=1}^{\infty} (-1)^h \frac{dx^h}{h} \sum_{n=1}^{\infty} \frac{1}{(x+n)^h}. \tag{5.6}$$

He did not write the infinite expression of $\log G(x)$ by deriving it from (5.5); he merely sought the differential of $\log G(x)$ and did not consider the expression of $\log G(x)$ of crucial importance.

Finally, from (5.4), Euler derived

$$\log S(x) = x \log a(1) + x \sum_{h=1}^{\infty} \log \frac{a(h+1)}{a(h)} - \sum_{h=1}^{\infty} G_h x^h,$$

where

$$G_h = \sum_{n=1}^{\infty} \left[\frac{d^h \log a}{h!\, dx^h} \right]_n$$

and $\log a_{\infty+1} - \log a_\infty = 0$, and then

[35] Other generalizations are possible (cf. [17, 2: Section 399]).

$$\frac{dS}{S} = dx \log a(1) + dx \sum_{h=1}^{\infty} \left[\log \frac{a(h+1)}{a(h)} \right] - \sum_{h=1}^{\infty} hG_h x^{h-1}\, dx.$$

If $G(x) = 1 \cdot 2 \cdot 3 \cdots x$, he obtained

$$\log G(x) = \sum_{h=1}^{\infty} \frac{(-1)^h}{h} C_h x^h$$

and

$$\frac{dG}{G} = \sum_{h=1}^{\infty} (-1)^h C_h x^{h-1}\, dx, \tag{5.7}$$

where C_1 is Euler's constant and

$$C_h = \sum_{n=1}^{\infty} \frac{1}{n^h}, \quad \text{for } h = 2, 3, 4, \ldots.$$

However, even in this case, the power series of $\log G(x)$ is only an intermediate step for arriving at (5.7) and in particular for calculating C_1.[36]

In conclusion, the problem of interpolation led to the invention of a new kind of function, i.e., new formulas involving letters and numbers. These functions can be considered as solutions, in the same way that any problem may be considered solved if reduced to trigonometric or logarithmic functions. Euler's work lies wholly within the principles of 18th-century mathematical analysis, based on the conception of the calculus as algebraic analysis, as has been illustrated by Fraser [24] and Panza [32]. The fact, discussed above, that *Euler did not define inexplicable functions in terms of infinite expansions*, but rather conceived of infinite expansions as useful technical tools for the investigation of inexplicable functions, reflects this same conception.

Euler was not entirely satisfied by this exposition of the theory of inexplicable functions and later tried to clarify it in [19]. Despite his efforts, during the second half of the 18th century, inexplicable functions did not actually enter into analysis in the same ways as the other known transcendental (logarithmic and trigonometric) functions. The reasons for this are closely connected with the limits and crisis of the algebraic view of analysis. I plan to analyze this set of issues in a sequel that

[36] By applying the sum formula (4.6) to the inexplicable functions $S(x) = \sum_{n=1}^{x} a_n$, Euler [17, 2: Sections 386–388] inferred

$$dS = a(x)\, dx + \frac{1}{2} da(x) + \sum_{n=1}^{\infty} (-1)^{n-1} \frac{B_{2n}}{(2n)!} \frac{d^{2n-1}a(x)}{dx^{2n-1}}\, da(x).$$

As a particular case, he considered $G = 1 \cdot 2 \cdot 3 \cdots x$ and, using the logarithms, derived

$$\frac{dG}{G} = \log x\, dx + \frac{dx}{2x} - \sum_{n=1}^{\infty} (-1)^{n-1} \frac{B_{2n}}{2n} \frac{dx}{x^{2n}}.$$

The latter is more suitable than (5.6) or (5.7) for calculating dS/S when x is very large.

will illustrate the transition from the Eulerian theory of inexplicable functions to Gauss's theory of the gamma function as formulated in [27]. The latter marked a turning point toward the modern theory of series.

ACKNOWLEDGMENTS

I thank the referees for comments on the first draft of the present article and Karen V. H. Parshall for improving my English.

REFERENCES

1. Daniel Bernoulli, Observationes de seriebus quae formantur ex additione vel subtractione quacunque terminorum se mutuo consequentium, *Commentarii academiae scientiarum imperialis Petropolitanae* **3** (1728), 85–100.
2. Jacques Dutka, The Early History of the Hypergeometric Function, *Archive for History of Exact Sciences* **31** (1984–1985), 15–34.
3. Leonhard Euler, De progressionibus transcendentibus seu quarum termini generales algebraice dari nequent. *Commentarii academiae scentiarum imperialis Petropolitanae* **5** (1730–1731), 36–57, or *Leonhardi Euleri Opera omnia. Series I: Opera mathematica*, Bern, 1911–1975, (1) 14:1–24.
4. Leonhard Euler, De summatione innumerabilium progressionum, *Commentarii academiae scientiarum Petropolitanae* **5** (1730–1731), 91–105, or *Opera omnia*, (1) 14:25–41.
5. Leonhard Euler, Methodus generalis summandi progressiones, *Commentarii academiae scientiarum Petropolitanae* **6** (1732–1733), 68–97, or *Opera omnia*, (1) 14:42–72.
6. Leonhard Euler, De summis serierum reciprocarum, *Commentarii academiae scientiarum Petropolitanae* **7** (1734–1735), 123–134, or *Opera omnia,* (1) 14:73–86.
7. Leonhard Euler, De progressionibus harmonicis observationes, *Commentarii academiae scientiarum Petropolitanae* 7 (1734–1735), 150–156, or *Opera omnia*, (1) 14:87–100.
8. Leonhard Euler, Methodus universalis serierum convergentium summas quam proxime inveniendi, *Commentarii academiae scientiarum Petropolitanae* **8** (1736), 3–9, or *Opera omnia*, (1) 14:101–107.
9. Leonhard Euler, Inventio summae cuiusque seriei ex dato termino generali, *Commentarii academiae scientiarum Petropolitanae* **8** (1736), 9–22, or *Opera omnia*, (1) 14:108–123.
10. Leonhard Euler, Methodus universalis series summandi ulterius promota, *Commentarii academiae scientiarum Petropolitanae* **8** (1736) 147–158, or *Opera omnia*, (1) 14:124–137.
11. Leonhard Euler, De seriebus quibusdam considerationes, *Commentarii academiae scientiarum imperialis Petropolitanae* **12** (1740), 53–96, or *Opera omnia*, (1) 14:407–462.
12. Leonhard Euler, *Introductio in analysin infinitorum*, Lausannae: M. M. Bousquet et Soc., 1748, or *Opera omnia*, (1) 8–9.
13. Leonhard Euler, Methodus equationes differentiales altiorum graduum integrandi ulterius promota, *Novi commentarii academiae scientiarum imperialis Petropolitanae* **3** (1750–1751), 3–35, or *Opera omnia*, (1) 22.
14. Leonhard Euler, De serierum determinatione seu nova methodus inveniendi terminos generales serierum, *Novi commentarii academiae scientiarum imperialis Petropolitanae* **3** (1750–1751), 8–10 and 36–85, or *Opera omnia,* (1) 14:463–515.
15. Leonhard Euler, Consideratio quarundam serierum quae singularibus proprietatibus sunt praeditae, *Novi commentarii academiae scientiarum imperialis Petropolitanae* **3** (1750–1751), 10–12 and 86–108, or *Opera omnia*, (1) 14:516–541.
16. Leonhard Euler, De seriebus divergentibus, *Novi commentarii academiae scientiarum imperialis Petropolitanae* **5** (1754–1755), 19–23 and 205–237, or *Opera omnia*, (1) 14:583–588.
17. Leonhard Euler, *Institutiones calculi differentialis cum eius usu in analysi finitorum ac doctrina serierum*, 1755, or *Opera omnia*, (1) 10.

18. Leonhard Euler, De curva hypergeometrica hac aequatione expressa $y = 1 \cdot 2 \cdot 3 \cdots x$, *Novi commentarii academiae scientiarum imperialis Petropolitanae* **13** (1768), 3–66, or *Opera omnia*, (1) 28:41–98.

19. Leonhard Euler, Delucidationes in capita postrema calculi mei differentialis de functionibus inexplicabilibus, *Opera omnia* (1) 16_1:1–33.

20. Leonhard Euler, Methodus succincta summas serierum infinitarum per formulas differentiales investigandi, *Mémoires de l'Académie des Sciences de St. Pétersbourg* **5** (1812), 45–56, or *Opera omnia*, (1) 16_2:200–221.

21. Giovanni Ferraro, Some Aspects of 18th-Century Infinitesimal Analysis: Functions, Functional Relations and the Law of Continuity in Euler, unpublished manuscript.

22. Giovanni Ferraro, The Value of an Infinite Sum: Origin of Euler's Notion of the Sum, *Sciences et techniques en perspective,* to appear.

23. Giovanni Ferraro, Rigore e dimostrazione in matematica alla metà del settecento, *Physis*, to appear.

24. Craig Fraser, The Calculus as Algebraic Analysis: Some Observations on Mathematical Analysis in the 18th Century, *Archive for History of Exact Sciences* **39** (1989), 317–335.

25. Michael C. Friedman, *Kant and the Exact Sciences*, Cambridge, MA/London: Harvard Univ. Press, 1992.

26. Paul H. Fuss, *Correspondance mathématique et physique de quelque célèbres géomètres du XVIIIème siècle*, St. Pétersbourg: Académie impériale des sciences, 1843.

27. Carl F. Gauss, Disquisitiones generales circa seriem infinitam

$$1 + \frac{\alpha\beta}{1 \cdot \gamma} x + \frac{\alpha(\alpha+1)\beta(\beta+1)}{1 \cdot 2 \cdot \gamma(\gamma+1)} xx + \frac{\alpha(\alpha+1)(\alpha+2)\beta(\beta+1)(\beta+2)}{1 \cdot 2 \cdot 3 \cdot \gamma(\gamma+1)} x^3 + \text{etc.},$$

Werke, Göttingen: 1863–1933, 3:124–162.

28. Christian Goldbach, Specimen methodi ad summas serierum, *Acta eruditorum* (1720), 27–31.

29. Christian Goldbach, De Terminis generalibus serierum, *Commentarii academiae scientiarum imperialis Petropolitanae* **3** (1728), 164–173.

30. Victor J. Katz, The Calculus of the Trigonometric Functions, *Historia Mathematica* **14** (1987), 311–324.

31. Stella Mills, The Independent Derivations by Leonhard Euler and Colin Maclaurin of the Euler–Maclaurin Summation Formula, *Archive for History of Exact Sciences* **33** (1985), 1–14.

32. Marco Panza, *La forma della quantità*, Cahiers d'historie et de philosophie des sciences, vols. 38 and 39, Nantes: Presses de l'Université, 1992.

33. Marco Panza, L'intuition et l'évidence: La philosophie kantienne et les géométries non euclidiennes: relecture d'une discussion, in *Les savants et l'épistémologie vers la fin du XIXeme siècle,* ed. Marco Panza and Jean-Claude Pont, Paris: Blanchard, 1995, pp. 39–87.

34. John Wallis, Arithmetica infinitorum, in *Opera mathematica*, Hildesheim: Georg Olms Verlag, 1972.