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Differentials and differential coefficients in the Eulerian foundations of the calculus

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Dedicated to P.L. Butzer, L. Carbone, F. Jongmans, F. Palladino, and the ethics of scientific research

Abstract

In the 18th-century calculus the classical notion of quantity was understood as general quantity, which was expressed analytically and was subject to formal manipulation. Number was understood as the measure of quantity; however, only fractions and natural numbers were considered numbers in the true sense of term. The other types of numbers were fictitious entities, namely ideal entities firmly founded in the real world which could be operated upon as if they were numbers. In this context Eulerian infinitesimals should also be considered as fictitious numbers. They were symbols that represented a primordial and intuitive idea of limit, although they were manipulated in the same way as numbers. This conception allowed Euler to consider calculus as a calculus of functions (intended as analytical expressions of quantities) and, at the same time, to handle differentials formally.
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Sommario

Nel diciottesimo secolo la classica nozione di quantità fu sviluppata fino ad essere intesa come quantità generale, la quale, analiticamente espressa, era soggetta a manipolazioni formali. Il numero era inteso come una misura della quantità; tuttavia solo le frazioni e i naturali erano considerati numeri nel vero senso del termine. Le altre specie di numeri erano entità fittizie, cioè entità ideali ben fondate nel reale che potevano essere manipolate come numeri. In tale contesto anche gli infinitesimi euleriani sono da intendersi come numeri fittizi. Essi erano simboli che rappresentavano una primordiale e intuitiva idea di limite e che potevano essere trattati come veri numeri. Tale concezione permetteva ad Euler di considerare il calcolo come un calcolo delle funzioni (intese come espressioni analitiche della quantità) e, allo stesso tempo, di operare formalmente con i differenziali.
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1. Introduction

This article examines the Eulerian notions of the differential and the differential coefficient with the aim of contributing to an understanding of the foundations of the calculus in the 18th century.[1]

In the initial part of the paper I shall deal with the notions of quantity, fictitious numbers, and formal manipulations, which are the basis of Euler's conception. Quantity was considered to be anything which could be augmented or diminished; it was connected with the idea of number as measurement and with the notion of the continuum, which was not reducible to points. However, quantity was considered in the calculus as a general quantity which had a symbolic nature and included fictitious numbers. Fictitious numbers were ideal entities which were useful for dealing with quantity and were handled as if they were true numbers (integers and fractions). They were well founded in nature but lacked a theoretical construction and differed ontologically from true numbers. Formal manipulations consisted in the fact that general quantities, analytically expressed, were handled regardless of the conditions of validity of the rules and the nature of their specific values.

In the central part of this paper, I shall show that the evanescent or infinitesimal quantity, to which Euler unquestionably reduces the differential,[2] is to be included in such a context. It was based upon a primordial idea of approaching a limit, which played the role of a basic intuition providing evanescent quantities with well-foundedness and semantic meaning. This idea was expressed by symbols which were operated upon by analogy with true numbers. For this reason evanescent quantities should be regarded as fictitious entities.

Last, in the final section of the article, I shall illustrate how this conception allowed Euler, on the one hand, to employ the differential coefficient as the ratio of differentials and, on the other hand, to state that the true object of the calculus was not differentials but differential coefficients and that the algorithm of the calculus did not transform differentials into differentials but functions into functions.

2. Quantities and fictions

Following the traditional approach, Euler conceived of mathematics as the science of quantity and defined quantity as something that could be increased or diminished.[3] It is clear that this definition of quantity is unsatisfactory when it is compared to modern mathematical definitions. It is even tautological: increasing means making the quantity larger while decreasing makes the quantity smaller. The definition of quantity recalls certain definitions of Euclid's *Elements*, such as 'A point is that which has no part'; 'A line is breadthless length,' which specify the sense of certain geometrical terms by referring to the pre-mathematical notions of part, breadth, length, and linking the mathematical theory with objects outside

[1] I have dealt elsewhere with other aspects of the Eulerian foundations of the calculus, in particular the notions of the sum of a series and of a function. See Ferraro [2000a, 2000b].

[2] The differential was regarded as an infinitesimal obtained by making the increment ω of a variable quantity tend to zero (see Euler [1755, 5–6, 84]). Euler often simply treats differentials and infinitesimals as the same thing (for instance, see Euler [1755, 70]).

[3] Erstlich wird alles dasjenige eine Größe gennent, welches einer Vermehrung oder einer Verminderung fähig ist, oder wozu sich noch etwas hizusetzen oder davon wegnehmen läßt... indem die Mathematic überhaupt nichts anders ist als eine Wissenschaft der Größen [Euler, 1770, 9].

it (in other words, ultimately, with empirical reality). In effect, the objects described by this kind of definition are understood by direct acquaintance (one could speak of an empirical intuition to use a Kantian expression) and the definition only determines certain characteristics that are essential for their use in mathematics.

It should be noted that in the writings of Euler and the other 18th-century mathematicians, the term "quantity" was used not only to denote a variable entity capable of increasing or diminishing but also to indicate specific determinations of this variable entity (the values of the quantity). To avoid confusion, I shall hereafter use the term "quantity" (or, also, "indeterminate quantity" or "variable quantity") to refer to an entity in the sense of its capability to increase or diminish while I use the term "quantum" or "determinate quantity" to denote a specific determination of quantity.

I stress the importance of the distinction between quantity and quantum in the Eulerian calculus: the calculus referred to indeterminate quantities, subject to possible variations, whether increases or decreases, rather than to specific determinations of quantities or determinate quantities. A quantity could assume different values or determinations, although a quantity was not reduced to the enumeration of these values (see Ferraro [2000a, 108]). Indeed a quantity possessed its own properties, which might be false for certain of its determinations. Thus, given any property P of quantity, *there might exist exceptional values at which the property fails* and a theorem involving certain quantities $x, y, \dots,$ was valid and rigorous as long as the variables $x, y, \dots,$ remained indeterminate; however, it might be invalid for certain specific determinations of those quantities, which were regarded as exceptional values. (On the treatment of exceptional values, see Engelsman [1984, 10–13], Fraser [1989, 321].)

The notion of quantity as intrinsically variable entity did not prevent quantities from being divided into constants and variables.[4] However, as Euler explained in the *Institutiones calculi differentialis*, this distinction did not depend on the nature of quantities but on specific questions, quantities being variable in themselves:

> [T]his calculus deals with variable quantities, even though every quantity, by its very nature, can be increased or diminished *in infinitum*; however, as long as the calculus is addressed toward a certain goal, some quantities are designed to maintain the same magnitude constantly while others are truly changed for each amount of increase and decrease: ... the former quantities are usually termed constants, the latter variables, so that this difference is not expressed so much in terms of the nature of the thing as in the character of the question to which the calculus refers.[5]

As an example, Euler observed that the trajectory of a bullet was determined by four quantities: the amount of gunpowder, the angle of fire, the range, and the time. Each of them was a quantity in the sense that it could be increased or reduced. This property was never lost, though in certain calculations it was utilized and not in others: in this sense, a quantity could be imagined as a variable or constant according to the specific calculation.

[4] In the *Introductio in analysin infinitorum*, Euler defined: "A constant quantity is a determinate quantity which always retains the same value ... A variable quantity is an indeterminate or universal quantity, which comprises all determinate values." "Quantitas constans est quantitas determinata, perpetuo eudem valorem servans... Quantitas variabilis est quantitas indeterminata seu universalis, quae omnes omnino valores determinatos in se complectitur." [Euler, 1748, 17].

[5] "[H]ic calculus circa quantitates variabilis versatur: etsi enim omnis quantitas sua natura in infinitum augeri et diminui potest; tamen dum calculus ad certum quoddam institutum dirigitur, aliae quantitates costanter eandem magnitudinem retinere concipiuntur, aliae vero per omnes gradus auctionis ac diminutionis variari: ... illae quantitates costantes, hae vero variabiles vocari solent; ita ut hoc discrimen non tam in rei natura, quam in quaestionis, ad quam calculus refertur, indole sit positum." [Euler, 1755, 3].

Quantities were usually distinguished as being continuous or discrete. In the classic sense, discrete quantity is made up of discontinuous parts, meaning there is no common boundary at which they join. A continuous quantity consists of parts whose position is established by reference to each other, so that the limit of the one is the limit of the next. The ancient Greeks considered there to be several types of continuous quantities, such as time, movement, and various geometrical quantities (on geometrical quantities in Euclid, see Grattan-Guinness [1996, 363]). After Descartes had showed how dimensional homogeneity could be circumvented (see Bos [1974, 7]), it was assumed that any quantity could be represented by lines. This idea was also shared by Euler (see *Institutiones calculi differentialis* [1755, 65]): in his writings quantity was modeled on the segment of a straight line (or a piece of a curved line, for certain properties of quantities, such as the way in which a quantity goes to zero).

Euler did not discuss the properties of continuous quantity explicitly; he tacitly assumed that continuous quantity behaved as a segment of a straight line or a piece of a curved line. Thus, the Eulerian continuum is a slightly modified version of the Leibnizian continuum, as described by Breger [1992a, 76–84], which, in turn, has many aspects in common with the classical Aristotelian conception.[6] I point out some features of this conception.

First, a segment was divisible into parts, each of which was similar in kind to the original quantity, but it could not be reduced to an aggregate of points. Thus, the continuum was given as a whole and was not regarded as a set of points, even though it was possible to determine specific points in it.

Second, for the precise reason that a segment was not considered as a set of points it was impossible to distinguish between an open and a closed segment: a segment is always thought of as including its endpoints. Breger stated: “One cannot, e.g., consider the interval from 0 to 1 without the point zero. Imagine a meter long thread without the left extremity of the thread. It is clearly an absurdity. Precisely in the same way, the point zero is not a part of continuum ... but its extremity on the left: the point cannot be suppressed, not even in thought.”[7]

Third, a curve or a relation between quantities was not defined pointwise. An equation, such as $y = x^2$, was viewed as a relation that assigned an interval on the y-axis to an interval on the x-axis in an appropriate way [Breger, 1992a, 77]. Curves were generated by motion; they were not plotted.[8]

It is possible to draw a distinction concerning the way continuous quantity was treated, a point which is of crucial importance in Euler’s calculus.[9] Continuous quantities could be referred to a concrete and perceptible representation in a diagram and investigated, at least partially, by means of the diagram itself. In this sense, one can speak of *geometrical quantities or figural quantities*. Continuous quantities could also be investigated in abstract and general form by means of a symbolic notation. In the latter sense, one can speak of *general quantities* or *abstract quantities* or *analytical quantities or nonfigural quantities* (the different terms underline different features of the notion). Geometrical quantities were the main subject-matter of Leibniz’s and Newton’s calculus. (This does not mean that they did not use analytical

[6] On the Aristotelian notion of continuum, see Barreau [1992, 3–15], Panza [1989, 39–80].

[7] “... on ne peut par exemple pas considérer l’intervalle de 0 à 1 sans le point zéro. Imaginez un fil d’un mètre de long sans l’extrémité gauche du fil. Cela est visiblement une absurdité. Exactement de la même façon, le point zéro n’est pas une partie du continu ... mais seulement son extrémité à gauche : le point zéro ne peut donc être supprimé, même en pensée.” [Breger, 1992b, 77].

[8] This expression is derived from Mahoney’s description of the notion of curve in Fermat [Mahoney, 1994, 82]. See also Ferraro [2000a, 121].

[9] This distinction is treated in Ferraro [2001].

expressions but simply that these expressions were embodied in a geometrical context.) In contrast, Euler thought that the calculus was independent of geometry (see Fraser [1989, 328–331], Bos [1974, 4]) and of its figural representation and that it only dealt with general quantity.

Unlike geometrical quantity, general quantity was not represented by a line in a diagram (in this sense one can term it nonfigural quantity). It was, however, closely connected with geometrical quantity and has to be imagined as an abstract entity made up of what all the geometrical quantities have in common (one therefore can refer to it as an abstract quantity). For this reason, even if general quantity was not represented in a diagram, it was assumed to have the properties of a "nice" or "well-behaved" curved line. This implies that the basic notions of continuous geometrical quantities were immediately transferred to the calculus. For instance, a variable quantity x always varied continuously and when it moved from a value x_1 to a value x_2, it was impossible to think of this variation without the initial and final values.

General quantity was investigated symbolically by analytical expressions (in this sense it can be termed analytical quantity). I specify that the decisive aspect of analytical symbolism was not the use in itself of certain signs but the fact that those signs were the objects of manipulation in their own right. For instance, I can write $a \perp b$ to indicate that the straight line a is perpendicular to b. However, if in the proof of a theorem of elementary geometry, for instance "Given a point A and a straight line a, there exists one and only one straight line b perpendicular to a and passing through A," I write $\perp$ in place of "perpendicular to," I do not really manipulate the symbol $\perp$ by itself, but work with the concept of "perpendicular to." The sign $\perp$ is employed as a mere shorthand symbol, unless one establishes a calculus upon $\perp$ and operates according to the rule of this calculus.

In symbolic expressions, such as $(a + b)^2 = (a + b)(a + b) = a^2 + 2ab + b^2$, the letters $a, b, \ldots,$ are used as the concrete objects of a calculation (see Panza [1992, 68–69]). According to Leibniz, a calculation is *cogitatio caeca*, blind reasoning. It can be compared to moving pebbles in an abacus: what is of importance is that the concrete objects of manipulation (pebbles or graphical signs) are handled according to certain rules (syntactically, in modern terms), not their meaning. Of course, algebra and analysis cannot be reduced to the mechanical or blind manipulation of letters. It is not only a matter of the inventiveness necessary to derive formulas that are not reduced to a simple exercise, as in the example; instead, the point is that doing mathematics does not merely consist of deriving formulas but of deriving formulas that have an interest or a sense in a certain context.

This is also true for modern mathematics. A theorem T of a formal theory is the last proposition of a sequence of propositions P_i, $i = 1, \ldots, n$, where $\mathrm{P}_n = \mathrm{T}$ and P_i, $i = 1, \ldots, n-1$, is an axiom or is deduced by a rule of inference from the preceding propositions. While all derivable propositions in the given theory are theorems in this sense, in mathematical praxis, only some propositions (significant for whatever reason) are theorems. The decision that P_n is a theorem, while P_{n-1} is not, is not part of the formal structure of theory. However, the goal of a formal theory is to yield theorems in this more restricted sense (see Panza [1997, 366–367]).

I would argue that the nature of analytical or algebraic derivations is necessarily syntactical and, as such, one handles signs associated with certain rules regardless of the meaning of the objects of calculation; however, the syntactical rules that govern analytical signs must make sense for the mathematician and must yield results that make sense or have some interest.

Eulerian general quantity surely has a symbolic nature in the above sense: it was reified into concrete signs, which were dealt with according to certain fixed transformations. However, the way in which the syntactical structure was constructed differed profoundly from the way it is conceived today. Today the rules[10] used in a theory are explicit axioms, which in principle are freely chosen, or, to use a widely employed term, arbitrary. Within the limits of the given system of axioms, mathematical objects can freely be created by arbitrary definitions.[11] In this way, the development of a theory is entirely syntactical and it is possible to make a distinction between syntactical correctness and semantic truth.

This is not the case for Euler. The idea of the free creation of mathematical objects was lacking in Eulerian analysis. Analytical objects were always connected with reality, directly or indirectly.[12] The rules of manipulation were not arbitrary: they were derived from the notion of quantity and expressed properties of quantities (or of numbers). For instance, $a + b = b + a$ is not an arbitrary axiom associated with the operation $+$ (which we may or may not choose, according to the objectives of our theory); it was a mere consequence of the concept of joining two quantities.

A system of explicit axioms in the modern sense and an accurate construction of certain mathematical objects (e.g., the construction of the different species of numbers) were lacking. In their place, Eulerian mathematics admitted the reference to the intuitive knowledge of the mathematical notions drawn from premathematical experiences.[13]

Moreover, even though signs were manipulated syntactically (blindly), analysis mirrored reality and it was impossible to distinguish a syntactically correct theory from a semantically true theory: a theory was acceptable only if it conformed to the reality.[14] Since reality is *unique*, alternative theories based on alternative definitions of certain notions (e.g., the sum of a series and limit of a sequence) could not exist.

Another fundamental aspect of the Eulerian conception, which has so far been left implicit in my argument, is the relationship between quantity and numbers. In his *Vollständige Anleitung zur Algebra*, Euler stated that all the determinations or measures of any quantity are reduced to determining the relation that a given quantity has with a certain quantity of the same kind taken as a measure or unity: "[this relation] is always indicated by numbers, so that a number is nothing but the relation of a quantity to another quantity, taken arbitrarily as a unity."[15] According to Euler, numbers were taken into account in analysis as they represented quantities considered in general without regard of the difference that existed between the special types of quantities (other parts of mathematics, he says, concern the specific types of

[10] It is clear that by "rules of manipulation" I do not intend rules of inference, but rules of the type $ab = ba$, which in modern formal theory are axioms (or theorems derived from axioms).

[11] Here freedom and arbitrariness do not mean that one chooses the system of axioms and gives definitions without reason; rather, it means that axioms and definitions are fixed by an act of will determined by the targets that one wants to achieve, without other restraint than the achievement of such targets. Axioms and definitions have no intrinsic necessity, nor do they consist in a description of physical or geometrical reality; however, they must have the capability of representing certain concepts adequately.

[12] If no intuitive interpretation of them was known—e.g., imaginary numbers—they were viewed as tools for improving the analytical theory of quantity—which, as a whole, was a mirror of reality—in the same manner as the sign 0 improves the notation of natural numbers that count objects even though it denotes no object.

[13] An example is the definition of quantity discussed above. Another is the rule $dy = y(x + dx) - y(x)$, the use of which we shall see later. According to Euler $dy = y(x + dx) - y(x)$ is not a definition of continuous function, but an intuitively obvious property of continuous quantity.

[14] As noted in Footnote 12, certain analytical objects did not represent reality in any obvious or straightforward way; however, they were indirectly connected with reality in the sense that they were part of a theory that aimed to conform to reality.

[15] "[Verhältnis] jederzeit durch Zahlen angezeigt wird, so dass eine Zahl nichts anders ist als das Verhältnis, worinnen eine andere, welche für die Einheit angenommen wird, steht." [Euler, 1770, 10].

quantities) [1770, 10]. Quantity was considered as an entity that logically precedes number and number was viewed as a tool for treating quantity.

The concept of number as the measure of quantity was a commonplace at least from the seventeenth century.[16] It allowed mathematicians to go beyond the Greek concept of number as "number of ...," multiplicity of unities, and made it possible to think of numbers as abstract and symbolic entities and to introduce new species of numbers in addition to natural numbers (see Klein [1968]). Euler also considered natural numbers as abstract and symbolic entities: a number, such as 7, was not considered as the attribute of a group of material or ideal objects; instead it was an abstract entity that expressed what all the things that are seven times the unity had in common. The number 7 was also a symbol that reified an ideal entity into ciphers upon which one manipulates directly.

In his treatise, after having defined numbers as the measure of quantity, Euler observed that the sequence of natural numbers is generated from repeatedly adding the unity starting from nothing [1770, 14].[17] Euler did not give an (explicit or implicit) definition of natural numbers: the relationship between a measurement of quantity and natural numbers was understood substantially intuitively.

Euler considered a fraction a/b to be the result of the division of two whole numbers a and b. Fractions were introduced in a formal way, namely, without explaining what the division of a by b means when a is not a multiple of b, although they had an exact meaning. Euler stated that we can have a clear idea of $7/3$ by considering a segment 7 feet in length and by dividing it into 3 parts [1770, 30]. Unlike fractions, irrational numbers did not represent a process of measurement in a precise sense (measuring meant repeating the operation of comparing with unity or one of its parts successively and finitely). Euler observed that the root of 12 is not a fraction. Nevertheless, it is a determinate quantity, which is greater than 3; $\frac{24}{7}$; $\frac{38}{11}$; $\frac{45}{13}, \ldots$, and smaller than 4; $\frac{7}{2}$; $\frac{52}{15}$; $\ldots$. Therefore, $\sqrt{12}$ is a new species of number. He then added that a correct idea of $\sqrt{12}$ can be gathered by observing that $\sqrt{12}$ is the number that, when multiplied by itself, makes 12 and that the value of $\sqrt{12}$ can be approximated as desired [1770, 50–51].[18]

Irrational numbers were significantly different from natural and fractional numbers. The latter had a meaning in terms of unity of measure and consequently were numbers in the strict sense of the term, or "true numbers." In contrast, irrational numbers were not true numbers since they were thought to measure quantity only in an approximate way.

This conception was a widely shared one during the 18th century. For instance, according to d'Alembert (see [1773, 188]), the extension of the term "number" to incommensurable ratios was considered incorrect because "number" presupposes an exact and precise denotation. Nevertheless, incommensurable ratios could be viewed as numbers because they were similar to "numbers"; they could be approached as closely as desired by "numbers" and could be represented geometrically.

The domain of true numbers was not sufficient to describe all the determinations of geometrical quantity and, moreover, other numbers apart from rational or irrational ones were necessary to investigate quantity: apart from infinitesimals and infinite numbers, which I shall deal with in the next section, there

[16] For instance, it can be recognized in Newton [1707, 2], Stevin [1585, 1], and Wallis [1657, 183]. On the notion of number from Stevin to Wallis I refer to Klein [1968, 186–224].

[17] I note that Euler shared the idea that 1 is a number, which was finally accepted only in the 17th century (see, e.g., Klein [1968, 194]).

[18] In Euler one may be able to grasp a distinction between irrational and transcendental numbers, where transcendental numbers are those nonrational numbers that derived from the application of transcendent operations (such as a logarithm) to a rational number. However, this distinction is not of importance for my purposes and I shall not take it into account.

are negative numbers, zero, and imaginary numbers. Euler [1770, 12–15] introduced negative numbers simply by stating that they were entities less than the nothing and that were represented by numbers with the sign – (in opposition, positive numbers were numbers greater than nothing and had the sign $+$). Like the other species of numbers above mentioned, they also had an intuitive meaning. Euler stressed that they could denote debts, or proceeding backward (e.g., the sequence $-1, -2, -3, \ldots$, proceeds backward with respect to the sequence $1, 2, 3, \ldots$). They can be represented by directed segments; however, they did not correspond to a notion of measurement of a quantity in the strict sense of the term.

In [1770, 14] zero is introduced merely as the absence of quantity; it is the name given to the "nothing." Euler did not list zero as an integer (integers were the natural numbers $+1, +2, +3, \ldots$, which are greater than nothing, and negative numbers were $-1, -2, -3, \ldots$, which are less than nothing).[19]

According to Euler, expressions such as $\sqrt{-1}$, $\sqrt{-2}$, $\sqrt{-3}$, $\sqrt{-4}$ are impossible or imaginary numbers: nevertheless they could be represented in our understanding and take a place in our imagination. We can gain a sufficient idea (hinlänglichen Begriff) of them based on the fact that, e.g., $\sqrt{-4}$ meant a number that multiplied by itself equals -4 [1770, 56]. In *Vollständige Anleitung zur Algebra*, Euler emphasized the similarity in the introduction of imaginary and irrational numbers (as formal instruments for obtaining the roots of certain numbers). However, imaginary numbers cannot be reduced to the measurement of quantity, not even in an approximate sense. They differed from other numbers and were generated by the symbolic mechanism of analysis; they had no meaning on their own but assumed a meaning within the overall context of analysis and were useful for dealing with quantity.

At this point it is clear that even though all numbers were abstract and symbolic entities, only some adequately reflected the concept of number as the exact result of a process of measurement and were "true" numbers. Other types of numbers did not fit the notion of a number (although for different reasons). In the strict sense of the term they were not true numbers. I shall term them "fictitious numbers" or "fictions."

The idea of false or fictitious numbers is an old one. For instance, many mathematicians, such as Cardano and Descartes,[20] referred to negative numbers as false numbers. In various cases Leibniz attempted to justify infinitesimal and infinite numbers as fictions[21] similar to other fictions used in mathematics (imaginary numbers, the power whose exponents are not true numbers, etc.) (see, e.g., Leibniz [GM IV, 92–93]). In effect, the idea of false numbers is at the basis of much of mathematical terminology regarding numbers, which we still partially retain today.

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[19] In his [1770, 14], following Stevin and Wallis, Euler seems to consider nothing, zero, as the principle of natural numbers, in the same sense as the point is the principle of the line.

[20] See Cardano [1545], Descartes [1637]. This is what Descartes wrote when commenting upon the roots of an equation: "But often it happens, that some of these roots are false, or less than anything, as if one supposes that x indicates also the defect of a quantity, which is 5, one has $x + 5 = 0$, which being multiplied by $x^3 - 9xx + 26x - 24 = 0$ is $x^4 - 4x^3 - 19xx + 106x - 120 = 0$, for an equation in which there are four roots, namely three true ones, which are 2, 3, 4, and a false one which is 5." "Mais souvent il arrive que quelque une des ces racines sont fausse ou moindres que rien ; comme si on suppose que x désigne aussi le défaut d'une quantité qui soit 5 on a $x + 5 = 0$ qui multiplié par $x^3 - 9xx + 26x - 24 = 0$ fait $x^4 - 4x^3 - 19xx + 106x - 120 = 0$, pour une équation en laquelle il y a quatre racines, à savoir trois vraies qui sont 2, 3, 4 et une fausse qui est 5." [Descartes, 1637, 56].

[21] On Leibniz's conception, see Horvàth [1982, 1986] and, above all, a very stimulating paper by Knobloch [1999].

Unlike Leibniz, Euler did not use the term "fiction" explicitly.[22] Nevertheless, I shall employ this expression because it expresses the nature of the Eulerian approach, in particular because it implies an ontological difference between that which is fictitious and that which is true. Eulerian mathematics effectively presents *an ontological difference* between natural and rational numbers (true numbers) and the other species of numbers (which did not correspond to the idea of numbers and therefore were fictitious numbers). To put it more clearly, nowadays $\sqrt{-1}$ is an element of the set of complex numbers C and exists in the same way as any other number in C, such as 1, 2, $1/2$, etc. In Euler's opinion, $\sqrt{-1}$ was a useful symbol for studying certain aspects of quantity; it did not have an existence in the same sense as true numbers. Similarly, 0 was the symbol that represented the absence of quantity, the nothing, the nonexistence; it was not a number because it did not measure quantity and did not denote anything, however 0 could be treated as a number. *Mutatis mutandis*, the same holds for irrational (unspeakable, inexpressible) numbers and negative numbers. Fictions had the following characteristics.

(a) Fictions were a useful tool for shortening the path of thought and arriving at new results. It was of no importance whether fictions appeared in nature or not, namely if they represented physical or geometrical objects. Irrational numbers appeared in nature (they represented the length of a segment); imaginary numbers did not appear.
(b) Fictions, however, were always connected with reality, directly or indirectly. They were not arbitrary creations of the human mind but had to be well-founded in reality and were needed for investigating reality (this is true even for imaginary numbers; see Euler [1770, 57]).

By the phrase "well-founded in reality or in nature"[23] I intend to emphasize the fact that certain mathematical objects did not originate from arbitrary definitions given in a theory based upon an arbitrary system of axioms; instead, they originated (1) from the need to express certain properties of quantities and (2) from the need to manipulate objects that directly expressed quantities or properties of quantities (an example would be the *casus irreducibilus* case of the equation of third degree). In the first case, a well-founded object had an intuitively obvious interpretation (e.g., irrational numbers but also, as we shall see, infinitesimals). For this reason I would say that they were directly connected to reality.[24]

The second case was that of imaginary numbers, which did not have an intuitively obvious meaning (see also Footnote 12). They were introduced in a merely formal way but they made up for rational and irrational numbers when these did not suffice: they were always connected to reality, even though only indirectly.

In any case, well-foundedness, used in this sense, excludes the possibility that mathematical objects could originate from a free act of will and required them to be rooted in reality, directly or indirectly, as elements of a theory that aimed to interpret the real.

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[22] Only in *Institutionum calculi differentialis Sectio III* (Euler had intended this work, which was published posthumously in 1862, to form the third part of the treatise on differential calculus, where he applied the calculus to geometry) is an appeal made to the *fictio animi* that a segment does not so much represent itself, so to speak, but an infinitesimal part of itself, dx (see Euler [A]).

[23] I derive this expression from Leibniz's statement that imaginary quantities have their foundation in nature [Leibniz, 1849–1850, IV:93]; see also Leibniz [1875–1890, VII:263–264].

[24] However, this does not mean that there exists a geometrical or physical object corresponding to it; for example, Euler did not assume the existence of physical or geometrical infinitesimals. See Section 3.

It should also be emphasized that fictions were not of interest in themselves, but only insofar as they allowed one to solve problems concerning quantities. They were auxiliary instruments for dealing with quantities.

(c) Fictions *were manipulated as if they were true numbers*. This means that a fiction was treated by analogical extensions of rules valid for true numbers or geometrical quantities.[25] Therefore, a fictitious number was more than a mere *façon de parler* or a shorthand way of denoting a certain operation upon true numbers: it was symbolic entity that formed part of the symbolic nature of true numbers and quantities.
(d) *An adequate theoretical construction* for moving from fictions as a sign for shortening the path of thought to the analogical use of fictions as true numbers *was completely lacking*. Thus, well-foundedness in the above sense was the only justification for fictions.
(e) Even though quantity is an entity abstracted from geometrical quantity and has the same properties as lines, it could be determined by fictitious values; in other words, one could assign fictitious numbers to a variable x.

Like other 18th-century mathematicians, Euler used the term “irrational quantities” to refer to irrational numbers or irrational determinations of quantity. Similarly he referred to negative quantities, imaginary quantities, etc. I shall maintain this terminology and, more generally, I use the term “fictitious quantities” by referring to fictitious determinations of quantity or fictitious numbers.

A general quantity has some determinations that can be represented by a nondirected segment, whereas others cannot. I shall use the term “real quantity” to denote a quantity which only assumes these determinations and which corresponds to the mental image of the geometrical or physical quantity. I do not therefore intend this term in opposition to fictitious quantity, since a real quantity can have both true numbers and certain fictitious numbers as its determinations.[26]

The above discussion allows a more precise characterization of general quantity. General quantity was an abstract entity that had the same properties as geometrical quantities but was capable of assuming any value, even fictitious values. It was represented by graphic signs which were manipulated according to appropriate rules, which were the same rules that governed geometrical quantities or true numbers. The principle of the generality of algebra held: the rules were applied in general, regardless of their conditions of validity and the specific values of quantity. (I shall later use the term “formal manipulation” to refer to

[25] This statement is to be understood as follows. The principle was assumed that if an operation (not only algebraic operations—sum, product, etc.—but also transcendental operations—logarithm, etc.) had true numbers or geometrical quantities as operands then it could have fictitious numbers as operands. Of course, some adjustments might be necessary. They took the form of specific rules inherent to the peculiar nature of every distinct species of fictitious numbers. For instance, the rule of signs was a specific rule for negative numbers. These specific rules were what distinguished a calculation involving a particular species of fictitious numbers from a calculation involving true numbers or a different species of fictitious numbers.

[26] Irrational numbers do not correspond to the idea of number and, therefore, are fictions; however, they have a very special nature with respect to other fictitious numbers since they can be represented by means of a nondirected segment and answer the question: What is the measure of a given (real) geometrical quantity? Rational numbers answer the same question (though in a more precise way) and thus these rational and irrational numbers might be grouped together to form the class of (positive) real numbers (and in effect rational and irrational numbers were often taken together, for instance as opposed to imaginary numbers; see Euler [1748, 18]). By so doing one obtains a different classification, which considers the capacity of numbers to express the determinations of geometrical quantities directly, but this is not relevant to my purposes in this paper.

the fact that general quantity, expressed in an analytical form, was handled regardless of the nature of its specific values and the condition of validity of the rules of manipulation.)

This concept was at the heart of the Eulerian notion of function. An Eulerian function was a relationship between general quantities. Because of the symbolic nature of general quantity, this relation was always understood to be symbolically represented. Symbolic representation, which was usually known as an analytical expression, could assume any value: it was handled regardless of the nature of these values and the condition of validity of the rules of manipulation. I have not dwelt upon the concept of a function in the 18th century, since it has been investigated elsewhere in other works to which I refer (see Ferraro [2000b], Fraser [1989], and Panza [1996]). I restrict myself to mentioning an example of this concept: the extension of the rules of the function $\log x$ to negative or complex values of x. Euler never defined "$\log x$" for negative or complex numbers but merely assumed in an unproblematic way that the properties of the analytical expression "$\log x$" lasted beyond the initial interval of definition, even when x is negative or imaginary. Thus, in *Institutiones calculi differentialis*, once he had established that $d(\log x) = dx/x$, Euler did not hesitate to apply this formula to the case where the variable was negative or imaginary, without making any distinction between real and imaginary variable. For instance, in Euler [1755, 124], he found that the differential of the function

$$y = \frac{1}{\sqrt{-1}} \log\bigl(x\sqrt{-1} + \sqrt{1-x^2}\bigr)$$

is

$$dy = \frac{dx}{\sqrt{1-x^2}}.$$

More explicitly, in [1749] Euler had stated: "For, as this calculus concerns variable quantities, that is quantities considered in general, if it were not generally true that $d(\log x) = dx/x$, whatever value we give to x, either positive, negative, or even imaginary, we would never able to make use of this rule, the truth of the differential calculus being founded on the generality of the rules it contains."[27]

To conclude this section, I wish to make some simple consequences of the above described notions of quantity and numbers explicit. First, since a single number was a specific determination of quantity, a single number expressed a quantum rather than a quantity. Second, even though each specific determination of real quantity can be represented by means of numbers, the idea that quantity might be reduced to a set of numbers was not taken into account. Third, more generally, numbers were not conceived of as elements of a set, if by "set" one means an extensional entity, which is arbitrarily defined, entirely characterized by the list of its elements, and having a certain cardinality. Instead, Euler classified numbers into different classes or species, where a "species" of numbers is intended as an intensional entity, which cannot be reduced to an enumeration of objects: it is given by a nonarbitrary and nontrivial property and is not necessarily associated with cardinality.

[27] "Car, comme ce calcul roule sur des quantités variables, c'est-à-dire sur des quantités considérées en général, s'il n'etoit pas vrai généralement qu'il tût $d.lx = dx/x$, quelque quantité qu'on donne à x, soit positive ou négative, ou même imaginaire, on ne pourrait jamais se servir de cette règle, la vérité du calcul différentiel étant fondée sur la généralité des règles qu'il renferme." [1749, 143–144].

3. Infinitesimals as fictitious quantities

Eulerian notions of infinitesimals and infinities should be included within the conception described in the previous section: infinitesimals and infinities were to be viewed as fictitious numbers, which were well founded in real quantities—they symbolized the process of making quantities approach a limit—but were formally manipulated in the context of the theory of functions as analytical expressions.

To justify this statement, I start by observing that Euler considered infinitesimals as evanescent quantities, although he never offered a definition, not even a vague or imprecise one, of the notion of "evanescent quantity." In [1755, 69] he restricted himself to stating: "There is no doubt that every quantity can be diminished until it vanishes completely and is reduced to nothing."[28] This basic principle is the expression of a property of geometrical or physical quantities: one may associate it with the mental image of a physical entity (such as the quantity of gunpowder, in the initial example of the *Institutiones calculi differentialis*) which we can imagine as becoming increasingly smaller; otherwise, still remaining within the field of mathematics, we may imagine an evanescent quantity as a segment which increasingly diminishes until it becomes a single point and disappears as a segment.

Similarly, in [1755, 65] the idea that quantity can be infinitely increased is regarded simply as part of the concept of quantity and has no need of further explanation or clarification. Euler restricted himself to exemplifying this idea by observing that the sequence $1, 2, 3, \dots,$ can always be increased or a straight line can always be continued.[29]

Euler repeatedly stated that an evanescent quantity is zero. For example, in Chapter 3 of *Institutiones calculi differentialis*, he wrote: "an infinitely small quantity is simply an evanescent quantity and therefore actually equal to zero."[30] This statement, which seems strange to modern eyes, was rooted in the 18th-century concept of number and quantity. Any number was a determination of quantity and was generated from the flow of quantities; in particular, zero was generated from a quantity that became nothing. Thus, when zero or another number was used in an analytical expression, it could be thought of as the value of any variable. For instance, in *De progressionibus transcendentibus* [1730–1731, 11–12], Euler sought the value of

$$\frac{1 - x^{\frac{g}{f+g}}}{g}$$

for $f = 1$ and $g = 0$, namely the value of $\frac{1-x^0}{0}$ (see Ferraro [1998]). He interpreted 0 as a value of a

[28] "Nullium [...] est dubium, quin omnis eousque diminui queat, quoad penitus evanescat atque in nihilum abeat." [Euler, 1755, 69].

[29] In Euler's terms: "Sic nemo facile reperietur, qui statuerit seriem numerorum naturalium 1, 2, 3, 4, 5, 6 etc. ita usquam esse determinatam, ut ulterius continuari non possit. Nullus enim datur numerus, ad quem non in super unitas addi sicque numerus sequens maior exhiberi queat, hinc series numerorum naturalium sine fine progreditur neque unquam pervenitur ad numerum maximum, quo maior prorsus non detur. Simili modo linea recta numquam eousque produci potest, ut in super ulterius prolongari non posset. Quibus evincitur tam numerus in infinitum augeri quam lineas in infinitum produci posse. Quae cum sint species quantitatum, simul intelligetur omni quantitati, quantumvis sit magna, adhuc dari maiorem hacque denuo maiorem sicque augendo continuo ulterius sine fine, hoc est in infinitum, procedi posse" [Euler, 1755, 65].

[30] "quantitas infinite parva nil aliud est nisi quantitas evanescens ideoque revera erit = 0." [Euler, 1755, 69].

variable quantity z and, by applying l'Hôpital's rule, found that the value of $\frac{1-x^z}{z}$ as z vanishes (*cum z evanescint*) was

$$\left(\frac{1-x^z}{z}\right)_{z=0} = \left(-\frac{(x^z \log x)\,dz}{dz}\right)_{z=0} = -\log x.$$

Similarly, in order to calculate the value of

$$\frac{x^\alpha - x^{\alpha+m\beta}}{1-x^\beta} \quad \text{and} \quad \frac{\pi^2}{6x(x-1)} + \frac{1}{x(x-1)^2} - \frac{(2x-1)\left(1+\frac{1}{2}+\frac{1}{3}+\cdots+\frac{1}{x}\right)}{x^2(x-1)^2}$$

at the point $x = 1$, Euler took $x = 1-\omega$, with ω infinitesimal (see Euler [1732–1733, 44], Fuss [1843, 2:229–231]).

One may be struck by the similarity between this procedure for "finding" the value of $\frac{1-x^0}{0}$ and the modern problem of extending the function $F(z) = \frac{1-x^z}{z}$ in a continuous way by putting

$$F(0) = \lim_{z\to 0} \frac{1-x^z}{z} \quad \text{for } z = 0.$$

Indeed there is a coincidence of results which might lead one to think that there was a substantial identity between evanescent quantities and limits. Nevertheless, these results are drawn from different assumptions and it is appropriate to clear the field of possible misinterpretations. From a modern perspective, the problem of extending the function $F(z) = \frac{1-x^z}{z}$ in a continuous way means that:

(a) for every fixed value $x > 0$, one considers the function (function in the modern sense of the term, not as an analytical expression) $f(z) = \frac{1-x^z}{z}$ defined for $z \neq 0$;
(b) the domain D of $f(z)$ has a point of accumulation at 0 so that we can attempt to calculate the limit as $z \to 0$, where by $\lambda = \lim_{z\to c} f(z)$ we mean:
given any $\varepsilon > 0$ there exists a $\delta > 0$ such that if z belongs to D and $|z| < \delta$ then $|f(z) - \lambda| < \varepsilon$;
(c) the application of l'Hôpital's rule, under whose hypotheses our case falls, makes it possible to state that such a limit exists and is equal to $-\log x$;
(d) finally, one defines a new function $F(z)$, which will be continuous at the point 0, by setting

$$F(z) = \begin{cases} \frac{1-x^z}{z} & z \neq 0, \\ -\log x & z = 0. \end{cases}$$

This procedure is substantially meaningless for Euler for the reasons that have been seen in the previous section. He did not consider a function as a pointwise correspondence between numerical sets but as a rule that linked two variables quantities and was embodied in one single analytical expression. He had no set of points or numbers, did not separate an interval of values (a segment) from its endpoints, etc., nor could he formulate the notion of extension of a function, but instead considered $\frac{1-x^0}{0} = -\log x$ necessarily to be the value of $\frac{1-x^z}{z}$ when the variable z flows.

However, apart from these crucial differences, there is something in common between the Eulerian procedure and the modern one based upon the notion of limit: evanescent quantities and endlessly increasing quantities were based upon an intuitive and primordial idea of two quantities approaching each other. I refer to this idea as "protolimit" to avoid any possibility of a modern interpretation.

The protolimit, which derives from the observation of physical and geometrical quantities, was the empirical intuition that guaranteed the well-foundedness of infinitesimals, namely the fact that they were not mere creations of our mind but were rooted in reality. Starting from such an intuition, Euler developed the notion of an infinitesimal merely by introducing a symbolism and assuming that one could operate upon them as if they were numbers. Euler posed the question of using different symbols to denote evanescent quantity, which ultimately all became equal to zero. In his own language: "[W]hy do we not always characterize the infinitely small quantities by the same sign 0, instead of using particular symbols to designate them? Since all zeros are equal among themselves, it seems superfluous to discriminate among them by means of different signs."[31]

Euler [1755, 70] justified the use of different signs by assuming that quantities vary and vanish in different ways: there exists a diversity between zeros depending on their origin and the signs $dx, dy, \dots,$ denote how the variables $x, y, \dots,$ vanish.

Although infinitesimals seem to be introduced as shorthand symbols, a *façon de parler*, they were not used exclusively in this way. If the latter were true, one could replace any occurrence of the sign dx by the expression "the variable x goes to zero," in the same manner as one can substitute the symbol "$\perp$" for the expression "perpendicular to" in certain theorems of elementary geometry. This is effectively possible in many cases (and Euler often replaced the infinitesimal dx with a variable $x \to 0$ or merely with 0, and vice versa), but in general it is not possible since dx was treated as a number.

From a modern standpoint, what appears to be critical is that $dx, dy, \dots,$ were neither signs governed by axiomatic and arbitrary rules nor entities constructed from other mathematical entities, as, in contrast, are hyperreal numbers.[32] Euler merely moved from the consideration of variables that vanish or endlessly increase to the consideration of infinitesimal and infinite numbers *in an immediate and natural way* as if *there were no difference between finite quantities, small or large as desired, and infinitesimal or infinite numbers*. Well-foundedness was all that the mathematics of the time required to transform an idea into a symbol governed by the same rules as true numbers. According to the terminology of the previous section, infinitesimals were fictitious numbers.

To found infinitesimals well Euler had to explain the principle of cancellation of differentials, namely the rule according to which

$$dx^n + dx^m = dx^n \quad \text{if } n < m. \tag{1}$$

This was the specific rule characterizing infinitesimals (see Footnote 25). In *Institutiones calculi differentialis*, he stated that, given two quantities a and b, the equality $a = b$ can be understood in an arithmetic sense (in other words, $a = b$, if $a - b = 0$) and in a geometric sense ($a = b$, if $a/b = 1$) [1755, 70, 74]. The arithmetic equality coincides with the geometric one for finite quantities but the situation is

[31] "[C]ur quantitates infinite parvas non perpetuo eodem charactere 0 designemus, sed peculiares notas ad eas designandas adhibeamus. Quia enim omnia nihila sunt inter se aequalia, superfluum videtur variis signis ea denotare." [Euler, 1755, 70].

[32] Modern hyperreals are the elements of a rich and well-organized algebraic structure *R which encodes how a sequence approaches a limit by a complex construction. Here is a possible construction of *R. Let m be a finitely additive measure on the set N of the positive integers such that: for all $A \subset N$, $m(A)$ is 0 or 1; $m(A) = 0$ if A is finite, $m(N) = 1$. (For a proof of the existence of this measure, see Lindstrøm [1988, 84–85].) Now, consider the equivalence relation $\sim$ on the set **S** of the sequence of numbers: $\{a_n\} \sim \{b_n\}$ iff $m\{n\colon a_n - b_n = 0\} = 1$. The set *R of the hyperreal numbers is defined by *R $= \mathbf{S}/\sim$. The classes of equivalence of the sequences $\{0\}$, $\{1/n\}$, $\{1/n^2\}$ are elements of *R and are examples of infinitesimals; the classes of equivalence of the sequences $\{n\}$, $\{n^2\}$, $\{n^3\}$ are three examples of infinite numbers.

different for infinitesimals. For infinitesimals, the arithmetic equation, which is always verified, does not imply a geometric one; namely, $0 = 0$ does not imply $0/0 = 1$.

Then Euler observes that $a \pm n\,dx = a$ (n being an arbitrary number) is true since it is not only verified in an arithmetical sense ($(a \pm n\,dx) - a = n\,dx$, where $n\,dx = 0$) but also in a geometric sense. Indeed, $(a \pm n\,dx)/a$ is equal to 1 and this means that infinitesimals vanish before any finite quantity. The situation is analogous for the powers $dx^2, dx^3, dx^4, \ldots$. Euler observes that "the infinitely small quantity dx^2 vanishes before dx," since the quantities $dx + dx^2$ and dx (both evanescent: $dx + dx^2 = dx = 0$) go to zero in the same way $(dx + dx^2) : dx = 1 + dx = 1$. More generally, if $m < n$, then $a\,dx^m + b\,dx^n$ was equal to $a\,dx^m$, since

$$a\,dx^m + b\,dx^n = dx^m = 0 \quad \text{and} \quad \frac{a\,dx^m + b\,dx^n}{a\,dx^m} = 1 + \frac{a}{b}\,dx^{n-m} = 1,$$

namely the arithmetical and geometrical equalities were verified.

We could briefly say: if $A = B + C$, and if C goes to zero before B (in other words, $A/(B + C) = 1$), then $A = B$. In modern terms, A and B are asymptotically equal or have the same asymptotic behavior.

I do not use this anachronistic terminology to vindicate Euler by attributing to him modern asymptotic notions. I merely wish to point out that the protolimit possessed many different facets and that it should be considered as a basic notion from which later mathematics has derived various modern notions. The crucial point, however, is that once he showed the principle of cancellation as well founded, Euler then used it as a tool for formal manipulations; in other words, a calculation involving this principle is no longer referred to the meaning of (1) in terms of approaching, but (1) is considered as a specific rule for the formal manipulation of infinitesimals.[33]

In order to illustrate the above discussion more clearly, I now investigate one of Euler's proofs, the derivation of the series expansion of the exponential function, recently investigated in Laugwitz [1989], McKinzie and Tuckey [1997] from a different point of view (see below). In Chapter V of the *Introductio in analysin infinitorum* [1748, 122] Euler stated that if a is a number greater than one and ω and ψ are infinitesimals, then $a^\omega = 1 + \psi$. Then he assumed that the infinitesimal ψ is equal to $k\omega$ and that $a^\omega = 1 + k\omega$. In [1748, 123–124] he considered a finite number x, set $i = x/\omega$, and observed that

$$a^x = a^{i\omega} = (1 + k\omega)^i = \sum_{r=0}^{\infty} \binom{i}{r} (k\omega)^r = \sum_{r=0}^{\infty} \binom{i}{r} \left(\frac{kx}{i}\right)^r.$$

Euler asserted that $\frac{i-1}{i} = 1$, $\frac{i-2}{i} = 1$, $\frac{i-3}{i} = 1, \ldots$, for an infinitely large number i. Thus he obtained

$$a^x = \sum_{r=0}^{\infty} \frac{1}{r!}(kx)^r. \tag{2}$$

There are some critical steps in this proof. First of all, Euler justified the relation

$$a^\omega = 1 + \psi \tag{3}$$

by making a reference to what he had stated in the preceding Chapter IV:

[33] Note that Euler's justification of (1) consists in explaining what is meant by the ratio ψ/ξ between quantities ψ and ξ that vanish or endlessly increase. In the calculus, however, he handles the infinitesimal and infinite quantities ψ and ξ as separate entities.

Let ω be an infinitely small number, or such small fractions that they are almost equal to nothing, then

$$a^{\omega} = 1 + \psi,$$

where ψ is a number infinitely small number, as well. Indeed, from the preceding chapter, it was established that if ψ was not an infinitely small number, then neither could ω be an infinitely small number.[34]

In reality, in Chapter IV, exponential and logarithmic functions had been introduced without making any mention of infinitesimals [Euler, 1748, 1:103–105]. He suggested the idea that the difference between a^{z_1} and a^{z_2} of the exponential function a^z might be made equal to a tiny finite quantity, provided that z_1 and z_2 are taken very close together (in other words, at a tiny finite distance). In the following chapter this idea was expressed by (3), as if infinitesimals were only shorthand symbols.

Similarly Euler justified the relation $\frac{i-1}{i} = 1$, where i is an infinite number, by an intuitive, direct consideration of the process of growth of a finite variable i. Indeed, he stated: "However much larger the number that we substitute for i, the more the value of the fraction $\frac{i-1}{i}$ comes closer to unity. Therefore, if i is a larger number than any assignable one, the fraction $\frac{i-1}{i}$ equals unity."[35]

However, in proving (2), Euler used i as an infinite number and ω as an infinitesimal number, namely, he assumed that one could operate upon symbols by expressing the fact that the finite number i increased beyond all limits or that the finite quantity ω vanishes as if i and ω were numbers. In this way $\frac{i-1}{i} = 1$ is transformed into a rule for formal manipulation, specific for the fictitious number i and analogous to (1): it should not be intended as[36] $\frac{i-1}{i} \approx 1$ nor as the limit $\lim_{i\to\infty} \frac{i-1}{i} = 1$.

Laugwitz has repeatedly pointed out that in the proof of (2) there is a gap (for instance, see Laugwitz [1989, 210–211]), which he argues consists in the implicit assumption that the sum of infinitely many infinitesimals is an infinitesimal. It is evident that Laugwitz's remark arises from the interpretation of $\frac{i-1}{i} = 1$ as $\frac{i-1}{i} \approx 1$. This interpretation contrasts with the Eulerian statement that $a + dx = a$ is an exact equality and not an approximate one. According to Euler, "geometric rigor is averse to even the slightest error"[37] and the exactness of mathematics required that the differential dx should be precisely equal to 0 (even though the meaning of this expression is to be interpreted in the above sense, as a variable going to zero). He also observed that, by ignoring infinitely small quantities but not naughts, it was still possible to commit extremely serious errors[38] [1755, 6].

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34 "Quia est $a^0 = 1$, atque crescente exponente ipsius a simul valor potestatis augetur, si quidem a est numerus unitate major; sequitur si esponens infinite parum cyphram excedat, potestatem ipsam quoque infinite parum unitatem esse superaturam. Sit ω numerus infinite parvus, seu fractio tam exigua, ut tantum non nihilo sit aequalis; erit

$$a^{\omega} = 1 + \psi,$$

existente ψ quoque numero infinite parvo. Ex praecedente enim capite constat, nisi ψ esset numerus infinite parvus, neque ω talem esse posse." [Euler, 1748, 1:122].

35 "Cum autem i sit numerus infinite magnus, erit $\frac{i-1}{i} = 1$; pater enim, quo maior numerus loco i substituatur, eo proprius valorem fractionis $\frac{i-1}{i}$ ad unitatem esse accessurum; hinc si i sit numerus omni assignabili maior, fractio quoque $\frac{i-1}{i}$ ipsam unitatem adaequabit." [Euler, 1748, 124].

36 By $a \approx b$, I mean that the difference $a - b$ is an infinitesimal hyperreal number.

37 "[R]igor geometricus etiam a tantillo errore abhorret" [Euler, 1755, 6]. These words echo Newton's and Berkeley's statements. See Newton [1707, 334], Berkeley [1734, Sections 4 and 9].

38 This observation seems to be an implicit answer to Berkeley's view on the calculus. In Berkeley's opinion, the calculus achieved correct results only thanks to a compensation of errors [Berkeley, 1734].

Euler knew that the sum of infinitely many infinitesimals need not itself be an infinitesimal.[39] However, he did not see gaps in the proof of (2), and this was due to the fact that he understood $\frac{i-1}{i}=1$ as a formal equality involving fictitious entities. The proof of (2) is not to be read as a sequence of numerical equalities, but as a sequence of formal manipulations: the function a^x can be transformed into $\sum_{r=0}^{\infty}\frac{1}{r!}(kx)^r$ by formal manipulations.[40]

Euler did not hesitate to pursue this approach to its most extreme consequences. For instance, in the Chapter VII of *Institutiones calculi differentialis* Euler derived

$$\begin{aligned} S(x) &= 1^n - 2^n + 3^n - 4^n + \cdots + (-1)^{x+1}x^n \\ &= (-1)^{x+1}\left(\frac{1}{2}x^n + \sum_{m=1}^{s}(-1)^{m+1}\binom{n}{2m-1}\frac{(2^{2m}-1)B_{2m}}{2m}x^{n-2m+1}\right) + C, \end{aligned} \tag{4}$$

where $n \geqslant 0$, $s=[(n+1)/2]$ is the integral part of $(n+1)/2$, B_n are the Bernoulli numbers,[41] and the constant C is determined by the condition $S(0)=0$. This constraint implies that if n is even, then $C=0$; if $n=0$, then $C=1/2$; if n is greater than 0 and is odd, then $C=(-1)^s\frac{(2^{n+1}-1)B_{n+1}}{n+1}$.[42]

Consequently $1^n - 2^n + 3^n - 4^n + \cdots + (-1)^{x+1}x^n$ can be expressed as $C+(-1)^x f(x)$, where C is a constant and $f(x)$ is an appropriate function of the index x. Euler thought that one could make $(-1)^x f(x)$ equal to zero for $x=\infty$ since "if x is an infinite number, which is neither even nor odd, this consideration [he means the alternating sequence of $+$ and $-$ in $(-1)^x f(x)$] has to end and, therefore, the sum of the ambiguous terms should be rejected.[43] Hence one derives that the sum of the series to infinity is expressed by means of the only constant quantity."[44] This implies that $(-1)^\infty = 0$ and that the sums are

$$1^n - 2^n + 3^n - 4^n + \cdots = C = (-1)^{[n/2]}\frac{(2^{n+1}-1)B_{n+1}}{n+1}. \tag{5}$$

Equation (5), which can also be derived differently (see, e.g., Euler [1761]), is the consequence of the conscious acceptance of formal manipulations and of the principle of generality in algebra.[45]

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[39] In this regard, the discussion about the use of infinitesimals for defining integrals in [1768–1770, 1:183–184] is very interesting. Here Euler stated that it was possible to take into account non-null infinitesimals but they constituted only an imprecise and approximate version of the notion of infinitesimal, which nevertheless has useful applications.

[40] Of course, the result was also read as a numerical relation between the quantity a^x and the series $\sum_{r=0}^{\infty}\frac{1}{r!}(kx)^r$, but this is an a posteriori interpretation of the formal derivation (see Ferraro and Panza [2003]).

[41] There are several definitions of Bernoulli numbers B_r. Here I refer to the definition $\frac{t}{e^t-1}=1+\sum_{r=1}^{\infty}(-1)^{[r/2]+1}\frac{B_r}{r!}t^r$, ($|t|<2\pi$, $[x]$ is the integral part of x), $B_0=1$, which is closer to Eulerian use.

[42] Euler's derivation of this result is illustrated in Goldstine [1977, 131–135].

[43] The most probable interpretation of this statement is the following. Given that ∞ is equal $\infty+1$ (geometric equality), it is impossible to distinguish the behavior at infinity of the variables i and $i+1$ and it is impossible to distinguish even and odd infinite numbers. Indeed since $(-1)^\infty=(-1)^{\infty(\infty+1)/\infty}=(-1)^{\infty+1}=-(-1)^\infty$, one has $(-1)^\infty=0$.

[44] "Quodsi se ergo x fuerit numerus infinitus, quoniam is est neque par neque impa, haec consideratio cessare debet ac propterea in summa termini ambigui sunt reiiciendi; unde sequitur huiusmodi serierum in infinitum continuatarum summam exprimi per solam quantitatem constantem adiiciendam" [Euler, 1755, 384]. Euler's approach recalls Leibniz's one in [Leibniz GM, V, 396–387] (see Ferraro [2000c, 63–67]).

[45] It should be noted that not all 18th-century mathematicians accepted the idea of assigning a meaning to $(-1)^\infty$. For instance, Nikolaus II Bernoulli stated that the partial sums of $1-3+5-7+\cdots$ are $-n(-1)^n$ and, therefore, the sum of

At this juncture, I would like to observe explicitly that some recent papers, such as Laugwitz [1989, 1992], McKinzie and Tuckey [1997], interpret 17th- and 18th-century texts by translating them using modern notions and thus approach the question of the nature of Eulerian infinitesimals in an essentially different way from the present article. In particular, Laugwitz, McKinzie, and Tuckey believe that it is possible to vindicate Euler on the basis of a modern version of infinitesimals. However, I think that there is an irreducible difference between Eulerian infinitesimals and modern hyperreal numbers.[46] In their investigations, Laugwitz, McKinzie, and Tuckey do not use Robinson's infinitesimals but make "more naïve" assumptions [McKinzie and Tuckey, 1997, 48]. Although these assumption are weaker than the ones in Robinson's theory, one can see in operation in their writings a conception of mathematics which is quite extraneous to that of Euler.

For instance, the starting point of Laugwitz's theory "is a generalization of field extension. A symbol $\Omega \ldots$ is adjoined to the real numbers. If a formula $F(n)$ containing the variable n (for natural numbers) is true for all sufficiently large n, then $F(\Omega)$ is defined to be true in extended theory" Laugwitz [1987, 273].

Similarly McKinzie and Tuckey employ "the more or less axiomatic introduction of infinite and infinitesimal numbers" [McKinzie and Tuckey, 1997, 48]. In a footnote, they explain: "In modern terms, the Eulerian continuum is an ordered field; the natural numbers are a subset of the real numbers, which contains $0, 1, \ldots$. The fundamental equations or axioms are exactly what one would use to axiomatize the mathematics necessary for basic algebra and trigonometry, with additional assumption that there is an infinite natural number. By the field axioms, this implies the existence of infinitesimals." [McKinzie and Tuckey, 1997, 48].

These commentators use notions such as set, real numbers, continuum as a set of numbers or points, functions as pointwise relations between numbers, axiomatic method, which are modern, not Eulerian. Furthermore, they do not pay attention to the notion of formal manipulation, which plays a decisive role in Euler's analysis.

Finally, it should be noted that the translation of Eulerian notions in the terminology of nonstandard analysis eliminates aspects that were regarded as unitary. For instance, in Laugwitz's theory, since $(-1)^n = -1 \vee (-1)^n = 1$ is true for a natural number n, $(-1)^\Omega = -1 \vee (-1)^\Omega = 1$ is a true formula for a hyperinteger Ω (see Laugwitz [1992, 147]). By contrast, Euler's use of infinity also includes (5) and $(-1)^\infty = 0$: these formulas cannot be eliminated as fringe features or curiosities of Eulerian analysis. Nor can one vindicate Euler by reformulating (5) using the theory of summability. Summability theory is based upon the idea that arbitrary definitions of the sum of a series and the limit of a sequence can

the series is not 0, although $1 - 3 + 5 - 7 + \cdots$ is generated from the expansion of $\frac{1-x}{(1+x)^2}$ for $x = 1$. In his terms: "Seriei $1 - 3 + 5 - 7 +$ etc. summa exprimitur per ultimum terminum hujus seriei $1 - 2 + 3 - 4 +$ etc. et quando nullus concipi potest hujus seriei ultimus terminus, nulla etiam concipi poterit summa prioris seriei, aut se velis illa summa erit $= -\infty - 1^\infty$, non autem $= 0$, a quo valore series $1 - 3 + 5 - 7 +$ etc. tanto magis recedit, quanto magis continuatur, quamvis illa formetur ex quantitate $\frac{1-1}{1+2+1} = 0$." [Fuss, 1843, 2:709].

[46] In his [1974] Bos has already discussed some differences between 17th- and 18th-century infinitesimals (especially Leibnizian infinitesimals) and nonstandard analysis (see Bos [1974, 81–86]). In particular, he insisted that 'the most essential part of nonstandard analysis, namely the proof of the existence of the entities it deals with, was entirely absent in Leibnizian infinitesimal analysis" [Bos, 1974, 83]. This is true not only for Leibniz but also for Euler (as was mentioned above). However, it is worth noting that the absence of such a proof is the result of a conception of mathematics for which the term "existence" had a meaning only if, in the final analysis, it referred to reality. For this reason, 18th-century mathematicians did not feel the need to prove the existence of any species of numbers by providing a mathematical construction of them: the possibility of grasping them intuitively and evaluating their well-foundedness was sufficient.

be given: it presupposes a conception of mathematics as a set of theories which are syntactically derived from arbitrary axioms and definitions. However, Euler never considered alternative theories based on alternative definitions of the sum and limit. In conclusion, the attempt to specify Euler's notions by applying modern concepts is only possible if elements are used which are essentially alien to them, and thus Eulerian mathematics is transformed into something wholly different. I am not claiming that 18th-century mathematics should be investigated without considering modern theories. Modern concepts are essential for understanding 18th-century notions and why these led to meaningful results, even when certain procedures, puzzling from the present views, were used. However, 18th-century analysis had its own principles, different from those of modern analysis: an unproblematic translation of certain chapters in the history of mathematics into modern terms tacitly assumes that the same logical and conceptual framework guiding work in modern mathematics also guided work in past mathematics. So it implicitly assumes the point of view of modern mathematics to be the only possible point of view and the growth of mathematical knowledge to be a linear process.[47]

4. Infinitesimals in the presentation of the calculus

In this section we shall see how the notion of the infinitesimal described above enters into the actual presentation of the algorithm of calculus. Euler claimed that functions were the subject-matter of the calculus and that differentials were mere tools for dealing with functions. In *De usu functionum discontinuarum in Analysi*, Euler stated: "Differential calculus ... is not concerned with investigating the magnitude of differentials, which is nothing, but with defining their mutual ratio, which has a determinate quantity in any case. It certainly investigates not so much the differential dy of the function y as its ratio with the differential dx."[48]

He subsequently made it clear that the value of the fraction dy/dx gives rise, for any possible case, to a determinate (variable) quantity which can be considered a new function of x.[49] Similar statements can be found in *Institutiones calculi differentialis*[50] and in *Institutiones calculi integralis*.[51] According to

[47] See the interesting remark of K.H. Parshall: "Traditionally, historians of mathematics have most often adopted a presentistic approach to their subject. From the vantage point of the state of the discipline in their own times, they have tended to portray the development of mathematics as fundamentally linear in nature. In other words, looking back into mathematical history, they have picked and chosen from among the various contributions and constructed a logical, straight line progression from the past to the present. This kind of history serves to anchor contemporary mathematics in the past by providing it with a clear sense of direction, but at the same time it profoundly distorts the view of the mathematical climate at any given time in history. In the search for predecessors of a particular type of equation, theorem, or idea, other concepts which may have been of prime importance to the authors under scrutiny tend to be ignored or trivialized. Furthermore, competing approaches and underlying philosophies often fall into total obscurity" [Parshall, 1988, 128].

[48] "Neque ... calculus differentialis in quantitate differentialum, quae nulla est, indaganda occupatur, sed in eorum ratione mutua definienda, quae ratio utique certam obtinet quantitatem. Functionis scilicet y non tam ipsum differentiale dy quam eius ratio ad differentiale dx investigatur" [1765, 80].

[49] "[Valor fractionis dy/dx] quovis casu determinatam quantitatem sortitur et ipse tanquam nova functio ipsius x spectari potest" [1765, 80].

[50] "[C]alculus igitur differentialis non tam in his ipsis incrementis evanescentibus, quippe quae sunt nulla, exquirendis, quam in eorum ratione ac proportione mutua scrutanda occupatur et cun hae rationes finitis quantitabus exprimantur, etiam hic calculus circa quantitates finitas versari est censendus." [Euler, 1755, 5].

[51] "In calculo differentiali iam notavi questionem de differentialibus non absolute sed relative esse intelligendam, ita ut, si y fuerit functio quaecunque ipsius differentiale dy quam eius ratio ad differentiale dx sit definienda. Cum enum omnia

Euler, given the function, for example $y = x^3$ whose differential is $dy = 3x^2\,dx$, the genuine object of the calculus was the study of the differential coefficient (in the example, $3x^2$) and not of the differential $3x^2\,dx$. The algorithm of the calculus transformed functions into functions and not differentials into differentials: this viewpoint was part of the general rewriting of analysis as a theory of functions.

In the preface to the *Institutiones calculi differentialis* and in the *De usu*, Euler provided examples where the calculation of the differential coefficient seems to look forward to the modern definition of the derivative. In [1755, 5], he considered the function $y = x^2$ and observed that

$$y(x+\omega) : \omega = (2x+\omega) : 1$$

(where $y(x+\omega)$ is the increment, $2x\omega + \omega^2$, of x^2). So the smaller ω is, the closer is $(2x\omega + \omega^2) : \omega$ becomes to $2x : 1$, even though it actually reaches $2x : 1$ only when the increment has completely vanished.[52] In [1765, 80], Euler considered the function $y = ax^2 + bx + c$ and obtained $\Delta y/\Delta x = (2ax\omega + a\omega^2 + b\omega)/\omega = 2ax + a\omega + b$ and $dy/dx = 2ax + b$, when ω is evanescent.

The similarity with the modern notion of derivative is striking, but it must not deceive us. Indeed, in Chapter 4 of *Institutiones calculi differentialis*, where the rules of the calculus are formulated,[53] Euler tried to base the analysis of infinities on the method of finite differences, considering it as a special case of this method "which occurs when the differences, which were previously assumed to be finite, are taken to be infinitely small."[54]

Even though the calculus dealt with finite quantities of the form dy/dx, Euler did not introduce differential coefficients directly as the limits of certain ratios: first he introduced differentials $dx, dy, \ldots$, and then he considered dy/dx as the ratio of the fictitious quantities dy and dx. The idea of approaching was incorporated into that of infinitely small numbers. Thus, in the calculation of the differential of x^n (which is illustrated below), there is no explicit reference to the limit process, in contrast to the examples presented in the preface to *Institutiones* and in *De usu*. This leads Bos to state in his [1974] that the definition of the analysis of infinities in Chapter 4 of the *Institutiones calculi differentialis* "is rather at variance with" some of Euler's remarks. Bos referred to two passages from the preface of the same work

differentialia per se sint nihilo aequalia, quaecunque functio y fuerit ipsius x, sempre est $dy = 0$ neque sic quicquam amplius absolute quaeri posset. Verum quaestio rite proponi debet, ut, dum x incrementum capit infinite parvum adeoque evanescens dx, definiatur ratio incrementi functionis y, quod inde capite, ad istud dx; etsi utrumque est $= 0$, tamen ratio certa inter ea intercedit, quae in calculo differentiali proprie investigatur. Ita si fuerit $y = xx$, in calculo differentiali ostenditur esse $\frac{dy}{dx} = 2x$ neque hanc incrementorum rationem esse veram, nisi incrementum dx, ex quo dy nascitur, nihilo aequale statuatur. Verum tamen hac vera differentialium notione observata locutiones communes, quibus differentialia quasi absolute enunciatur, tolerari possunt, dummodo semper in mente saltem ad veritatem referantur. Recte ergo dicimus, si $y = xx$, fore $dy = 2x\,dx$, tametsi falsum non esset, si quis diceret $dy = 3x\,dx$ vel $dy = 4x\,dx$, quoniam ob $dx = 0$ et $dy = 0$ hae aequalitates aeque subsisterent; sed prima sola rationi verae $\frac{dy}{dx} = 2x$ est consentanea." [Euler, 1768–1770, 1:6].

[52] "Interim tamen perspicitur, quo minus illud incrementum ω accipiatur, eo [the ratio $2x : 1$] proprius ad hanc ratione accedi; unde non solum licet, sed etiam naturae rei convenit haec incrementa cogitatione continuo minora fieri concipiantur, sicque eorum ratio continuo magis ad certum quendam limitem appropinquare reperietur, quem autem tum demum attingant, cum plane in nihilum abierint. Hic autem *limes*, qui quasi rationem ultimam incrementorum illorum constituit, verum est obiectum Calculi differentialis" [Euler, 1755, 7, my emphasis].

[53] In the *De usu*, Euler briefly discusses the nature of calculus, restricting himself to the examples mentioned here. On the preface to *Institutiones calculi differentialis*, see below.

[54] "Erit ... analysis infinitorum, quam hic tractare coepimus, nil aliud nisi casus particularis methodi differentiarum in capite primo expositae, qui oritur, dum differentiae, quae ante finitae erant assumptae, statuantur infinite parvae" [Euler, 1755, 84].

[1755]. In the first, quoted in Footnote 52, the Swiss mathematician explicitly used the term "limit."[55] The second is the following:

> Although the rules, as they are usually presented, seem to concern evanescent increments, which have to be defined; still conclusions are never drawn from a consideration of increments separately, but always their ratio ... But in order to comprise and represent these reasonings in calculations more easily, the evanescent increments are denoted by symbols, although they are nothing; and since these symbols are used, there is no reason why certain names should not be given to them.[56]

According to Bos, there is "a contradiction which shows that his arguments about the infinitely small did not really influence his presentation of calculus" [Bos, 1974, 68–69]. However, I would argue that one may see a contradiction in the *Institutiones* only if, in contrast to Euler, one distinguishes between limits and infinitesimals and neglects the nature of evanescent quantities as fictions, the role of formal manipulations and the absence of a separation between semantics and syntax in the Eulerian calculus. In reality the idea of a limit (or, rather, protolimit) functioned as the basic intuition upon which infinitesimals were founded: it provided the semantic meaning of the infinitesimals (quantities that vanish) and allowed one to justify the principle of cancellation. However, infinitesimals were employed as fictitious entities that were subject to formal manipulations.

In my opinion, the definition of the calculus in Chapter 4 differs from that of the preface only in its emphasis. The latter highlights the intuitive and semantic aspect of calculus (the idea of approaching a limit), while the former presents the formal and syntactic aspect of calculus (infinitesimal numbers). This is due to the different contexts in which the two definitions are inserted. Indeed, the preface to *Institutiones calculi differentialis* aimed to give a preliminary explanation of the nature of the differential calculus to readers who had no acquaintance with this discipline. For this reason, Euler introduced some basic notions of calculus (variables, functions, infinitesimals and differential coefficients) and stressed their intuitive aspects. Thus all the definitions of the preface are different from those that Euler gave elsewhere in a more formal manner (in [1748]), for variables and functions, and in Chapters 3 and 4 of the first part of *Institutiones calculi differentialis*—i.e., in the treatise in the strict sense of the word—for infinitesimals and differential coefficients (on this, see also Ferraro [2000a, 113–114]).

To clarify the above discussion, I will illustrate how Euler derived the rules of differentiation in Chapter 1 of the *Institutiones*. Put $y^{(n)} = y(x + n\omega)$, for a nonnegative integer n, and $y = y^{(0)}$, Euler [1755, 16–20] defined

$$\Delta y = y^{(1)} - y, \qquad \Delta y^{(n)} = y^{(n+1)} - y^{(n)}, \qquad \Delta^m y = \Delta^{m-1} y^{(1)} - \Delta^{m-1} y,$$
$$\Delta^m y^{(n)} = \Delta^{m-1} y^{(n+1)} - \Delta^{m-1} y^{(n)} \quad \text{for } m > 1 \text{ and } n > 0.$$

He set out the rules of the sum and the product of finite differences and, then, calculated the differences of algebraic, exponential, logarithmic, trigonometric functions. For example, in [1755, 28–29] Euler used

[55] Euler generally avoided the use of the term "limit," in contrast to other 18th-century mathematicians (e.g., d'Alembert [1754, 1765]). As far as I am aware, only in this passage did Euler use "limit" to mean "approaching to a limit."

[56] "Quamvis enim praecepta, uti vulgo tradi solent, ad ista incrementa evanescentia definienda videantur accomodota, nunquam tamen ex iis absolute spectatis, sed potius semper ex eorum ratione conclusiones deducuntur ... Quo autem facilius hae rationes colligi atque in calculo repraesentari possint, haec ipsa incrementa evanescentia, etiamsi sint nulla tamen certis denotari solent; quibus adhibitis nihil obstat, quominus iis certa nomina imponantur." [Euler, 1755, 5].

the expansion of the logarithm

$$\log(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \tag{6}$$

and obtained

$$\Delta y = y^{(1)} - y = \log(x+\omega) - \log x = \log\left(1 + \frac{\omega}{x}\right) = \frac{\omega}{x} - \frac{\omega^2}{2x^2} + \frac{\omega^3}{3x^3} - \frac{\omega^4}{4x^4} + \cdots. \tag{7}$$

On the basis of the study of the elementary functions (these and their composition, however, constituted the universe of Eulerian functions (see Fraser [1989, 325])), he stated that the difference Δy, for every function y, could be expressed in the form

$$\Delta y = P\omega + Q\omega^2 + R\omega^3 + S\omega^4 + \cdots. \tag{8}$$

Analogously he asserted that higher-order differences could be written in the form

$$\Delta^2 y = P\omega^2 + Q\omega^3 + R\omega^4 + \cdots, \tag{9}$$

$$\Delta^3 y = P\omega^3 + Q\omega^4 + R\omega^5 + \cdots, \tag{10}$$

etc. Differential calculus[57] originated by letting $\omega = dx$, where dx is infinitesimal, in (8). Since x and y are continuous quantities, Euler [1755, 85] considered it obvious that if ω is an infinitesimal, Δy also became an infinitesimal (see Footnote 13). By neglecting the powers of ω (which vanish before ω) one obtained $\Delta y = P\omega$ or, in a different notation, $dy = P\,dx$, which could also be written as $dy : dx = P : 1$ (see Euler [1755, 86]).[58]

For instance, in order to determine the differential of $y = x^n$, Euler considered

$$dy = y^{(1)} - y = (x+dx)^n - x^n = nx^{n-1}\,dx + \frac{n(n-1)}{1\cdot 2}x^{n-2}\,dx^2 + \cdots.$$

By neglecting $dx^2, dx^3, \ldots$, he had $dx^n = nx^{n-1}\,dx$.

Similarly, since $dy = \log(x+dx) - \log x = \log\bigl(1 + \frac{dx}{x}\bigr)$, he derived

$$dy = \frac{dx}{x} - \frac{dx^2}{2x^2} + \frac{dx^3}{3x^3} - \frac{dx^4}{4x^4} + \cdots. \tag{11}$$

By applying (6) in formula (11), the terms dx^n/nx^n, for $n > 1$, vanish in comparison with dx/x, and he obtained $d(\log x) = dy = dx/x$ [Euler, 1755, 122].

In the above calculation, Euler did not take specific values of the functions x^n and $\log x$ into account.[59] He derived $d(x^n) = nx^{n-1}\,dx$ and $d(\log x) = dx/x$ by considering x^n and $\log x$ as general quantities.

[57] This modality to introduce differentials allowed one to connect the differential dy with a sequence $x^{(n)} = x + n\,dx$ of values of x, in the same manner as the first difference Δy is connected with a sequence $x^{(n)} = x + n\omega$ (the differential is the first difference of the sequence $y^{(n)} = y(x + n\,dx)$. By so doing a strong link is established with the Leibnizian calculus; however, some problems dependent on the choice of the sequence defining first differences are transferred to differentials. Euler explicitly referred to the sequences $x + n\,dx$ at p. 88 of [1755] when he dealt with second differentials.

[58] Note that if one takes the examples given in the preface and the *De usu* into account, he could have defined the differential coefficient as $\frac{\Delta y}{\omega} = \frac{P\omega+Q\omega^2+R\omega^3+S\omega^4+\cdots}{\omega} = P + Q\omega + R\omega^2 + S\omega^3 + \cdots$ for ω as an evanescent quantity. However, Euler avoided a direct use of the approaching idea and preferred to define first the differential dy and then the differential coefficient.

[59] On the global nature of Eulerian functions, see Fraser [1989, 329], Truesdell [1956, p. XLI].

Thus, even if the quantity is handled by assuming that it has the property of real quantity, the formulas of differentiation were understood to be valid also for imaginary values of x (see the example at the end of Section 2).

This situation does not differ significantly from that regarding functions with more than one variable. Euler systematically used the equivalent of the modern partial derivative, which he denoted by the symbols of the type $\left(\frac{dF}{dx}\right)$, $\left(\frac{dF}{dy}\right)$, $\left(\frac{dF}{dz}\right)$, where $F(x, y, z)$ is a function of the variables x, y, z. Similarly to differential coefficients of functions of one variable, the symbols $\left(\frac{dF}{dx}\right)$, $\left(\frac{dF}{dy}\right)$, $\left(\frac{dF}{dz}\right)$ were defined as ratios of formally manipulated differentials dF, dx, dy, dz.

In [1755] the differential of the function $V(x, y, z)$ was introduced by Euler by letting

$$dV = V(x + dx, y + dy, z + dz) - V(x, y, z).$$

By analyzing two examples, he observed that dV can be expressed as $dV = p\,dx + q\,dy + r\,dz$ where p, q, r are functions of $x, y, z, \ldots$. He also noted that if y and z were taken as constants then $dy = 0$, $dz = 0$, and $dV = p\,dx$. Similarly, if x and y were constants then $dx = 0$, $dy = 0$, and $dV = r\,dz$; if x and z were constants, then $dx = 0$, $dz = 0$, and $dV = q\,dy$. Consequently, dV is obtained by calculating the differentials of V supposing, on each occasion, that two of the variables are constant [Euler, 1755, 144–146].

Euler then demonstrated the theorem on mixed differentials which, in the case of functions with two variables, could be formulated as follows:

> if $dV = P\,dx + Q\,dy$ then the differential of P for variable y and constant x and the differential of Q for variable x and constant y are equal [Euler, 1755, 153–154].[60]

Subsequently [1755, 156–157], Euler set $dP = r\,dy$ (constant x) and $dQ = q\,dx$ (constant y), and observed that $dP\,dx = r\,dx\,dy$ and $dQ\,dx = q\,dx\,dy$. Since the mixed differentials are equal, he had $r = q$. Only at this point did Euler decide to introduce a symbolism to indicate the functions r and q in a convenient and unambiguous way. He denoted r by means of the symbol $\left(\frac{dP}{dy}\right)$, which meant *the differential of P for variable y and constant x* (that is, considering P as a function of the single variable y) *divided by dy*. Similarly $\left(\frac{dQ}{dx}\right)$ indicated *the differential of Q for variable x and constant y divided by dx*.[61] Therefore the condition that linked the finite quantities P and Q in the differential $dV = P\,dx + Q\,dy$ could be expressed as

$$\left(\frac{dP}{dy}\right) = \left(\frac{dQ}{dx}\right).$$

Finally, I would like to mention the subject of higher-order differentials. In the preface to *Institutiones calculi differentialis*, Euler observed that, since infinitesimals were equal to zero, higher-order differentials were never considered per se, but only in relation to each other. More precisely,

[60] Euler first published it in [1734–1735]. A hand-written version was published in Engelsman [1984, 205–213]. This proof is well known, it is therefore not illustrated here (for instance, see Engelsman [1984, 128–130], Fraser [1989, 319–321]).

[61] "Brevitas, gratia autem hoc autem capite quantitates r et q ita commode denotari solent, ut r indicetur per $\left(\frac{dP}{dy}\right)$, qua scriptura designatur P ita differentiari, ut sola y tanquam variabilis tractetur atque differentiale istud per dy dividatur; sic enim prodibit quantitas finita r. Simili modo significabit $\left(\frac{dQ}{dx}\right)$ quantitatem finitam q, quia hac ratione indicatur functionem Q sola x posita variabili differentiari tumque differentiale per dx dividi habere." [Euler, 1755, 157].

given a function $y = f(x)$, whose differential coefficient is a certain function p, second differentials were obtained by considering the ratio of the increment of the function p with other increments. He also asserted that the symbols of differentials serve only to give a convenient representation of certain finite quantities [Euler, 1755, 8]. However, this presentation was not developed in the chapters of *Institutiones* devoted to this topic, where Euler avoids giving a direct definition to the second differential as $d(q\,dx)$, where $q = dy/dx$. Similarly to the first-order differentials, he preferred to present higher-order differentials as a special case of higher-order finite differences, even though, as discussed below, some complications arose as a consequence.

Indeed, in Chapter 3 of the *Institutiones*, Euler [1755, 84 and 88] stated that higher-order differentials derived from higher-order finite differences $\Delta^n y = P\omega^n + Q\omega^{n+1} + R\omega^{n+2} + \cdots$ with $\omega = dx$ in the same way in which first differentials derived from $\Delta y = P\omega + Q\omega^2 + R\omega^3 + \cdots$ with $\omega = dx$. For example, as regards the second differential, the terms $Q\omega^3, R\omega^4, \ldots,$ of (9) vanished before $P\omega^2$ and, therefore, $d^2y = P\,dx^2$, where dx^2 was the square of dx. According to Euler, d^2y was equal to 0 while the ratio between d^2y and dx^2 was finite and equal to $P : 1$ [Euler, 1755, 88].

Given a function $y = y(x)$, whose first and second differentials are $dy = p\,dx$ and $d^2y = q\,dx^2$, the problem arose of establishing the connection between p and q. Having used (9) to introduce second differentials, there is no a priori guarantee that there exists a simple relationship between p and q. In order to determine such a relationship (namely, the second differential coefficient is obtained by differentiating the first coefficient), Euler observed that one could set $dp = q\,dx$ (since all the differentials of functions possessed this form) and that $n\,dp = nq\,dx$, where n represented a constant quantity.[62] If one then let $n = dx$ (therefore dx is constant[63]), one obtained $dp\,dx = q\,dx^2$. Remembering that $dy = p\,dx$ and $dp = q\,dx$, one obtained

$$d^2y = d(p\,dx) = dp\,dx = q\,dx^2,$$

namely the second differential of y had a finite relationship with dx^2, which coincided with the differential coefficient of p [Euler, 1755, 89]. Naturally, the reasoning can be repeated; if $dq = r\,dx$, then $d^3y = d(q\,dx^2) = r\,dx^3$, if $dr = t\,dx$, then $d^4y = d(r\,dx^3) = t\,dx^4, \ldots$. Therefore the higher-order differentials of y can be calculated one after another by differentiating p, q, r, t, etc.

Although higher-order differentials could be viewed as fictitious entities and be subjected to formal manipulations in the same way as first-order differentials, Euler thought that they "were utterly unsuitable for analysis" ("prorsus ad Analysin esse inepta" [Euler, 1755, 174]). This judgment expressed the fact that formulas involving higher-order differentials were not univocally determined. For instance, consider the formula

$$\frac{y\,d^2x + x\,d^2y}{dx\,dy}$$

in which the differentials d^2x and d^2y occur. The meaning of this formula varies according to which differential is taken as constant and which variable is chosen as independent. If one considers dx as a

[62] Euler justifies this step by appealing to finite differences; however, its extension to differentials is not a source of further difficulties.

[63] Euler justified this assertion by stating that the variable quantity x received equal increments, or rather that the sequence of values x, $x^{(1)} = x + dx$, $x^{(2)} = x + 2\,dx$, $\ldots$, $x^{(n)} = x + n\,dx$, was assigned to the variable x (see Footnote 57). Consequently, d^2x was everywhere equal to zero.

constant (namely, x is the independent variable) then

$$\frac{y\,d^2x + x\,d^2y}{dx\,dy} = \frac{x\,d^2y}{dx\,dy},$$

if dy is taken to be a constant (y is the independent variable), then one obtains a different result:

$$\frac{y\,d^2x + x\,d^2y}{dx\,dy} = \frac{y\,d^2x}{dx\,dy}.$$

In the special case where $y = x^2$, one has

$$\frac{y\,d^2x + x\,d^2y}{dx\,dy} = 1$$

for dx constant and

$$\frac{y\,d^2x + x\,d^2y}{dx\,dy} = -\frac{1}{2}$$

for dy constant (see Euler [1755, 170]).

The indeterminacy of higher-order differentials was an intrinsic aspect of the Leibnizian calculus, which was not based on functions but on curves analytically expressed by an equation $f(x, y) = 0$. This equation 'was considered as one entity, not a combination of two mutually inverse mappings $x \to y(x)$ and $y \to x(y)$' [Bos, 1974, 6]. The independent variable was not chosen a priori and therefore it was not established a priori that dx was a constant: thus the formulas containing higher-order differentials did not possess a meaning per se.

By contrast, the Eulerian calculus dealt with functions which had a directional character based on a clear distinction between dependent and independent variable. However, the indeterminacy of higher-order differentials resulted from the way of presenting calculus as a special case of the theory of finite differences. Already in Chapter 1 of *Institutiones calculi differentialis*, Euler[64] had noted that the first difference $\Delta y = y(x_1) - y(x) = y(x + \omega) - y(x)$ was not influenced by the sequence x_i, while the second differences changed according to the nature of x_n (see Euler [1755, 18]).[65] In Chapter 4, he stated: "For the same reason nothing can be said with certainty about the second differentials, unless the first differentials, with which the variable quantity x is conceived to increase continually, proceed according to a given law."[66]

Euler felt this ambiguity (vagueness, in his terms) made higher-order differentials different from first-order differentials and unsuitable for analysis. He therefore tried to eliminate them by a technique already known to Johann Bernoulli[67] which, in his opinion, showed that higher-order differentials did not have an effective use in analysis [Euler, 1755, 174]. This technique consisted in replacing them by

[64] The question of choice of the progression of variables and of the indeterminacy of higher-order differentials in the 18th-century calculus is treated in detail by Bos (see, in particular Bos [1974, 25–31, 66–77]).

[65] Indeed, in general it is $\Delta^2 y = \Delta^{(1)} y - \Delta y = y(x_2) - 2y(x_1) + y(x)$, and, for $x_n = x + n\omega$, $\Delta^2 y = \Delta^{(1)} y - \Delta y = y(x + 2\omega) - y(x + \omega) - y(x + \omega) + y(x) = y(x + 2\omega) - 2y(x + \omega) + y(x)$.

[66] "Ob eandem ergo rationem de differentiabus secundis nihil certi statui poterit, nisi differentialia prima, quibus quantitas variabilis x continuo crescere concipitur, secundum datam legem progrediantur" [Euler, 1755, 89].

[67] See Johann Bernoulli [1742, 77–79]. For Bernoulli, however, this technique was not a tool which could—once and for all—eliminate higher-order differentials from calculus.

differential coefficients. For instance, given a formula in which the interdependent variables x and y occur, if one assume dx to be a constant and introduces the differential coefficients $p, q, r, \dots,$ by the relations $dy = p\,dx$, $dp = q\,dx$, $dq = r\,dx, \dots$. By putting $d^2y = q\,dx^2$, $d^3y = dq = r\,dx^3, \dots,$ the differentials of y could be eliminated. Similarly one has to operate in more complex cases. Thus, if one considers $\sqrt{dx^2 + dy^2}$ as a constant (a case, Euler says, that is often found in the applications of the calculus), then one sets $dy = p\,dx$ and $dp = q\,dx$. In this way one obtains

$$dx\sqrt{1+p^2} = \text{constant}, \quad d^2x\sqrt{1+p^2} + \frac{pq\,dx^2}{\sqrt{1+p^2}} = 0 \quad \text{and} \quad d^2x = -\frac{pq\,dx^2}{1+p^2}.$$

The second differential of y is derived by considering

$$d^2y = q\,dx = q\,dx^2 + p\,d^2x = q\,d^2x - \frac{p^2q\,dx^2}{1+p^2} = \frac{q\,dx^2}{1+p^2}.$$

Analogously one could derive d^3x, $d^4x, \dots, d^3y$, $d^4y, \dots$ [Euler, 1755, 178].

5. Conclusion

This article has attempted to emphasize some aspects of the Eulerian foundations of the calculus. The Eulerian calculus was not based on the notion of set nor on that of real number but on that of quantity. Euler's concept of quantity was a modified version of the classical one and was connected with the idea of the continuum, which was not reducible to points. Quantities were investigated in abstract and general form, without referring to concrete and specific representations in a diagram. They had a symbolic nature; namely, they were reified in concrete signs which were dealt with according to certain fixed transformations.

Numbers were understood as the measure of quantity; however, only natural numbers and rational numbers were considered numbers in their own right. Irrational, negative, and imaginary numbers and zero were viewed as fictions. They were ideal entities useful for dealing with quantities, firmly founded in the real world (directly or indirectly), and subject to manipulation as if they were numbers.

Eulerian infinitesimals should be placed in this context. They, when interpreted using the conceptual instruments available to modern mathematics, seem to be an ambiguous mixture of different elements, a continuous leap from a vague idea of limit to a confused notion of infinitesimal. In reality, Euler does not confuse the modern notion of limit and the modern concept of infinitesimal: he simply did not possess such notions, but merely a primordial idea (directly derived from the physical world) of two variable quantities approaching each other. This intuitive idea was transformed into a fiction by expressing evanescent quantities by symbols which were operated upon in analogy with true numbers, without a theoretical construction.

This conception allowed Euler to conceive the calculus as a calculus of finite quantities, having as an object not the differentials $dy, dx, \dots,$ but the differential coefficients dy/dx. Nevertheless, the first-order differentials not only served to introduce differential coefficients but, as fictions, could be used per se and played an important role in the calculus.

There are many other aspects connected with the Eulerian foundations of the calculus that are worthy of investigation. The consideration given to Eulerian concepts in the 18th century, their various interpretations, and their influence on the developments of calculus is of particular interest. However, these are topics for another paper.

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