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Functions, Functional Relations, and the Laws of Continuity in Euler

Giovanni Ferraro

via Nazionale 38, 80021 Afragola, Naples, Italy

ALLA MEMORIA DI MIO PADRE
IN THE MEMORY OF MY FATHER

La nozione euleriana di funzione presenta un duplice aspetto: essa era, a un tempo, relazione funzionale tra quantità e formula composta da simboli operazionali, da costanti, e da variabili. Queste ultime erano concepite come universali e pertanto godenti di particolari proprietà. Anche se il calcolo di Euler era basato sulla manipolazione delle formule, egli non esitò a ricorrere alle relazioni funzionali se necessario. Inoltre le relazioni funzionali erano essenziali per la costruzione o definizione delle formule analitiche e per le applicazioni dei risultati. Ovviamente un tale modo di intendere le funzioni provocò ambiguità tra l'aspetto intuitivo, geometrico, o empirico dei concetti e la loro rappresentazione simbolica in analisi. © 2000 Academic Press

Eulerian functions had two aspects: they were both functional relations between quantities and formulas composed of constants, variables, and operational symbols. The latter were regarded as universal and possessed extremely special properties. Even though Eulerian calculus was based upon the manipulation of formulas, mathematicians did not hesitate to use functional relations when it was necessary. Besides, functional relations were essential to the construction or definition of analytic formulas and application of the results of calculus. This concept of function led to ambiguity between the intuitive, geometrical, or empirical nature of concepts and their symbolic representation in analysis. © 2000 Academic Press

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1. INTRODUCTION

If one were to adopt a presentistic approach and look back at 18th-century calculus, one would observe a nonrigorous corpus of manipulative techniques, which succeeded in anticipating certain modern results thanks to a series of lucky circumstances and fortuitous cases. Effectively, the predecessors of certain equations or theorems were accurately selected in a sea of "errors" and "meaningless" assertions. However, such a puzzling image disregards the conceptual background and the reasons and philosophy underlying 18th-century mathematics and reduces the complexity of historical progress to a mere cataloguing of new conquests which develop according to an unproblematic and purely linear scheme (see [37]). This state of things is further aggravated by the fact that 18th-century and modern terminology are seemingly similar but, in reality, differ profoundly. An exemplary and important case is Euler's concept of function, which I investigate in this paper with the aim of achieving a better knowledge of the fabric of 18th-century analysis.

In particular, I discuss the notion of a variable and analytical expression and argue that Eulerian functions have to be considered as two-leveled. There is an intuitive and geometrical (or empirical, according to circumstances) level—the functional relation between

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quantities—and there is a formalized level—the analytical expression connecting variables. Even though infinitesimal calculus manipulated analytical expressions, not only were functional relations essential to the construction of analytic formulas and their application, but also Euler did not hesitate to use them when it was necessary. My inquiry helps to explain the main aspects of Euler's concept of a function and, in particular, the apparently contradictory definitions and uses of the term "function," the permanence of geometric notions, especially the law of continuity, in the analytical concept of function, the peculiar role of elementary functions, and the insignificance of exceptional values.

2. VARIABLES AS ABSTRACT QUANTITIES

While reading the *Introductio in analysin infinitorum* [14], one immediately notes that Euler first defines variable quantities, in Section 2, and only later introduces the concept of a function, in Section 4, and that the latter presupposes the former. This is puzzling to the modern reader, who is accustomed to think of a function $f(x)$ as a rule that assigns a unique element y of a set B to each element x of another set A. One now considers two sets A and B and a law f that relates the objects belonging to A and B. The notion of variable is of no importance: x and y merely denote the generic elements of A and B, respectively. However, according to Euler, one initially considered the variables $x, y, \ldots$, and then the analytical expression that related them. In a sense, variables, as such, played the basic role of objects belonging to given sets: they were the primary objects of analysis. Sets, though, were lacking. Of course, Euler knew well that aggregates, classes, or sets could be formed by grouping objects, but mathematical theories were not based upon sets. The crucial point, for my purpose, is that a set is *the mere sum* of individual objects of arbitrary nature, whereas a variable refers only to quantities and is a universal or abstract entity, which can never be reduced to the mere sum of individual objects. As a consequence, a modern function is a relation between *individual* objects of any nature, while Eulerian functions related universal objects.

In order to clarify these points, it should be remembered that the notion of a variable derived historically from the variable geometric quantity. In the 17th century, the curve was the fundamental object of inquiry in analysis and embodied relations between several variable geometric quantities defined with respect to a variable point on the curve (see [4, 5]). Geometrical quantities were therefore lines or other geometrical objects connected to a curve, such as ordinate, abscissa, arc length, subtangent, normal, and areas between curves and axes. In the first works on calculus, analysis was an instrument for studying curved lines and variables were simply considered as lines denoted by the letters $x, y, \ldots$ Euler instead endeavored to eliminate any reference to geometry and, therefore, could not give to variables a meaning that did not immediately reduce them to lines. Thus, he resorted to the notion of abstract or universal quantity. In the *Introductio*, he gave the following definitions:

> A constant quantity is a determinate quantity which always retains the same value ... A variable quantity is an indeterminate or universal quantity, which comprises all determinate values.[1]

A variable quantity was therefore conceived of as a universal or abstract quantity. This means that a variable quantity did not refer to a particular geometric quantity (e.g., abscissa

[1] "Quantitas constans est quantitas determinata, perpetuo eundem valorem servans. ... Quantitas variabilis est quantitas indeterminata seu universalis, quae omnes omnino valores determinatos in se complectitur" [14, 17].

or arc length of a given curve)[2] but to the geometric quantity in general. According to Euler, the notion of a variable quantity was generated from particular geometrical quantities by means of a process of abstraction, which consists in rendering as a variable what is common to all quantities, just as "greenness" consists of the specific shared attributes of all green individual objects, such as trees and grass. In the *Introductio*, he indeed stated: "In the same way as the ideas of species and genera are formed from the ideas of individuals, so a variable quantity is the genus, within which all determinate quantities are included."[3] The meaning Euler gave to terms such as abstraction, genus, species can be deduced from the following excerpt from *Letters to a German Princess*:

> There are moreover other types of concept that are also formed by abstraction. These provide the mind with the most significant subjects to expand its forces: they are the ideas of genus and species. When I see a pear tree, a cherry tree, an apple tree, an oak, a fir tree, etc., all these ideas are different; however I note several things that are common to them ... I only stop to consider the things that the different ideas have in common, and I give the term *tree* to the object for which these qualities are appropriate. Thus, the idea of a tree I form in this way is a *general notion* ...[4]

A general notion included the characteristics constituting the essence of this notion (see [20, 203]). The notion of a variable therefore concerned the *essence* of quantity, namely the capability of being increased or decreased: "every quantity can be increased or decreased by it own nature indefinitely."[5] For this reason, more directly, the usual 18th-century definition of a variable stressed this essential characteristic.[6]

A crucial aspect of the Eulerian conception is that a quantity was a variable insofar as one *made the values and other specific characteristics of this quantity abstract*. Consequently, when one investigated a function with respect to any of the variables of which it was composed, one considered only the way that this variable entered into the function, namely, how it combined with itself and the other variables. An abstract quantity was "merely characterized by its operational relations with other abstract quantities" [36, 241] and not by its specific content (which, apart from anything else, was identical for all the variables). The form of the relation was investigated and the study of quantities was reduced to the

[2] A line gave an idea of a variable. At the beginning of the second part of the *Introductio* Euler stated: "Quoniam quantitas variabilis est magnitudo in genere considerata omnes quantitates determinatas in se complectens, in Geometria hujusmodi quantitas variabilis convenientissime repraesentabitur per lineam rectam indefinitam RS. Cum enim in linea indefinita magnitudinem quamcunque determinatam abscindere liceat, ea pariter ac quantitas variabilis eadem quantitatis ideam menti offert." [14, 2: Section 1].

[3] "Quemadmodum . . . ex ideis individuorum formantur ideae specierum et generum, ita quantitas variabilis est genus, sub quo omnes quantitates determinatae continentur" [14, 17]. Euler used the terms "genus" and "species" in an Aristotelian sense. There are other Aristotelian notions in Euler (see [34]); however, I do not investigate in this paper how they came to Euler.

[4] "Il y a encore une autre espèce de notions qui se forment aussi par l'abstraction, et qui foumissent à l'âme les plus importants sujets pour y déployer ses forces: ce sont les idées des *genres* et *espèces*. Quand je vois un poirier, un cerisier, un pommier, un chêne, un sapin, etc., toute ces idées sont différent, mais cependant j'y remarque plusieurs choses qui leur sont communes ... je m'arrête uniquement à ces choses que les différent idées ont de commun, et je nomme un arbre l'objet auquel ces qualités conviennent. Ainsi, l'idée de arbre que je me suis formée de cette façon est une *notion générale*..." [20, 93].

[5] "omnis quantitas sua natura in infinitum augeri et diminui potest" [17, 3].

[6] For instance, Lacroix stated: "Quantities, considered as changing in value or capable of changing, are said to be variables, and the name constant is given to those quantities that always maintain their value during the calculation" [31, 1:82]. In the preface of *Institutiones calculi differentalis*, Euler himself preferred to give a simpler definition of this type (see footnote 17).

modality of the combinations of the symbols x, y, ... There was no other solution since two different abstract quantities were distinguishable only by means of the symbols denoting them [36].[7]

3. FUNCTIONS AND FUNCTIONAL RELATIONS

It is well known that Euler based calculus upon the notion of a function, which he defined in the following manner. "A function of a variable quantity is an analytical expression composed in whatever way of that variable and numbers or constant quantities."[8] This definition, which appeared in the *Introductio*, was anticipated by the definition of the general term of a series, which appeared in one of Euler's earlier papers [11], submitted to the Academy of St. Petersburg in 1729: "A general term is a formula that consists of constant quantities or any other quantities like n, which gives the order of terms; thus, if one wishes the third term, 3 can be set in the place of n."[9]

Analytical expressions or formulas (for Euler, these terms were substantially synonymous) played a crucial role in the above definitions. The following example helps to clarify how Euler used this notion. In [12], he stated: "It is derived from the nature of differential calculus that if y is given in whatever way by means of x and constants and we replace x by $x + dx$, then we shall have $y + dy$ in place of y."[10] He assumed a generic function $y(x)$ is subject to the law of continuity and can be differentiated. From this assumption, he concluded that every function can be expanded in a Taylor series, namely

$$y(x+a) = y(x) + \frac{a dy(x)}{1!dx} + \frac{a^2 d^2 y(x)}{2!dx^2} + \frac{a^3 d^3 y(x)}{3!dx^3} + \frac{a^4 d^4 y(x)}{4!dx^4} - \cdots. \tag{1}$$

Euler even applied this result to two discrete functions, the nth term $X = X_n$ and the nth partial sum $S(x) = \sum_{n=1}^{x} X_n$ of a series, because "both S and X, *in the case that the series is determined*, are composed of x and constants."[11] He indeed considered $S(x-1) = \sum_{n=1}^{x-1} X_n = S(x) - X(x)$ and wrote: "If we compare this with the above formula [namely, (1)], then $S = y$ and $a = -1$ and the value of transformed S, namely $S - X$, is

$$= S - \frac{dS}{1dx} + \frac{ddS}{1 \cdot 2dx^2} - \frac{d^3S}{1 \cdot 2 \cdot 3dx^4} + \text{etc.''}^{12}$$

[7] It is therefore no wonder that the 18th-century definitions of a variable often stressed symbolism, which served to transform the abstract concept of a variable into a concrete and manipulable sign (for instance. cf. [33, 1]).

[8] "Functio quantitatis variabilis, est expressio analytica quomodocunque composita ex illa quantitate variabili, et numeris seu quantitatibus constantibus" [14, 1:Section 4].

[9] "Terminus ... generalis est formula, quam ingrediuntur tum quantitates constantes tum alia quaepiam non constans ut n, quae ordinem terminorum exponit, ut, si tertius terminus desideretur, oporteat loco n ponere 3." [11, 4]

[10] "Ex natura calculis differentialis sequitur, si fuerit y quomodocunque per x et constantes datum atque loco x ponatur $x + dx$, tum abiturum y in $y + dy$." [12, 109].

[11] "Sit igitur series quaecunque $A + B + C + D + \cdots + X$, in qua A denotat primum terminum, B secundum et X eum, cuius index est x, ita ut X sit terminus generalis seriei propositae. Ponatur autem summa huius progressionis $A + B + C + D + \cdots + X = S$; erit S terminus summatorius atque tam S quam X, *si series fuerit determinata*, ex x et constantibus erunt composita." [12, 112, my emphasis]

[12] "Comparentur ergo haec cum superiore formula; erit $S = y$ et $a = -1$, quamobren valor ipsius S transmutatus seu $S - X$, erit

$$= S - dS/1dx + ddS/1 \cdot 2dx^2 - d^3S/1 \cdot 2 \cdot 3dx^4 + \text{etc.'' [12, 112].}$$

Some key concepts emerge from this example. What characterizes S as a function of x is two things: S is composed of x and constants and S is effectively determined. Moreover an Eulerian function, as such, is continuous, is differentiable, and can be expanded in Taylor series, namely, continuity and differentiability are intrinsic properties of functions.

I shall return to continuity and differentiability in Section 7. Now I observe that Eulerian functions cannot be reduced to purely analytical expressions. For instance, in [14], Euler transformed the analytical expression $y = \frac{1-z^2}{1+z^2}$ into $y = \frac{2x}{1+x^2}$ by the substitution $z = \frac{1-x}{1+x}$. However, Euler did not view this as a mere transformation of an analytical expression; he instead felt the need for an explanation in terms of a correspondence between pairs of numbers: "If we give any determinate value to x, then we find the determinate values of z and y. Thus, we obtain the value of y corresponding to z and, at the same time, derive z. Since if $x = 1/2$, then $z = 1/3$ and $y = 4/5$, we also find $y = 4/5$, if we put $z = 1/3$ in $\frac{1-zz}{1+zz}$, y being equal to this expression."[13]

This example shows how the idea of a correspondence or functional relation is hidden behind analytical expressions. Another interesting example is furnished from the following excerpt from *Introductio in analysin infinitorum*:

> 77. Even though we have so far examined more than one variable quantity, they were connected so that each of them was the function of only one variable and once the value of one variable was determined, the others would be simultaneously determined at the same time. We shall now consider certain variable quantities that do not depend on one another; if a determined value is given to one of these variables, the others remain indeterminate and variable. It would be convenient to denote such variables with x, y, z, because they comprise all determined values; if they are compared with each other, they will be completely unconnected, since it is legitimate to replace any value of one of them such as z, and the others, x and y, remain entirely free as before. This is the difference between dependent variable quantities and independent variable quantities. In the first case, if we determine one, all the others are determined. In the second case, the determination of a variable in no way restricts the meanings of the others.
>
> 78. Therefore a function of two or more variable quantities x, y, z is an expression composed of these quantities in whatever manner.[14]

Euler first, in Section 77, spoke of "dependence" among variables; he later, in Section 78, defined a function of more than one variable as an analytical expression. At a first glance, this seems to be a contradiction. I think however that the contradiction is only apparent and that Euler's concept of function effectively contained both the idea of dependence or relation among variables and the idea of analytical expression. The dependence or relation

[13] "Sumpto ... pro x valore quocunque determinato ex eo reperientur valores determinati pro z et y sicque invenitur valor ipsius y respondens illi valori ipsius z, qui simul prodiit. Uti, si sit $x = 1/2$, fiet $z = 1/3$ et $y = 4/5$; reperitur autem quoque $y = 4/5$, si in $\frac{1-zz}{1+zz}$, cui expressioni y aequatur, ponatur $z = 1/3$" [14, 1:59].

[14] 77.Quanquam plures hactenus quantitates variabiles sumus contemplati, tamen eae ita erant comparatae, ut omnes unius essent Functiones, unaque determinata reliquae simul determinarentur. Nunc autem ejusmodi considerabimus quantitates variabiles, quae a se invicem non pendeant, ita ut quamvis unae determinatus valor tribuatur, reliquae tamen nihilominus maneant indeterminatae ac variabiles. Ejusmodi ergo quantitates variabiles, cujusmodi sint x, y, z, ratione significationis convenient, cum quaelibet omnes valores determinatos in se complectatur; at, si inter se comparentur maxime erunt diversae, cum, licet pro una z valor quicunque determinatus substituatur, reliquae tamen x et y aequae late pateant, atque ante. Discrimen ergo inter quantitates variabiles a se pendentes, et non pendentes, in hoc versatur, ut priori casu, si una determinetur, simul reliquae determinentur, posteriori vero determinatio unius significationes reliquarum minime restringat.

78. Functio ergo duarum pluriumve quantitatum variabilium, x, y, z, est expressio quomodocunque ex his quantitabus composita. [14, 1:91]

was only the first, unanalytical, intuitive level of the concept of a function (I shall later refer to this aspect of Euler's notion of function as the *functional relation*, for the sake of clarity). At a second level, the intuitive concept of a functional relation was made analytical by appropriate symbols (I shall refer to this as the *formula* or *analytical expression*). In the above quotation, Euler referred to the first level of the notion of function, the functional relation, in Section 77, while the second level or formula was referred to in Section 78. In my opinion, not only were formulas and functional relations not contrasted with each other but they were closely intertwined. A formula was a function since it embodied a functional relation; conversely, a functional relation could be the object of study in calculus only insofar as it was expressed by a formula.

Before I investigate this in detail, I wish to make clear that the generic observation of functionality in nature, among empirical objects, which is probably as ancient as man, is one thing, while the mathematical treatment of functionality is quite another. Indeed, it is in no way certain that an empirical functional relation can be studied by mathematics; even if it could be studied mathematically, this could be done by a geometric or tabular representation. In the 17th century, certain functional relations were indeed objectified in curves and studied geometrically. Symbolic written expressions, on which one could operate using specific rules, were only later used to denote the relations among *geometrical quantities*. Therefore, in the 18th century, the real novelty of the notion of function was not the appearance of functionality in mathematics but the fact that functionality was subjected to calculations by means of formulas or analytical expressions.

In the *Introductio* Euler mainly intended to investigate this newer aspect of functionality and, therefore, defined a function as a formula, however, it is not possible to eliminate the idea of a functional relation in his text. This means that the definition of a function did not characterize this mathematical object entirely and some of its aspects were tacit. In effect, mathematical definitions play different roles in Eulerian and modern mathematics. In Euler's mathematics, a definition did not necessarily exhaust the defined notion; it could have an implicit meaning, which, in a sense, was considered as obvious in a given context. Euler's concept of definition is, however, beyond the scope of this paper (on this subject, see [23]).

It should also be noted that formulas played a crucial role only in analysis. In geometry and mechanics, the objects of inquiry were functional relations between certain geometrical or physical entities. For instance, while investigating a curve, one had to study functional relations between certain variable geometric quantities (abscissa, ordinate, tangent, normal, arclength, etc.) connected with a curve. According to Euler, the analytical investigation of curved lines was possible insofar as the functional relations concerning quantities embodied in a curve (such as the relationship between abscissas and ordinates) were incorporated into appropriate formulas. After formulas had been manipulated, it was possible to apply the results to geometry and mechanics if and only if analytical expressions were reinterpreted as functional relations. Thus, in the second part of the *Introductio in analysin infinitorum*, where Euler applied various analytical notions, which he had introduced in the first part, to the study of geometry, he reinterpreted analytical expressions as functional relations in order to represent them geometrically and used them to investigate certain curves.[15]

[15] For example, he stated: "Sit y functio quaecunque ipsius x, quae ergo valorem determinatum induat, si pro x valorem determinatus substituatur. Sumta recta indefinita RAS ad valores ipsius x denotandos, cuilibet valori

For these reasons, Euler focused either on analytical expressions or on functional relations according to the circumstances. He emphasized the formula in analytical manipulation. The functional relation was mainly stressed in arithmetical, geometrical, and physical applications, where the context made an intuitive discussion possible (as we shall see in Section 4) and where a reference to the aspects of a function that usually were tacit was appropriate or necessary (for instance, as we saw above, in treating the transformation of functions Euler explained that the transformation of analytical expressions can also be viewed as the composition of functional relations). I shall discuss how analytical expressions and functional relations were connected in analysis in Sections 5, 6, and 7.

4. AN ALTERNATIVE DEFINITION OF FUNCTION

The two-leveled aspect of a function explains the presence of an apparently differing definition in Euler's *Institutiones calculi differentialis* [17], where a function is also defined as a functional relation. Some historians have recognized "a very general formulation of the concept of function" [4, 10] and even the first emergence of "a new, general definition of function" [39, 39] in Euler's definition of *Institutiones calculi differentialis* and have identified a direct thread that would link the latter to Dirichlet's definition, passing by way of Condorcet's and Lacroix's definitions.[16] At the same time, the same authors are forced to admit that such a seemingly new and extremely general concept of function had no consequence in the *Institutiones* (see, for instance, [39, 70]) and that 18th-century calculus was always a calculus of analytical expressions. It is therefore appropriate to explore why Euler preferred an alternative definition of a function in [17] (on the use of the term "function" in [18], see Section 7).

I believe that the difference between [14] and [17] was mainly a matter of emphasis that depended on the particular context in which the 1755 definition was presented, namely in the preface of the *Institutiones calculi differentialis*. In this preface, Euler discussed the epistemological nature of differential calculus for readers with no preliminary acquaintance with this discipline. He noted that calculus could not be defined using everyday notions and even that branch of the analysis of finite quantities from which the differential calculus is developed is not sufficient for this purpose. Therefore he had to introduce the basic notions of the calculus (variables, functions, infinitesimals, and differential ratios) in an intuitive way.

ipsius x determinato AP normaliter applicetur recta PM valori ipsius y respondenti aequalis. §.6 Si igitur hoc modo pro omnibus valoribus determinatis ipsius x definiantur valores ipsius y respondentes, ad singula rectae RS puncta P constituentur rectae normaliter applicatae PM valores functionis y exprimentes ... Singulae ergo applicatarum extremitates M repraesentabunt lineam quampiam, sive rectam, sive curvam; quae igitur hoc modo per y functionem determinabitur. Quare, qualibet ipsius x functio, hoc modo ad Geometriam translata, certam determinabit lineam, sive rectam sive curvam, cuius natura a functionis y pendebit. §.7. Hoc autem modo linea curva, quae ex functione y resultat, perfecte cognoscitur, quoniam omnia eius puncta ex functione y determinantur ... Quomodocunque autem linea curva fuerit comparata, ex eius singulis punctis rectae normales ad rectam RS duci possunt, sicque obtinentur intervalla AP, quae valores variabilis x exhibent, et longitudines applicaturam PM, quae valores functionis y repraesentant. Hinc nullum curvae extabit punctum, quod non hac ratione per functionem y definiatur." [14, Sections 5–7]

[16] "[T]he classical definition of function included in almost every current treatise on mathematical analysis is usually attributed either to Dirichlet or to Lobatchevsky (1837 and 1834, respectively). However, historically speaking, this general opinion is inaccurate because the general concept of a function as an arbitrary relation between pairs of elements, each taken from its own set, was formulated much earlier, in the middle of the 18th century" [39, 38].

Thus the definitions of the 1755 preface are different from those that Euler gave elsewhere in a formal or analytical manner (in [14], for variables and functions, and in Chapters III, IV, and V of the first part of *Institutiones calculi differentialis*—i.e., in the treatise in the strict sense of the word—for infinitesimals and differential ratios). In the 1755 preface, Euler initially defined a variable simply as a continually increasing or decreasing quantity.[17] He then illustrated this notion with a nonanalytical example (the trajectory of a bullet) which should not have been included in the treatise in a strict sense, since it dealt with pure analysis. Euler considered four quantities (the amount of gunpowder, the angle of fire, the range, and the time) and noted that each of them could be conceived as a variable or constant according to circumstances and that the variation of any of these quantities produces variations in the others. For instance, if the amount of gunpowder was fixed and one changed the angle of fire, then the range and time of the trajectory also changed. One could interpret the range and time as two variable quantities dependent (*pendentes*) on the angle of fire. It is precisely a dependence of this kind that characterizes a function: "Quantities that depend on others in this way (whereby, when the latter are changed, the former are changed as well) are referred to as functions of the latter. This definition is extremely broad and covers all ways in which one quantity can be determined by others. If, therefore, x denotes a variable quantity, then all quantities which depend upon x in any way or are determined by it are called functions of x."[18]

By this definition, Euler was simply explaining that there was a mathematical term for denoting the idea of dependence between empirical quantities. The intuitive meaning of the word "function" (in my terminology, the functional relation) was sufficient for the scope of the preface of [17] (but not for analytical investigation). However, when mechanical phenomena and geometric problems needed to be converted into analytical terms, the intuitive relationships between empirical or geometrical quantities had to be translated into symbols and conceived of as formulas. It is more worthwhile noting the similarity between, on the one hand, the 1755 definition and Section 77 of Chapter V of [14], and, on the other hand, the 1748 definition and Section 78 of [14]. In conclusion, the 1755 definition can be interpreted as marking the emergence of a new notion of function only if one extrapolates it from its context.

5. CONDITIONS FOR THE REPRESENTABILITY OF FUNCTIONAL RELATIONS AS FUNCTIONS

At this juncture, it is necessary to answer the following questions: (Q1) Given a functional relation R, what were the conditions that made it a function according to Euler? Conversely: (Q2) Given certain signs (such as $\sin x$, 2^x), what was it that made them functions?

In general, one can answer (Q1) by stating that a functional relation R was considered a function if one was able to associate with it an algorithm consisting of symbols (*signi*) and

[17] "[E]tsi enim omnis quantitas sua natura in infinitum augeri et diminui potest; tamen dum calculus ad certum quoddam institutum dirigitur, aliae quantitates costanter eandem magnitudinem retinere concipiuntur, aliae vero per omnes gradus auctionis ac diminutionis variari: ad quam distinctionem notandam illae quantitates costantes, hae vero variabiles vocari solent" [17, 3].

[18] "Quae autem quantitates hoc modo ab aliis pendent, ut his mutatis etiam ipsae mutationes subeant, eae harum functiones appellari solent; quae denominatio latissime patet, atque omnes modos, quibus una quantitas per alias determinari potest, in se complectitur. Si igitur x denotet quantitatem variabilem, omnes quantitates, quae utcunque ab x pendent seu per eam determinatur, eius functiones vocantur" [17, 4].

rules of calculation (*praecepti*). No function was given without a special calculus concerning it. Conversely, the answer to the second question is that a string of signs, syntactically correct as regards the rules of elementary algebra and calculus, which denoted numbers, constant quantities, variable quantities, operations, was conceived as a function only if it represented a functional relation at least for an interval of values of the variable.

In order to make these points clear, let us observe that trigonometric functions, intended as formulas involving letters and numbers, were introduced into calculus about 1740 (see [30, 312]). In [14], Euler constructed the analytical functions $\sin x$ and $\cos x$ by assuming as known their geometric meanings as functional relations between lines in a circle and their properties such as $\sin(x+y) = \sin x \cos y + \cos x \sin y$ and $\sin^2 x + \cos^2 x = 1$. These functional relations were conceived as functions when a special calculus (i.e., a group of rules that enabled the signs sin and cos to be both algebraically and differentially manipulated) was associated with them. In [16], Euler wrote:

> The different kinds of quantities, which Analysis deals with, generate different types of calculus, where rules have to be adapted to any kind of quantities. Thus one teaches the special algorithm of both fractions and irrational quantities in elementary Analysis. The same use occurs in higher Analysis. There, since logarithmic and exponential quantities, which form a new kind of transcendental quantities, enter into computations, one usually teaches a special type of algorithm concerning both symbols and rules. It was termed *exponential calculus* by the inventor Joh. Bernoulli and also includes the theory of logarithms and their differentiation and integration. In addition to the logarithmic and exponential quantities there occurs in analysis a very important type of transcendental quantity, namely the sine, cosine, and tangent of angles, whose use is certainly the most frequent. Therefore this type rightly merits, or rather demands, that a special calculus be given, whose invention in so far as the special signs and rules are comprised, the celebrated author of this dissertation [Euler], is able rightly to claim all for himself, and of which he gave examples in his *Introduction to Analysis* and *Institutions of Differential Calculus*.[19]

The calculus of the function $f(x)$ implied a knowledge of $f(x)$ as an analytical expression and functional relation. One had to possess algorithmic rules related to the analytical expression $f(x)$, such as the differentiation rule; but it was also necessary to be able to calculate the quantity $f(x)$ corresponding to a given value of quantity x (for instance, by mean of a table of values), at least when x varied in a certain interval. Only if these conditions occurred was a symbol associated with a given functional relation accepted as a function.[20]

[19] "Diversa genera quantitatum, circa quas Analysis versatur, diversas etiam species calculi gignunt, in quo praecepta ad quodcunque quantitatum genus accommodari debent. Ita in Analysi elementari peculiaris traditur algorithmus tam pro fractionibus quam irrationalibus quantitatibus tractandis. Idem usu venit in Analysi sublimiori, ubi cum logarithmi et quantitates exponentiales, quibus novum quantitatum genera revera transcendens constituitur, in computum ingrediuntur, peculiaris species algorithmi tam signi quam praeceptis distincta tradi solet, quae ab inventore Joh. Bemoullio *calculus exponentialis* est vocata, siquidem ibi quoque doctrina de logarithmis eorumque differentiatione et integratione tractatur. Praeter logarithmos et quantitates exponentiales aliud in Analysi occurit amplissimum genus quantitatum transcendentium, angulorum scilicet eorumque sinuum, cosinuum et tangentium, cuius usus omnino est frequentissimus. Pari igitur iure hoc genus meretur ac potius postulat, ut ei peculiaris calculus tribuatur, cuis inventionem, quatenus quibus peculiaribus signis et praeceptis continetur, Cel. Auctor huius dissertatione omni iure sibi vindicare potest, cuius insignia specimina in Introductione sua in analysin et in Institutionibus calculi differentialis dedit" [16, 542–543].

[20] These are precisely the conditions that allowed the object "function" to be accepted as the solution to a problem. Generally speaking, in order to solve a problem it is necessary to exhibit a known object. In analysis, an object was considered as known if it had an analytical expression on which one could operate and if one could at least partially calculate its values. Functional relations by themselves are not acceptable as the solutions to problems, because a functional relation is not generally easy either to calculate or to handle.

Not all functional relations were therefore viewed as functions and the number of functions was fixed at a given moment, even if, in principle, it could be increased. When Euler wrote "any function," he referred precisely to *one of the known functions or a composition of known functions*. This poses a new question: What were the functional relations that were effectively recognized as functions?

In order to answer this question, let us consider the classification of functions in [14, 19]. Here, Euler subdivided all operations into two classes, algebraic and transcendental.[21] The functions composed solely of algebraic operations on variables were termed algebraic (for instance, $\pi + z$, $4z^{\pi}$, and $\frac{a+bz-c\sqrt{2z-z^2}}{a^2z-3bz^3}$), while the others were referred to as transcendental.[22] This classification gives rise to various problems. The first problem concerns algebraic operations, which indeed comprised not only the six elementary operations (addition, subtraction, multiplication, division, raising to a power, extraction of a root) but also the *resolutio aequationum*, namely the solution to algebraic equations. Euler did not explain why he introduced the *resolutio aequationum*; however, in the first chapter of [1748], we find sentences such as: "[A]lgebraic functions can often not be exhibited explicitly; a function of z of this type is Z if it is defined by an equation such as $Z^5 = azzZ^3 - bz^4Z^2 + cz^3Z - 1$. Indeed, although this equation cannot be solved, it is known that Z is equal to an expression composed of the variable z and constants and, therefore, Z is a certain function of z."[23]

Euler hypothesized that, given an algebraic equation $F(x, y) = 0$, y was always expressible as a function of x. I think that this hypothetical function served to justify the use of algebraic equations $F(x, y) = 0$ as (implicit) functions in analysis even when one was not able to transform equations into explicit functions.

Other problems concern transcendental functions. According to Euler, some transcendental functions (logarithmic, exponential, and trigonometric functions) had a status similar to algebraic ones, as they could be manipulated as easily as the algebraic quantities: "even if [logarithmic and circular functions] are transcendental, now they are so common in analysis that they can be treated in the same easy way as algebraic quantities."[24] Initially, this class of peculiar transcendental functions consisted solely of the exponential and logarithmic functions. Thus in [11, 3], Euler distinguished these functions from transcendental ones that were connected with the quadrature of curves. In [14], when he enumerated transcendental functions, Euler still did not explicitly mention the trigonometric ones; however, he

[21] "[O]perationes sunt additio et subtractio, multiplicatio et divisio, evectio ad potestates et radicum extractio, quo etiam resolutio aequationum est referenda. Praeter has operationes, quae algebricae vocari solent, dantur complures aliae transcendentes, ut exponentiales, logarithmicae, atque innumerabiles aliae, quas Calculus integralis suppeditat" [14, 19].

[22] Some doubts concerned the functions of the kind z^c, c being an irrational number: somebody, Euler said, preferred to term it "*interscendentes*" [14, 1:20].

[23] "[F]unctiones algebraicae saepenumero ne quidem explicite exhiberi possunt, cuiusmodi functio ipsius z est Z, si definiatur per huiusmodi aequationem

$$Z^5 = azzZ^3 - bz^4Z^2 + cz^3Z - 1.$$

Quanquam enim haec aequatio resolvi nequit, tamen constat Z aequari expressioni cuipiam ex variabili z et constantibus compositae ac propterea fore Z functionem quandam ipsius z" [14, 1:19–20].

[24] "[logarithmi et arcus circulares] etiamsi transcendentes, nunc quidem in Analysin ita sunt receptae, ut aeque facile tractari quaent ac ipsae quantitates algebraicae" [21, 522].

provided a broad treatment of them in this text (cf. [31]). After [14], the set of elementary (i.e., algebraic, exponential, logarithmic, and trigonometric) functions, characterized by simple rules and procedures, was established and played a fundamental role in analysis.

Are there other functions in Euler's writings? The answer is complex. Some scholars (see [24, 40; 25, 322; 35, 200; 36, 251]) emphasized that the set of commonly accepted functions was only constituted by elementary functions and their composition in the 18th century. I substantially agree even though, at first glance, it would seem that Euler took many other transcendental functions into consideration. For instance, in *Institutiones calculi integralis*, we find many transcendental functions expressed by an integral, for instance the logarithmic integral $\int_0^z \frac{dz}{\lg z}$ or the functions originating as elliptic integrals. In *Institutiones calculi differentialis*, he investigated inexplicable functions[25] including the gamma function.[26] Even though there were partial successes, the result of the investigation of new transcendental functions was never really satisfactory and Euler did not put these functions and elementary ones on the same plane.[27] Thus, starting from the middle of the 18th century, Euler often called two types of objects "functions": the first type consisted of elementary functions; the second type consisted of other transcendental functions. He considered the objects of the first type as functions in a strict sense of the term, while he did not consider the objects of the second type as true functions because their knowledge was incomplete. Indeed, as we saw at the beginning of this section, it was not sufficient that one associated an analytical expression, such as an integral, and a functional relation to obtain a function. According to Euler, one could not invent a new function $F(x)$ merely by offering a definition, such as $\int_0^z \frac{dz}{\lg z}$ or the function interpolating $x!$. A function was an entirely known object to such a degree that it could be accepted as the final solution to a problem. Tables of values for it and a special calculus concerning it were necessary so that one could determine the numerical value of $F(x)$ and manipulate it directly (as occurred for $\sin x$ or $\cos x$) and not indirectly, by resorting to the general properties of integrals or series. Nonelementary transcendental functions either partially or completely lacked the simple rules of calculus that governed elementary transcendental functions and, therefore, differed from the latter. Euler realized that new functions were of essential importance in the development of analysis and believed that they could be accepted as true functions when our knowledge of them was improved. In Euler's writings, nonelementary transcendental functions were, *de facto*, objects to be investigated and made known, rather effectively given functions. For instance, in *Institutiones calculi integralis*, the first nonelementary transcendental function we find is precisely $\int_0^z \frac{dz}{\lg z}$ (Euler used an indefinite integral), but Euler observed: "These integrations

[25] Euler termed inexplicable those functions "quae neque espressionibus determinatis, neque per aequationum radices explicari possunt; ita ut non solum non sit algebrice, sed etiam plerumque incertum sit, ad quod genus trascendentium pertineat. Huiusmodi functio inexplicabilis est $1+1/2+1/3+\cdots+1/x$, quae utique ad x pendet, at nisi x sit numerus integer nullo modo explicari potest. Simili modo haec expressio $1\cdot 2\cdot 3\cdot 4\cdot\cdots\cdot x$, erit functio inexplicabilis ipsius x, quoniam si x sit numerus quicunque eius valor non solum non algebricae, sed ne quidem per ullum certum quantitatum trascendentium genus exprimi potest" [17, Sect. 367]. Euler effectively dealt with the functions interpolating the sums $S(x)=\sum_{n=1}^{x} a_n$ and the products $P(x)=\prod_{n=1}^{x} a_n$.

[26] Euler considered them difficult to study since they lacked any closed expression; however, he succeeded in finding various infinite expressions. For instance, he expressed $P(x)=\prod_{n=1}^{x} a_n$ if x was not an integer, by the infinite product $P(x)=a_1^x\prod_{n=1}^{\infty}\frac{a_{n+1}^x a_n^{1-x}}{a_{n+x}}$ [17, 2: Sect. 382].

[27] For instance, see the above quotation from [16] or [19, 1:13–14], where Euler argued that logarithmic and trigonometric functions (differently from other transcendental functions) were comparable to algebraic functions.

[of the functions $\frac{x^{m-1}}{(\lg x)^n}$, $n = 2, 3, 4, \ldots$] depend on the formula $\int \frac{x^{m-1}}{\lg x} dx$. Put $x^m = z$, hence $x^{m-1} dx = (1/m)\, dz$ and $\lg x = (1/m) \lg z$; this formula is reduced to the very simple form $\int \frac{dz}{\lg z}$. If the integral of this kind could be assigned, it would be of a very wide use in Analysis ... It therefore seems that this formula $\int \frac{dz}{\lg z}$ furnishes a peculiar type of transcendental function, which however merits more careful investigation."[28] However, for the time, "the nature of this function is not known enough."[29]

It was only in the last years of his life that Euler briefly mentioned the possibility that certain objects could actually be accepted as functions. In a short note, published in 1784, Euler [21, 522–523] observed that quantities concerning the rectification of conics had been analyzed to such a degree that if a problem was reduced to these quantities, which were included in certain integral formulas of the type

$$\int dx \sqrt{\frac{f + gx^2}{b + kx^2}},$$

then it could be regarded as quite solved.[30] This reference, however, was isolated and the use of elliptic integrals as actual functions remained a mere suggestion without practical consequences in Euler's work.

There is another very important aspect of the representability of a functional relation as a function to which I referred several times above. A functional relation could be expressed by means of an analytical expression if and only if it was a relation between *quantities*. This meant that, when a functional relation was turned into an analytical expression $f(x)$, both x and $f(x)$ were conceived of as abstract quantities or variables; in Euler's words: "A function of a variable quantity is also a variable."[31] Of course, if functions are variables, then they are universal or abstract quantities[32] and enjoyed all the properties of variables. This had various important consequences in Euler's calculus. First, since variables necessarily varied, a formula expressed a function if it transformed variable quantities into another variable quantity: for example, $y = a$ (with a constant) is not a function. Second, since variables could assume every value, in principle, functions carried $\mathbb{C}$ onto $\mathbb{C}$, to use an anachronism (cf. [35, 432]). Third, since a variable was a universal quantity, there might exist exceptional values at which a theorem involving functions failed. Fourth, since variables varied in a continuous way, functions were intrinsically continuous.[33]

[28] "Hae ... integrationes pendent a formula $\int \frac{x^{m-1}}{\lg x} dx$ quae posito $x^m = z$ ob $x^{m-1} dx = (1/m)\, dz$ et $\lg x = (1/m)$ $\lg z$ reducitur hanc simplicissimam formam $\int \frac{dz}{\lg z}$ cuius integrale si assignare possit, amplissimum usus in Analysi esset allaturum ... Videtur ergo haec formula $\int \frac{dz}{\lg z}$ singularem speciem functionum transcendentium suppeditare, quae utique accuratiorem evolutionem meretur" [19, 1:122].

[29] "natura huis functionis transcendentis parum cognoscitur" [19, 128].

[30] "[Q]uantitates, quae rectificationem sectionum conicarum involvunt, a Geometris iam ita sunt exploratae, ut problemata, quae ad eas fuerint perducta, pro perfecte solutis haberi soleant. Continentur autem istae quantitates transcendendentes in huiusmodi formulis integrabilus $\int dx \sqrt{\frac{f + gxx}{b + kxx}}$" [21, 522].

[31] "Functio ergo quantitatis variabilis ipsa erit quantitas variabilis" [14, 1:18]. In some cases, Euler directly defined a function as a quantity [17; 18]. The emphasis on quantity corresponded to applied contexts.

[32] In [35; 36] Panza placed particular emphasis upon this aspect and characterized 18th-century functions as forms expressing quantities or quantities expressed by forms.

[33] I shall return to different meanings of this term in Section 7. Here I intuitively refer to "continuity" as a variation without jumps.

In order to clarify the first three points, I observe that the particularization of a variable was problematic in Euler's conception. Today the symbol of a variable x is a mere sign denoting one of the elements $a, b, \ldots$ of the set S, on which $f(x)$ is defined; the properties of x, as a generic element of S, are the same properties that every element of S possesses for the simple reason that it belongs to S. According to Euler, a variable was instead a universal object and a universal object was always different from its particular occurrences, each of which was accidental and transient. A variable did not consist of the enumeration of its values but substantially differed from them. When a given value was attributed to an abstract quantity, one descended from the general to the particular; the variable lost its essential character of indeterminacy and its nature was altered. Consequently, a function, as such a variable, must vary (see [27, 47]): it could not assume the same value and, therefore, $y = a$ (a constant) was not conceived as a function. Euler stated: "Sometimes even merely apparent functions occur, such as z^0, 1^z, $\frac{aa-zz}{a-z}$, which nevertheless maintain the same value, however one varies the variable quantity. Although they give the misleading appearance of functions, they are actually constant quantities."[34]

Besides, given a function $y = y(x)$, since a variable quantity included all numbers (Euler emphasized in [14, 1:18]: "even zero and imaginary numbers"), both x and y assumed complex values. He indeed stated:

> "Since every determinate value can indeed take the place of the variable quantity, a function assumes innumerable values and there is no determinate value which the function can not assume, as a variable quantity also involves imaginary values. Thus, the function $\sqrt{(9-zz)}$ can not assume a value greatest than 3 if we replace z by a real number, however, if we attribute imaginary values to z, such as $5\sqrt{-1}$, then no determinate value is given that can not be derived from the formula $\sqrt{(9-zz)}$."[35]

It should also be noted that a constant quantity *was not a specific case* of a variable quantity, as the latter was an abstract, general quantity. A variable indeed enjoyed its own properties, which might be false for certain determinate values. What is legitimate for the variable could not be legitimate for all its occasional values. Consequently, given any property P of x, *there might exist exceptional values at which the property fails.* A proof involving the variables $x, y, \ldots$ was valid and rigorous as long as the variables $x, y, \ldots$ remained indeterminate; but this was no longer the case if one gave a determinate value to $x, y, \ldots$ Thus, if one expanded $f(x)$ into a power series $\sum_{n=1}^{\infty} a_n x^n$ and made no assumptions concerning the individual values of variables, then the equality $\sum_{n=1}^{\infty} a_n x^n = f(x)$ was considered globally valid even if there might exist certain occasional values at which the general relation $\sum_{n=1}^{\infty} a_n x^n = f(x)$ did not furnish a numerical equality: these points were "not significant" [25, 331].[36] (On the treatment of exceptional value in Euler, see Engelsman [10, 10–13]).

[34] "Si Z ejusmodi fuerit Functio multiformis ipsius z ut z^0, 1^z, $\frac{aa-zz}{a-z}$ perpetuo nonnisi unicum valorem exhibeat realem; tum Z Functionem uniformem ipsius z mentientur, ac plerumque loco Functionis uniformis usurpari poterit" [14, 1:18–19].

[35] "Cum enim loco quantitatis variabilis omnes valores determinatos substituere liceat, hinc functio innumerabiles valores determinatos induet; neque ullus valor determinatus excipietur, quem functio induere nequeat, cum quantitas variabilis quoque valores immaginarios involvat. Sic, etsi haec functio $\sqrt{(9-zz)}$ numeris realibus loco z substituendis nunquam valorem ternario maiorem recipere potest, tamen ipsi z valores imaginarios tribuendo, ut $5\sqrt{-1}$, nullus assignari poterit valor determinatus, quin ex formula $\sqrt{(9-zz)}$ elici queat" [14, 18].

[36] Consequently it is very difficult to undermine Eulerian calculus by means of counterexamples derived from assigning a particular value to a variable, for the simple reason that an Eulerian theorem was a theorem that

In order to clarify the fourth point, consider Euler's construction of the exponential function in [14, 1:103–105]. At a first sight, it would seem that Euler defined the exponential function a^z by associating a real value with the symbol $y = a^z$ for each real number z. Indeed, he initially considered the case in which z is a natural number and then when z is a negative integer or zero. He, later, observed that, if z is a fraction, such as $z = 5/2$, the quantity a^z assumes a unique positive real value ($a^2\sqrt{a}$), which lies between a^2 and a^3. A similar situation occurs if z is irrational: for example,[37] the quantity $a^{\sqrt{7}}$ has a determined values lying between a^2 and a^3. But, in the absence of a theory of real numbers,[38] what is the actual sense of this construction?

Euler did not really define a^z but sought analytically to characterize a quantity y represented by the symbol a^z, by assuming the existence of this quantity.[39] The use of the symbol a^z immediately implies that the *quantity* y has to be subjected to certain conditions, i.e.,

(a) it must assume the values $\ldots, a^{-3}, a^{-2}, a^{-1}, a^0, a^1, a^2, a^3, \ldots$;
(b) it must be governed by the law of powers $a^{z+x} = a^z \cdot a^x$.

This is sufficient for characterizing the exponential function analytically. Indeed, since a^z must be a quantity, it varies continually (flows, in Newton's terms) and Euler can state the relation

(c) $a^\omega = 1 + \psi$, where ω and ψ are infinitesimal,[40]

without any special explanation. Thus a^z was entirely characterized by (a), (b), and (c) and this allowed Euler to develop the calculus of exponential functions.

The arithmetical functional relation a^n, for $n = \ldots, -3, -2, -1, 0, 1, 2, 3, \ldots$, is only the starting point for the construction of $y = a^z$. What was important for Euler was the relation between the continuous quantities y and z. In modern terms, we could say that he was searching for a continuous function $f(z)$ such that $f(x+z) = f(x) \cdot f(z)$ and $f(1) = a$; but it is better to think of the construction of a^z as a Wallis interpolation [22], i.e., as the solution to the problem: find a *quantity* $y = a^z$ that interpolates $\ldots, a^{-3}, a^{-2}, a^{-1}, a^0, a^1, a^2, a^3, \ldots$. In the final analysis, the construction of the exponential function refers to a curved line that

concerned abstract quantities (variables) and not their values. Only after Cauchy did this point of view change and a theorem become falsified by a single counterexample derived from assigning a particular value to a variable.

[37] "Eodem modo res se habet, si exponens z valores irrationales accipiat, quibus casibus cum difficile sit numerum valorum involutorum concipere, unicus tantum realis consideratur. Sic $a^{\sqrt{7}}$ erit valor determinatus intra limites a^2 et a^3 comprehensus." Euler [14, 1:104]

[38] Only integers and fractions were indeed numbers in the strict sense of the term in the 18th century, while irrational numbers were the ratios of two given quantities of the same kind. Mathematicians were naturally accustomed to working with the decimal representation of real numbers or their approximating sequences (see, e.g., [14, 2: Section 510]). However, a sequence could approximate an irrational number but did not define it. The extension of the term "number" to incommensurable ratios was considered as incorrect because "number" presupposes an exact and precise denotation. Nevertheless, an incommensurable ratio was similar to a number and could therefore be viewed as a number because (1) it could be approached by numbers as closely as desired; (2) it had many properties that were common to numbers; (3) even though it could not be represented rigorously by means of arithmetic, it could at least be represented geometrically (e.g., $\sqrt{2}:1$ could be represented as the diagonal and the side of a square) (for instance, see [2, 188]).

[39] When Euler introduced the function a^z, he merely stated: "Sit igitur proposita huismodi quantitas exponentialis a^z, quae est potestas quantitatis constantis a exponentem habens variabilem z" [14, 103].

[40] "Quia est $a^0 = 1$, atque crescente exponente ipsius a simul valor potestatis augetur, si quidem a est numerus unitate major; sequitur si esponens infinite parum cyphram excedat, potestatem ipsam quoque infinite parum unitatem esse superaturam. Sit ω numerus infinite parvus, seu fractio tam exigua, ut tantum non nihilo sit aequalis, ω erit $a^\omega = 1 + \varphi$, existente φ quoque numero infinite parvo." [14, 1:122].

passes through the point (n, a^n) and this guaranteed the existence of the function. In order to satisfy this geometric intuition, Euler excluded values of a which made jumps in a^z [14, 1:104–105].

We thus arrive at a crucial aspect of Euler's analysis: the intuitive image of a function was the segment line or piece of a curved line described by means of other lines. Analytical symbols hide a *geometric* perception of relationships. By this, I do not mean that Euler never referred to relations between objects other than quantities but that he analytically represented only relations between quantities. Functions connected quantities rather than numbers, which were present in analysis only as particular determinations of quantities (and, as we saw, did not have an independent existence, except for the two more elementary types of numbers). Although a table of the values of a given function was one of the tools which mathematicians had to possess in order to know this function, a table of values was not the image of a function. To use the language of computer science, Eulerian analysis was analogical rather than digital. In the realm of analysis only the continuous, irreducible to the numerical, actually existed. Not only did the numerical fail to precede the continuous logically but on the contrary the discrete could be derived from the continuous and be regarded as an interruption of the continuous.

6. LOCAL AND GLOBAL VIEWPOINTS

Today we have a *local* conception of differential calculus. A rule concerning a function $f(x)$ is derived in the neighborhood of a number under conditions of continuity, differentiability, etc., and is then considered valid for the points of the domain of $f(x)$ which are subject to the same conditions. The Eulerian conception was different. It was based upon the principle of the generality of algebra, which was rooted in the notion of variables as universal: anything involving the universal object variable *was universally valid* and could not be limited to a particular range of its values. Euler expressed this principle as follows:

> "For, as this calculus concerns variable quantities, that is quantities considered in general, if it were not generally true that $d(\log x) = dx/x$, whatever value we give to x, either positive, negative, or even imaginary, we would never be able to make use of this rule, the truth of the differential calculus being founded on the generality of the rules it contains."[42]

[41] 99. Si fit $a = 0$, ingens saltus in valoribus ipsius a^z deprehenditur, quamdiu enim fuerit z numerus affirmativus seu major nihilo, erit perpetuo $a^z = 0$: si $z = 0$ erit $a^0 = 1$; sin autem feurit z numerus negativus, tum a^z obtinebit valorem infinite magnum. Si enim $z = -3$; erit $a^z = 0^{-3} = 1/0^3 = 1/0$, idoque infinitum. Multo majores autem saltus occurrent, si quantitas constans a^z habeant valorem negativum, puta -2; tum enim ponendis loco z numeris integris valores ipsius a^z alternatim erunt affirmativi et negativi, ut ex hac Serie intelligitur

$$a^{-4}; a^{-3}; a^{-2}; a^{-1}; a^0; a^{-1}; a^2; a^3; a^4; \text{ec.}$$

$$+\frac{1}{16}; -\frac{1}{8}; +\frac{1}{4}; -\frac{1}{2}; +1; -2; +4; -8; +16, \text{ec.}$$

Praeterea vero si Exponenti z valores tribuantur fracti, Potestas $a^z = (-2)^z$ mox reales mox imaginarios induet valores: erit enim $a^{1/2} = \sqrt{-2}$ imaginarium, at erit $a^{1/3} = \sqrt[3]{-2} = -\sqrt[3]{2}$ reale: utrum autem, si exponenti z tribuantur valores irrationales, Potestas a^z exhibeat quantitates reales an imaginarias, ne quidem definiri licet.

100. Hic igitur incommodis numerorum negativorum loco a substituendorum commemoratis, statuamus a esse numerum affermativum, et unitate quidem majorem, quia huc quoque illi casus, quibus a est numerus affermativus unitate minor, facile reducuntur."

[42] "Car, comme ce calcul roule sur des quantités variables, c'est-à-dire sur des quantités considérées en général, s'il n'etoit pas vrai généralment qu'il tût $d.\text{l}x = dx/x$, quelque quantité qu'on donne à x, soit positive ou negative,

A function, such as log x, was viewed as a whole and its behavior was a global matter, which could not be reduced to the sum of the behavior of the points of its domain: it could not have a property P here, and a different property there. This does not mean that Euler merely considered functions that had the property P in every point: rather they obeyed rules that were valid over an interval I_x (or, more precisely, for certain values that this variable x assumed moving with continuity) and hence were also globally valid. Thus, if one proved that a function $f(x)$ had the property P in the interval I_x, then one could extend this property beyond the interval I_x, where it had initially been derived.[43] This conception, which can be called a *generalized local conception*, derived from the double role of functions, as an analytical expression and a relation. A functional relation between quantities had a “natural” domain D for which its properties were valid. When this functional relation was analytically expressed and was conceived of as an analytical expression, it was not restricted to its original domain D: the results concerning a formula were derived substantially by using certain local properties of the functional relation, only then was it conceived globally, without considering any constraints. For instance, given the analytical expression $\log x$ constructed from a relation valid for positive values of quantity x, the principle of generality of algebra allowed $\log x$ to be considered when x is negative and even imaginary. Euler did not define $\log x$ for x as negative or complex numbers but merely assumed in an unproblematic way that the properties of the analytical expression $\log x$ (such as the differentiation rule) lasted beyond the original interval of definition. Of course, if what was valid in an interval was generally valid, not only did a function possess the same properties everywhere but it also maintained the same analytical expression everywhere since the analytical expression embodied all its properties. Therefore *one function necessarily consisted of one single formula* (cf. [25; 36]) and a relation such as

$$f(x) = \begin{cases} 2x & \text{if } x \text{ is a positive quantity} \\ x^2 & \text{if } x \text{ is a nonpositive quantity} \end{cases}$$

was never considered as a function.

Such an approach did not enable Euler to appreciate the difference between complex and real variables and, therefore, between complex and real analysis. His attention was focused on functions of real variables. For instance, in [14, 1:24], after dividing functions into many-valued and single-valued, Euler stated that an equation $Z^n - PZ^{n-1} + QZ^{n-2} - RZ^{n-3} + SZ^{n-4} - \text{etc.} = 0$ (with P, Q, R, S, etc. single-valued functions of z) is a many-valued function Z of z but observed that if Z assumes one real value, then it behaves as a single-valued function of z and generally can be used as a single-valued function.[44] Thus $\sqrt{P}$ was a many-valued function because it assumed two real values, whereas $\sqrt[3]{P}$ had to be considered a single-valued function because it assumed one real value and two complex values. Real functions were really of interest; complex functions were not an autonomous

ou même imaginaire, on ne pourrait jamais se servir de cette régle, la verité du calcul differential étant fondée sur la généralité des régles qu’il renferme” [15, 143–144].

[43] According to Fraser [25, 329]: “The existence of an equation among variables implies the global validity of the relation in question.”

[44] “15. Si Z ejusmodi fuerit Functio multiformis ipsius z ut perpetuo nonnisi unicum valorem exhibeat realem; tum Z Functionem uniformem ipsius z mentientur, ac plerumque loco functionis uniformis usurpari poterit.”

object of study, but were useful tools for the theory of real functions and their use seemed to be restricted to exceptional circumstances.[45]

Finally, it is also worthwhile noting that the generality of algebra was restricted to analysis, where functions were studied without *a priori* restrictions concerning variables. In arithmetic, geometry, and mechanics, functions and variables have natural ranges and therefore mathematicians were obliged to take into consideration the restrictions which the nature of the specific problem under examination imposed. When the results derived from the use of generality were applied to other sciences, they had to be subjected to appropriate reinterpretations which adapted them to concrete circumstances. This approach is an aspect of the mathematical method for studying natural science in the 18th century, which Dhombres [9] referred to as the "functional method." By solving a problem mathematically, appropriate symbols replaced concrete quantities and their relations come to be conceived as formulas and equations. The solutions to these equations were to be interpreted in relation to the specific problem and by eliminating anything that was meaningless for this particular problem. The most systematic example of this conception is Euler's series theory, where the convergence was studied *a posteriori* as a condition for applicability of series theory (cf. [23; 35]). Results were obtained without any restriction concerning the convergence of series; only at the moment of application was the numerical meaning of series (and therefore convergence) of importance.

7. THE LAW OF CONTINUITY

Until now, I have often referred to continuity (e.g., when referring to quantities that increases continuously) in a sense close to the modern local point of view. According to Euler, continuity was, however, a global matter and was viewed as equivalent to uniqueness. This conception was grounded in the idea that an object was continuous if it was an unbroken object, i.e., if it was not broken into two objects and was therefore *one* object. (On the origin of this conception, see [34].) I shall call Euler's concept of continuity G-continuity for short. With respect to this global view *one* function had to be G-continuous.

If one applies such a concept to a curve, a continuous, unbroken curve is characterized by means of the *connectedness* or continuity of its run. Thus global and local viewpoints seem be connected in a simple manner provided we consider a curve as an empirical object, immediately capable of being grasped by our intuition and not represented analytically. The global point of view (uniqueness, absence of break) can then be regarded locally as the absence of jumps in the course of the curve or as the assumption of any intermediate state between two given states or gradual change (these notions were considered equivalent at the time). I shall call this concept of continuity L-continuity for short.

The intuitive idea of a curved line (such as a mark made by a pencil) implies L-continuity: one can imagine that a curve consists of more than one branch, each of them L-continuous, but the idea of a completely discontinuous curve does not belong to geometric intuition. Since a function was an abstract representation of a curved line, it was necessarily L-continuous. A function was L-continuous or was not a function. According to Euler, each function $y(x)$ possessed the following property: $\Delta y = y(x+\omega) - y(x)$ is infinitesimal if ω is infinitesimal.[46] Unlike Cauchy's approach [6], this property *was not the definition of*

[45] See also the last example of Section 7.

[46] He stated "augmentum illud ω, quo quantitatem variabilem x crescere sumpsimus, statuemus infinite parvum ... manifestum est, incrementum seu differentiam functionis y quoque fore infinite parvam" [17, 82].

continuity but only a trivial consequence of the application of the idea of L-continuity to formulas (see also footnote 10). Indeed, the problem of the definition of L-continuity never arose during the 18th century.

It should be stressed that mathematicians could imagine an L-discontinuous functional relation (and Euler surely considered discrete functional relations, such as sequences), but only if a functional relation was L-continuous at least over an interval was it considered acceptable in order to construct a function. Of course, the generalized local conception allowed mathematicians to consider L-continuity as a global property of the analytical expression. L-continuity was, in a sense, incorporated into the analytical expression, as has been seen in the case of the exponential function (see Section 5). Thus, in the second volume of the *Introductio in analysin infinitorum*, in the chapter devoted to transcendental curves, Euler [14, 2: Section 51] examined the "equation" (significantly, this term, and not "function," was used by Euler) $y=(-1)^x$ and refused to consider it as a function. He referred to $y=(-1)^x$ as paradoxical because its graph is totally discontinuous: there are pairs of points whose distance is smaller than any assignable quantity and, at the same time, no segment of the straight lines $y=1$ and $y=-1$ belongs to it (to the 20th century eyes, it is composed of two everywhere dense sets of isolated points).

While the expression $(-1)^x$ was paradoxical, the expressions $x^{\sqrt{2}}$ and x^x were not considered problematic although they give rise to a similar case for negative values of x. The difference between $(-1)^x$, $x^{\sqrt{2}}$, and x^x is continuity over an interval: $x^{\sqrt{2}}$ and x^x are functions because they could be conceived of as (continuous) quantities in certain intervals (and, as we saw, the properties of a function were determined in their entire range by an arbitrary interval). Instead $(-1)^x$ was paradoxical as it could never be viewed as a (continuous) quantity, or, if preferred, it represented a continuous functional relation in no interval. Analysis dealt with those expressions that guaranteed regularity *a priori* and avoided paradoxical phenomena.

Furthermore, the image of quantity as a piece of a curved line implied further considerable regularities, such as the existence of tangents and of radius of curvature, and this suggested not only that functions were intrinsically continuous but even that the existence of differentials and higher-order differentials was intrinsically connected to their nature (see, for instance, [12, 109]). An undifferentiable function was a contradiction in terms.

Let us now return to G-continuity. I have already stated that *one* function was G-continuous merely because it was one. For the same reason, *one* curved line and *one* functional relation were G-continuous. However, if one regarded functions, functional relations, and curves as different aspects of the same object, then G-continuity became problematic and the simple connection between the local and global points of view began to crumble. For instance, the function $y=k/x$ is G-continuous since it is one, but its geometrical counterpart, the hyperbola of the equation $y=k/x$, is broken into two pieces: it is then very natural to ask whether the hyperbola is continuous, i.e., whether its two pieces form a unique curve. Put in more general terms, *how does one recognize that an object is one*? The most obvious answer is that an object is one if it retains its properties. Now, if we study a curve analytically, its properties are included within its analytic expression. If we accept this view, then it is entirely natural that the criterion of uniqueness must be applied to the analytical expression, as Euler did in classifying curves [14, 2: Section 8]. Indeed he stated that although some curves could be described mechanically, he aimed to study curves insofar as they originated as functions because this method was the most general and best suited to calculus.[47] According to Euler, from such an idea about curved lines, it immediately follows that they should be divided into continuous and discontinuous or mixed. A curve was continuous if its nature was determined by only one function, and discontinuous or mixed if it was described piecewise by more than one function and, consequently, was not formed according to a unique law.[48] Uniqueness did not apply to the course of a curve, which was seen as an outward manifestation, but to the function itself as a primary object. The number of the branches of a curve was therefore of no importance.

Euler also subdivided curves into complex and noncomplex ones using a similar criterion. He noted that the equations of certain algebraic curves could be broken down into rational factors:

> Such equations include not one but many continuous curves, each of which can be expressed by a particular equation. They are connected with each other only because their equations are multiplied mutually. Since their connection depends upon our discretion, such curved lines cannot be classified as constituting a single continuous line. Such equations (referred to above as complex) do not give rise to continuous curves, although they are composed of continuous lines. For this reason, we shall call these curves complex.[49]

The complex curves (like mixed ones) were discontinuous because their equation was characterized by arbitrariness; in other words, they are not determined by *exactly one* analytical law. Their difference is that the complex curves were composed of more than one whole curve, whereas mixed curves were composed of pieces of more than one curve.[50]

In [14], Euler only considered G-discontinuous curves. However, in [18], a paper written after the controversy with d'Alembert about the vibrating string (see [38]), he tried to extend the notion of discontinuity to functions. In his [1] d'Alembert described the motion of a stretched elastic string by equations equivalent to a partial differential equation

[47] "Quamquam complures linae curvae per motum puncti continuum mechaniche describi possunt, quo pacto, tota linea curva simul oculis offertur, tamen hanc linearum ex functionibus originem hic potissum contemplabimur, tanquam magis analyticam latiusque patentem, atque ad calculum magis accommodatam. Quaelibet ergo functio ipsius x suppeditabit lineam quandam, sive rectam sive curvam, unde vicissim linea curvas ad functiones revocare licebit. Cuiusque ergo lineae curvae natura exprimetur per ejusmodi functionem ipsius x" [14, 2: Section 8].

[48] "Ex hac linearum curvarum idea statim sequitur earum divisio in continuas et discontinuas seu mixtas. Linea scilicet curva *continua* ita est comparata, ut ejus natura per unam ipsius x functionem definitam exprimatur. Quodsi autem linea curva ita sit comparata, ut variae ejus portiones BM, MD, DM, etc. per varias ipsius x functiones exprimantur; ita ut, postquam ex una functione portio BM fuerit definita, tum ex alia functione portio MD describatur; hujusmodi lineas curvas *discontinuas* seu *mixtas* et *irregulares* appellamus: propterea quod non secundum una legem constantem formatur, atque ex portionibus variarum curvarum continuarum componuntur" [14, 2: Section 9].

[49] "Hujusmodi ... aequationes in factores resolubiles non unam sed plures curvas continuas in se complectuntur, quarum qaevis peculiari aequatione exprimi queant; et quae aliter inter se non sunt connexae; nisi quod earum aequationes in se mutuo sint multiplicatae. Qui cum sit nexus ab arbitrio nostro pendens, ejusmodi lineae curvae non unam continuam lineam constituere censeri possunt. Tales ergo aequationes, quas supra complexas vocavimus, producent lineas curvas non continuas, attamen ex continuis compositas, quas propterea complexas vocabimus" [14, 2: Section 61].

[50] On the basis of these subdivisions the curve of the equation $y = \sqrt{x^2}$ is not continuous. Although it appears to be a G-continuous curve since it derives from one two-valued function, it is in reality the complex curve corresponding to the (implicit function) equation $y^2 - x^2 = 0$. According to Euler, uniqueness did not refer to the 'apparent', complex form, but to the essential, irreducible form. In the light of this observation, Cauchy's objection to Euler's classification in [7] should also be considered.

$\partial^2 z/\partial x^2 = a^2 \partial^2 z/\partial t^2$. He solved this equation and found $z = f(t+x) + F(t-x)$, for $a = 1$, f and F being two arbitrary functions. D'Alembert thought that the solution to the problem had to be interpreted only by means of G-continuous functions, because calculus was grounded in functions derived from *one* functional relation (see [38]). In contrast, Euler tried to eliminate this restriction in geometric or mechanical applications but without prejudicing the nature of calculus [14]. In the summary of *De usu functionum discontinarum in Analysi* Euler explained: "The solutions that Geometers gave to the problem of the vibrating motion of strings include nothing but the assumption that the figure, which is given to the string at the beginning of the motion, is regular and can be represented by a certain equation. Instead they denied that the other case (if this figure is discontinuous or irregular) was of relevance for analysis or that the motion that originated from this configuration might be reasonably defined."[51] He thought that similar problems involved the use of discontinuous functions necessarily[52] but merely added the new G-discontinuous functions to old continuous functions, without changing the concept of the latter. Euler obtained this result by a change in terminology and a peculiar interpretation of the constants resulting from the integration of partial differential equations.

In [14], the term *function* always denoted an analytical written expression (embodying a functional relation) and the word *curve* had an obvious geometrical meaning. Any function could be represented geometrically by a curve; the converse was not true, since some curves were not analytically expressible. For this reason a function had to be continuous and a curve could be discontinuous. In [18], every curve was instead viewed as analytically expressible by a function[53] and Euler denoted the analytical expression by the term *equation*, while he indicated the functional relation by the terms *curve* and *function* (the one was often used in place of the other in the paper). In this way, Euler could introduce the notion of a discontinuous function: curves or functions were said to be discontinuous if they were unions of more than one equation.[54]

[51] "Qui problematis de motu cordarum vibratorio solutiones dederunt Geometrae, non nisi illum casum contemplati sunt, quo figura, cordae ab initio motus impressa, regularis et certa quadam aequatione comprehensa esse supponitur; alterum vero casum, si haec figura fuerit discontinua sive irregularis, negarunt ad Analysin pertinere aut motus inde secuturos posse ulla ratione definiri" [18, 7].

[52] "Quod autem de hoc problemate [Euler referred to a geometric problem that he studied by means of a partial differential equation in *De usu functionum discontinarum in Analysi*] est ostensum, simul de omnibus aliis eiusdem generis valet, quorum scilicet solutio functiones binarum variabilium implicat, ex quo quaestio initio proposita de usu functionum discontinuarum in Analyseos ita est resoluta, ut in Analysi quidem communi, quae circa functiones unius variabilis tantum versatur, huiusmodi functionibus nullus locus sit concedendus, in sublimioribus autem Analyseos partibus, ubi functiones binarum pluriumve variabilium tractantur, tales functiones ita necessario ad calculi essentiam pertinere sint censendae, ut nulla integratio pro absoluta et completa haberi queat, nisi simul functio maxime indefinita, atque adeo etiam discontinua, in calculum introducatur" [18, 27].

[53] "[Q]uomodocunque quantitas y per x determinatur, seu quaecunque fuerit functio y ipsius x, semper curva describi potest, cuis abscissae cuicunque x conveniat ea ipsa applicata ... vicissim proposita linea curva quaecunque, eius applicatae certas quasdam functiones abscissarum exibent" [18, 3].

[54] In [18, 4–5] Euler stated: "Iam vero notissimum est, in Geometria sublimiori alias lineas curva considerari non solere, nisi quarum natura certa quadam relatione inter coordinatas, per quampiam aequationem expressa definiatur, ita ut omnia eius puncta per eandem aequationem tanquam legem determinentur. Quae lex cum principium continitatis in se complecti censeatur, quippe qua omnes curvae partes ita vinculo arctissimo inter se coharent, ut nulla in illis mutatio salvo continuitatis nexu locum invenire possit; hanc ob rem istae lineae curvae continuae appellantur, nihilque interest, sive aequatio illarum naturam continens sit algebrica sive trascendens, sive cognita sive etiamnun incognita, dummodo intelligamus dari quandam aequationem, qua natura huismodi

Since the aim of [18] was the application and interpretation of certain results of the calculus, Euler now emphasized the intuitive aspect of functional relation by the word "function" (as in the preface of [17]), and resorted to "equation" to denote the formal aspect. Euler's conception did not, however, change substantially: the tension between the formal and intuitive aspects of functionality was not eliminated but produced a change in terminology [35, 259]. G-discontinuity did not regard the analytical expression, i.e., the formal aspect of a function: it concerned the functional relation, i.e., the informal aspect, however it was termed. According to my terminology, only functional relations were G-discontinuous and could be thought of as arbitrary or as lacking a definite law of formation (e.g., the relation between the Cartesian coordinates of a curve traced by a free stroke of the hand). A formula was instead always associated with a definite law. For this reason, when he spoke of G-discontinuity, Euler was obliged to refer to curves and to use the term *function* as synonymous with the term *curve*.

After having defined discontinuous functions, Euler had to explain how these new functions entered into calculus (he indeed agreed that calculus concerned single analytical expressions, i.e., continuous functions). He began with the two-leveled notion of a function and resorted to a special interpretation of the constants produced by integration. He indeed observed that these new functions, absolutely indefinite and dependent upon our discretion, originated from the integration of a function of two variables, a new and little developed field of the integral calculus,[55] which "differs very much from the common integral calculus, where functions of a single variable only occur. It demands entirely special rules, even if it also uses the devices of the first part [of calculus]."[56]

In [19, 2:35–37], Euler explained that if one integrates a function $X(x)$ of one variable x, one obtains $\int X(x)\,dx = F(x) + C$, where $F(x)$ is a function such that $\frac{dF(x)}{dx} = X(x)$ and the constant C is determined by the nature of the problem of which the integration gives the solution. In the same way, if one integrates a function $Z(x, y)$ of the variables x and y with respect to x, one obtains $\int Z(x, y)\,dx = F(x, y) + f(y)$, where $F(x, y)$ is a function such that $\frac{dF(x,y)}{dx} = Z(x, y)$ and $f(y)$ is an arbitrary quantity dependent on y.[57] The character of the quantity $f(y)$ is determined by the nature of the problem and could even be a quantity that is not expressible by a formula but can be thought of as the ordinate of a curve whose abscissa is y (i.e., a G-discontinuous functional relation).

Thus, if one considers the wave equation $\partial^2 z/\partial y^2 = a^2 \partial^2 z/\partial x^2$, by a change of variable $t = x + ay$, $u = x - ay$, one obtains $\partial^2 z/\partial t \partial u = 0$. By integrating with respect t, one has a

linearum curvarum exprimatur. ... Constituto continuitatis criterio sponte patet, quid sit functio discontinua, seu lege continuitatis destituita: omnes enim linae curvae per nullam certam aequationem determinatae, cuiusmodi libero manus tractu delineari solent tales functiones discontinuas suppeditant, quandoquidem in iis valores applicatarum nulla certa lege abscissis definire licet."

[55] "Verum haud diu est, ex quo haec pars Analyseos coli est caepta, ita ut vix adhunc prima eius elementa satis sint evoluta." [18, 20].

[56] "plurimum differt a calculo integrali communi, ubi non nisi functiones unius varibilis occurunt, et praecepta omnino singularia postulat, praeterquam quod in eo omnia quoque artificia prioris partis [namely, of calculus of a function of one variable] sint in usum vocanda" [18, 20].

[57] In [18, 20], Euler formulated this idea as follows: "Quemadmodum [...] calculi integralis communis vis in eo consistit, ut qualibet integratione nova quantitas constans arbitrio nostra permissa in calculum introducatur: ita in hac parte, circa functiones binarum occupata, singulis integrationibus, non solum nova quantitas constans, sed adeo nova functio cuiuspiam variabilis prorsus indeterminata, in calculum invenitur, quae ita ab arbitrio nostro pendet, ut eius loco etiam functiones discontinuae assumi quaent."

function $\partial z/\partial u = h(u)$, hence $z = \int h(u)\,du + f(t) = F(u) + f(t)$ and

$$z = f(x + ay) + F(x - ay). \tag{2}$$

According to Euler, the functions f and F could be discontinuous.

Since integration naturally contains an element of arbitrariness, Euler believed that the integral calculus of functions of more than one variable could directly provide a functional relation, without the intermediate step of the formula. Of course, in order to give a sense to this interpretation of integration, it was necessary to explain what the differential ratio (or derivative, in modern terms) of a G-discontinuous function is. Euler merely used the geometric meaning of a function and stated that if $f(x)$ represented a curve, then $f'(x)$ was the slope of the tangent whereas, if $f(x)$ was interpreted as an area, then $f'(x)$ was a curve (he used precisely the symbol $f' : x$ [19, 3:69]). This geometrical interpretation was problematic since the manipulation of G-discontinuous functions required specific rules which were never formulated. In [19, 3:192–193], Euler was, however, obliged to admit that the use of an immediately geometrical notion in an analytical context gave rise to a remarkable deficiency. He indeed observed that if one applied (2) to the equation $\frac{\partial^2 z}{\partial y^2} + a^2 \frac{\partial^2 z}{\partial x^2} = 0$, then one obtained the complex solution $z = f(x + ay\sqrt{-1}) + F(x - ay\sqrt{-1})$. Euler passed to an equation having a complex coefficient without any special hypothesis: as I had already noted in Section 7, he did not appreciate the difference between complex and real analysis. An interpretation of this solution, which was obviously influenced by a weak knowledge of the conditions of differentiability of a function of a complex variable, is beyond the scope of this paper. I limit myself to illustrating how Euler derived ”real solutions” from $z = f(x + ay\sqrt{-1}) + F(x - ay\sqrt{-1})$ provided f and F were continuous.

He indeed observed that if f and F are continuous, then they can be reduced to the form $P \pm Q\sqrt{-1}$. Hence it is easy, he said, to obtain solutions in the real form

$$\begin{aligned} z = {} & \frac{1}{2}[f(x + ay\sqrt{-1}) + f(x - ay\sqrt{-1})] \\ & + \frac{1}{2\sqrt{-1}}[F(x + ay\sqrt{-1}) - F(x - ay\sqrt{-1})].^{58} \end{aligned} \tag{3}$$

He probably realized that if $P \pm Q\sqrt{-1}$ satisfies $\partial^2 z/\partial y^2 + a^2 \partial^2 z/\partial x^2 = 0$, then

$$\begin{aligned} P + Q = {} & \mathrm{Re}[f(x + ay\sqrt{-1}) + F(x - ay\sqrt{-1})] \\ & + \mathrm{Im}[f(x + ay\sqrt{-1}) + F(x - ay\sqrt{-1})] \\ = {} & \frac{1}{2}[f(w) + \overline{f(w)} + F(\bar{w}) + \overline{F(\bar{w})}] + \frac{1}{2\sqrt{-1}}[f(w) - \overline{f(w)} + F(\bar{w}) - \overline{F(\bar{w})}] \end{aligned}$$

[58] “[Q]uoties autem functiones f et F sunt continuae, cuiuscunque demum fuerint indolis, semper earum valores ad hanc formam $P \pm Q\sqrt{-1}$ reduci possent, unde sequens forma ex illa facile deducenda semper valorem realem exhibebit

$$z = \frac{1}{2} f : (x + ay\sqrt{-1}) + \frac{1}{2} f : (x - ay\sqrt{-1}) + \frac{1}{2\sqrt{-1}} F : (x + ay\sqrt{-1}) - \frac{1}{2\sqrt{-1}} F : (x - ay\sqrt{-1})\text{” [19, 192].}$$

also does (here, I take $w = x + ay\sqrt{-1}$ and denote the conjugate, real part, and imaginary part of the complex number w by $\bar{w}$, $\mathrm{Re}(w)$, and $\mathrm{Im}(w)$, respectively).

Euler assumed that $\overline{h(w)} = h(\bar{w})$ for every continuous function h and therefore

$$P + Q = \frac{1}{2}[f(w) + f(\bar{w}) + F(\bar{w}) + F(w)] + \frac{1}{2\sqrt{-1}}[f(w) - f(\bar{w}) + F(\bar{w}) - F(w)].$$

Since f and F are two generic continuous functions, the latter expression furnishes (3).

Euler justified the equality $\overline{h(w)} = h(\bar{w})$ as follows. Put $x = s\cos\varphi$ and $ay = s\sin\varphi$; one has

$$(x \pm ay\sqrt{-1})^n = s^n(\cos n\varphi \pm \sqrt{-1}\sin n\varphi)$$

and since h is a continuous functions, namely one composed of analytical (algebraic or elementary transcendental) operations, its values can be exhibited by means of the sine and cosine (every continuous function, in Euler's sense, can be expanded in a power series with real coefficients).

If f and F are discontinuous, then they cannot be reduced to a real form: "In any curve traced by a free stroke of the hand, what meaning will one give the ordinates corresponding to the abscissas

$$x + ay\sqrt{-1} \quad \text{and} \quad x - ay\sqrt{-1}$$

according to the nature of imaginaries and their real sums [the real part of their sums] or the difference which will also be real if it is divided by $\sqrt{-1}$? Therefore we note this not slight lack of calculus, for which one can make up in no way yet."[59]

Despite this fact, Euler's solution to the problem of the vibrating string was substantially accepted in the 18th century. G-discontinuous functions were considered as tools which made up for a local insufficiency of calculus, just as imaginary quantities made up for local insufficiencies of real quantities. Calculus remained a calculus of single analytical expressions and G-discontinuous functions were never really considered. With hindsight, the controversy over the vibrating string posed the question of the lack of analytical tools for describing certain more complicated phenomena: it actually showed the restricted nature of 18th-century analysis and its overall inadequacy for more sophisticated investigations rather than its local inadequacy. To avoid a "return to geometry" [29, 11] and to make G-discontinuous functions actually analytical objects, it was necessary to restructure analysis; but Euler did not realize this.

[59] "Quis autem in curva quacunque libero manus ductu descripta applicatas abscissis

$$x + ay\sqrt{-1} \quad \text{et} \quad x - ay\sqrt{-1}$$

respondentes animo saltem imaginari ac summam earum realem assignare valuerit aut differentiam, quae per $\sqrt{-1}$ divisa etiam erit realis? Hic ergo haud exiguus defectus calculi cernitur, quem nullo adhuc modo supplere licet" [19, 193].

8. CONCLUSION

In this paper, I have tried to show that Euler's analysis mainly concerned a two-leveled mathematical object that can be characterized as an analytical expression embodying a functional relation between quantities. Euler usually termed this object "function"; in the context of the solution of partial differential equations, he instead called it "continuous function." However, the distinction between continuous and discontinuous functions remained isolated from the mainstream of contemporary mathematical analysis, and "function" was the prevailing name of this object. In this paper, I have followed this use.

Of course, the effective content of this object depended on the notions of analytical expression and functional relation between quantities. A functional relation between quantities was substantially viewed as a relation between quantities connected to a "nice" curve; hence a function enjoyed all the properties of a "nice" curve, such the absence of jumps and existence of tangents, namely L-continuity and differentiability. A geometric image underlay a function.

An analytical expression was viewed as an appropriate string of variables, constants, and symbols of operations. This string had to be exhibited explicitly. Every symbol in an operation was ruled by its own laws and every function had a special calculus (if it could not be reduced to other simpler functions). One can state that, according to Euler, a function was an entirely known object, even if the precise meaning of this remained vague and was not made clear. From the 1750s Euler used the term "function" for certain mathematical objects that lacked this property. He, however, thought that these objects substantially differed from effective functions since only the latter could be manipulated and, therefore, accepted as solutions to a problem.

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