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Charles L. Dodgson's Geometric Approach to Arctangent Relations for Pi

FRANCINE F. ABELES

Department of Mathematics and Computer Science, Kean College of New Jersey, Union, New Jersey 07083

Approximating π and attempting to square the circle have a long and interesting history. In 1875, C. L. Dodgson began work on a computationally simple approximation method for would-be circle squarers that would convince them of the futility of their attempts. Relating the earlier geometric and the newer analytic approaches in a practical way, this method produces an accurate approximation for π efficiently. © 1993 Academic Press, Inc.

Les tentatives d'approximation de π et de quadrature du cercle ont une longue et intéressante histoire. En 1875, soucieux de convaincre les "quadrateurs de cercle" de la futilité de leurs tentatives, C. L. Dodgson chercha à mettre au point un nouveau procédé simple du point des calculs d'approximation de π. Combinant de façon adaptée à la réalisation pratique des calculs des idées anciennes et modernes fondées sur la géométrie et sur l'analyse, la méthode de Dodgson permet de déterminer efficacement une approximation précise de π. © 1993 Academic Press, Inc.

Näherungsweise Berechnungen von π und Versuche, den Kreis zu quadrieren, haben eine lange und interessante Geschichte. 1875 begann C. L. Dodgson, an einer rechnerisch einfachen Approximationsmethode für Möchtegern-Kreisquadrierer zu arbeiten die sie von der Nutzlosigkeit ihrer Versuche überzeugen konnte. Indem diese die früheren geometrischen und neueren analytischen Herangehensweisen in praktischer Weise miteinander verbindet, führt diese Methode in wirksamer Form auf eine genaue Approximation von π. © 1993 Academic Press, Inc.

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BACKGROUND

Until the development of the calculus, all the methods for calculating π depended on inscribed and circumscribed regular polygons within and about a circle. By doubling the number of sides of a hexagon four times, Archimedes was able to approximate π as 3.1418. His method, however, was slow and cumbersome. The new techniques of the calculus replaced the older geometric methods with analytic expressions capable of much finer approximations. Using the inverse trigonometric functions, which are given by integrals of quadratic functions that characterize the curvature of a circle, both Newton and Leibniz calculated approximations for π using series expansions of these functions. Newton used the inverse sine function while Leibniz preferred the inverse tangent function.

The superiority of the inverse tangent function expanded as a series was shown by John Machin in 1706 [Schepler 1950, 223]. That series, in which each term is

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evaluated by James Gregory's series (1668), converges more rapidly to π. Machin's series, where for convenience the arccotangent function, notation arccot $x = [x]$, is substituted for the arctangent function, is

$$\frac{\pi}{4} = 4[5] - [239].$$

Gregory's series is

$$[x] = 1/x - 1/3x^3 + 1/5x^5 - \dots.$$

The smaller the value of x, the more rapidly Machin's series converges to the value of the arccotangent. Perhaps the best known calculation of the digits of π using this method was that of William Shanks who, in 1853, gave an approximation to 607 places and to 707 places 20 years later [Schepler 1949–1950, 227, 279]. The discovery in 1945 of an error beginning with the 527th place does not diminish the magnitude of his accomplishment. In fact, the first computer calculation for π, 2037 digits, which ENIAC produced in 1949, used Machin's series [1].

Arctangent equalities, particularly the problem of expressing π as a linear combination of angles whose tangents are reciprocals of integers and whose coefficients are integers, have had a long history in Diophantine equations. Around 1836, Carl Friedrich Gauss, who was concerned more generally with finding a base for the rational and the integral arctangents, found these two series for π [2]:

$$\frac{\pi}{4} = 12[18] + 8[57] - 5[239]$$

$$\frac{\pi}{4} = 12[38] + 20[57] + 7[239] + 24[268].$$

One way to compare arccotangent relations is to observe that when Gregory's series is used, the number of terms needed to obtain a particular approximation varies inversely as the logarithm of x. Using this observation, D. H. Lehmer developed a measure of efficiency, the amount of labor necessary to evaluate arccotangent relations of n terms of the type

$$\frac{k\pi}{4} = \sum_{i=1}^{n} a_i[m_i],$$

the quantity,

$$\sum_{i=1}^{n} 1/\log m_i$$

[Lehmer 1938, 657–658].

Machin's series has measure $1/\log 5 + 1/\log 239 = 1.8511$, while Gauss's two series have measures 1.7866 and 2.0348, respectively. The most efficient of the arccotangent relations for approximating π is due to Samuel Klingenstierna (1698–1765),

$$[1] = 8[10] - [239] - 4[515],$$

which has measure 1.2892 [3]. The million-digit computation for π achieved in 1973 was generated by Gauss's first series.

Shadowing the work of obtaining efficient approximations for π were the attempts by amateurs to solve the quadrature-of-the-circle problem. To square a circle is to produce an exact value for the area of a circle in terms of the square on its radius. James Gregory had proved in 1668 that the geometrical quadrature (determining the circumference of a circle using only straight lines and circles as prescribed by Euclid) was impossible. Despite the refusals of both the French Academy of Sciences in 1755 and the British Royal Society a few years later to consider quadrature problems, fruitless attempts persisted. The impossibility of the arithmetical quadrature (no two numbers can represent the ratio of the diameter to the circumference of a circle) was proved by Johann H. Lambert in 1761. This, too, did not deter the circle-squarers because quadratic irrationalities are constructible.

Some would-be circle squarers were able to derive reasonable approximations for π from their constructions, while others argued for abysmally poor ones. In the first category was John Parker who in 1851 gave 3.141594269 . . . , while in the second category was James Smith who used 3.125 in 1860 and refused to believe his methods incorrect even when this was demonstrated by Augustus DeMorgan, William Whewell, and William Rowan Hamilton. The proof by Ferdinand Lindemann in 1882 that π is transcendental and circle quadrature therefore impossible still did not altogether put an end to these attempts. As late as 1914, T. M. P. Hughes gave a geometric construction for 3.14159292035 [4].

Probably the chief recipient of failed quadrature attempts in the 19th century was Augustus DeMorgan, who lamented this situation in *A Budget of Paradoxes,* a book published after his death in 1871. The problem of the quadrature of the circle has long had a following among exuberant, nonmathematically trained dilettantes, the inability to distinguish an approximate construction from an ideal solution being the main source of confusion. When DeMorgan died, Charles Dodgson (Lewis Carroll) took up the burden of refuting the circle-squarers' faulty arguments. Not wanting to involve himself in the details of each argument, he set out to devise a method whereby each novice circle-squarer could convince himself that his method was flawed. Dodgson never finished the work, but enough of it remains in manuscript form to demonstrate that his approach was unusual. He introduced a method to compute approximations for π that was efficient and simpler than the prevailing method using Machin's and Gregory's series, and he provided an intuitively appealing setting that demonstrated the strong connection between the early geometric attempts at quadrature and the newer analytic ones.

THE LIMITS OF CIRCLE-SQUARING

Dodgson began his treatise on circle-squaring in 1875, working on it until 1893, 5 years before his death. The introductory chapter, written in 1882, was printed in a limited edition in *The Lewis Carroll Centenary, Special Edition* (1932) from the manuscript in the Parrish Collection in the Princeton University library. Dodgson intended to call his pamphlet "The Limits of Circle-Squaring: Simple Facts for Circle-Squarers." Four undated proof sheets containing two theorems undoubtedly intended for an early chapter are in the Warren Weaver Collection in the Harry Ransom Humanities Research Center at the University of Texas. A problem submitted to *The Educational Times* in 1892 contains the essentials of Dodgson's method for approximating π using inverse tangent series. Only in the Parrish Collection do we find the theorems linking this approximation method with the Euclidean constructions that would have enabled a misguided circle-squarer to apprehend his own errors.

In the proposed introductory chapter Dodgson wrote,

> . . . if there is one fact in Geometry more certain than another, it is that the area of a circle is less than its circumscribed square and greater than its inscribed square; and that these two squares are respectively four times, and twice, the square of the radius. . . . But the numbers proposed are in no case so wide of the mark as this, and if an answer of this kind is to be given to their proposers, the limits fixed must be very much closer together than the numbers 4 and 2. [Refer to Proposition I below.]
>
> And this, it has occurred to me, it is possible to do, without using more than the very simplest facts in Mathematics. . . . To measure the area of the circle itself is a complicated matter, and the processes by which the value 3.14159 has been calculated, are long and abstruse;
>
> The course I propose to take is briefly this:—first, to give a list of the elementary truths I shall afterwards have occasion to quote: then to prove by very simple methods . . . that, whatever be the exact value of the area, it is at any rate less than 3.1417 times, and greater than 3.1413 times, the square on its radius.
>
> The method, by which they [the above-named limits] are obtained, is one that he [any circle-squarer] can easily carry further for himself, and find new pairs of limits, each pair closer together than the preceding pairs so that, even if he has adopted a value a little within the limits 3.1417 and 3.1413, he may still find limits which will exclude the possibility of his value being true. [Dodgson 1932, 123–125].

The probable date of the galleys of Propositions I and II that were meant for this monograph on circle squaring is April 1882 when the introductory chapter was written. These two propositions read:

> PROPOSITION I. *The area of a circle is less than four times, and greater than twice, the square on its radius.*
>
> PROPOSITION II. *The area of a circle is less than* 3.4 *times, and greater than* 2.7 *times, the square on its radius.* (Dodgson crossed out 3.4 and 2.7, writing in 3 1/3 and 2 2/3, respectively.)

These limits are obviously not useful, but the theorems establish the general direction of the development. The key theorems are in manuscript form in the

Parrish Collection, which for clarity we refer to as Theorems A, B, C, D. The first is trigonometrical and simplified by the notation $\{x\} = \arctan x$.

THEOREM A. $\{1/k\} = \{1/(k + x)\} + \{1/(k + y)\}$, *where* $xy = k^2 + 1$. [Dodgson 1884, 5–16]

The next three are in the section, "Limitaries for the Area of a Circle." The first of these is:

THEOREM B. *If a sector of a circle has the vertical angle arctan* $1/k$ *then the isosceles triangle contained by the two radii and the chord of the sector equals* $1/(2\sqrt{k^2 + 1})$ *times the square of the radius, and the tetragon contained by the two radii and the tangents drawn at their extremities equals* $1/(k + \sqrt{k^2 + 1})$ *times the same square.* [Ibid., 5–16]

The next proposition in this section, Theorem C, reads:

If each octant of a circle is divided into sectors such that a of them have the vertical angle arctan $1/\alpha$, *b of them have arctan* $1/\beta$, *and so on, then the area of the inscribed polygon contained by the chords of the sectors equals* $4((a/\sqrt{\alpha^2 + 1}) + (b/\sqrt{\beta^2 + 1}) + \ldots)$ *times the square of the radius, and the circumscribed polygon contained by the tangents drawn at the extremities of the radii equals* $8((a/(\alpha + \sqrt{\alpha^2 + 1}) + (b/(\beta + \sqrt{\beta^2 + 1}) + \ldots)$ *of the same square.* [Ibid., 5–17]

Finally, Theorem D restates the hypothesis of Theorem C and generalizes the conclusion to the limits of the area of the circle. Concluding the section Dodgson wrote:

The above process may be extended without limit. By the help of Prop.2 [Theorem A] the angle 45° may be broken up into an unlimited number of angles, each of the form arctan $1/k$; and thus an octant may be divided into an unlimited number of sectors whose vertical angles are of this form. In Appendix [D] it will be found extended until it is proved that the area of a circle is greater than 3.141583 of the square of the radius, and less than 3.141597 of the same square [Ibid., 5–17].

To achieve these limits, the octant is divided into 385 sectors. (1)

By contrast, 66 sectors produce the limits, 3.14145 and 3.14167 (2)

while 25 sectors yield 3.14064 and 3.14212. (3)

The corresponding series expansions and efficiency measures are

$$71\{1/317\} + 5\{1/327\} + 19\{1/342\} + 71\{1/351\} + 19\{1/443\} + 71\{1/456\} + 10\{1/489\} + 10\{1/593\} + 39\{1/1252\} + 51\{1/2855\} + 19\{1/5618\}\,[3.9504] \tag{1'}$$

$$20\{1/57\} + 19\{1/68\} + 12\{1/117\} + 10\{1/268\} + 5\{1/327\}\,[2.4083] \tag{2'}$$

$$7\{1/18\} + 5\{1/38\} + 10\{1/47\} + 3\{1/57\}\,[2.5972]. \tag{3'}$$

In each expansion the coefficients sum to the number of sectors in the octant. Dodgson did not use either Machin's or Gregory's series, opting instead for expansions based on Theorem A which he proved as an identity in this way [5]:

Proof. Using the lemma $\{a\} + \{b\} = \{(a + b)/(1 - ab)\}$ where $a = 1/(k + x)$, $b = 1/(k + y)$,

$$\begin{aligned}\{1/(k + x)\} + \{1/(k + y)\} \\ &= \{(1/(k + x) + 1/(k + y))/(1 - 1/(k + x) \cdot 1/(k + y)\} \\ &= \{(2k + x + y)/(k^2 + k(x + y) + xy - 1)\}\end{aligned}$$

Substituting k^2 for $xy - 1$ gives

$$\{(2k + x + y)/(2k^2 + k(x + y))\} = \{1/k\}.$$

This theorem was known to William Wallace, professor of mathematics at the University of Edinburgh, who included it in his article "Algebra" in the seventh edition of the *Encyclopedia Britannica* (1839). Dodgson posed the theorem as a question requiring investigation in the *Educational Times* to elicit responses from the mathematical community. H. J. Woodall provided a discussion linking Dodgson's formula to one by Gauss [6]. Lehmer attributes the theorem to Dodgson.

We can use Gregory's series expansion to compare the accuracy of the approximations of π for which we have already computed efficiency measures. The number of terms in the arctangent series is given last:

Machin	3.157866845	1.8511	2
Klingenstierna	3.141592640	1.2892	3
Gauss 1	3.141592642	1.7866	3
Gauss 2	3.141592637	2.0348	4
Dodgson 1	3.077143544	3.9504	11
Dodgson 2	3.141592561	2.4083	5
Dodgson 3	3.141592641	2.5972	4

Particularly interesting is Dodgson's five-term series. Deriving it from the four-term series, he substituted the sum of two arctangents for one arctangent, a method that Lehmer points out is generally to be avoided except when one of the new arctangents already is present in the relation. For example, substituting $\{57\} + \{268\}$ for $\{47\}$ improves the efficiency measure by $1/\log 47 - 1/\log 268$, a decrease of .186212371322. The least accurate of these three series, the one with 11 terms, is the most inefficient [7].

To appreciate what Dodgson developed, we turn to the geometric theorems that motivated the series he constructed. I argue that Dodgson's use of Euclidean methods to relate inscribed and circumscribed regular polygons, an octant divided into sectors, and arctangent series produced the most accurate and efficient approximation of π based on "common sense" that was known in the nineteenth century.

Theorem B, the key step relating polygons and the arctangent of a sector of a circle in the proposed monograph, is Proposition 1 of Section 2, "Limitaries for the Area of a Circle." It bears the date 28 July 1884:

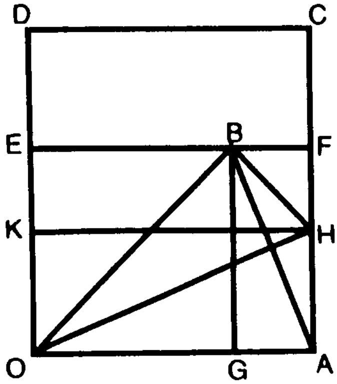

FIGURE 1

If a sector of a circle has vertical angle arctangent $1/k$, the isosceles triangle contained by the two radii and the chord of the sector is $1/(2\sqrt{k^2+1})$ of the square of the radius; the tetragon contained by the two radii and the tangents drawn at their extremities is $1/(k+\sqrt{k^2+1})$ of the same square.

Proof. Let the sector be OAB, so that tan BOA $= 1/k$. On OA describe the square OACD. EF is drawn through B parallel to OA. BG is perpendicular to OA; OH bisects angle BOA; HK is drawn parallel to OA. HA and HB are tangents to the radii OA, OB. Then AF/AO = area OAFE/area OACD; BG/OB = sin BOA $= 1/\sqrt{k^2+1}$. Hence the area of rectangle OAFE $= 1/\sqrt{k^2+1}$ of the area of square OACD. So the area of triangle OAB $= 1/(2\sqrt{k^2+1})$ area OACD. [See Fig. 1.]

Similarly, AH/AC = area OAHK/area OACD; AH/OA = tan BOA/2 $= 1/(k+\sqrt{k^2+1})$. Hence the area of rectangle OAHK $= 1/(k+\sqrt{k^2+1})$ of the area of square OACD. Since tetragon OAHB = 2 triangle OAH, the area of OAHB $= 1/(k+\sqrt{k^2+1})$ of the area of square OACD. [Dodgson 1884, 5–16]

To illustrate the connection between arctangent series and the geometric methods of inscribed and circumscribed polygons, Dodgson provided examples where an octant is a sector whose vertical angle is arctangent 1; each octant of a circle is divided into three sectors, of which two have the vertical angle arctangent $\frac{1}{3}$, and one has arctangent $\frac{1}{7}$.

The second theorem of the section on "Limitaries for the Area of a Circle," Theorem C, is a generalization of Theorem B that permits the division of an octant into an unlimited number of sectors:

If each octant of a circle is divided into sectors such that a of them have the vertical angle arctan $1/\alpha$, *b of them have* arctan $1/\beta$, *etc., then the area of the inscribed polygon, contained by the chords of the sectors, is equal to* $4((a/\sqrt{\alpha^2+1}) + (b/\sqrt{\beta^2+1}) + etc.)$ *of the square of the radius; and the area of the circumscribed polygon, contained by the tangents drawn at the extremities of the radii, is equal to* $8((a/\alpha+\sqrt{\alpha^2+1}) + (b/(\beta+\sqrt{\beta^2+1}) + etc.)$ *of the same square.*

Proof. The circle consists of $8(a + b +$ etc.) sectors such that $8a$ of them have the vertical angle arctan $1/\alpha$, $8b$ of them have arctan $1/\beta$, and so on. So the inscribed polygon consists of $8(a + b + \text{etc.})$ isosceles triangles such that $8a$ of them have an area equal to $1/(2\sqrt{\alpha^2 + 1})$ of the square of the radius, $8b$ of them have an area equal to $1/(2\sqrt{\beta^2 + 1})$ of it, and so on. Hence the area of the inscribed polygon is $4((a/\sqrt{\alpha^2 + 1}) + (b/\sqrt{\beta^2 + 1}) + \text{etc.})$ of the square of the radius. (1)

Similarly, the area of the circumscribed polygon is $8((a/(\alpha + \sqrt{\alpha^2 + 1}) + (b/(\beta + \sqrt{\beta^2 + 1}) + \text{etc.})$ of the square of its radius. [Dodgson 1884, 5–17] (2)

Theorem D simply generalizes these results, establishing that (1) < area of the circle < (2).

These four theorems provide a framework in which an arctangent relation yielding an approximation for π can be linked directly with the division of an octant of a circle into sectors such that the sum of the coefficients of the arctangent terms is the number of sectors of that division. By choosing the number of sectors appropriately, a reasonably efficient and accurate approximation results.

CONCLUSION

Dodgson's method of approximating π, evolving as a response to the circle-squaring dilemma, provided a "do-it-yourself-kit" approach for the mathematical dilettante. Rooted in Euclidean geometry, a common background for all late-nineteenth-century British circle-squarers, the method could produce a relatively accurate and efficient approximation that was intuitively sensible. The limitation of having only positive terms in the arctangent series prevents further improvement in accuracy without sacrificing a great deal in efficiency. But this is the trade-off necessary to link these series to the division of a circle into sectors, giving the method its intuitive geometric appeal. Dodgson stopped after constructing six series, the last having 11 terms and the most accuracy when Theorem C is used to obtain the limits. Obviously, a dilettante would not wish to work with a series having many more terms nor, given the lack of modern calculating machines, with arctangents of very large numbers.

Until very recently, to actually compute π to a large number of places by machine, Gauss 1 was best. Generally, the rate of convergence of arctangent series requires $O(n)$ full-precision operations to compute n decimals. That is, a maximum of cn addition, multiplication, division, or square-root operations, where c is a positive constant, are necessary for n-decimal accuracy. For example, to improve accuracy by a factor of 10 decimals requires a 10-fold increase in the number of full-precision operations, each of which requires a 10-fold increase in time.

In 1976, R. P. Brent and E. Salminen, using the much faster convergence of the defining recursive sequences for the arithmetic-geometric mean of two positive real numbers, were able to reduce the number of full-precision operations to $O(\log n)$. Since $\log 10n$ is not much different from $\log n$ for large n, the number of operations to obtain $10n$ digits of π is about the same as for n digits. Brent and Salminen's work uses ideas that had been developed by Gauss and Adrien-Marie Legendre. Curiously, Archimedes' inscribed and circumscribed regular polygon method for approximating π can be cast as a double-sequence recursion, one that is almost the same as the defining sequences for the arithmetic–geometric mean of $\frac{1}{2}$ and $\frac{1}{4}$, but converging much more slowly than the defining recursion for the arithmetic–geometric mean of 1 and $1/\sqrt{2}$ used for π [Borwein & Borwein 1984, 351–366].

NOTES

1. For a history of π, the reader is referred to [Schepler 1949–1950] and [Beckmann 1971].

2. Gauss discovered that $[x/y]$ can be written as a linear combination of a basic set, [1], [2], [3/2], [4], [5/2], [6], [5/4], ... This appears in [Gauss 1876]. See also [Wrench 1938].

3. Klingenstierna's relation, which appears in the article by Lehmer, does not seem to be well known. To my knowledge, it is not listed in the standard references on approximations for π.

4. This appeared in his article in *Nature,* "The Diameter of a Circle Equal in Area to any Given Square," vol. 93, p. 110. For a more complete history of the circle-squaring problem, the reader is referred to the discussion in [Hobson 1953].

5. When Theorem A is used, the larger angle is chosen as small as possible. For example, when $k = 3$, x and y are chosen as 2 and 5, rather than 1 and 10, respectively, so $\{\frac{1}{3}\} = \{\frac{1}{5}\} + \{\frac{1}{8}\}$, not $\{\frac{1}{4}\} + \{\frac{1}{13}\}$.

6. [Woodall 1893]. A wide spectrum of British and continental mathematicians contributed problems and solutions to the Mathematical Questions and Solutions section of the *Educational Times* which was published separately.

7. Dodgson's four-term series compares favorably in accuracy to Gauss 1 when expanded by Gregory's series. However, Dodgson used Theorem C and obtained limits for his series by calculating the reciprocals of the square roots of integers, a method yielding accuracy levels inverse to those produced by the former method.

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