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What Descartes knew of mathematics in 1628

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Abstract

The aim of this paper is to give an account of Descartes' mathematical achievements in 1628–1629 using, as far as is possible, only contemporary documents, and in particular Beeckman's *Journal* for October 1628. In the first part of the paper, I study the content of these documents, bringing to light the mathematical weaknesses they display. In the second part, I argue for the significance of these documents by comparing them with other independent sources, such as Descartes' *Regulae ad directionem ingenii*. Finally, I outline the main consequences of this study for understanding the mathematical development of Descartes before and after 1629.

Résumé

Cet article se proposer d'étudier le développement mathématique de Descartes en 1628–1629 en restant au plus près des sources existantes. Le principal témoignage est donné par un passage du *Journal* de Beeckman daté d'Octobre 1628. Dans la première partie de l'article, j'analyse en détail ces documents en insistant sur les nombreuses faiblesses qu'ils présentent. Dans la seconde partie, j'essaye de montrer quelles raisons nous pouvons avoir de prendre ces faiblesses au sérieux en les confrontant à des sources indépendantes, en particulier les *Regulae ad directionem ingenii*. J'essaye ensuite d'en tirer les conséquences quant à l'évaluation d'ensemble du développement mathématique de Descartes avant et après 1629.

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1. Introduction

The starting point of this paper is a puzzling testimony given in his *Journal*[1] by Isaac Beeckman, a Dutch scholar whom Descartes met in 1618 and considered "the promoter of his studies" [AT I, 161]. In October 1628, the French philosopher came to Dordrecht to visit his old friend, for whom he had written the *Compendium musicae* ten years before and to whom he presented his project of reforming all sciences, not long before leaving Holland in 1619. He described to Beeckman the results of his past nine years of study, especially in mathematics, during which, he said "he had made as much progress as was possible for a human mind" [Beeckman, 1939–1953, Vol. III, p. 94, fol. 333*r*]. As proof, he gave Beeckman a specimen of a "general algebra." He also promised to send a treatise on algebra, by which he claimed one could bring geometry to perfection and, more than that, all human knowledge [Beeckman, 1939–1953, Vol. III, p. 95, fol. 333*r*]. But when one looks at the specimen that was transcribed by Beeckman under the title *Algebrae Des Cartes specimen quoddam* and which I will study in Section 2.3, one is struck by its very poor and apparently confused mathematical content.

The discovery of these documents at the beginning of the 20th century has not discouraged commentators from proposing many reconstructions of Descartes' philosophical and mathematical development in which he is supposed to have reached a high level of technique before 1628, sometimes as early as 1619–1620. What I intend in this article is a bit of deflationist history. As a methodological exercise, I shall stick as far as possible to the texts to see if it is possible to build a coherent interpretation of them without making assumptions for which we have no evidence. Hence my first task will be to provide reasons we might have for taking Beeckman's testimony seriously, and thus consider that in 1628 Descartes was not mathematically well advanced in the topics that he discussed with his friend. This implies the necessity of looking for confirmation in other sources, and this is what I shall do in the second part of the paper. The main document in this regard is the *Regulae ad directionem ingenii*,[2] in which Descartes presented a schematism (i.e., a spatial representation of algebraic operations) very close to that presented to Beeckman in 1628, and which seems to have been written, at least in part, during the same period.[3] As I will

[1] The manuscript of Beeckman's *Journal* was recovered by Cornelis de Waard in the Provincial library of Zeeland in 1905, so Charles Adam could insert the passages related to Descartes in Volume X of *Les Oeuvres de Descartes* (1908), which he launched with Paul Tannery before the death of the latter in 1904. See [Descartes, 1964–1974, pp. 17–78, 151–169 and 331–348. From now on, AT]. De Waard gave an edition of it in four volumes [Beeckman, 1939–1953].

[2] The *Rules for the direction of the mind* is an unfinished methodological treatise, which was not published and does not contain a clear indication of date [AT X, 359–469, from now on *Regulae*]. It was written in Latin and was left by Descartes in his papers at his death. It consists of a series of Rules, of which only the first 21 are extant, Rules XIX–XXI subsisting only through their planned titles. The original manuscript is now lost and all existing editions were made from copies. The first one was a translation in Dutch in 1684, followed by a Latin edition published in *Renati Descartes Opuscula posthuma* [Descartes, 1701]. For the tormented history of the treatise, see [Descartes, 1966].

[3] One of the main arguments is the fact that Descartes mentions, in Rule VIII, the study of the "anaclastic curve" and concludes, "Even though the anaclastic has been the object of much fruitless

try to show, the mathematical weaknesses in these two projects are strikingly similar. More than that, these weaknesses, which seem incompatible with the techniques employed in *La Géométrie*, are fully compatible with those employed in an early Cartesian treatise, often forgotten by commentators: The *Progymnasmata de solidorum elementis*.[4]

This strategy also requires giving a new account of the relationship between the passages in Beeckman's *Journal* of October 1628 and the text Descartes sent Beeckman a few months later and which Descartes described as his "most outstanding discovery": the "construction" of the third and fourth degree equation by the intersection of a parabola and a circle — a technique much more sophisticated than anything that could be expected from the "sample" presented in 1628.[5] I emphasize that these various documents were presented separately to Beeckman, as were the other works done on curves (conic sections), which were produced at that time by Descartes in the context of optics and *through purely geometrical means*. There is no question of studying curves through algebraic techniques in the documents produced in 1628–1629, and the program presented to Beeckman is not that of a new classification of curves. How are we to understand this fact if this classification of curves is supposed to have been at the core of Descartes' program since 1619?

My claim is that one should resist the temptation of projecting onto this set of documents the unified view permitted by the "geometrical calculus" presented in 1637 in *La Géométrie*.[6] In fact, this methodological requisite seems of importance if one wants to understand the breakthrough leading to the very idea of this "*calcul géométrique*," that is, how what were separated in 1628–1629 (the "general algebra," the "construction" of equations, and the study of curves) came to be gathered in a unified treatment. As I will try to show, it is not sufficient to paste together the different attempts presented in 1628 to regain a unified treatment, and there were deep reasons for these elements to remain separated at that time.

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research in the past, I can see nothing to prevent anyone who uses our method exactly from gaining a clear knowledge of it" [AT X, 395; transl. Descartes, 1985–1991, I, 29; on this example, see below Note 10]. Since Beeckman, explicitly answering a question from Descartes, found a demonstration that the hyperbola satisfies the condition and, after receiving Descartes' approbation, transcribed it in a page of his *Journal* dated 1 February 1629, this would mean that the passage could plausibly have been written between 1626 (when Descartes started his research on this topic) and 1629.

[4] The *Progymnasta de solidorum elementis* is a treatise on solid geometry and polyhedral numbers that exists now only in a (presumably partial) copy with no clear indication of date. It was reproduced in the Adam–Tannery edition [AT X, pp. 257–277]. Two more up-to-date editions are available, one with English translation [Descartes, 1982] and the other with French translation [Descartes, 1987].

[5] Fol. 339*v*, dated 1 February 1629. At the end of the text, Beeckman writes: *Hanc inventionem tanti facit D. des Chartes, ut fateatur se nihil unquam praestantius invenisse, imo a nemine unquam praestantius quid inventum* [Beeckman, 1939–1953, Vol. IV, pp. 138–139].

[6] See for instance [Israel, 1998]. To avoid anachronistic designations such as "analytical geometry" or "algebraic geometry," I shall use Descartes own terminology in *La Géométrie*: "Mais, parce que j'espère que dorénavant ceux qui auront l'adresse de se servir du calcul géométrique ici proposé, ne trouveront pas assez de quoi s'arrêter touchant les problèmes plans ou solides, je crois qu'il est à propos que je les invite à d'autres recherches, où ils ne manqueront jamais d'exercice" [AT VI, 390].

2. What Descartes said to Beeckman in October 1628

2.1. Situating the Cartesian program in 1628

According to Beeckman's *Journal*, in 1628 "Louis René Descartes du Perron" went first to Middleburg (the town where Descartes, in May 1619, had last sent letters to him) in order to visit him. But Beeckman was no longer in Middleburg and it was not until 8 October, 1628 that Descartes finally met his friend again in Dordrecht [Beeckman, 1939–1953, Vol. III, p. 95, fol. 333*r*]. As he would reiterate nine years later,[7] the Frenchman explained to his friend that he had worked a lot in arithmetic and geometry since their last exchange and made such great advances in these sciences that he had nothing more to expect from them (*se in arithmeticis and geometricis nihil amplius optare*).[8] To make his progress clear, he presented some "samples" of his work (*cujus rei non obscura mihi specimina reddidit*) and promised to send later a treatise of *Algebra* completed in Paris.

Even if Descartes did say that the treatise on algebra was complete, it was not yet ready for publication because, as he explained, he thought of it as a basis for a more ambitious project intended to embrace all human knowledge.[9] This ambitious project was, in fact, the main reason that Descartes gave for his visit. What he expected from Beeckman was to engage in this program, pursuing their former fruitful collaboration: "traveling through Germany, France and Italy, he had not found anybody else, he says, with whom he could discuss according to his heart (*secundum animi*) and from whom he could hope for aid in his researches".[10] This development is transcribed by Beeckman in his journal under the title

[7] In 1637 Descartes published the "Discours de la méthode pour bien conduire sa raison et chercher la vérité dans les sciences" as an introduction to a collection of "Essais de la Méthode" presenting his major scientific achievements and consisting of *La Dioptrique*, *Les Météores* and *La Géométrie* [AT VI].

[8] Compare with *Discours de la méthode*, AT VI, 20–21. Just after recalling the results he had obtained following the rules of his method, Descartes also explains in the *Discours* in what sense this could be seen as "perfect knowledge": "In this I might perhaps appear to you to be very vain if you did not remember that having but one truth to discover in respect to each matter, whoever succeeds in finding it knows in its regard as much as can be known" [Descartes, 2003, p. 94].

[9] *Paulo post Parisiis suam Algebram, quam perfectam dicit, quâque ad Geometriae scientiam pervenit, imo qua ad omnem cognitionem humanam pervenire potest*, *propediem ad me missurus* [Beeckman, 1939–1953, Vol. III, p. 95].

[10] Beeckman, 1939–1953, Vol. III, p. 95. This is not mere courtesy, as Descartes did indeed ask Beeckman for help in the resolution of an important problem in optics: the proof of the fact that the hyperbola is an anaclastic curve. "The line called the 'anaclastic' in optics," to repeat Descartes' own description in the *Regulae*, is "the line from which parallel rays are so refracted that they intersect at a single point" [AT X, 393–394; transl. Descartes, 1985–1991, I, 28–29]. Immediately after the sample of algebra, Beeckman also transcribed in his *Journal* other Cartesian works concerning optics and conic sections. The first fragment concerns the angle of refraction (*Angulus refractionis Des Cartes exploratus* fol. 333*v*) and contains the famous "law of sines." The second concerns "burning lenses," which Descartes discovered using the fact that the hyperbola is an anaclastic curve (*Quod attinet ad inventionem hyperbolicae sectionis ejus generis, per quam omnes radii in idem punctum refringantur, quod dictus DES CHARTES dicit se fecisse* fol. 334*r*). This last fact was apparently not proved by Descartes, who asked Beeckman if he could demonstrate it. This is testified by a fragment dated 1 February 1629, which begins with these words: "*Hanc de hyperbola propositionem D. des Chartes indemonstratam reliquerat, ac me rogavit ut ejus demonstrationem quaererem, quam cum invenissem, gravisus est ac genuinam esse judicavit*" [Beeckman, 1939–1953, Vol. III, pp. 109–110, fol. 338*v*].

Historia Des Cartes ejusque mecum necessitudo and is followed by a general commentary, entitled *docti cur pauci*, making reference to the fact that there were so few learned men at the time — a phrase which reminds us of the famous *physico-mathematici paucissimi* written by the same Beeckman when he first met Descartes in 1618 [Beeckman, 1939–1953, Vol. I, p. 244, fol. 100*v*].

The *Algebra* is hence presented in the context of a more ambitious project, which is very close to the one presented in the *Regulae*, in which Descartes warns,

> I would not value these Rules so highly if they were good only for solving those pointless problems with which arithmeticians and geometers are inclined to while away their time, for in that case all I could credit myself with achieving would be to dabble in trifles with greater subtlety than they. I shall have much to say below about figures and numbers, for no other disciplines can yield illustrations as evident and certain as these. But if one attends closely to my meaning, one will readily see that ordinary mathematics is far from my mind here, that it is quite another discipline I am expounding, and that these illustrations are more its outer garments than its inner parts. This discipline should contain the primary rudiments of human reason and extend to the discovery of truths in any field whatever. Frankly speaking, I am convinced that it is a more powerful instrument of knowledge than any other with which human beings are endowed, as it is the source of all the rest. [Rule IV, AT X, 373–374; transl. Descartes, 1985–1991, I, 17]

The whole treatise, according to a general plan exposed in Rule XII, was supposed to contain a description of a general method to solve any question and was intended to consist of three parts of twelve rules each, dealing respectively with "simple propositions," "perfect questions," and "imperfect questions."[11] In introducing the second part, Descartes gives a warning similar to that in Rule IV: "This part of our method was designed *not just for the sake of mathematical problems*; our intention was, rather, that the mathematical problems should be studied almost exclusively *for the sake of the excellent practice which they give us in the method*." [Rule XIV, AT X, 442; transl. Descartes, 1985–1991, I, 59: my emphasis.]

All these documents converge toward the idea that Descartes left France with a project that was supposed to go much further than mathematics, or even "physico-mathematics," and touch *ad omnem cognitionem humanam*. In the third part of *Discours de la méthode* (1637), he claims to have spent nine years after 1619 practicing "in the solution of mathematical problems according to the Method, or the solution of other problems which though pertaining to other sciences, I was able to make almost similar to those of mathematics" [AT VI, 30]. But Descartes then adds a very important comment: this occurred, he says, before he had taken "any definite part in regard to the difficulties as to which the learned are in the habit of disputing, or had commenced to seek the foundation of any philosophy more certain than the vulgar" [AT VI, 30]. As regard these "foundations," the correspondence of 1629–1630 makes it clear that they were related to his later investigations in physics.

In a letter of October 1629 to Mersenne, Descartes states indeed, just after mentioning his actual project of a small treatise of physics (on meteors), that he has "*now* taken a stand on *all the foundations* of philosophy."[12] In a famous letter of 15 April 1630, he answers a theological question from Mersenne by stating,

[11] A "question" is everything in which a truth can be sought (as opposed to "intuition" in which no falsity can occur). A question is "perfect" when everything that is sought can be deduced from the data and "imperfect" if not [AT X, 431].

[12] "Pour la Raréfaction je suis d'accord avec ce Médecin et ai *maintenant* pris parti touchant *tous les fondements* de la Philosophie" [AT I, 25: my emphasis].

> I think that all those whom God has given the use of this reason have an obligation to employ it principally in the endeavour to know him and to know themselves. That is the task with which I began my studies; and I can say that I would not have been able to discover *the foundations of* physics if I had not looked for them along that road. It is the topic which I have studied more than any other and in which, thank God, I have not altogether wasted my time. *At least I think that I have found how to prove metaphysical truths in a manner which is more evident than the proofs of geometry* — in my opinion, that is: I do not know if I shall be able to convince others of it. *During my first nine months in this country I worked on nothing else*. [AT I, 144; Descartes, 1985–1991, III, 22: my emphasis]

Hence it is clear that the first nine months that Descartes spent in Holland, in 1629, were dedicated to these kinds of inquiries (foundations of physics, connected more generally to the foundations of philosophy and the status of "metaphysical truths") and that they led to a decisive change in Descartes' thought ("une prise de parti," as he says). The very fact that the evidence of geometry could be presented in 1629 not as a basis for a reform of all human knowledge as in 1628, but as inferior to "metaphysical truths," is a striking sign of this change. It is in accordance with the famous thesis presented in the correspondence of 1630 with Mersenne concerning the creation of eternal (i.e., mathematical) truths by God.[13] In the letter of 6 May, Descartes emphasizes,

> Those who have no higher thoughts than these can easily become atheists; and because they perfectly comprehend mathematical truths and do not perfectly comprehend the truth of God's existence, it is no wonder they do not think *the former depend on the latter*. But they should rather take the opposite view, that since God is a cause whose power surpasses the bounds of human understanding, and since the necessity of these truths does not exceed our knowledge, these truths are therefore something less than, and subject to, the incomprehensible power of God. [To Mersenne, AT I, 150; transl. Descartes, 1985–1991, III, 25: my emphasis]

Moreover, Descartes explains that he is no longer interested in mathematics[14] and mentions some treatises begun in Paris that he left unfinished because he realized that they were dependent on other important issues that he had to deal with first [AT I, 138].

One should be careful not to conflate these different steps: a first project, presented to Beeckman at the end of 1628, was already supposed to lead to a more general program concerning "all human knowledge," but it did not imply research on the "foundations of philosophy"; when Descartes finally took a stand on these foundational issues, at the beginning of 1629, he also clearly *changed his mind* considering the role of mathematical certainty and the feasibility of a pure *Mathematica-Physica*.[15]

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[13] "Mais je ne laisserai pas de toucher en ma Physique plusieurs questions métaphysiques, et particulièrement celle-ci: Que les vérités mathématiques, lesquelles vous nommez éternelles, ont été établies de Dieu et en dépendent entièrement, aussi bien que tout le reste des créatures" [AT I, 145].

[14] "Pour des problèmes, je vous en enverrai un million pour proposer aux autres, si vous le désirez; mais je suis si las des mathématiques, et en fais maintenant si peu d'état, que je ne saurais plus prendre la peine de les soudre moi-même" [AT I, 139].

[15] In this regard, one should remember that on the occasion of his brief rupture with Beeckman, in 1630, Descartes criticized very violently what his friend had achieved under this particular program: "I have never learnt anything but idle fancies from your *Mathematico-Physica*, any more than I have learnt anything from the *Batrachomyomachia*" [AT I, 159; Descartes, 1985–1991, I, 27, translation modified]. On the complex relationship between Descartes and "Physico-mathematics" after 1630, see [Garber, 2000].

2.2. *Ars generalis, scientia penitus nova, Mathesis universalis, etc*

In the above I have drawn attention to the complexity and the richness of Descartes' projects during the years 1628–1629. As we have seen, documents exist, be they from this very period (Beeckman's *Journal*, correspondence of 1629–1630) or from later memories (*Discours de la méthode*, 1637), testifying that Descartes changed his mind in a significant way during this time. I now introduce another document often mentioned in the literature but to my mind less directly connected with our topic: the letter to Beeckman of 26th March 1619. In it, Descartes announced to his friend, not long before leaving Holland, that he wanted to launch a "completely new science (*scientia penitus nova*) by which all questions in general may be solved that can be proposed about any kind of quantity, continuous as well as discrete. But each according to its own nature." He then makes a parallel:

> In arithmetic, for instance, some questions can be solved by rational numbers, some by surd numbers, and others can be imagined but not solved. For continuous quantity I hope to prove that, similarly, certain problems can be solved by using only straight or circular lines, that some problems require other curves for their solution, but still curves which arise from one single motion and which therefore can be traced by the new compasses, which I consider to be no less certain and geometrical than the usual compasses by which circles are traced; and, finally, that other problems can be solved by curved lines generated by separate motions not subordinate to one another. [AT X, 157–158; transl. Bos, 2001, p. 232][16]

This program, still in infancy and merely analogical, is nonetheless very close to that of *La Géométrie*, in which Descartes succeeded in proposing a new classification of geometrical problems through a classification of curves (relying on the one hand on what he calls "continuous motion" and on the other hand on algebraic criteria). It is therefore very tempting to consider that it was the core of "the" Cartesian program from the encounter with Beeckman in 1618 to the *Discours de la méthode* and the *Essais* appended to it in 1637. The "general algebra" of 1628 could then be considered as a step on this continuous path.

But this kind of continuity seems to me a mere artefact coming from the tendency to go from the programs to a reconstruction of the practice and not from the actual practice to an evaluation of the programs. I shall indicate this more clearly in the following pages. But even staying at the level of declarations, one should emphasize the fact that neither in the documents given by Beeckman in 1628–1629, nor in the *Regulae*, is there *any* mention of a classification of geometrical problems depending on the classification of curves. How should we interpret this silence if this is supposed to be at the very core of Descartes' program? The study of curves (in fact, only conic sections) is presented in 1628 *separately* and *independent* of the "general algebra" in the context of works in optics. Moreover, Descartes does not use algebra in his geometrical analysis of these curves, as transcribed by Beeckman.

One should also be very cautious when identifying the program of the new discipline (*alia disciplina*) announced in the *Regulae*, in the first part of Rule IV, with the *mathesis*

[16] Descartes then continues, "But in due time I hope to prove which questions can or cannot be solved in these several ways: so that hardly anything would remain to be found in geometry. This is a truly infinite task, not for a single person. Incredibly ambitious; but through the dark confusion of this science I have seen some kind of light, and I believe that by its help I can dispel darkness however dense." In the margin, Beeckman writes *ars generalis ad omnes quaestiones solvenda quaesita*.

universalis mentioned in the second part of the same Rule IV, to ground a hypothetical proximity with the program of a *scientia penitus nova* of 1619 (which was supposed to "solve any question that can be proposed about any kind of quantity, continuous as well as discrete"). Without entering into the details of this controversial matter,[17] I simply recall what Descartes says in Rule IV. The general context is an investigation into what makes possible the unity of mathematics — which is indeed the traditional context of reflection on a possible "general" or "universal" mathematics. On this path, Descartes arrives at the observation that "the exclusive concern of mathematics is with questions of order or measure and that it is irrelevant whether the measure in question involves numbers, shapes, stars, sounds, or any other object whatever." This leads him to the conclusion "that there must be a general science (*scientia generalis*) which explains all the points that can be raised concerning order and measure irrespective of the subject-matter, and that this science should be termed not with an unusual name, but with a venerable one, with a well-established meaning: *mathesis universalis* — for it covers everything that entitles these other sciences to be called branches of mathematics" [AT 377, 9–378, 10; Descartes, 1985–1991, I, 19]. The *mathesis universalis* is therefore presented as a "venerable" discipline with a "well-established meaning," not a "completely new science." A few lines later, Descartes goes further by stating that everybody not only knows its name, but also its subject matter (*cum nomen ejus omnes norint, et, circa quid versetur... intelligant*) and he then wonders why people do not pay more attention to it [Descartes, 1985–1991, I, 20]. His answer marks the second (and last!) use of the term in the entire Cartesian corpus:

> Aware how slender my powers are, I have resolved in my search for knowledge of things to adhere unswervingly to a definite order, always starting with the simplest and easiest things and never going beyond them till there seems to be nothing further which is worth achieving where they are concerned.[18] Up to now, therefore, I have devoted all my energies to this *mathesis universalis*, so that I think I shall be able in due course to tackle the somewhat more advanced sciences, without my efforts being premature. [AT X, 378–379, 6; Descartes, 1985–1991, I, 20; transl. modified]

Mathesis universalis is therefore a discipline that Descartes studied before he dared to "tackle the somewhat more advanced sciences," according to a rule that states that one should never go beyond the easiest matter "till there seems to be nothing further which is worth achieving where they are concerned" — the exact contrary of a program indeed.

My aim here is to draw attention to the fact that there is no reason to merge what seem to be different projects (the "entirely new science" of 1619, the *alia disciplina* and the *mathesis universalis* of the *Regulae*, the "general algebra" of 1628) into a single grandiose view culminating in *La Géométrie*. In fact, it seems very dubious to ground any hermeneutic postulate in these texts at all, since they are very allusive and surrounded, as we saw in the previous section, by many other texts calling into question the continuity of Descartes' line of thought. In my opinion, the only way to solve the uncertainties raised by these texts is to confront them with Descartes' actual mathematical practice.

[17] For a more detailed analysis, see [Rabouin, 2009]. For an overview of the debates around this notion, see [Van de Pitte, 1991] and [Sasaki, 2003, Chap. IV § 3].

[18] *Donec in istis nihil mihi ulterius optandum superesse videatur*: note the proximity with the *Is dicebat mihi se in arithmeticis and geometricis nihil amplius optare*.

2.3. The general algebra

I turn now to the *Algebrae Des Cartes specimen quoddam* (henceforth the *Specimen*), the text transcribed by Beeckman, which describes a "general algebra" invented by Descartes (*dicit idem se invenisse Algebram generalem*). It relies on a simple geometrical schematism in which Descartes uses only plane and not solid figures (*se non uti corporum figuris, sed planis duntaxat*). This echoes a famous passage of the *Regulae* in which Descartes explains how he distanced himself from the traditional interpretation of algebraic powers corresponding to geometrical dimensions, because he understood that they were "nothing but magnitudes in continued proportion" and, as a consequence, "should never be represented in the imagination otherwise than as a line or a surface."[19]

But, as we shall see, this is not the justification of the notion of dimension given in our text. Moreover, the justification in the *Regulae* seems closely tied to what Descartes called the "number of relations," which is usually interpreted as connected with the use of an exponential notation. However, another surprising element of our text is that it still uses cossic notation.[20] The point is not that Descartes uses this notation in and of itself, but that he does so when introducing what is supposed to be the results of his great advances in mathematics during the nine preceding years — a very intriguing situation if he was in possession of his new notation at that time. One could of course attribute this "archaism" to the copyist and make the hypothesis that Beeckman changed Descartes' notation, but this would mean, in any case, that Descartes did not emphasize its importance. On the other hand, one should be very cautious when dealing with Descartes' notation in the *Regulae* and not forget that we possess only late copies of the text, made after the publication of *La Géométrie*. Besides, these copies are not coherent with regard to notation. For example, Rule XVIII does not satisfy the prescription presented in Rule XVI to represent indeterminate quantity by lower case and unknown by uppercase, and uses indifferently *a* and *A* to designate the same quantity. One striking example of the modifications introduced by editors is in the *Excerpta mathematica*, a collection of mathematical essays by Descartes that was published for the first time in the same volume as the *Regulae: Renati Descartes Opuscula posthuma* [Descartes, 1701]. We possess indeed other copies of some texts than the one given in the *Opuscula* and one can clearly see in them that x and xx were substituted for

[19] "We should note also that those proportions which form a continuing sequence are to be understood in terms of a number of relations; others endeavour to express these proportions in ordinary algebraic terms by means of many dimensions and figures (...). I realized that such terminology was a source of conceptual confusion and ought to be abandoned completely. For a given magnitude, even though it is called a cube or the square of the square, should never be represented in the imagination otherwise than as a line or a surface. So we must note above all that the root, the square, the cube, etc. are nothing but magnitudes in continued proportion which, it is always supposed, are preceded by the arbitrary unit mentioned above" [AT X, 456–457; Descartes, 1985–1991, I, 68].

[20] "Cossic" comes from Cristopher Rudolff's book *Coss* (1525) — which means "unknown" (like the Italian *cosa*), the "cossic art" being then the art of dealing with unknowns. This style of algebra spread in Germany in the 16th century (with authors like Stifel, Faulhaber, or Roth, to whom I shall come back later). It relied on a notational system in which a symbol was used for every power of the unknown. The notation transcribed by Beeckman is ℜ for the root, ʒ for the square, ℭ for the cube and ʒʒ for the square of the square (*biquadratum*).

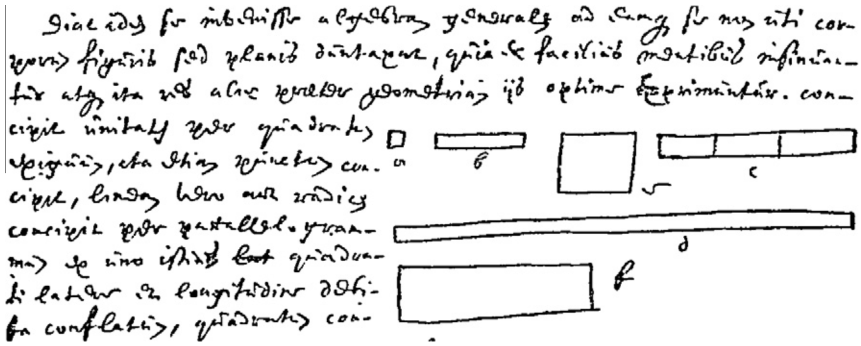


Fig. 1. Facsimile of fol. 333 of Beeckman's autograph. Excerpted from (Beeckman, 1939–1953, III).

what, in other sources, are cossic symbols.[21] One could also note that the *Opuscula* uses a cossic sign for addition ✠ in Rule XVI, which was changed in the Adam–Tannery edition. I shall come back to these issues later because the first surprise occurs before one even reaches them: in the presentation of the schematism.

As observed above, the schematism used by Descartes in the "general algebra" relied on two-dimensional figures — the reason invoked being that this representation is easier to grasp (*qui eae facilius mentibus insinuantur*). According to Descartes, one should represent the unit by a small square, the root by a parallelogram composed of as many times the square unit as needed, and then the square, the cube or the square of the square as rectangles composed of as many times the roots as implied in them. In the diagram drawn by Beeckman (Fig. 1), *a* is the square unit, *b* the root composed of three units, *c* the square composed of three times three units, *d* the cube composed of three times the square [Beeckman, 1939–1953, Vol. III, p. 95, fol. 333*r*].

Beeckman explains that the representation could be done with simple lines only (*per nudas lineas*), each representation being prefixed by the cossic symbol corresponding to it (Fig. 2).

This could be seen as an anticipation of the famous and groundbreaking opening of *La Géométrie*, in which Descartes explained that all the operations of arithmetic could be interpreted as manipulations on "simple lines."[22] However, certain details of the text pre-

[21] See [AT X, 298]. Descartes' discourse about the "number of relations" in the *Regulae* is perfectly compatible with other notation, such as Stevin's (which Descartes could have come across in [Van Roomen, 1597] — a plausible source for his concept of *mathesis universalis*; see [Crapulli, 1969]) or even an adaptation of [Clavius, 1609]. It should indeed be noted that Clavius (like German cossists: Stifel, Roth, Faulhaber) proposed, when introducing several unknowns, to mark them by capital letters A, B, C — in the same way as Descartes proposes in Rule XVI [Clavius, 1609, Chap. XV, p. 72 sq]. Moreover, what we now write as "2y^4" would be written by Clavius as "2A", expressing the power of an unknown and not an unknown in and of itself (p. 73). Since was referred to, at the very beginning of Clavius' treatise, as corresponding to the "exponent" 4, it would have been very natural (and convenient) to abridge this first notation as "2A4", or "2A(4)" (Girard's notation) if one wanted to avoid ambiguity (since in Clavius's notation "2A" could refer to "2y^4" or "2yx^4").

[22] [AT VI, 371]: "il est à remarquer que, par a^2 ou b^3 ou semblables, je ne conçois ordinairement que des lignes toutes simples, encore que, pour me servir des noms usités en l'algèbre, je les nomme des carrés, ou des cubes, etc." On the role of this "calcul des segments" in Descartes' *Géométrie*, see [Jullien, 1996].

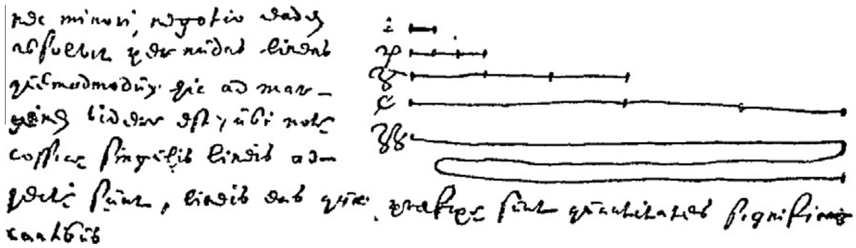


Fig. 2. Representation *per nudas lineas*. Facsimile of fol. 333 of Beeckman's autograph excerpted from (Beeckman, 1939–1953, III).

vent us from jumping to this conclusion. When introducing the representation through unit squares, Descartes did indeed add a strange *ita etiam punctum concipit* ("or he conceives of it in the same way with a point"). After completing the first sequence, Beeckman states more explicitly: "Or rather, he explains also all of this *by lines* so that *a* would represent a *point*, *b* a line, *c* a square and *d* a cube. In this manner *f* represented a cube produced by the multiplication of the square *e* by the number of roots." [Beeckman, 1939–1953, Vol. III, p. 95, fol. 333*r*: my emphasis.]

This whole development is very close to Stevin's ideas. In his *Arithmétique*, first published in 1585, and reedited by Albert Girard in 1625, Stevin insisted on the fact that one could interpret algebraic powers in terms of geometric magnitudes in a continuous proportion and therefore proposed a very simple geometric schematism, in which there would be no need to escape from the three dimensions of everyday space. The main difference between Descartes and Stevin is that Stevin then used a three-dimensional schematism (made of "beams", which he calls "plinthides" and "docides", and not of rectangles).[23] See Fig. 3.

But the fact that Descartes also allows us to represent the unit by a geometrical *point* is astonishing. It goes directly against what Stevin rightly designated as the great mistake of his predecessors, who considered the unit in arithmetic as somehow equivalent to the point in geometry [Stevin, 1955–1966, Vol. II B, p. 498].

The parallel with Stevin leads to another important remark: as in Stevin, the use of the schematism is here not operative, but foundational,[24] that is, there is nothing like a geometrical *calculus* in our text, and Descartes does not explain how to construct the different magnitudes through determinate operations. This is not merely an argument from silence since there seems to be, at first sight, no possibility of grounding a geometric algebra if the unit is represented as a point and the root as a line.

The apparent confusion of Descartes is reinforced by the following passage, in which he explains how one should understand the notion of algebraic dimension:

> He conceives the cube in particular through three dimensions, as do others. But he conceives a *biquadratum* as if a simple cube made of wood was transformed into a stone cube: in this manner, indeed, one dimension is added to the whole; but if another dimension is to be added, he considers a cube made of iron; then of gold, etc. — that which can be obtained not only with weight but also with colours and with all the other qualities. Cutting a wooden cube in three squares, he thus conceives that he is cutting a cube made only of wood, of stone etc. so that the iron cube is transformed into the wooden cube in

[23] This could provide a way of interpreting the opening *se non uti corporum figures, sed planis duntaxat*.

[24] "Description du fondement des nombres géométriques" is in fact the title of the section in which Stevin introduces his schematism [Stevin, 1625].

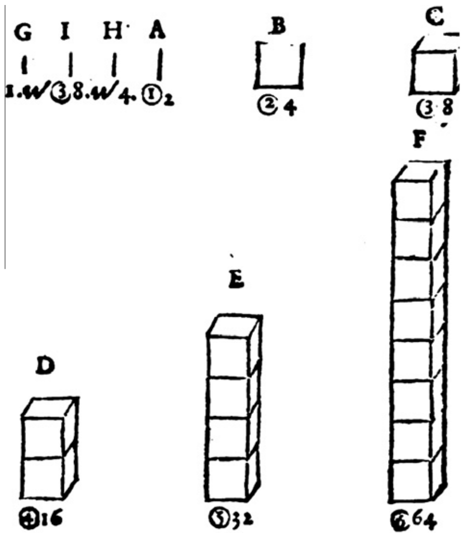


Fig. 3. Representation of the powers of 2 according to Stevin [from pp. 12 and 13 of his *Arithmetique* 1585; reproduced in Stevin, 1955–1966, Vol. IIB, pp. 511–512].

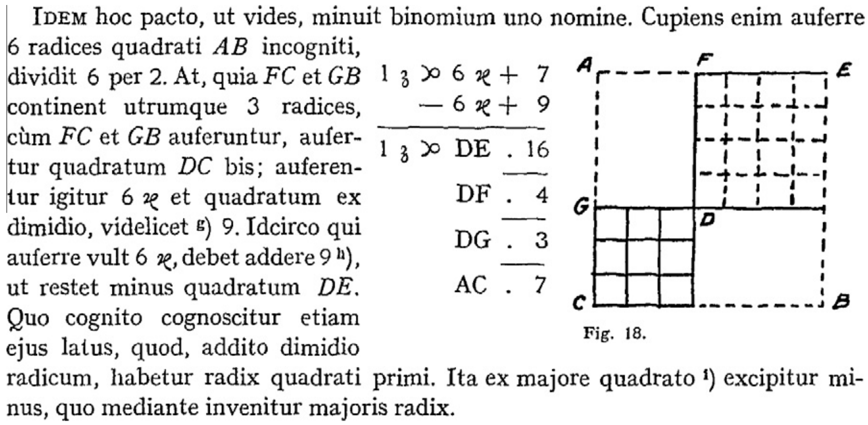
> the same manner than the simple cube is into the squares which are to be considered in each kind. [Beeckman, 1939–1953, Vol. III, p. 96, fol. 333*v*, my transl.]

One could consider that these developments provide just one among many other examples in the history of science of a powerful technique being attached to confused conceptual grounds. Unfortunately, we have here no evidence of a powerful technique. Indeed, in the following passage, Descartes shows his friend evidence of his technique. But what example does he choose? The resolution of a quadratic equation through a simple manipulation of squares, that is to say one of the most elementary techniques in algebra, known since Al-Khwarizmi!

As one can see in Fig. 4, to solve the equation $x^2 = 6x + 7$, Descartes constructs the square of side x, divides the coefficient by 2 to remove two rectangles of area $3x$, and thus obtains a new square DE. In this construction, the square of side GD is subtracted twice and one has to compensate for this by adding 9 to the whole. So if one subtracts $6x$ from the sought square x^2, one will obtain $7 + 9 = 16$. This gives us the area of square DE (16), and then its side DF = 4. Since DG = 3, one obtains the result AC = 7. *This* is the (only) "sample" of Descartes' technique given in 1628! And it is presented as the result of nine years of mathematical studies in which "he has made as much progress as was possible for a human mind"!

Then Beeckman notes, "The irrational numbers, which cannot be explained otherwise, he explains them by a parabola (*per parabolam*)." This allusive statement could be a reference to the solution of the third- and fourth-degree equations by the intersection of a parabola and a circle, which Descartes sent to his friend four months later and which he may have announced to him in October.[25] The end of the text contains a discussion of the different

[25] One should, however, keep in mind that the Greek name for "application of area" was *parabolê tôn choriôn*. This technique, exposed in Book II of Euclid's *Elements*, was presented by [Clavius, 1609] when dealing with the extraction of square roots (Chap. XII, p. 46 sq.). As we will see, it was naturally involved in the calculus presented in the *Regulae*, so one should not dismiss this interpretation of *per parabolam* too quickly.


Idem hoc pacto, ut vides, minuit binomium uno nomine. Cupiens enim auferre 6 radices quadrati *AB* incogniti, dividit 6 per 2. At, quia *FC* et *GB* continent utrumque 3 radices, cùm *FC* et *GB* auferuntur, aufertur quadratum *DC* bis; auferentur igitur 6 ℞ et quadratum ex dimidio, videlicet g) 9. Idcirco qui auferre vult 6 ℞, debet addere 9 h), ut restet minus quadratum *DE*. Quo cognito cognoscitur etiam ejus latus, quod, addito dimidio radicum, habetur radix quadrati primi. Ita ex majore quadrato 1) excipitur minus, quo mediante invenitur majoris radix.

1 ʒ ∞ 6 ℞ + 7
— 6 ℞ + 9
1 ʒ ∞ DE . 16
DF . 4
DG . 3
AC . 7

Fig. 4. The resolution of an equation in 1628. (Beeckman, 1939–1953, III p. 96).

denominations of the roots of an equation ("true", "implicit" (i.e., negative), and "imaginary") and the number of them, which Descartes derives, he says, *ex tabula vulgari* (fol. 333*v*).

By recalling the *Algebrae Des Cartes specimen quoddam* I want to emphasize the many weaknesses it presents: the geometric schematism does not seem well grounded, the algebraic technique is trivial, and the notion of algebraic dimension is metaphorically explained through a qualitative analogy. How are we to understand this collection of disappointing features if Descartes at that time was already in possession of an original and powerful technique? Was this the outcome of an "entirely new science"?[26] Not that a great mathematician cannot make errors, feel hesitations, or have zones of confusion in his conceptual foundations. But we have to keep in mind that we are not dealing here with a draft that Descartes kept for himself. What we have here is nothing less than a demonstration brought to make clear the progress made during the nine preceding years and to convince Beeckman to engage in a more ambitious program.

Of course, one could object that the poverty and the confusion that I have attributed to Descartes were in fact linked to the very specific situation in which this text arose. Maybe Descartes was simply hiding from Beeckman the real power of his technique; maybe he adapted a powerful technique to a simple example, taking into account the low mathematical level of his interlocutor; maybe Beeckman, who was certainly not an expert algebraist, simply did not understand what Descartes had explained to him; maybe he introduced his own confusion into Descartes' clear and distinct ideas, etc.

These arguments are very difficult to dismiss since they rely on the postulate that there is something behind the extant documents, about which we have no direct evidence at all. This is why I presented my interpretation as a methodological exercise. There are nonetheless positive arguments that indicate that Beeckman's mathematical weaknesses are insufficient to justify the disappointing features presented by the sources. It should be noted first that Descartes is truly asking Beeckman for help in the solution of a "physico-mathematical

[26] See for example [Sasaki, 2003, p. 166]: "What can be asserted from our examination of Beeckman's record 'A certain specimen of Descartes's Algebra' is that before October 1628, Descartes had already possessed his own system of algebra, which can be thought to be the product on an 'entirely new science.'"

problem" and that there is no reason to suspect that he is insincere when saying that he wants to launch a new ambitious program with him or that he is hiding something from him. Suppose, for the sake of the argument, that Descartes did not show the full extent of his skills to his friend. It would not change the essence of our interrogations: why did he not even mention what is supposed to be the real content of his techniques (the exponential notation, the classification of curves, the algebraic analysis of geometrical problems)? Remember that Descartes had no difficulty in announcing his ambitious program of a "new science" in 1619. It would be very natural to mention the outcome of this program, if there was any. Besides, why would Descartes think that the treatment of quadratic equations is a simple example of his techniques? Finally, it should be pointed out that Descartes had no difficulty sending a much more complicated example (the resolution of solid problems through the intersection of a circle and a parabola) to his friend a few months later; once again, as we shall see below, we find no trace of the exponential algebraic notation in this piece, nor of the classification of curves, nor of the algebraic analysis of geometrical problems. In this case, there is no reason to suspect that Beeckman made changes to Descartes' writings, since he explicitly states that he made a faithful copy of them (*quod ex illius scriptis ad verbum describo*). Last, but not least, we possess an independent source supporting our interpretation: the same confusions on similar topics appear in the *Regulae ad directionem ingenii.*

3. *Dimensio*, *unitas*, *et figura*: looking outside of the *Specimen*

3.1. The schematism of the Regulae

There is extensive literature on Descartes' *Rules for the Direction of the Mind*, but strangely, some passages seem to persistently escape the attention of readers. These passages precisely concern the three notions identified above: dimension, unity, and figure (*dimensio, unitas et figura*), and the fact that *two* schematisms are proposed by Descartes — depending on which representation one chooses for the unit. Therefore, I would like to put particular emphasis on these aspects. In Rule XIV Descartes famously introduces the foundation of his geometrical calculus (presented in Rule XVIII) by stating that "in all reasoning it is only by means of comparison that we attain an exact knowledge of the truth" and that "the business of human reason consists almost entirely in preparing this operation." This establishes the central role of proportion in the treatment of any "question" and the main goal of the method in this second part of the treatise: to reduce these proportions to simple comparisons, that is, equalities [AT X, 440; Descartes, 1985–1991, I, 57–58]. According to this general structure, any "question" can be expressed in terms of proportions between magnitudes and, in consequence, through some spatial representation "since no other subject [*scil.* than spatial extension] displays more distinctly all the various differences in proportions." Hence the conclusion that "perfectly determinate problems present hardly any difficulty at all, save that of expressing proportions in the form of equalities, and also that everything in which we encounter just this difficulty can easily be, and ought to be, separated from every other subject and then expressed in terms of extension and figures. Accordingly, we shall dismiss everything else from our thoughts and deal exclusively with these until we reach Rule twenty-five" [AT X, 441; Descartes, 1985–1991, I, 58].

These ideas are taken up again by Descartes in the *Discours de la méthode*, when he comments on the very first application of the Method. He starts by recalling, as in Rule IV, that he studied the common part of mathematics, "observing that, although their

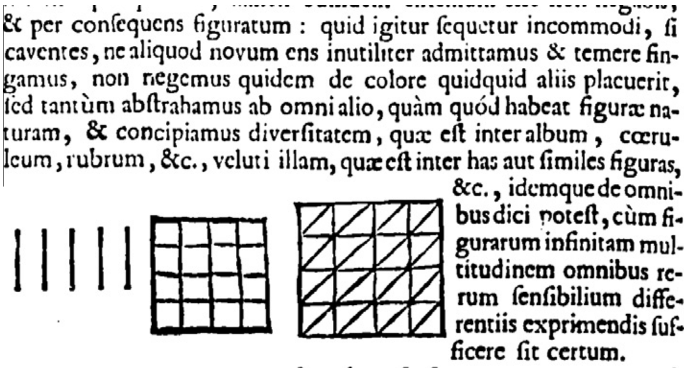

&c per conſequens figuratum : quid igitur ſequetur incommodi, ſi caventes, ne aliquod novum ens inutiliter admittamus & temere fingamus, non negemus quidem de colore quidquid aliis placuerit, ſed tantùm abſtrahamus ab omni alio, quàm quód habeat figuræ naturam, & concipiamus diverſitatem, quæ eſt inter album, cœruleum, rubrum, &c., veluti illam, quæ eſt inter has aut ſimiles figuras, &c., idemque de omnibus dici poteſt, cùm figurarum infinitam multitudinem omnibus rerum ſenſibilium differentiis exprimendis ſufficere ſit certum.

Fig. 5. How to represent the difference between white, blue, and red according to Rule XII [from Descartes, 1701, p. 34].

objects are different, they do not fail to agree in this, that they take nothing under consideration but the various relationships or proportions which are present in these objects, I thought that it would be better if I only examined these proportions in their general aspect." He then explains the way in which he could handle these proportions:

> Then, having carefully noted that in order to comprehend the proportions I should sometimes require to consider each one in particular, and sometimes merely keep them in mind, or take them in groups, I thought that, in order the better to consider them in detail, I should picture them in the form of lines, because I could find no method more simple nor more capable of being distinctly represented to my imagination and senses. I considered, however, that in order to keep them in my memory or to embrace several at once, it would be essential that I should explain them by means of certain formulas, the shorter the better. And for this purpose it was requisite that I should borrow all that is best in Geometrical Analysis and Algebra, and correct the errors of the one by the other.[27]

The main difference between the two documents is that, in the second case, the "method" is presented in the context of an application to pure mathematics or *physico-mathematica*, whereas the *Regulae* are concerned more with epistemological considerations (with explicit reference to the theory of imagination exposed in Rule XII). In particular, one should notice that the model presented in Rule XIV on the example of colors (see Fig. 5) is not yet that of a geometrical calculus *stricto sensu*, but that of a schematic representation used to code differences.[28] This is clear from the beginning of Rule XII, where Descartes treats the example of colors, which he alludes to in Rule XIV.

As I shall describe in the following paragraphs, Descartes explains in detail the epistemological grounds of his geometric schematism, expanding at large on the definition of "extension." He then arrives at "the characteristics of extension itself which may assist us in elucidating differences in proportion", namely our three key notions: dimension, unity, and figure.

Dimension, first, is defined in term of the number of parameters entering a question. Hence a triangle can be seen as having three dimensions, since it needs three parameters to fully determine it (the three sides, or two sides and an angle, or two angles and its area).

[27] [AT VI, 19–20; Descartes, 2003, p. 93].

[28] On this general project, see [Sepper, 1996, Chap. III] and [Fichant, 1998, pp. 1–28].

In the same way, a trapezium can be seen as having five dimensions; a tetrahedron, six, etc. The difference from the "qualitative" interpretation presented in the *Specimen* is that here the notion is more clearly limited to measurable parameters: "By 'dimension' we mean simply a mode or aspect in respect of which some subject is considered to be measurable. Thus length, breadth, and depth are not the only dimensions of a body: weight too is a dimension — the dimension in terms of which objects are weighed. Speed is a dimension — the dimension of motion; and there are countless other instances of this sort" [AT X, 447; Descartes, 1985–1991, I, 62]. This is consistent with the idea that Descartes' overall project was at that time still close to Beeckman's notion of a *Physico-mathematica*, and there is no clear distinction between the mathematical and the physical notions of dimension (what he says being close to what we now deal with in "dimensional analysis").

Of particular interest is the fact that Descartes then justifies the limitation to two dimensions not through mathematical requisites but according to his theory of knowledge [AT X, 449; Descartes, 1985–1991, I, 63]. But the most interesting development comes with the notion of unity considered as a "common nature which all the things which we are comparing must participate in equally." Descartes goes on to say that "we may adopt as unit either one of the magnitudes already given or any other magnitude, and this will be the common measure of all the others." And he then adds a significant comment: "We shall regard it as having as many dimensions as the extreme terms which are to be compared. We shall conceive of it either simply as something extended, abstracting it from everything else — in which case it will be the same as a geometrical point (the movement of which makes up a line, according to the geometers), or as some sort of line, or as a square" [AT X, 449–450; Descartes, 1985–1991, I, 63–64]. Notwithstanding the fact that Descartes states here that the unit should have "as many dimensions as the extreme terms which are to be compared" — an intriguing statement to which I shall return — he seems to repeat here again the "unfortunate" error pointed out by Stevin: a unit can be conceived either *as a geometrical point* or as a line segment or as a square. Furthermore, Descartes then justifies the double regime of representation in the following way:

> As for figures, we have already shown how ideas of all things can be formed by means of these alone. We have still to point out in this context that, of the innumerable different species of figure, we are to use here only those which most readily express all the various relations or proportions. There are but two kinds of things which are compared with each other: multiplicity and magnitudes. We also have two kinds of figures which we may use to represent these conceptually: for example, the points,
>
> •
> ••
> •••
>
> which represent a triangular number. . . . Figures such as these represent multiplicities; while those which are continuous and unbroken, such as △, □ etc., illustrate magnitudes. [AT X, 450; Descartes, 1985–1991, I, 64; transl. modified]

At the beginning of the next Rule (XV), Descartes repeats, "If we wish to form more distinct images of these figures in our imagination with the aid of a visual display, then it is self-evident how they should be drawn. For example, we shall depict the unit in three ways, viz. by means of a square, □, if we think of it only as having length and breadth; by a line, ——, if we regard it as having just length; or, lastly, by a point, ., if we view it as the element which goes to make up a multiplicity" [AT X, 453; Descartes, 1985–1991, I, 65–66, transl. modified].

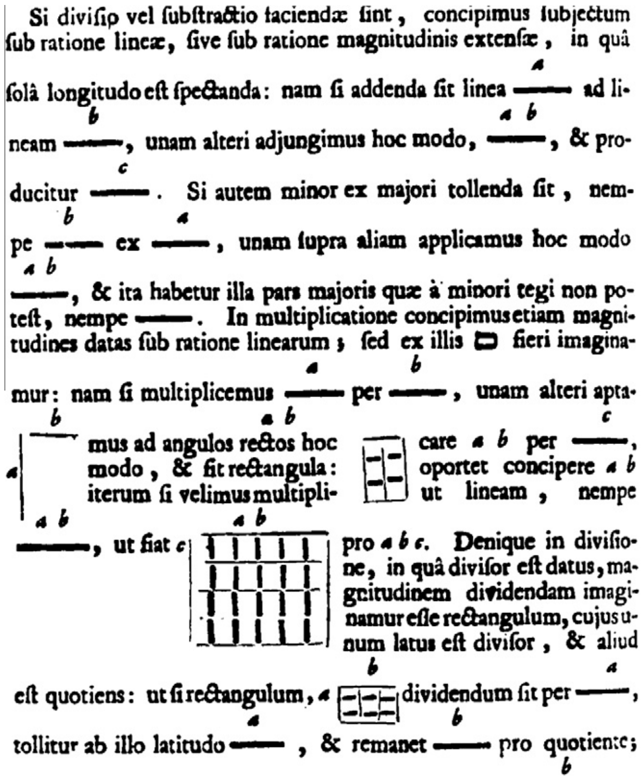


Fig. 6. The geometrical calculus of the *Regulae* (Rule XVIII), as presented in the *Opuscula Posthuma* [Descartes, 1701, p. 64].

In the *Regulae*, however, the schematism is not confined to a foundational role, but leads to an operative use presented in Rule XVIII.

As one can see in Fig. 6, the calculus is now presented only with segments and rectangles. This is certainly why many commentators did not pay much attention to the double regime of representation presented in the preceding rules. One should also notice that this schematism is quite different from the one presented to Beeckman. In the *Specimen*, Descartes presented what we now write as x^3 as x times a "square," a "square" being itself x times a rectangle composed of x "unit squares." He saw that this procedure could be accomplished only *per nudas lineas* so that the square would be x times a *line* composed of x unit *segments*. This procedure can be immediately applied to the product of any given magnitude, so that *abc* is *c* times *b* times a line composed of *a* unit segments. But this is not the path followed in Rule XVIII. There, Descartes is much closer to the schema in use since the time of the ancient Greeks (at least), in which one represents multiplication and division as forming rectangles and taking their side.[29] This discrepancy supports well the distinction between a foundational and an operative use of the schematism. Descartes used a certain schematism to figurate the quantities involved in a question (XIV–XV), but when it came to computations proper, he went back to the representation of multiplication as forming

[29] The main difference being, of course, that Descartes is not constrained by the problem of dimensionality and can express the product of three magnitudes *abc* by taking a segment equal to the rectangle *ab* and use it to form a rectangle with other side *c*.

rectangles (XVIII). This was, in fact, also the case in the *Specimen* when he solved the quadratic equation and represented $3x$ as a rectangle with sides x and 3.

The preceding passages should suffice to make it clear that the weaknesses of the *Specimen* were due neither to some misunderstanding or *lapsus calami* coming from Beeckman, nor to an oversimplification coming from Descartes. When Descartes was writing the *Regulae*, he fell into the same apparent confusions concerning the notion of dimension, the status of the unit and the way to represent it. This then raises two important questions, which I shall tackle in the remaining part of the paper: why did Descartes maintain a double regime of representation, which looks at first like a useless source of confusion? How are we to understand the compatibility between what I have called the "foundational" and the "operative" use of the schematism?

3.2. Points and numbers

To illustrate the fact that one can use dots to represent "multiplicities", Descartes gives the example of "triangular numbers" and of a family tree [AT X, 450]. These examples are very important if one wants to understand why Descartes could have been attached to this kind of representation. If we cease to read the development of the years 1628–1629 through the lens of his future mathematics and ask ourselves, "what kind of mathematics did Descartes do before?" we see immediately that Descartes actually spent a lot of time and energy reflecting on "figurate numbers" and on polyhedra. In fact, the *only* extant Cartesian mathematical treatise from this period (*Progymnasmata de solidorum elementis* [AT X, 257–277], which we possess in a partial copy made by Leibniz) deals precisely with these questions, as do some important passages of the *Cogitationes Privatae*.[30] This thus allows a simple answer to our first question: Descartes maintained a double regime of representation simply because it corresponded to the kind of mathematics he was doing.

In agreement with P. Costabel [Descartes, 1987], I consider the *Progymnasmata* to have been conceived at the beginning of the 1620s. A number of arguments have been advanced to support this hypothesis, although none of them are completely conclusive. The use of cossic notation seems to indicate an early writing (but there is an important exception in a text dated from 1638 [AT X, 297–299]). One can also add the fact that Descartes professed to Mersenne in 1638 that he had neglected geometry for "more than fifteen years" [AT II, 99]. Since the first part of the copy made by Leibniz deals precisely with geometry, this claim would make the *De Solidorum elementis* the outcome of studies made before 1623.[31] Finally, there has recently

[30] See [AT X, 241]: on triangular numbers; [AT X, 246–247]: on the rectangular tetrahedron; [AT X, 247–248]: on the pyramid, and [Sasaki, 2003, pp. 128–132], for an English translation and a description of these fragments.

[31] Descartes, 1987, pp. 106–109. This claim by Descartes does not preclude the fact that the treatise could have been *written* later. Neither does it limit the kind of geometry that Descartes was doing in the 1620s. In the letter to Mersenne, dated 15 April 1630, where Descartes confesses that he is now tired of mathematics, he gives examples of other kinds of (plane) geometrical problems, which he solved using only ruler and compass: "*Invenire diametrum sphæræ tangentis alias quatuor magnitudine et positione datas. Invenire axem parabolæ tangentis tres lineas rectas positione datas et indefinitas, cujus etiam axis secet ad angulos rectos aliam rectam etiam positione datam et indefinitam. Invenire stilum horologii in data mundi parte describendi, ita ut umbræ extremitas, data die anni, transeat per tria data puncta, saltem quando istud fieri potest*" [AT I, 139]. It should be noted that these problems are all plane problems and that it is interesting that Descartes brought up this kind of example if he was already dealing with much complex ones.

been a renewal of interest in the striking proximity between Descartes' preoccupations in *De Solidorum elementis* and research by contemporary German mathematicians such as Johannes Faulhaber, Johannes Remmelin, and Peter Roth.[32] Even if one remains cautious regarding the story (told by Lipstorp) of the encounter between Descartes and Faulhaber in Ulm in 1620, there are, as Schneider has summarized, "a number of astounding concurrences in the treatment of various mathematical problems besides the solution of the general quartic. These concern, for instance, the rule of signs for a determination of how many real roots an equation can have, a Pythagorean theorem for three dimensions, Faulhaber's extension of figurate numbers to polyhedral numbers of the five regular polyhedra and the content of Descartes' *De solidorum elementis*, which contains a relation between the numbers of corners, edges, and faces of regular polyhedra" [Schneider, 2008, p. 53].

To this set of converging arguments, I shall add another one. In the *Discours*, as we have seen, Descartes presents the first outcome of his method as a study of mathematics, more specifically of its unifying part ("ratios and proportions"). Then he adds,

> As a matter of fact, I can venture to say that the exact observation of the few precepts which I had chosen gave me so much facility in sifting out all the questions embraced in these two sciences [scil. Algebra and geometrical analysis], *that in the two or three months which I employed in examining them* — commencing with the most simple and general, and making each truth that I discovered a rule for helping me to find others — not only did I arrive at the solution of many questions which I had hitherto regarded as most difficult, but, towards the end, it seemed to me that I was able to determine in the case of those of which I was still ignorant, by what means, and in how far, it was possible to solve them. [AT VI, 21; Descartes, 2003, p. 93]

According to this story, Descartes obtained the first results of his methodological investigations in algebra and geometry as early as 1620. However, one should be cautious when evaluating this declaration, since Descartes wrote it more than 15 years after the event. In any case, we would have to explain how it is that Descartes was convinced in 1628 that "insofar as arithmetic and geometry were concerned, he had nothing more to discover." In what kind of "Geometry" and "Arithmetic" did Descartes obtain his results? As I have tried to argue in the preceding section, the many weaknesses of the contents of the *Specimen* and of the *Regulae* should convince us that there is not necessarily a "hidden" technique to uncover under this declaration. My hypothesis is that it most likely refers to works such as the *Progymnasmata*, which present the advantage of being compatible with the weaknesses remaining in the texts of 1628. The *De solidorum elementis* would indeed fit perfectly into Descartes' description in the *Discours*, not only because they contain some new and interesting results in geometry (including an *algebraic* demonstration of the fact that there are only five regular polyhedra — which was considered by many people at that time as the culmination of Euclid's *Elements*), but also because the study of "figurate numbers" was an important part of algebra in the German tradition.

3.3. Calculating with points

I will limit my study of the *Progymnasmata* to aspects of particular interest to us. The copy made by Leibniz does not contain any diagrammatic representation of figurate num-

[32] See the studies by [Schneider, 1993 and 2008; Mehl, 2001; Penchèvre, 2004; Manders, 1995 and 2006].

bers through dots, but this representation is implied by their definition and by the very aim of the second part of the text: to calculate the "weights" (i.e., the number of vertices) of polyhedral numbers.[33] These weights, which Descartes denotes by O, can be obtained by counting the number of polygonal faces opposite the solid angle chosen as origin (F), the number of edges (R, for "radix"), and the number of vertices (A, for "angles"). Using the formula for the gnomon, F – R + A, the total weight O can then be calculated [Descartes, 1987, pp. 38–39].

As one can see, this exercise is mainly a matter of *counting*, as is the first part of the text copied by Leibniz, which deals with solid geometry and which contains (although not in the modern "Euler formula" form) the famous relationship between the number of faces, edges, and solid angles for regular polyhedra. It thus corresponds to the part of the *mathesis* that Descartes ascribed to the consideration of "order" in the *Regulae*.[34] These enumerations are conducted by recourse to a representation in which (polygonal) surfaces are reduced to more basic figures (triangles) composed of lines, the extremities of which are the different *pondera*. Hence they satisfy the requirement that proportion (we shall see that the term is used by Descartes in the text) be studied through the consideration of decomposition into "simple lines," the extremities of which are points — even if in a meaning quite different from that of the "calcul géométrique." They also satisfy the requirement that these computations, when involving complex configurations, be treated with the help of algebraic formulas (written at that time in cossic notation).

At the beginning of the passage on polyhedral numbers, for example, Descartes begins by recalling how one can obtain the weight of any polygonal number by decomposing it into triangles. He gives the example of the tretragonal weight, the number of which can be calculated as twice the number of triangular numbers minus the number of edges in common.[35] He then gives the general formula for regular polyhedral numbers in a similar mixed form of cossic notation and rhetorical discourse involving a description of the spatial

[33] The first line of the second part of the text copied by Leibniz makes direct reference to this kind of representation: "*Omnium optime formabuntur solida per gnomones superadditos uno semper angulo vacuo existente, ac deinde totam figuram resolvi posse in triangula*" [AT X, 269].

[34] To those who object that "order and measure" should be interpreted in a much broader meaning, grounding a technique which culminated in the "calcul géométrique" of 1637 or referring to the methodical "order," I refer to a letter of Descartes to Ciermans, which has not yet received enough attention from Cartesian scholars. Perceiving the generality of the technique involved in *La Géométrie*, Ciermans proposed to call it a *mathesis pura* rather than a "geometry." Descartes responded negatively to this proposition by saying: "for I did explain in this treatise not one question pertaining properly to Arithmetic, *nor even one of those concerned with order and measure*, as did Diophantus; but, moreover, I didn't deal in it with movement, even if pure mathematics, at least the one which I have at most cultivated, has it for its principal object" [AT II, 70–71: my emphasis]. Note also the mysterious claim according to which the "principal object" of *pure* mathematics is in 1638 movement. On this aspect see [De Buzon, 1996] and [Domski, 2009].

[35] "$\frac{1}{2}$𝔷 + $\frac{1}{2}$♃ per 2 fit $\frac{2}{2}$𝔷 + $\frac{2}{2}$♃ unde sublato 1♃ fit 1𝔷" [Descartes, 1987, p. 4]. As one can see, the text uses the astrological symbol ♃ for the unknown, a modification that is thought to come from Leibniz. The general formula for polygonal numbers was already given by Roth, as were the ones for pyramidal numbers, which is used by Descartes in the last part of the passage without explanation [Schneider, 2008, p. 10].

configuration of these numbers.[36] His main goal here seems to be to extend these results, already obtained by Faulhaber and Roth at the beginning of the century to semiregular polyhedra, which he does in the last part of the text — the results of these different steps being collected into several tables.

This kind of practice fits very well into the idea that an important part of mathematics deals with the consideration of *order* and that these computations can be performed with the help of schematic spatial representations using *points, lines, and figures* (the basic figure here being not the rectangle or the square, but the triangle). Moreover, in the *De solidorum elementis*, "order" is contrasted with "measure", in accordance with what was written in the *Regulae*. This development reminds us very much of those we have already encountered concerning the notion of dimension in the *Specimen* and in the *Regulae*:

> When we will imagine these same figures as measurable, then all the units will be understood as being of the same ratio as the figures themselves. For example, triangles [are measured by] triangular units; pentagons are measured by a pentagonal unit, etc. Then the proportion between a plane [figure] and its radix would be the same as that between a square and its radix; and for a solid, [it will be] the same as that of a cube [to its radix]: for example, if the radix is 3, the plane will be 9, the solid 27, etc. This holds also for the circle and the sphere, and all other figures. If the circumference of a circle is three times larger than another, its area will be nine times larger. From which it is observed that the progressions involved in our *mathesis*, ♃, 𝔷, ℭ etc. are not tied to linear, square or cubic figures, but designated in a general way through these diverse kind of measure. [My translation of Descartes, 1987, pp. 4–5]

Even if this development is very intricate, the parallel with the *Regulae* seems no less striking: the first sentence repeats the idea (on the *same* examples of the triangle and the pentagon) that the notion of dimension can be seen as the number of parameters of the problem (in this case the linear dimension of the given figure).[37] This passage also echoes the fact that the unit should have "as many dimensions as the extreme terms which are to be compared" [AT X, 450]. Furthermore, we now have a way to understand the apparent contradiction between this last qualification and the identification of units with "points": it amounts to the difference between the consideration of order (counting the number of *pondera*, in which case the unit is a point) and of measure (measuring areas and volume, in which case the unit needs to have as many dimensions as the surface being considered). In the margin, Leibniz draws a picture (Fig. 7) that makes it easier to grasp what Descartes has in mind [Descartes, 1987, p. 39].

The last sentence of the above quotation expresses the fact that already at the time of the *De solidorum elementis*, Descartes had arrived at the idea, once again in perfect accordance

[36] "The algebraic expressions for these figurate numbers are found by multiplying the exponents of the face plus $\frac{1}{2}$♃ by $\frac{1}{2}$♃ + $\frac{1}{3}$, and then by the number of faces, which is done as many times as there are different types of faces in the given body; then add to or subtract from the result the number of radices multiplied by $\frac{1}{2}$𝔷 + $\frac{1}{2}$♃, and the number of angles multiplied by ♃" [Descartes, 1982, p. 108].

[37] Note that Descartes designates the identity of dimension by the expression "the same ratio" *ejusdem rationis*, and that he also talks of the "proportion" between the plane figure and its "side" — a very interesting choice of terminology if one wants to understand what he has in mind in other texts when talking about "ratio and proportion" and that might well be much broader that what is usually thought. This passage can be compared with the beginning of the development on *dimensio* in the *Regulae*, in which it is defined as "*modum* ***et rationem****, secundum quam aliquod subjectum consideratur esse mensurabile*" [AT X, 447, my emphasis].

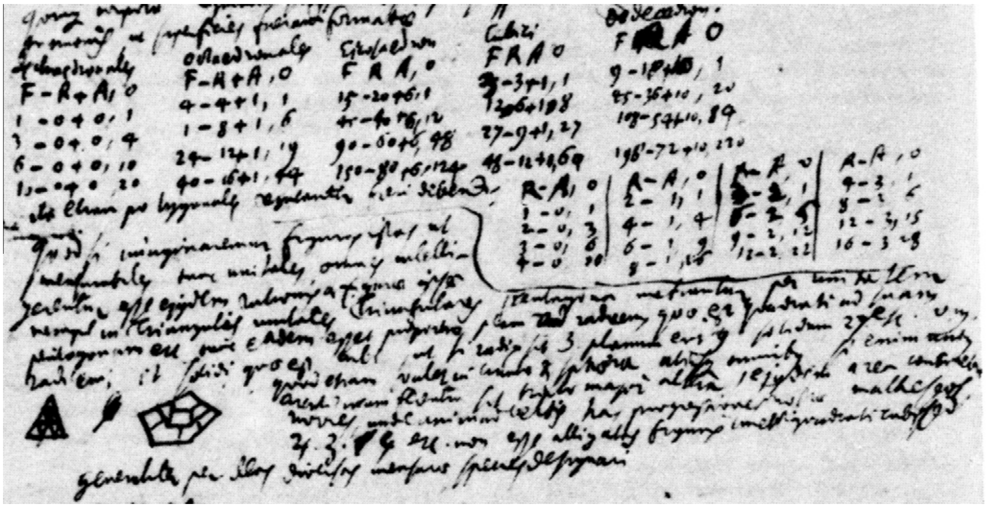


Fig. 7. In the margin, Leibniz draws a triangle and a pentagon, whose side is three units and whose area is nine units [from a facsimile of the Hanover manuscript, Descartes, 1987].

with the *Regulae*, that algebraic powers should not be interpreted in terms of a certain fixed geometrical representation, but in terms of proportions entering into a certain relation (in this case that of measure). This gives a very nice context for understanding how Descartes could at the same time have such clear ideas about the fact that algebraic degrees should not be confused with geometric dimensions *and* nonetheless be apparently so unclear when expressing the notion of dimension in and of itself.

Even if there is no reason to conflate *mathesis* and *mathesis universalis*, the expression “the progressions involved in our *mathesis*” (“*has progressiones notrae matheseos*”) is also very striking. It indicates that Descartes thought that he had already elaborated a proper *mathesis* in which the algebraic progressions played a central role — but *independently* of the exponential notation. It will not be necessary to insist on the fact that this merging of algebra and geometry then had a meaning totally different from the one it would later acquire. The reader should keep this in mind when reading Descartes’ claims that, in his first mathematical studies at the beginning of the 1620s, he did “borrow all that is best in Geometrical Analysis and Algebra, and correct the errors of the one by the other.”

In addition, the comparison between the *Progymnasmata* and the *Regulae* gives meaning to other intriguing passages in the latter, which reinforces the impression of proximity between the two texts. For example, in Rule X, Descartes states that: “In order to acquire discernment we should exercise our intelligence by investigating what others have already discovered, and methodically survey even the most insignificant products of human skill, *especially those which display or presuppose order*” (my emphasis). This text is therefore a privileged one in understanding what *ordo* meant for Descartes. Yet here are the examples given:

> [one] must first tackle the simplest and least exalted arts, and especially those in which order prevails — such as weaving and carpet-making, or the more feminine arts of embroidery, in which threads are interwoven in an infinitely varied pattern. *Number-games and any games involving arithmetic, and the like, belong here.* (...) For, since nothing in these activities remains hidden and they are totally adapted to human cognitive capacities, they present us in the most distinct way with innumerable instances of order,

each one different from the other, yet all regular. Human discernment consists almost entirely in the proper observance of such order. [AT X, 403–404; Descartes, 1985–1991, I, 34–35: my emphasis]

So at that time Descartes held an opinion regarding "Number-games and any games involving arithmetic" (compared with weaving, carpet-making, and embroidery) quite different to the one he held later when he mocked mathematicians who wasted their time with such useless matters [AT II, 91, to Mersenne, 31 March 1638]. In the next passage in the text he even gives the example of cryptography, an essential feature of the "number games" that German *Rechenmeister* were playing in their "algebra" (*Wortrechnung*).[38]

I am not claiming, of course, that the *De solidorum elementis* provide the one and only key to reading the *Regulae*. However, it is certainly of interest if one wants to understand some of its aspects and it should be noted that the comparison has not previously been made by Cartesian scholars. It has the tremendous advantage of relying on documents that are at the same time both extant *and* compatible with the mathematical weaknesses contained in the *Specimen*. If one follows this direction, how are we then to answer our second question?

3.4. Calculating with segments

As I emphasized in previous sections, Descartes had two different uses of geometrical schemas, which I have called "foundational" and "operative". This designation is not totally accurate since, as we have seen in the study of the *De solidorum elementis*, the "foundational" use is not solely a matter of justification, but can also support various kinds of computation. My point is that, even in this case, the figuration does not *represent* the computation in and of itself. Neither in the *Progymnasmata*, nor in the *Specimen*, nor in Beeckman's *Journal*, nor even in Rule XIV of the *Regulae* do we find examples of the second use strictly speaking. The schematism is merely an aid to solving certain problems, and this aid can take various forms, including the use of dots and of basic figures other than squares and rectangles (typically triangles, into which polygons can be decomposed). What I have called the "operative" use seems to appear only in Rule XVIII, in which Descartes tries to represent the basic operations between magnitudes in general. This coincides, however, with the sudden disappearance of points. How are we to understand this?

As should be clear now, my understanding is that Descartes is turning to another aspect of his project with Rule XVIII. But to distinguish these two aspects would not solve our main difficulty, that of the apparent *incompatibility* between them. It does not seem possible to consider dots as represented by the extremities of segments without losing the very efficiency of the calculus (just think what happens to the extremities when two segments are added). More generally — as pointed out by Stevin — the interpretation of units by points contradicts the very idea that an algebraic equation is a transcription of a series of magnitudes in continuous proportion.

It is possible that what we face here is a mere contradiction. The *Regulae* is an unfinished treatise and it stops abruptly precisely after the exposition of the calculus in Rule XVIII, the remaining Rules (until XXI) subsisting only through their planned titles. It is not totally implausible that Descartes was not satisfied with this sketch, or that he saw that he was

[38] There is also an interesting reference in Rule VII to the game consisting of transforming its name into an anagram [AT X, 391, 18]. This game played an important role in the fight between German cossists; see [Mehl, 2001, pp. 192–204].

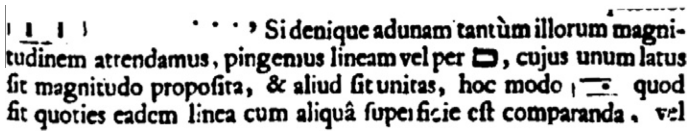


Fig. 8. A rectangle with a point in it? [Descartes, 1701, p. 57].

not taking it in the right direction. I do not think, however, that this type of explanation is totally satisfying, precisely because it leaves us with such a crude contradiction in Descartes' thought as the one mocked by Stevin. One should at least try to see if another interpretation is possible. And it turns out that it is. It rests, however, on a very small detail, which could easily escape our attention.

Recall that Descartes proposed in Rule XV to represent the unit in three ways: as a square, as a line, or as a point. When one considers two magnitudes at a time, they can be represented as a rectangle composed of unit squares or as a lattice of points, if they are commensurable, or as an indecomposable rectangle, if they are incommensurable. When we deal with only one of these magnitudes, we can picture it either as a line or a series of dots, or as a rectangle. In this last case, one has just to take a rectangle whose side is the given magnitude and the other the unit. But in the picture reproduced in the *Opuscula* (see Fig. 8), the rectangle contains… a point ⊡.

We have no way of knowing if this representation was Descartes' original one, or a suggestion of the copyist, or even a misprint. But since it offers an alternative proposition to the (disastrous) one consisting of interpreting dots as extremities of line segments, it should attract our interest. The passage is very intricate, but one thing at least seems to be clear: Descartes is fully aware of the fact that the representation through dots and that through continuous magnitudes are not isomorphic. He divides the representation through rectangles into two cases according to whether or not we can divide the rectangle into unit cells. In the case of two incommensurable magnitudes, it is not possible to find a unit square into which their "rectangle" can be decomposed (since, by definition, there is no common measure into which they can both be decomposed). The representation by dots is therefore seen as a *subclass* of the more general one through rectangles (and not an equivalent one, which would make us fall within Stevin's critique).

The nature of the relationship between the two representations is not clear: on the one hand, Descartes insists on the fact that there are two different kinds of schematism depending on whether or not we are dealing with *multitudines*; on the other hand, he proposes a *single* schematism based on segments and rectangles without making clear whether one has to interpret it in different ways depending on the objects under consideration, or if it is one among two *different* representations. But we have no reason to try to reduce this difficulty, which is inherent to the tension between the foundational and the operative use of the schematism. What I want to emphasize is that there is, in any case, a way to escape Stevin's critique and maintain the use of dots as "units."

Here we should also recall the proposition made to Beeckman in the *Specimen*. When introducing his schematism through unit squares and adding *ita etiam punctum*, Descartes *did not* develop a parallel representation through dots, but "by lines" involving "point, line, square and cube." We understand now how this very strange claim could nonetheless remain in accordance with the idea that algebraic powers are magnitudes in continuous proportion. We just have to represent points "inside" our rectangle to render it consistent (see Fig. 9).

There is, however, a price to pay for this solution, since we now lose the close connection with the computation on dots (as extremities of lines) presented in the *De solidorum elemen-*

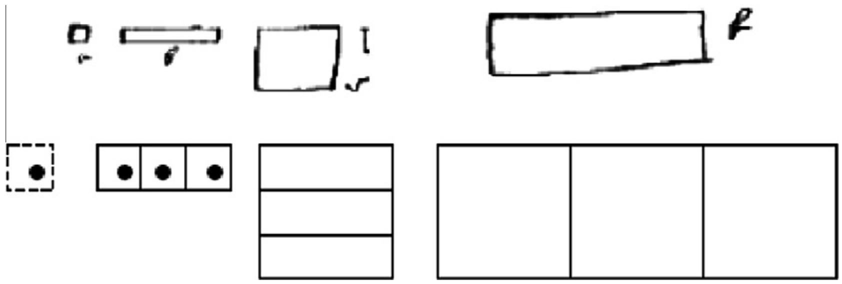


Fig. 9. *Punctum*, *lineam*, *quadratum*, *cubum*. Diagram adapted from a figure of the facsimile of fol. 333 of Beeckman's autograph published in (Beeckman, 1939–1953, III).

tis. My overall impression is that these two representations originate from two different lines of thought, which do not match where they overlap.

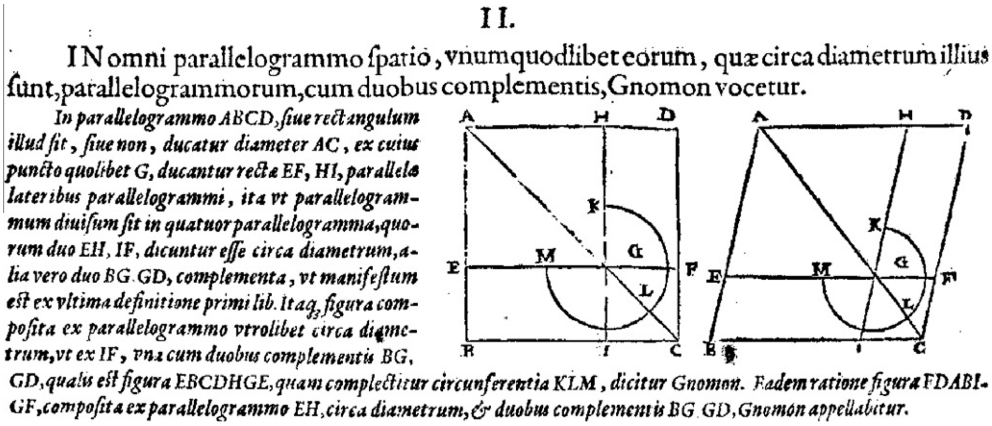
3.5. The limitation of the geometrical calculus

As an indication of the fact that the representation of the calculus presented in Rule XVIII is a sketch, which could correspond to a new line of thought for Descartes, I shall stress its limitations. They appear clearly in the question of mean proportionals, at which point the extant part of the treatise abruptly stops. The first remark we should make in this regard is that Descartes, at the time of the composition of the *Regulae*, isolated four basic operations, *not five*, as was usual at that time and as he would do in *La Géométrie*. This is due to the fact that root extraction is not seen as a basic operation but as a special case of division:

> As for those divisions in which the divisor is not given but only indicated by some relation, as when we are required to extract the square root or the cube root etc., in these cases we must note that the term to be divided, and all the other terms, are always to be conceived as lines which form a series of continued proportionals, the first member of which is the unit, and the last the magnitude to be divided. We shall explain in due course how to find any number of mean proportionals between the latter two magnitudes. [AT X, 467; Descartes, 1985–1991, I, 75]

The last sentence is very striking. It means that at that time Descartes thought that he had a *general* procedure to insert any number of mean proportionals between two given magnitudes. It could be a reference to his famous proportional compasses discovered in 1619, which do precisely this job. This has to remain a conjecture, since the last part of the treatise in which Descartes was supposed to present this general procedure is not extant (lost or never written).[39] In any case, Descartes has to explain first how to perform this operation in the simple case of *one* mean proportional, an operation equivalent to the extraction of the square root.

[39] The last Rules of the second part, of which we just have the titles, give us no clue about this crucial issue. They just say that one must reduce any problem to an equation (XIX: "Using this method of reasoning, we must try to find as many magnitudes, expressed in two different ways, as there are unknown terms, which we treat as known in order to work out the problem in the direct way. That will give us as many comparisons between two equal terms"), and then carry out the operations (XX: "Once we have found the equations, we must carry out the operations which we have left aside, never using multiplication when division is in order") and try to reduce a system of equations to a single one (XXI: "If there are many equations of this sort, they should all be reduced to a single one, viz. to the equation whose terms occupy fewer places in the series of magnitudes which are in continued proportion, i.e. the series in which the order of the terms is to be arranged").



II.

IN omni parallelogrammo spatio, vnumquodlibet eorum, quæ circa diametrum illius sunt, parallelogrammorum, cum duobus complementis, Gnomon vocetur.

In parallelogrammo ABCD, siue rectangulum illud sit, siue non, ducatur diameter AC, ex cuius puncto quolibet G, ducantur rectæ EF, HI, parallelæ lateribus parallelogrammi, ita vt parallelogrammum diuisum sit in quatuor parallelogramma, quorum duo EH, IF, dicuntur esse circa diametrum, alia vero duo BG GD, complementa, vt manifestum est ex vltima definitione primi lib. Itaq; figura composita ex parallelogrammo vtrolibet circa diametrum, vt ex IF, vna cum duobus complementis BG, GD, qualis est figura EBCDHGE, quam complectitur circunferentia KLM, dicitur Gnomon. Eadem ratione figura FDABIGF, composita ex parallelogrammo EH, circa diametrum, & duobus complementis BG GD, Gnomon appellabitur.

Fig. 10. Definition of the gnomon and the complements at the beginning of Euclid's *Elements* II [in Clavius, 1611–1612, Vol. 1, p. 83]. Just after this definition, Clavius presents the general procedure of Book II as the foundation of the rules of algebra.

As we have seen, a key feature of Descartes' calculus in the *Regulae* is that one works on segments and on rectangles at the same time. This implies that one is able to transform any rectangle into a line and vice versa. Since we already know how to represent any line by a rectangle (whose sides are the given line and a unit), what we need is a general procedure to transform a given rectangle into another one. This is the general problem on which the *Regulae* abruptly stops:

> It is therefore important to explain here how every rectangle can be transformed into a line, and conversely how a line or even a rectangle can be transformed into another rectangle, one side of which is specified. Geometers can do this very easily, provided they recognize that in comparing lines with some rectangle (as we are now doing), we always conceive the lines as rectangles, one side of which is the length which we adopted as our unit. In this way, the entire business is reduced to the following problem: given a rectangle, to construct upon a given side another rectangle equal to it. The merest beginner in geometry is of course perfectly familiar with this; nevertheless I want to make the point, in case it should seem that I have omitted something. [AT X, 468; Descartes, 1985–1991, I, 76]

The rest is missing.

"The merest beginner in geometry" would certainly know how to proceed in this case, since it involves one of the basic procedures explained in Book II of Euclid's *Elements* and known as "application of areas." The list of identities exposed in Book II is indeed based on the fact that the "gnomon" constructed around the diagonal of a rectangle comprises two equivalent rectangles called by Euclid the "complements" (see Fig. 10).

It is therefore easy to use this property and construct a rectangle equivalent to another one, for example, a "line" c (i.e., "line-rectangle") equivalent to a "product" ab [Bos, 2009, see Fig. 11].

But our "merest beginner" would also know that there is a particular case in which this operation is not so easy to perform: when one of the sought-for rectangles is a square. In this case, which is equivalent to the insertion of a mean proportional, one needs, as Euclid explained in the last proposition of Book II (II, 14: "to construct a square equivalent to a given figure"), the help of an auxiliary *circle*. But neither in the schematism presented in the *Regulae*, nor in the one presented to Beeckman in 1628, is there any mention of a circle. In contrast, the *Géométrie* would not only propose another schema for the basic operations on "simple lines,"

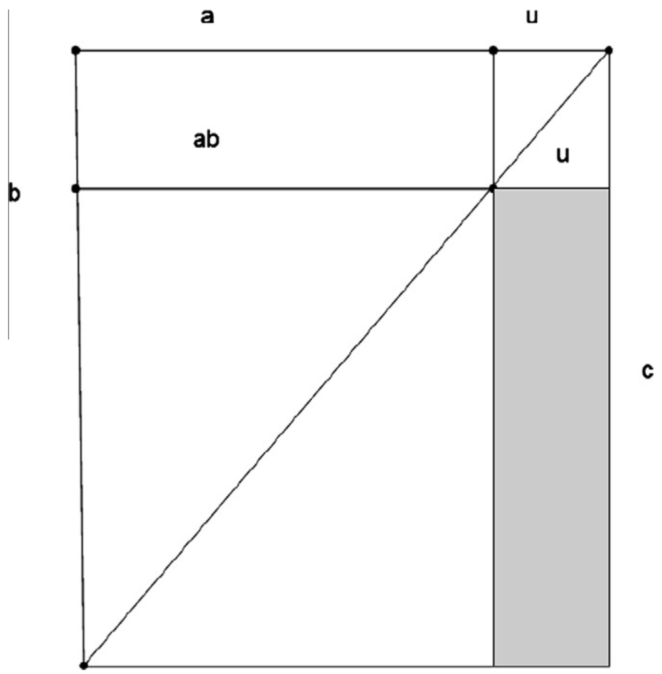


Fig. 11. How to construct a "line-rectangle," whose side c is equal to the product ab, u being the unit.

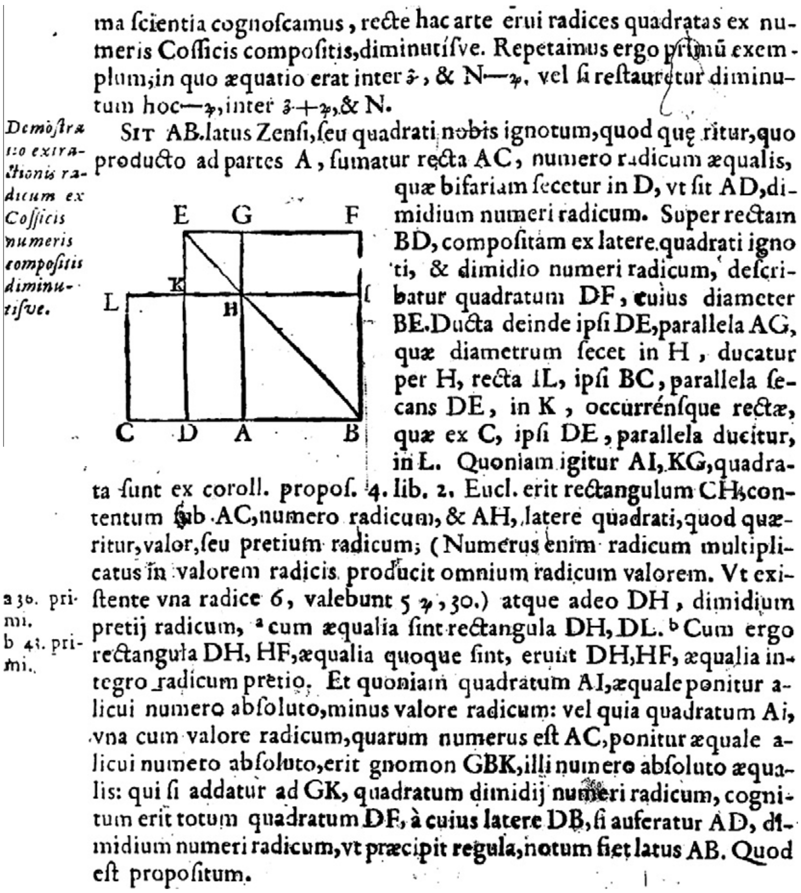
but also would introduce a fifth basic operation, the extraction of a square root, which would be represented, in the traditional Euclidean way, by resorting to a semicircle.

This discrepancy fits very well into the general context which I have tried to reconstruct above. Indeed, the representation of the extraction of square roots by a circle, which was very well known as a geometric procedure, does not have an immediate correspondent in the representation of computation on *multitudines*. Thus it is no particular surprise that Descartes simply "forgot" to include the circle as a basic element of his schematism in 1628. Moreover, as mentioned earlier, Clavius used "application of area" in the part of his *Algebra* dedicated to the extraction of roots, but did so without resorting to the circle as shown in Fig. 12. This was because Clavius's goal was not constructing the square root, but determining its value. He could therefore proceed by first drawing the sought-for square and then determining the value of its side by using a gnomon property (in a way that is very close to the kind of technique that Descartes presented to Beeckman in his *Specimen*).

If this interpretation is right, it would have another important consequence for the study of Cartesian mathematical texts of the late 1620s. It would mean that the inclusion of the solutions of the third- and fourth-degree equations and of the *Specimen* into the same group of documents is far from obvious — the reason being that the first one was done through the intersection of a parabola *and a circle*.

3.6. Per parabolam

In February 1629, Beeckman transcribed in his *Journal* the content of new documents which he had received from Descartes, presumably after he himself had sent his demonstration of the fact that the hyperbola was an anaclastic curve. These documents comprise first, after Beeckman's demonstration, a solution to the problem of the insertion of two mean



ma ſcientia cognoſcamus, recte hac arte erui radices quadratas ex numeris Coſſicis compoſitis, diminutiſve. Repetamus ergo primũ exemplum, in quo æquatio erat inter ʒ, & N—℞. vel ſi reſtauretur diminutum hoc—℞, inter ʒ+℞, & N.

Demõſtratio extractionis radicum ex Coſſicis numeris compoſitis diminutiſve.

SIT AB latus Zenſi, ſeu quadrati nobis ignotum, quod quæritur, quo producto ad partes A, ſumatur recta AC, numero radicum æqualis, quæ bifariam ſecetur in D, vt ſit AD, dimidium numeri radicum. Super rectam BD, compoſitam ex latere quadrati ignoti, & dimidio numeri radicum, deſcribatur quadratum DF, cuius diameter BE. Ducta deinde ipſi DE, parallela AG, quæ diametrum ſecet in H, ducatur per H, recta IL, ipſi BC, parallela ſecans DE, in K, occurrénſque rectæ, quæ ex C, ipſi DE, parallela ducitur, in L. Quoniam igitur AI, KG, quadrata ſunt ex coroll. propoſ. 4. lib. 2. Eucl. erit rectangulum CH, contentum ſub AC, numero radicum, & AH, latere quadrati, quod quæritur, valor, ſeu pretium radicum; (Numerus enim radicum multiplicatus in valorem radicis producit omnium radicum valorem. Vt exiſtente vna radice 6, valebunt 5 ℞, 30.) atque adeo DH, dimidium pretij radicum, [a] cum æqualia ſint rectangula DH, DL. [b] Cum ergo rectangula DH, HF, æqualia quoque ſint, erunt DH, HF, æqualia integro radicum pretio. Et quoniam quadratum AI, æquale ponitur alicui numero abſoluto, minus valore radicum: vel quia quadratum AI, vna cum valore radicum, quarum numerus eſt AC, ponitur æquale alicui numero abſoluto, erit gnomon GBK, illi numero abſoluto æqualis: qui ſi addatur ad GK, quadratum dimidij numeri radicum, cognitum erit totum quadratum DF, à cuius latere DB, ſi auferatur AD, dimidium numeri radicum, vt præcipit regula, notum fiet latus AB. Quod eſt propoſitum.

a 36. primi.
b 43. primi.

Fig. 12. Extraction of square root in Clavius [1609, Chap. XIII, p. 52]. One first constructs the square ABHI and then determines the value of its side.

proportionals. This solution, which was achieved through the intersection of a circle and a parabola, was demonstrated geometrically by a "Parisian mathematician" (presumably Mydorge) after Descartes discovered its construction.[40] The second text presents a general method for constructing all solid problems — a "universal secret" to solve equations involving three and four dimensions.[41] We know from other documents that Descartes

[40] [Beeckman, 1939–1953, Vol. IV, p. 156]. *Cum D. des Chartes invenisset per parabolam duo media proportionalia inveniri, hoc mathematicus quidam Galus Parisiis geometrice demonstravit hoc modo.* Beeckman then adds: *quod ad verbum descripsi.*

[41] *Auxilio parabolae omnia solida problemata generali methodo construere. Quod alio loco vocat D. Des Chartes secretum universale ad aequationes omnes tertia vel quarta dimensione involutas lineis geometricis exponendas* [Beeckman, 1939–1953, Vol. IV, pp. 138–139]. Once again Beeckman writes, *Quod ex illius scriptis ad verbum describo*. Remember that Descartes promised in 1628 to send Beeckman his Parisian *Algebra*. A passage from the letter of 1630 would indicate that he did so (*Scribis enim Algebram, quam tibi dedi, meam amplius non esse* [AT I, 159]). It could then be thought that the solution of the third degree equation was part of this package. On the other hand, Descartes states in the letter dated 25 January 1638 to Mersenne that nobody has a copy of his *Algebra* [AT I, 501]. If this is true, it would mean that he just sent Beeckman an excerpt of it, perhaps his resolution of the "solid problem," as is suggested by the fact that his construction is presented just after the demonstration by "a certain French mathematician."

discovered his solution of the insertion of two mean proportionals in around 1625 [Bos, 2001, p. 255; Serfati, 2002, pp. 72–75]. We do not know, however, how he discovered the generalization to the solution of any equation of degree 3 or 4 — for which, once again, he just gives the construction. Neither do we know if the "general" solution was discovered before, after, or at the same time as the particular one.

Henk Bos has proposed an elegant reconstruction of the demonstration of the generalization using the method of "indeterminate coefficients" [Bos, 2001, p. 257]. By just writing the general equation of a circle and using the fact that the points should also lie on a parabola, one immediately obtains a fourth-degree equation in one variable expressed only in terms of the coordinates of the center of the circle. By then identifying these coefficients with that of a general equation of degree 4, one obtains the different magnitudes involved in Descartes' construction. The method of "indeterminate coefficients" was known and used by cossist algebraists, but in an arithmetical context. By using it in a geometrical context in which curves are represented and manipulated through their equations, Descartes would have made his first step in what would be the core of *La Géométrie*'s new technique. In fact, most commentators, when confronted with the text of 1629, simply replace it by that of *La Géométrie*, in which the third part is dedicated to the solution of solid problem.[42]

If these reconstructions are correct, Descartes was certainly doing algebraic analysis in 1628–1629. But unless new elements are brought into the debate, it would remain based on a *petitio principii* (if Descartes was doing algebraic analysis, the reconstructions could be correct, and if they are, Descartes was doing algebraic analysis). As confessed by Bos, "it may be that Descartes arrived at the general construction by the method of indeterminate coefficients. The proof which I have added above shows that such a technique leads directly to the construction. Moreover, in his notes to the 1659 Latin edition of the Geometry Van Schooten added a derivation of the construction by indeterminate coefficients. However, it may also be that Descartes found the general solution by successive generalization of his construction of two mean proportionals" [Bos, 2001, p. 258].

For my part, I would like to insist once again on the actual content of the text. One should first note that this development is entitled (maybe by Beeckman) *Parabolâ aequationes cossicas lineis exponere*. As we have seen, Descartes does seem to have still been using cossic notation at the time. The matter is one of importance, since many reconstructions would not fit with this situation. For example, it is not easy to express the general equation of a circle using cossist symbols.[43] The argument could, however, seem to be of little value, since in fact Beeckman's transcription does not use symbolic notation *at all*. But the fact that the text, which is supposed to be transcribed *ad verbum*, is purely rhetorical should be, in and of itself, very interesting. In Descartes' description, the coefficients of the general equation of fourth degree are described not by letters but by expressions such as "the number of squares," "the number of roots," and "the absolute number," in the same way as they were described, for example, in Clavius [1609].

[42] See for example [Serfati, 2002, pp. 95–104]. According to Serfati, Beeckman "s'est contenté de transcrire dans son langage cossique archaïque, sans pouvoir véritablement comprendre ni l'intérêt authentique de la découverte cartésienne, ni surtout le système symbolique mathématique employé par son ami qui le lui exposait" (p. 101).

[43] Not easy, but not impossible considering the elements given in [Manders, 2006]. In Faulhaber's notation, it was possible to introduce a second variable and to deal with indeterminate quantities — even if, to my knowledge, he did not do both at the same time.

But the main argument against this type of reconstruction is that it assumes that Descartes, in 1628–1629, was studying curves through their equations. This claim is immediately problematic with regard to the circle, since Descartes does not use this equation *even in later works*, instead relying on the Pythagorean theorem [see Galuzzi and Rovelli, forthcoming, Chap. 2.6]. With regard to the parabola, one has to keep in mind that Descartes also sent Beeckman in 1628 some texts concerning conic sections, a topic he had studied in the course of his works of optics (see Note 10). The main document concerns ovals, which Descartes studied in an algebraic way in a fragment preserved in the *Excerpta mathematica* [AT X, pp. 310–324] and in which he developed his first attempt at what would later be known as the "method of normals" [Maronne, in this issue]. In this context, it is very striking that the documents preserved by Beeckman present a purely geometrical analysis with no use of algebraic techniques at all.

Finally, one should take into account the fact that the general context of this demonstration is not the study of curves, but the theory of equations and more specifically of their "construction."[44] To reiterate the point made earlier: to argue that, with respect to the teleology leading towards the "calcul géométrique" of 1637 (i.e., the algebraic analysis of curves), Descartes did not reach the achievements usually ascribed to him, does not amount to saying that he was *ipso facto* a lesser mathematician. With regard to his *Algebra*, in particular, the document sent to Beeckman shows that Descartes was already well advanced in many aspects of the subject (the study of a "general equation"; the reduction of degree; the use of factorization, which seems necessary in any case to obtain the magnitudes entering into his construction). The attention brought by scholars to the German cossists active at the beginning of the 17th century has shown how far their algebraic skills were developed, and how close their work is to what can be found in the theory of equations provided by Descartes in the third book of *La Géométrie* [Manders, 2006].[45] It is hence plausible that Descartes was already well advanced in these techniques at the beginning of 1620, and the documents that we have studied do not contradict this hypothesis.

But, following the arguments given in the preceding sections, there are no cogent reasons for thinking that the *Specimen* was part of the same project (i.e., the *Algebra*, dealing with the general theory of equations and their construction) and that all of these elements were pasted together in a coherent enterprise. On the contrary, these arguments show that there are cogent reasons to think that they are not. This would confirm Bos's judgement that the Pappus problem, studied at the end of 1631 on Golius' suggestion, was for Descartes "*the crucial catalyst*; it provided him, in 1632, with a *new* ordered vision of the realm of geometry and it shaped his convictions about the structure and the proper methods of geometry" [Bos, 2001, p. 283: my emphasis]. In my opinion, it is at this moment that Descartes went back to his 1619 project (on the classification of geometrical problems in analogy with arithmetical ones) and merged it with that of the *Regulae* (to treat all problems as equations and to use a geometrical calculus to solve them).

[44] On this theory, see [Bos, 1984].

[45] One should also keep in mind that when Mersenne asked Descartes to send his "old algebra" to Mydorge, he replied that he would find a much better version of it in the third part of *La Géométrie* [AT I, 501].

4. Conclusion

The line of interpretation I have presented offers to my mind three important advantages: first, it relies, as far as is possible, on contemporary documents without projecting onto them a hidden practice for which we have no evidence, direct or indirect; second, it gives meaning to many texts which remain in the retrospective view mysterious; finally, it allows us to put these texts into a coherent narrative without jumping too quickly to the diagnosis of crude contradictions in Descartes' thought.

On the other hand, as I have tried to show, to say that Descartes was not well advanced in 1628 with regard to the teleology leading to the *Géométrie* does not amount to saying he was a bad mathematician, persuaded that confused ideas and poor technique could ground a grandiose reform of "all human knowledge." The *De solidorum elementis* provides impressive results for a young man who had not received a very advanced mathematical education, even if they appear in a context which is not the one we would recognize as the "modern" ("Cartesian"!) one. These results were, without doubt, a way of borrowing "all that is best in Geometrical Analysis and Algebra, and correct[ing] the errors of the one by the other."

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