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Descartes and the cylindrical helix

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To Henk, in gratitude for all he has taught us.

Abstracts

In correspondence with Mersenne in 1629, Descartes discusses a construction involving a cylinder and what Descartes calls a "*helice*." Mancosu has argued that by "*helice*" Descartes was referring to a cylindrical helix. The editors of Mersenne's correspondence (Vol. II), and Henk Bos, have independently argued that, on the contrary, by "*helice*" Descartes was referring to the Archimedean spiral. We argue that identifying the *helice* with the cylindrical helix makes better sense of the text. In the process we take a careful look at constructions of the cylindrical helix available to Descartes and relate them to his criteria for excluding mechanical curves from geometry.

Résumé

Dans sa correspondance avec Mersenne en 1629, Descartes discute d'une construction qui fait intervenir un cylindre et ce que Descartes appelle une "hélice". Mancosu a argumenté que Descartes faisait référence à une hélice cylindrique. Les éditeurs de la Correspondance de Mersenne (vol. 2), et Henk Bos, ont indépendamment affirmé que, au contraire, par "hélice" Descartes faisait référence à la spirale d'Archimède. Nous affirmons qu'identifier l'hélice avec l'hélice cylindrique est plus cohérent avec le texte. Dans le même temps nous examinons soigneusement les constructions de l'hélice cylindrique que Descartes avait à disposition et nous les mettons en relation avec son critère d'exclusion des courbes mécaniques de la géométrie.

Despite the painstaking studies of generations of scholars, Descartes' motivation and rationale for the distinction between geometrical and mechanical curves are still not com-

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pletely understood.[1] Descartes offers several extensionally equivalent criteria for distinguishing geometrical from mechanical curves: geometrical curves are generated by (uniform) pointwise construction; generated by regulated continuous motions; or given by an algebraic equation. By contrast, a curve is mechanical, Descartes writes, if it is generated by arbitrary points; generated by unregulated motions or by strings that are sometimes like lines and sometimes like curves; or given by a nonalgebraic equation.[2] Since the identity conditions of curves are extensional, that is, they do not depend on how the curves are constructed or presented, providing a mechanical construction of a curve is an insufficient ground for judging that curve is mechanical. Such a procedure would be hostage to the possibility that an alternative construction, carried out using only acceptable geometrical construction principles, could show the curve to be geometrical after all. The kind of impossibility proof that would be required to show that no such alternative construction was forthcoming was not available to Descartes. Nevertheless, Descartes confidently excluded certain curves, such as the Archimedean spiral and the quadratrix, from the realm of geometrical curves. Hence it should be explained why Descartes believed he was justified in this exclusion.

In recent work (Mancosu, 1992, 1996, 2007) one of us has proposed an interpretation that appeals to Descartes' belief that the quadrature of the circle is impossible geometrically as a way to explain how Descartes could confidently judge these curves ungeometrical.[3] Mancosu's suggestion is that Descartes could confidently exclude certain curves by using criteria (*sufficient but not necessary* for nongeometricality) such as the one related to the quadrature of the circle: if a curve—in combination with other geometrical curves and constructions—allows the quadrature of the circle, then it cannot be geometrical, for the quadrature of the circle is impossible geometrically.

[1] For more on Descartes' conception of geometricity and more particularly of non-geometricity, cf. (Bos, 2001; Mancosu, 1996). On some occasions, Descartes deviates from his own favorite account of geometricity and means by 'geometrical curves' those constructible by ruler and compass; in a letter to Mersenne dated April 15, 1630 (AT I, p. 139), he singles out this latter mode of construction by calling it "ordinary geometry" [*la Geometrie simple*]. We will discuss such a case in Section 1, concerning a division of circles into 27 and 29 parts. This does not affect our argument, as on both construals of geometricity the curves that are of interest to us (the quadratrix, spiral, and cylindrical helix) turn out to be nongeometrical.

[2] Since these criteria and their relation to Descartes' classification of problems into plane, solid, and linear have been discussed at length in the literature (see bibliography in note 1), we will not rehearse them here. We will illustrate instances of geometrically legitimate and illegitimate constructions when discussing constructions for the cylindrical helix later in the paper.

[3] We should point out that the impossibility of squaring the circle geometrically was a widely held opinion at the time. For instance, Isaac Beeckman, at one time Descartes' mentor, notes in his *Journal*, "Quadratura circuli estne possibilis? Respondeo: Si physicè dicas, maximè. Nulla enim res physica infinitè secatur; primordia igitur physica erunt communis mensura circuli et quadrati, ergo aequalis numerus talium mensurarum circulum et quadratum perficiunt. Verùm, quoniam haec eadem primordia physica ⟨non⟩ infinitè secari possunt, dubitatur mathematicè, quamquam quadratum majus et minus dari possit, aut physicè aequale cogitari possit. Nec mirum. Recta enim rectae, et rectilineum rectilineo, est incommensurabile. Quidni ergo circularis linea ad rectam et circulus ad rectilineum ἀσύμμετρος dici posset? Et quod magis moveat: angulus comprehensus a tangente et peripheriâ planè incommensurabilis est ad angulum rectilineum, licet rectilineus illo et major et minor dari possit." (Beeckman, 1939–1953, Vol. 1, p. 26, around 1613–1614.)

In support of this interpretation, among other things, Mancosu appeals to a letter from Descartes to Mersenne dated November 13, 1629. He claims that in that letter, Descartes mentions the cylindrical helix as an example of a mechanical curve.[4] In the *Géométrie* (1637), Descartes only explicitly mentions the Archimedean spiral and the quadratrix as examples of mechanical curves. The cylindrical helix would thus be a welcome addition to the list, as it would help us better understand the grounds upon which Descartes rejects the mechanical curves and the criteria followed in doing so. The inclusion of the cylindrical helix was a supporting piece of evidence for Mancosu's interpretation on account of the fact that the only curves used since antiquity to square the circle were the Archimedean spiral, the quadratrix, and the cylindrical helix. Relying on the impossibility of the geometrical quadrature of the circle, Descartes would have had a *sufficient* criterion for excluding these three curves confidently from the realm of geometricality.

Mancosu's claim that the cylindrical helix is the curve mentioned by Descartes in his letter to Mersenne is challenged implicitly by two alternative interpretations of what the letter actually says. The first interpretation is due to the editors of Mersenne's correspondence, Cornelis de Waard and René Pintard (Mersenne, 1945, p. 309). The second has been proposed by Henk Bos in his magisterial book *Redefining Geometrical Exactness* (Bos, 2001). Both alternative interpretations claim that the curve mentioned in the letter is the ordinary Archimedean spiral. However, neither of them makes an attempt to eliminate the possibility that the cylindrical helix might actually be the curve mentioned in Descartes' letter to Mersenne. Our paper is devoted to discussing what arguments can be adduced in favor of each alternative interpretation and to argue that the evidence favors reading the November 1629 letter as referring to the cylindrical helix as opposed to the Archimedean spiral. While establishing this claim gives independent support to Mancosu's interpretation of the geometrical/mechanical divide in Descartes' work up to 1637, defending that interpretation is not a task of this paper. We begin by reviewing the central pieces of evidence, namely Descartes' letters to Mersenne dated October 8, 1629 and November 13, 1629, respectively.

1. Gaudey's invention

The relevant passages in Descartes' correspondence with Mersenne concern the discussion of an "invention" that had apparently been communicated in a letter from Mersenne to Descartes which is now lost. We learn about the problem from Descartes' reply to Mersenne in a letter dated October 8, 1629. Descartes says:

> Dividing circles in 27 and 29 I believe can be done mechanically but not in geometry. It is true that it can be done in 27 by means of a cylinder, even though very few know how; but not in 29, nor in all others and if you want to send me the procedure, I dare promise you that I will show that it cannot be exact.[5]

[4] For a contemporary treatment of the cylindrical helix and its special properties, cf. (Gomes Teixeira, 1971, Vol. II, pp. 373–388).

[5] "De diviser les cercles en 27 et 29, ie le croy, mechaniquement, mais non pas en Geometrie. Il est vray qu'il se peut en 27 par le moyen d'un cylindre, encore que peu de gens en puissant trouver le moyen; mais non pas en 29, ny en tous autres, & si on m'en veut envoyer la pratique, i'ose vous promettre de faire voir qu'elle n'est pas exacte." (Descartes, AT I, pp. 25–26; letter to Mersenne, October 8, 1629.)

This text raises several problems that we might as well begin to enumerate. The first, perhaps minor, puzzlement emerges from the plural "*cercles*." In general, one speaks in the singular of the division of the circle or the construction of an equilateral triangle over any given segment. The plural in the statement might perhaps indicate that the problem was not simply one of partitioning an arbitrary circle into 27 or 29 equal parts. Regardless, the editors of Mersenne's correspondence interpret the problem as that of dividing the circle into 27 or 29 equal parts and Bos seems to agree by classifying the problems as one of angular division. Bos, in particular, speaks of general multisection of angles.

Note, however, that the problem must have mentioned other divisions, in addition to 27 and 29, since Descartes says to Mersenne "ny en tous autres." We have no idea of course of what other numerical divisions might have been mentioned in Mersenne's letter. The list certainly could not have included division by 2 or any power of 2, since those are easily obtainable by Euclidean means.

Another important issue here is the contraposition between mechanical and geometrical. How does Descartes understand the terms in this context? At first sight, it seems that he cannot mean geometrical as understood in the *Géométrie*, for a division of an angle of 360° (i.e., a division of the circle) into 27 or 29 parts is geometrical according to the criteria of geometricality proposed in the *Géométrie*. In the first case one has to solve an equation of at most degree 27 (indeed in this case it is sufficient to solve two cubic equations corresponding to a repeated trisection) and in the other one of at most degree 29.[6]

The other possibility is that Descartes' notion of geometricality here is just construction with circles and straight lines. This would account for why Descartes claims that both problems are mechanical, although he then seems to imply that there is quite a difference between the two. Obviously the division into 27 parts can be obtained by repeated trisection of angles, whereas the division into 29 parts cannot. But Descartes adds that the division into 27 parts can be achieved through a cylinder, whereas the division into 29 parts cannot.

The editors of Mersenne's correspondence suggest that Descartes had in mind a repeated trisection through the use of a cylindrical section, that is, an ellipse. However, they do not provide the construction (here they follow, without explicit mention, Tannery, 1893, p. 18). Tannery says that the letter to Mersenne from November 13, 1629 "proves" that Descartes had in mind such a trisection. While we believe that the construction involved a repeated trisection, it is unclear to us that the letter in question "proves" that an ellipse, namely a cylindrical section, was involved.[7]

Upon receiving from Mersenne the details of the construction (this letter is also lost; regrettably, for otherwise it would be easy to solve the problem of the present paper), Descartes replies, on November 13, 1629, as follows:

[6] That Descartes takes for granted the geometrical constructibility of all algebraic equations is evident in the *Géométrie* at AT VI, p. 485, where after constructing fifth- and sixth-degree equations, he writes that the same procedure used for these cases will permit the construction of equations of arbitrarily high degree ["il ne faut que suivre la même voie pour construire tous ceux qui sont plus composés à l'infini"].

[7] The mathematician E. Lucas in 1875 proposed an interpretation of the passage concerning the division of the circle into 27 parts by providing a trisection of the angle by means of a construction involving the intersection of a cylinder and two spheres having their centers on the surface of the cylinder which is then iterated. Cf. (Lucas, 1875) for the original article and (Gomes Teixeira, 1971, Vol. III, pp. 352–353) for a description of Lucas's method.

> Mr Gaudey's invention is very good and very exact in practice. However, so that you will not think that I was mistaken when I claimed that it could not be geometric, I will tell you that it is not the cylinder which is the cause of the effect, as you had me understand and which plays the same role as the circle and the straight line; rather, the whole thing depends on the helix line that you had not mentioned to me and that is not a line that is accepted in geometry any more than that which is called the quadratrix, for the latter, just as much as the former, can be used to square the circle and even to divide an angle in all sorts of equal parts and has many other uses as you can see in Clavius's Commentary to Euclid's Elements.[8]

With this part of the letter we discover that the construction in question was due to Gaudey, a geometer about whom very little is known. Descartes triumphantly remarks that the construction is not geometrical on account of the use of the *helice*, a curve that is not geometrical. Descartes tells Mersenne that the *helice* is a curve that is excluded from geometry just as the quadratrix is, for both can be used to square the circle and to divide an angle into an arbitrary number of parts. The reference is to Clavius's commentary on the quadratrix, which has been extensively investigated in the secondary literature (cf. Bos, 2001; Gäbe, 1972; Garibaldi, 1995; Knobloch, 1995; Mancosu, 1992, 1996, 2007).

The reader familiar with the *Géométrie* might be tempted immediately to infer that the *helice* is the Archimedean spiral, for the quadratrix and the Archimedean spiral are the only two mechanical curves explicitly mentioned in the *Géométrie*. But before drawing any conclusions, let us read the remaining passage relevant to our issue.

> For although one could find an infinity of points through which the helix or the quadratrix must pass, one cannot find geometrically any one of those points which are necessary for the desired effect of the former as well as of the latter. Moreover, they cannot be traced completely except by the intersection of two movements which do not depend on each other; or also the helix by means of a thread [*filet*] for revolving a thread obliquely about the cylinder it describes exactly this line; but one can square the circle with the same thread so precisely that this will not give us anything new in geometry. This does not stop me from admiring Mr Gaudey's invention and I do not think that one could find a better one for the same effect.[9]

[8] "L'invention de Mr Gaudey est tres bonne & tres exacte en prattique; toutesfois affin que vous ne pensiés pas que ie me fusse mespris de vous mander que cela ne pouvoit estre Geometrique, ie vous diray que ce n'est pas le cylindre qui est cause de l'effait, comme vous m'aviés fait entendre, et qu'il n'y fait pas plus que le cercle ou la ligne droitte, mais que le tout depend de la ligne helice que vous ne m'aviés point nommee & qui n'est pas une ligne plus receue en Geometrie que celle qu'on appele *quadraticem*, pource qu'elle sert a quarrer le cercle & mesme a diviser l'angle en toutes sortes de parties esgales aussy bien que celle cy & a beaucoup d'autres usages que vous pourrés voir dans les elemans d'Euclide commantés par Clavius."(AT I, pp. 70–71.)

[9] "Car encore qu'on puisse trouver une infinité de points par ou passe l'helice & la quadratrice, toutefois on ne peut trouver Geometriquemant aucun des poins qui sont necessaires pour les effaits desirés tant de l'une que de l'autre, et on ne les peut tracer toutes entieres que par la rencontre de deus mouvemans qui ne dependent point l'un de l'autre; ou bien l'helice par le moyen d'un filet, car tournant un filet de biais autour du cylindre, il decrit justemant cete ligne la; mais on peut avec le mesme filet quarrer le cercle, si bien que cela ne nous donne rien de nouveau en Geometrie. Ie ne laisse pas d'estimer bien fort l'invention de Mr Gaudey, & ne croy pas qu'il s'en puisse trouver de meilleure pour le mesme effait." (AT I, p. 71.)

We enter now the heart of the dispute. Is the *helice* an Archimedean spiral, as Bos and the editors of Mersenne's correspondence claim, or the cylindrical helix? We will attempt to show that the latter is the case. The editors of Mersenne's correspondence have no doubt that Gaudey's construction was carried out, following Pappus (*Collectio* IV, problem 11, prop. 35), by means of an Archimedean spiral (Mersenne, 1945, p. 309). However, they ascribe to Gaudey the construction of a machine for the mechanical generation of the spiral that, they claim, finds an analogue in a machine described by Jacques Besson in his successful 16th-century compendium of machines, *Theatrum Instrumentorum et Machinarum* (Besson, 1578). Bos also claims that the *helice* is an Archimedean spiral but perceptively notices that in the description of his machine, Besson emphasizes as a positive feature that his machine does not require the use of strings. Bos' candidate for the kind of construction Gaudey might have provided uses a machine described by Huygens in 1650. Before we describe such mechanisms we will raise and dispose of a preliminary issue.

Can't the indecision about whether the Archimedean spiral or the cylindrical helix is the curve mentioned in the letter be resolved on linguistic grounds by looking at the use of "*helice*" in the mathematical literature of the time? Unfortunately, that strategy will not work. Indeed, during this period both the word "spiral" (Fr: *spirale*; Lat: *spiralis*) and the word "helix" (Fr: *helice*; Lat: *helix*) are used indifferently to talk about the Archimedean spiral (a planar curve) and the cylindrical helix (a spatial curve). For instance, in his translation of Pappus, Commandinus uses 'linea spiralis' for both (cf. Pappus, 1588, pp. 90–91). Torricelli in his investigations of plane spirals uses indifferently "spirales" and "helices".[10] Finally, Mersenne in *Cogitata Physico Mathematica* (Mersenne, 1644) uses "helix" for both the plane spiral (*Hydraulica*, p. 129) and the cylindrical helix (*Mechanica*, pp. 56–60).

Thus, out of context "*la ligne helice*" can refer either to a plane spiral or to a cylindrical helix. Our attempt to show that in the context of the 1629 letter Descartes has in mind a cylindrical helix is thus consistent with the fact that on other occasions Descartes uses "*helice*" to refer to the plane spiral.[11]

2. The interpretation of de Waard and Pintard

As we have mentioned above, de Waard and Pintard identify the *helice* with the Archimedean spiral. However, the multisection of the angle in an arbitrary number of parts by means of the Archimedean spiral was already achieved by Pappus in *Collectio* IV, probl. XI, prop. 35. Thus, de Waard and Pintard claim that Gaudey had built an instrument for the generation of the spiral "of which one finds the analogue in Besson" and his original contribution would have consisted, presumably, in the invention of this machine. Let us

[10] *De Infinitis Spiralibus*: "Different etiam helices nostrae ab Archimedeis secundum numerum revolutionum. Spirales enim Archimedis..." (cf. Torricelli, 1919–1944, Vol. I, part II, p. 362).

[11] "Ie n'ay pas loysir d'examiner ce que vous me mandez de l'helice & de la parabole; mais si on avoit trouvé une ligne droite egale a une hyperbole, comme vous avez escrit a Mr de Zuylichem, ie le trouverois bien plus admirable." (AT III, p. 642, Descartes to Mersenne, March 23, 1643). Conversely, Descartes uses "spirales" also in the context of the cylindrical helix as in his discussion of the screw with Huygens (AT IV, p. 761, Descartes to Huygens, November 15, 1643).

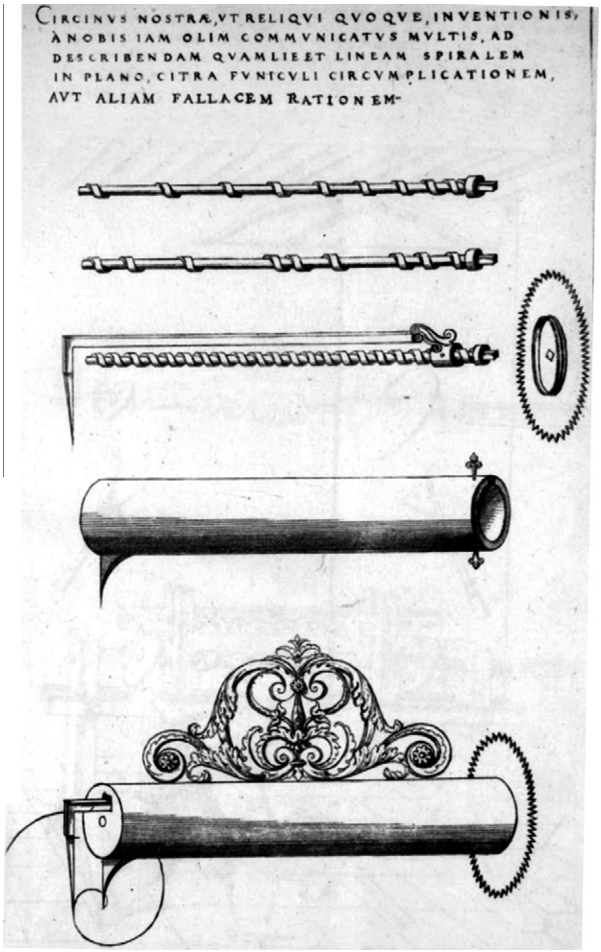

Fig. 1. Besson's machine.

thus look at the instrument described by Besson (depicted in Fig. 1). We translate here for the first time the description found in Besson's book.[12]

> Statement of the author on the sixth figure.
>
> This is a compass also invented by us just as much as the others, which invention we have communicated long ago to many people. By means of it one can draw any spiral line in the plane without any twisting of strings or any other fallacious way of proceeding.
>
> Explanation of the same figure VI.
>
> A compass of our invention (as the others are also), and long ago communicated to many, for describing any spiral line in a plane without winding of strings or any other false procedure.

[12] The diagram in the Latin version of the text does not indicate, as the Italian and French versions do, which are the North, South, East, and West sides that the author refers to in his description. Looking at the diagram (see Fig. 1) the top of the page is North, the bottom South, the left West, and the right East.

> The whole machine of this compass is [the figure closest to] South. The remaining things towards the North are its parts, which I will now explain. The part which is round, long, and hollow, is called Cannon on account of its similarity to the Bombarda. It is a case [*theca*], with a pointed end towards the West, around which the compass turns in order to delineate the spiral. The part after that is a screw in whose external part is attached a ruler at the Western tip of which is a moving pointed end. The remaining parts to the North are two types of interior screws useful for drawing different types of spirals and can be inserted, now one now the other, in the external part of the screw.
>
> Now in the center of the toothed wheel, found in the East, one places the squared tip of the interior part of the screw so that once the compass is assembled with all its parts, the screw will move with the aid of the wheel so that little by little the pointed tip will come out, with the ruler (which pushes it) always placed in the square hole found in the upper part of the cannon. Which is what we proposed. (Besson, 1578, unnumbered)[13]

When this is compared with the text of the letter from Descartes to Mersenne we see that de Waard and Pintard are committed to interpreting the reference to the cylinder and the *filet* as a reference to an instrument concocted for the generation of the Archimedean spiral. Moreover, they claim, this instrument finds its analogue in an instrument described by Besson. We question both assumptions. First of all, it is not clear that Descartes is describing Gaudey's construction. He could instead be justifying his claim that the *helice* is not a geometrical curve. By the same token, we should not infer that Gaudey's construction was by means of arbitrary points or unregulated motions just because Descartes mentioned these as justifying the nongeometricity of both the quadratrix and the *helice* in the same

[13] "Circinus nostrae ut reliqui quoque inventionis a nobis iam olim communicatus multis, ad describendum quamlibet lineam spiralem, in plano citra funiculi circumplicationem aut aliam fallacem rationem.

Declaratio 6. Figurae.

Huius circini in meridie est integra machina. Reliqua septentrionalia partes illius sunt, quas libet explicare. Pars rotunda longa & cava, quam placet dicere a similitudine bombardae canonem, theca est in qua occidentem versus cuspis est circa quam volvitur circinus ad spiram describendam, proxima pars cochlea est, in cuius parte exteriore adhaeret regula, in cuius fine occidentali cuspis est mobilis, reliquae aliae quae partes septentrionales sunt cochleae interiores partes ad pingendam multifariam lineam spiralem, & imponuntur vicissim in parte exteriori cochleae. Iam in rotulae orientalis, cuius ultima ora denticulata est, centrum infigitur extremitas quadrata cochleae partis interioris, ut circino omnibus partibus suis compositio, illius rotulae auxilio moveatur cochlea, & paulatim emittatur cuspis mobilis, regula (quae eam defert) semper existente in scissura quadrata, quae in canonis superiori parte est. quod est propositum."

The Italian and French translations of the work add the following "Additione": "The usefulness of this compass is not less than that of the others, for it often happens that one needs this sort of line in construction. This is normally done with the ordinary compass with great difficulty. Indeed, one needs to open and close it many times and draw many partial lines. For this reason the spiral line is never so naturally drawn as when it is designed with this compass. It must also be noted that the slot or groove that the cannon has on top must be as long as the cannon is and it must have a sparrow tail form and the ruler that enters in it must be equivalent [*a l'equipollente*]. The screw must lengthen itself inside the cannon and must have free motion around its pivot that is put in that round hole which is in the Western part of the completely assembled instrument. Finally, the two small screws which are in the cannon on the East side are needed to fasten the toothed wheel by means of a moving circle which is [attached] there and that being fixed with the cannon allows the wheel to have its proper movement." (Besson, 1582, p. 18.)

letter just a few lines before the mention of the cylinder and the *filet*. But even supposing that the reference to the cylinder and the *filet* refers to Gaudey's construction, there remains an insurmountable problem for this interpretation. Namely, there is no *filet* in Besson's machine. Besson in fact proudly announces that his machine does not make use of any twisting of strings. Thus, Besson's machine seems like a poor candidate for the machine provided by Gaudey, if he indeed had provided one.

While it is clear that de Waard and Pintard identify the *helice* with the Archimedean spiral, we would like to consider a modification of their interpretation that accords better with what is said in the November 1629 letter. Might not Descartes be referring by "*helice*" in the November 1629 letter to the screw in Besson's machine? This is consistent with his reference to *filets* later in the letter, since later writers (whom we will discuss in Section 5.1) speak of the "filet de la vis" in reference to the threads of a screw.

We believe this is unlikely. That is because Descartes says that the *helice* is obtained by "revolving a thread obliquely about the cylinder" [*tournant un filet de biais autour du cylindre*]. It would be incorrect to say that the threads of a screw are revolved about a cylinder, because the threads of a screw are partly *constitutive* of the screw. They are not threads until the revolving has been completed. But notice that even if Descartes were referring to the screw in Besson's machine—and, we want to emphasize, de Waard and Pintard offer no argument for identifying Gaudey's invention with Besson's machine—it would follow that "*helice*" refers to the cylindrical helix, as we claim, rather than to the Archimedean spiral, as de Waard and Pintard claim. That is because the screw's thread follows the path of a cylindrical helix. Thus even if this modified version of de Waard and Pintard's reading were correct, the evidence would still favor our view of what curve is mentioned in the November 1629 letter.

3. Bos's interpretation

Bos also identifies the *helice* of the November 1629 letter with the Archimedean spiral, reasoning as follows (Bos, 2001, p. 345). In his October 1629 letter to Mersenne, Descartes responds to Mersenne's description of Gaudey's construction. Bos believes that this construction is a procedure for the general multisection of angles. After receiving a reply from Mersenne, Descartes replies in the November 1629 letter, pointing out that in the previous letter Mersenne had omitted to mention the *helice* when describing Gaudey's achievement. As Bos reads the exchange, what Descartes' November 1629 letter conveys is that Gaudey's construction involved a cylinder, a string, and a *helice*. Bos points out (cf. Bos, 2001, pp. 347–348) that a machine meeting this description, used for constructing Archimedean spirals, was known to a contemporary of Descartes, Christiaan Huygens. He also remarks that Huygens as a child spent time with Descartes and so could plausibly have learned the machine from Descartes. (We'll describe this machine next.) Since the Archimedean spiral can be used to solve the general angle multisection problem[14], Bos concludes that Gaudey's "invention" was a machine for tracing the Archimedean spiral, and hence Descartes meant by "*helice*" the Archimedean spiral.

[14] Pappus had shown how to solve the general angle multisection problem using the Archimedean spiral and the quadratrix in *Collectio*, Book IV, §45–46 (Pappus, 1876, Vol. 1, pp. 284–289). Bos describes Pappus' solution in Bos [2001, pp. 43–44].

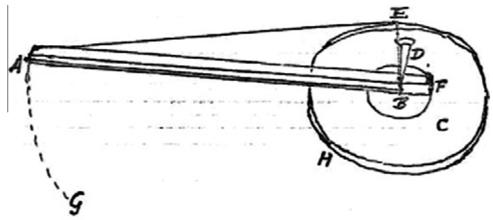

Fig. 2. Huygens's spiral-tracing machine.

Huygens' machine (depicted in Fig. 2; Huygens, 1888–1950, Vol. 11, p. 216) consists of a cylinder *C* of small height fixed on a plane; a smaller cylinder *BF* that is set upon *C* but can rotate on *C*; a ruler *AF* that is attached to cylinder *BF* so that *AF* can be rotated, with its end *A* a pulley; and a string *EAB* affixed to *C* at *E*, fit through the pulley *A* and then along *AF*, and affixed at its other end to a stylus *DB*, so that the string is pulled taut. Huygens leaves a few details unexplained, but they seem clear enough from what he wrote so that they can be stated confidently. First, the diagram suggests that *B* is the center of cylinder *BF*, and the description suggests that *BF* is rotated around *B*, which remains fixed relative to *C*. Second, there must be some means of ensuring that the stylus *DB* slides rectilinearly along *AF*, perhaps by following a groove carved into the ruler *AF*. Third, the diagram suggests that in the initial position of the machine, the ruler *AB* is orthogonal to the line *BE* (though this line does not correspond to any physical part of the machine).

The machine is operated by rotating ruler *AF* so that the angle *EBA* increases (moving *A* toward the point *G* in the diagram). As a result, the segment *EA* is lengthened, and more string is needed along that segment. Consequently, the stylus *DB* is pulled along *AF*, toward *A*. Meanwhile, the string along the segment *EA* wraps around *C*. The path of the stylus *DB* traces an Archimedean spiral.[15]

Bos suggests that Gaudey's invention was a machine like Huygens's. We now want to raise some points against Bos's interpretation. First, as we said regarding de Waard and Pintard's case, it is not clear that Descartes is describing Gaudey's construction in the letter, rather than justifying his claim that the "*helice*" is not a geometrical curve. Second, the connection between Huygens's machine and Descartes is tenuous. It is possible that Huygens learned of this machine from Descartes, but there is no evidence that this is so. Third, Descartes describes the string wrapping around the cylinder "obliquely" [*de biais*], but it is not clear what this means for Huygens's machine. When the handle is turned, the string wraps around the cylinder, but it is not clear what is oblique about this wrapping. Last, note that Descartes suggests in the letter that the cylinder is geometrical (it "plays the same role as the circle and straight line"). If Descartes is thinking of the cylinder as a component of a machine, this seems to imply that we can judge components of machines to be geometrical or ungeometrical; e.g., the stylus could be geometrical or not. But this is absurd.[16]

Still, we have not shown definitively that Bos's interpretation is wrong. We will give our own interpretation of the November 1629 letter, and show that the evidence supports it better than

[15] Our description of Huygens's machine differs slightly from Bos's, but not in a way that affects his argument.

[16] This last objection also applies to de Waard and Pintard's interpretation, since it too involves a machine.

it does Bos's. But first we will describe the various constructions of the cylindrical helix available to Descartes through the mathematical and mechanical literature of his time.

4. Cylindrical helix constructions

4.1. Double motion constructions

One type of construction found in the classical literature traces the cylindrical helix by two simultaneous uniform motions. For instance, Heron traces the cylindrical helix as follows (depicted in Fig. 3 in the left figure). Let a cylinder be given. Take a line along the edge of this cylinder and rotate it around the cylinder's circumference. In the time that it takes for this edge to make one full rotation, move a point uniformly along this line, from the base of the cylinder to its top (adjusting the speeds of each motion so that each—a full rotation of the edge and a full traversal of the edge by the point—is possible). The path of this point traces a cylindrical helix.[17]

Simplicius' construction is quite similar, the main difference being that it constructs the cylinder rather than taking it as given. He proceeds as follows (depicted in Fig. 3 in the right figure). Let a rectangle be given. Pick one side of this rectangle as an axis of rotation, and rotate it to generate a cylinder. While doing so, simultaneously move a point along the side of the rectangle parallel to the axis side, in such a way that when the rotation begins, this point is at the base of the rectangle, and when the first rotation is complete, this point is at the top of the rectangle. This point traces a cylindrical helix.[18]

4.2. Projection from the Archimedean spiral

In *Collectio* IV, §34, Pappus constructs a quadratrix using an Archimedean spiral. Wilbur Knorr has pointed out that in the course of this construction, a cylindrical helix is also traced.[19] We will focus just on the part of this construction yielding the cylindrical helix. This construction takes a cylinder as given, constructs a planar Archimedean spiral, and using this spiral projects a cylindrical helix onto the cylinder. It proceeds as follows (depicted in Fig. 4). Let a circle be given with center B, with AC a quarter arc of this circle, and let a cylinder with this circle as base be given. An Archimedean spiral AGB is traced by a double motion construction, uniformly rotating through the arc AC a radius BD coinciding initially with BC and terminally with BA, while simultaneously moving a point G uniformly along BD, so that G begins at B and reaches D at the same time that BD completes its rotation through the arc AC. The path of the point G traces the spiral. During this

[17] Cf. (Heron, 1894, p. 102). Heron's (Heron, 1894) was not known in the Renaissance and the 17th century, but Proclus and Pappus, in works that were known at Descartes' time, give essentially the same construction (cf. Proclus, 1992, p. 85, and Pappus, 1876, Vol. 3, Book VIII, pp. 1110–1111). Descartes read Pappus in Commandinus' edition (Pappus, 1588).

[18] Cf. (Simplicius, 2002, p. 33). Like Proclus and Pappus, this text was known in the Renaissance and 17th century. In addition, the same definition of the cylindrical helix was available at the time in Dasypodius [1570] as part of Definition 7 in Heron's "Definitiones" (cf. Heron, 1912, Vol. 4, pp. 18–21).

[19] For the original proof, cf. (Pappus, 1876, Vol. 1, pp. 262–265). For Knorr's remarks, including a description of Pappus' constructions, cf. (Knorr, 1993, pp. 166–167); also cf. (Knorr, 1978, pp. 63–65). Heath's remarks on this construction are also useful; cf. (Heath, 1981b, pp. 381–382).

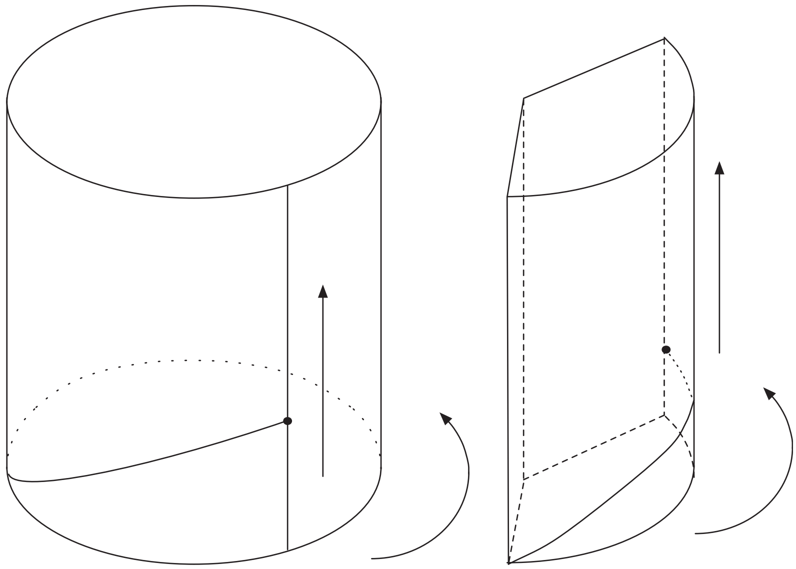

Fig. 3. Heron and Simplicius' cylindrical helix constructions (respectively).

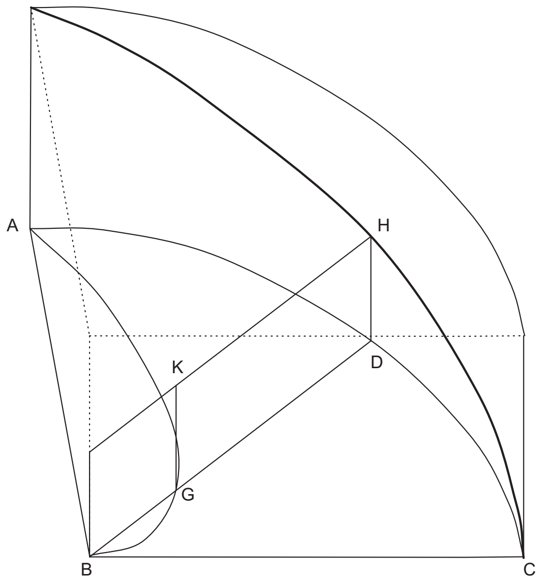

Fig. 4. Pappus' cylindrical helix construction by projection.

tracing of the spiral, a line segment *GK* is extended orthogonally to the plane of the spiral such that its length is equal to the length of BG, as is a line *KH* passing through *K* and parallel to *BD*. The path traced by *H* on the surface of the cylinder is a cylindrical helix.

4.3. Triangle wrapping constructions

In addition to the double motion construction explained above, Heron gave the following construction of the cylindrical helix:

> When we want to trace this line on the surface of the cylinder, we operate in this manner: we give ourselves on a plane two lines perpendicular to each other, one equal to the edge

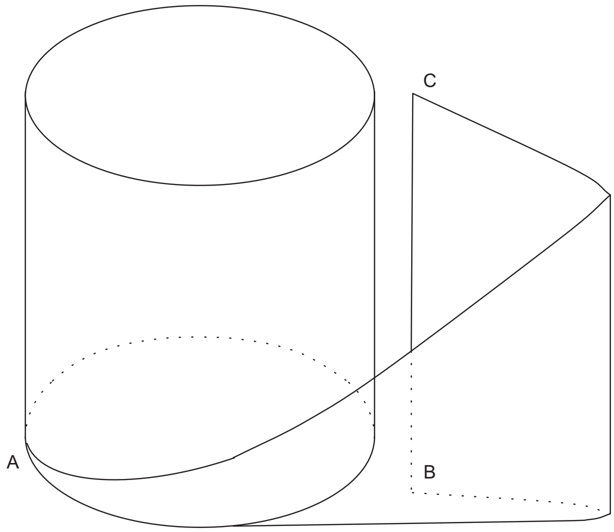

Fig. 5. Heron's cylindrical helix construction by triangle wrapping.

> of the cylinder, the other equal to the circumference of the cylinder, i.e. with the circumference of its base; and we join the two ends of these two lines which include the right angle, by a line subtending the right angle. We apply [in the sense of fit over] the line equal to the edge of the cylinder to this edge, and the line equal to the circumference of the base of the cylinder to this circumference. The line which subtends the right angle is wrapped around the surface of the cylinder and traces a path of the screw.[20]

We may reconstruct Heron's construction as follows (depicted in Fig. 5). Let a cylinder be given. Construct a right triangle ABC such that one side, AB, is equal to the circumference of the cylinder's base, and the other side, BC, is equal to the height of the cylinder. Next, wrap this triangle around the cylinder so that BC is orthogonal to the cylinder's base, and AB lays along the cylinder's base. The hypotenuse AC of this triangle traces a cylindrical helix.[21]

4.4. String constructions

String constructions are another well-known type of cylindrical helix construction. Theon of Smyrna gives an informal example of such a construction in the course of describing planetary motions, as follows:

[20] Cf. (Heron, 1894, p. 102). Carra de Vaux's French translation of this passage (from the Arabic; the Greek original is lost) reads, "Lorsque nous voulons tracer cette ligne sur la surface du cylindre, nous opérons de cette manière : nous nous donnons sur un plan deux lignes perpendiculaires l'une à l'autre, l'une égale à l'arête du cylindre, l'autre égale à la circonférence du cylindre, c'est-à-dire à la circonférence de sa base; et nous joignons les deux extrémités de ces deux lignes qui comprennent l'angle droit, par une ligne qui soutend l'angle droit. Nous appliquons la ligne égale à l'arête du cylindre sur cette arête, et la ligne égale à la circonférence de la base du cylindre sur cette circonférence. La ligne qui soutend l'angle droit s'enroule sur la surface du cylindre et décrit un tour de vis."

[21] As noted earlier, Heron's *Mechanics* was not known in the Renaissance and 17th century, but Pappus gives essentially the same construction in *Collectio* VIII, §28 (cf. Pappus, 1876, Vol. 3, pp. 1109–1111; also cf. Drachmann, 1963, pp. 75–76). In the 16th century, Guidobaldo gives this construction again, citing Pappus as his source (cf. Monte, 1577, pp. 120–122).

> The planets trace a helix by accident on account of the fact that there are two movements in opposite directions to each other [. . ..] They [the planets] describe a helix similar to the tendril of vines; just as the ephors [governors] wound their dispatches as strips around the Spartan batons.[22]

The *locus classicus* of string construction of the helix, however, is Vitruvius, *De Architectura* X.6, wherein Vitruvius describes how to construct a water screw [*cochlea*].[23] The first step in doing so yields a construction of a cylindrical helix. Vitruvius describes this component of the construction as follows:

> Take a beam, as many digits thick as it is feet long. This should be rounded out to an exact circle. At each end, the circumference will be divided by the help of a compass into eight segments, in such a way that the intersecting diameters of each circle, when the beam is laid flat, will correspond perfectly with one another on the level, then score circles around the beam along its entire length at intervals equal to one-eighth the circumference. Once the beam has been laid horizontal, lines should be drawn from one end to the other so that they are perfectly level. In this way equal intervals have been drawn along the length, the transverse scorings create intersections, and these intersections determine specific points. Once all these things have been drawn carefully, take a strip of slender willow or cut agnus castus which, once it has been dipped in liquid pitch, is fixed in place at the first point formed by the intersections. Then it is carried obliquely to the next point of intersection between length and circumference, and proceeding row by row in this fashion, as the strip passes individual points, winding around, it is fastened at each intersection so that, by the time it reaches the eight point away from the beginning and is fixed in place, it has arrived again at the same line in which it was fastened down in the first place.[24]

[22] For the original Greek see (Smyrna, 1892, p. 328). That he is talking of the cylindrical helix becomes even more clear, if need be, in the following passage (p. 330): "The planets trace still another spiral but not as if it were drawn around a cylinder from one end to the other, but as if it were drawn on a planar surface."

[23] *De Architectura* was widely available and translated into many languages in Descartes' time; cf. (Vitruvius, 2001, pp. xii–xiv), for some details on its reception in the Renaissance.

[24] "Tignum sumitur, cuius tigni quanta paratur pedum longitudo tanta digitorum expeditur crassitudo. Id ad circinum rotundatur. In capitibus circino dividentur circumitiones eorum tetrantibus et octantibus in partes octo, eaeque lineae ita conlocentur ut, plano posito tigno, utriusque capitis ad libellam lineae inter se respondeant, et quam magna pars sit octava circinationis tigni, tam magna spatia decidantur in longitudinem. Item, tigno plano conlocato lineae ab capite ad alterum caput perducantur ad libellam convenientes. Sic et in rotundatione et in longitudine aequalia spatia fient. Ita quo loci describuntur, lineae quae sunt in longitudinem spectantes facient decusationes et in decusationibus finita puncta. His ita emendate descriptis, sumitur salignea tenuis aut de vitice secta regula, quae uncta liquida pice figitur in primo decusis puncto. Deinde traicitur oblique ad insequentis longitudinis et circumitionis decusis; item ex ordine progrediens singula puncta praetereundo et circum involvendo conlocatur in singulis decusationibus, et ita pervenit et figitur ad eam lineam, recedens a primo in octavum punctum, in qua prima pars est eius fixa. Eo modo quantum progreditur oblique spatium et per octo puncta, tantundem et longitudine procedit ad octavum punctum. Eadem ratione per omne spatium longitudinis et rotunditatis singulis decusationibus oblique fixae regulae per octo crassitudinis divisiones involutos faciunt canales et iustam cocleae naturalemque imitationem." (Vitruvius, 1986, 10.6, pp. 22–23; English translation from Vitruvius, 2001, p. 125).

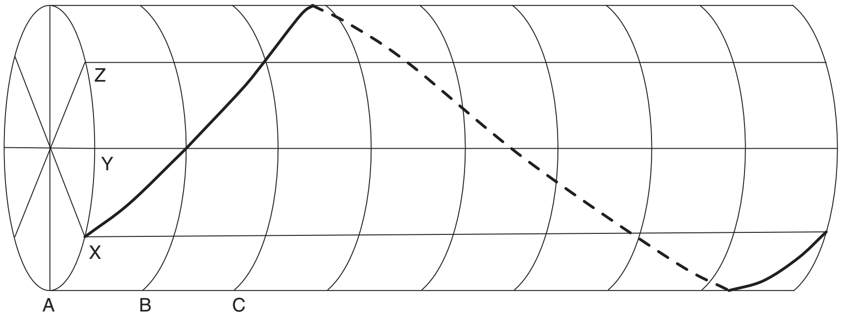

Fig. 6. Vitruvius' helix construction.

We may reconstruct Vitruvius' construction of a cylindrical helix as follows (depicted in Fig. 6). Let a cylinder be given, such that its length is 16 times its diameter. Divide the circle A, constituting one of the cylinder's ends, into eight equal arcs (using a compass). Let one of these arcs be denoted by arc XY, where X and Y are the endpoints of this arc, and another be denoted by YZ, where its endpoints are Y and Z. Let a denote the length of these (equal) arcs. Next, trace another circle B around the circumference of the cylinder so that the distance between A and B along the cylinder is a. Then trace another such circle so that its distance from B along the cylinder is a, and continue to trace such circles along the entire length of the cylinder. Next, trace lines from the endpoints of each arc on A, along the cylinder, in such a way that each line is orthogonal to A; let X, Y, Z also denote the lines originating from these endpoints, respectively. We may now trace a helix around this cylinder as follows. Take a strip of willow and fasten it to point X, then lay the strip around the cylinder so that it passes through the intersection of Y and B, and fasten it to this point; then continue to lay the strip so that it passes through the intersection of Z and C, and so on, through the following intersections of lines and circles along the cylinder. Once the strip has been fastened to its eighth point of intersection, it will have intersected X again (the length-to-diameter ratio ensures that the cylinder is long enough for at least one full rotation). This strip traces a cylindrical helix.

It is worth pointing out that a similar pointwise construction for the cylindrical helix is found in Dürer, who describes the procedure in connection to the tracing of helices on columns, an important problem in architecture (Dürer, 1977, pp. 67–68, 193, 200). In this way we have clearly established that the cylindrical helix was not some kind of esoteric curve in the early modern era but rather a curve of central importance to mathematics[25], mechanics (in connection with the Archimedean screw)[26], and architecture.[27] As such it cannot possibly have escaped Descartes' attention.

[25] For the importance of the cylindrical helix in the debates on the ancient and medieval classification of curves, see (Rashed, 2005). A "proportional" version of the cylindrical helix, under the name of "linea gyrativa," was also extensively discussed in connection to infinity in natural philosophy, especially in the XIVth century. See (Dewender, 2002, pp. 82–89) and in particular note 59 (p. 82) for an extensive list of sources mentioning the cylindrical helix (and its "proportional" version) including, among others, Averroes, Albertus Magnus, Thomas Aquinas, Buridan, Marsilius of Inghen, and Lawrence of Lindores.

[26] For a scholarly investigation of the theory of the Archimedean screw, including a discussion of its reception in the modern era, see (Koetsier and Blauwendraat, 2004).

[27] Beautiful examples of columns with cylindrical helices can be found at http://home.nordnet.fr/~ajuhel/Surfaces/quad_archi_hel.html.

Building on the previous descriptions, we now propose two possible interpretations of the November 1629 letter, both of which we find superior to the alternative interpretations offered by de Waard-Pintard and Bos.

5. Our interpretations

5.1. *Our first proposal*

Our first proposal is that, in the passage concerning the helix in Descartes' November 1629 letter to Mersenne quoted on p. 5 of this paper, Descartes is simply justifying his view that the cylindrical helix is not a geometrical curve. To support this contention, we will show that Descartes would have objected to all of the constructions of the helix that we have given, and that these objections reflect the objections in the November 1629 letter.

In the letter, Descartes first says that the helix is not geometrical because while one could find an infinity of points through which the helix must pass, these points cannot all be found geometrically. In the *Géométrie*, Descartes writes that a geometrical procedure for finding the points of a curve is one in which any point of the curve can be found by this procedure, not just a subset of the curve's points (AT VI, pp. 411–412). On this criterion, both Vitruvius and Dürer's constructions are ungeometrical because they only permit the determination of points obtained by dividing the base of the cylinder (a circle) into a number of parts that is a power of two (in contemporary terms, the construction yields only countably many points, rather than arbitrarily (continuum) many points as needed).

Next, he says that the helix can only be traced "completely" either by intersecting two movements that are independent of each other, or by strings, and hence in neither case is geometrical. Let us begin with double-motion constructions. This remark would apply both to the constructions of Heron, Simplicius, et al. described in Section 4.1, and to the projection constructions of Pappus described in Section 4.2. In the case of the constructions in Section 4.1, the problem is that the two motions, one a rotational motion and the other a rectilinear motion, are to be carried out independent of each other, so that somehow the point moved rectilinearly along the cylinder's edge is moved from one end of the cylinder to the other in exactly the time it would take to complete a single rotation of the edge. Descartes repeatedly dismisses such constructions as ungeometrical, in the *Géométrie* and elsewhere (for instance, at AT I, p. 71; AT I, pp. 233–234; AT II, p. 517; AT VI, p. 390; AT X, p. 157; and AT X, p. 223). His case against the construction in Section 4.2 would be similar. Since it constructs the cylindrical helix in the course of constructing an Archimedean spiral, which is itself constructed by two independent motions, its helix construction would be judged ungeometrical as well.

Let us turn next to the string constructions of Section 4.4. In the letter Descartes also judges the helix ungeometrical because when it is traced by a string, this string can be used to square the circle, which is sufficient for the helix being ungeometrical (cf. Mancosu, 1996,

p. 78, and Mancosu, 2007, pp. 120–121).[28] Pappus showed (*Collectio* IV, §33) that the cylindrical helix can be used to construct the quadratrix; he also showed that the quadratrix can be used to square the circle (*Collectio* IV, §31–32; cf. Mancosu, 2007, p. 121). Aware of this, Descartes could confidently judge the cylindrical helix to be ungeometrical.

In the letter, Descartes only mentions strings while asserting that the helix is ungeometrical. This suggests that in describing the helix as traced by a string revolved obliquely around a cylinder, Descartes might simply be giving an informal illustration of how to think of the cylindrical helix: as in Vitruvius, take a string and wrap it obliquely around a cylinder in such a way that the string's angle to the cylinder's base remains constant. In favor of this reading is the fact that this intuitive way of thinking about the cylindrical helix, as generated by a string wrapped around a cylinder, is natural enough to be quite common in the literature. We find it, for instance, in a 1752 *Cours de mathématique* by Charles Étienne Louis Camus, who writes, "The screw is a right cylinder on the outside of which is dug a spiral, so that it has the shape of a spiraling cord twisted around a right cylinder...".[29] When the spiral constructed this way has a constant angle to the base of the cylinder, Camus calls it the "Filet de la Vis" (thread of the screw), corresponding to our ongoing talk of "threaded" screws.[30] In the 19th century intuitive descriptions of the helix mentioning the wrapping of strings are found, for instance, in texts by Hachette and Lacroix.[31] Hachette says that "a string bent freely on a cylinder forms a helix."[32] Similarly,

[28] By saying that the same string could be used to square the circle, Descartes could have had in mind the following procedure. Begin by flattening onto a plane the cylinder on which the cylindrical helix is drawn (the reverse of the procedure carried out in Section 4.3). Then the cylindrical helix corresponds to the hypotenuse of a right triangle, one of whose sides is the height of the cylinder. Taking the cylindrical helix as given, this triangle can be used to find the other side of the triangle, which is the circumference of the base of the cylinder. Finally, by Archimedes' strategy (in Proposition 1 of *Measurement of the Circle*), this triangle can be used to square the circle. There is reason to think Descartes was familiar with this procedure, for in a letter to Mersenne dated May 27, 1638, he writes, "You ask me if I think that a sphere which rotates on a plane describes a line equal to its circumference, to which I simply reply yes, according to one of the maxims I have written down, that is that whatever we conceive clearly and distinctly is true. For I conceive quite well that the same line can be sometimes straight and sometimes curved, like a string" (AT II, pp. 140–141; translated with commentary in Mancosu [2007, p. 118ff]). The suggestion in this 1638 letter is that one could take a string, wrap it around the circumference of a sphere and mark the point where it overlaps its beginning, and then unwrap the string and lay it straight. By this procedure the string can be used to rectify and then square the circle. Of course, we cannot exclude that Descartes might also have had in mind something more precise and that appealing to the "*filet*" was simply shorthand for a statement to the effect that given the cylindrical helix it is possible (perhaps passing through the quadratrix) to square the circle.

[29] "La vis est un cylindre droit creusé extérieurement en spirale, en sorte qu'elle a la figure d'un cordon spiralement entortillé autour d'un cylindre droit..." (Camus, 1752, p. 275).

[30] To ensure that the angle remains constant, Camus identifies the "filet de la vis" with the hypotenuse of a right triangle that is wrapped around a cylinder (pp. 275–276), thus identifying his string construction with the triangle wrapping constructions we discussed in Section 4.3.

[31] In addition to these, we find a similar construction in an 1841 treatise on carpentry by Émy (Émy, 1841, Vol. 2, p. 569). One interesting twist in Émy's construction is that he suggests dyeing the string when wrapping it around the cylinder, presumably in order to mark its path more clearly.

[32] "Un fil plié librement sur un cylindre, forme une hélice" (Hachette, 1822, p. 140). In (Hachette, 1811, pp. 80–89), Hachette refers to the threads of the Archimedean water screw (as in Vitruvius) as "les filets de la vis."

Lacroix says that “we call *helices* the curves formed by wrapping a straight line on a cylindrical surface”[33], and adds that

> To have an idea of this curve, it is enough to suppose that the string has a certain width, like that of a ribbon; then one will see that there is a manner of wrapping it around the [cylinder], without twisting it: the line specified by where the ribbon touches this surface forms precisely the curve that we have in mind.[34]

These examples make plausible, we submit, that Descartes' mention of string with respect to the helix in the letter is meant to illustrate how to think of the cylindrical helix, rather than to reflect the string's being a component of Gaudey's invention.

Finally, although Descartes does not mention this in the letter, he also would object to the triangle-wrapping constructions described in Section 4.3 because the hypotenuse of the triangle used to trace the helix begins straight, and after being wrapped around the cylinder, is curved. It is reasonable that Descartes would say that this line is like a string in that it is sometimes curved, sometimes straight, and hence that this construction is not geometrical (cf. AT VI, p. 412).

We have said nothing yet about Descartes' equational criterion of geometricity, because Descartes does not mention it in the letter (indeed, it is unclear that Descartes had a general equational criterion of geometricity in 1629). We will return to the equational criterion in a later section.

5.2. *Our second proposal*

Our first proposal, then, simply consists in reading the passage on the helix in Descartes' November 1629 letter to Mersenne as explicating Descartes' reasons for claiming that the cylindrical helix cannot be accepted as a geometrical curve. We now want to put forth a second, more speculative proposal that offers a possible conjecture as to what Gaudey's result might have been. While even in this case, the mention of the “*filet*” on Descartes' part is limited to providing a reason for not accepting the *helice* as geometrical, this interpretation has the advantage, in comparison to those of Bos and de Waard–Pintard, of accounting for the result by Gaudey by only appealing to the two elements that are *indisputably* part of the construction Mersenne sent to Descartes, namely the cylinder and the helix. According to this proposal, Gaudey would have found a way to obtain the multisection of angles in a way that improved upon Pappus' solution in the *Collectio*. Recall that Pappus provides two solutions for the multisection of an angle, one by means of the quadratrix and the other by means of the Archimedian spiral (these are found in Pappus [1933, Vol. I, pp. 222–223]). But Pappus also showed how to construct the quadratrix by means of a cylindrical helix (cf. Pappus, 1933, Vol. I, pp. 199–201). Thus it is immediate that one could solve the problem of the multisection of angles by starting with the cylindrical helix as

[33] “On appelle *hélices*, les courbes formées par l'enveloppement d'une ligne droite sur un surface cylindrique” (Lacroix, 1812, p. 96).

[34] “Pour se faire une idée de cette courbe, il n'y a qu'à supposer que le fil ait une certaine largeur, comme celle d'un ruban; alors on verra qu'il y a une maniére de l'envelopper autour de la surface proposee, sans le tordre: la ligne suivant laquelle le ruban touche cette surface, forme precisément la courbe que nous avons en vue” (Lacroix, 1812, p. 97). Evidently Lacroix envisions the constant angle of the resulting helix as resulting from the ribbon's not twisting as it is wrapped.

given, using that curve to construct the quadratrix, and then solving the problem through the latter curve.

We submit that Gaudey might have found a way to solve the multisection of angles directly without going through the quadratrix. The proof we offer was, as far as we know, first published by Vincenzo Viviani in a small treatise that appeared in 1674 (Viviani, 1674). Viviani's construction is the second construction offered in "Modi vari meccanici, lineari, e solidi per le construzioni de' due illustri problemi" (pp. 273–274).[35] It is a rather simple corollary of the definition of the main property of the cylindrical helix but provides a direct construction of the multisection of angles without a detour through other mechanical curves.

We now explain how the construction works. Consider a cylindrical helix generated on a cylinder. For simplicity of illustration we will focus only on the first half of the helix and consider a division of the angle of at most 180° in any given proportion.

Let the base of the cylinder be the circle with diameter AB with O denoting the center of the circle (as depicted in Fig. 7). Consider two points D, E onto the cylindrical helix and project them on the circumference AB. Let the projections be F and G, respectively. The cylindrical helix, in whichever way we might construct it, is defined by the following property:

$$AF : FD = AG : GE. \tag{1}$$

Indeed, notice that in Book IV of the *Collectio*, Pappus defines the cylindrical helix simply by appealing to this property. Only in Book VIII does he provide a construction by a method found in Heron that involves the superposition of a rectangular triangle on the cylinder, which we have described in Section 4.3.

In order to divide a given angle in a given ratio we now proceed as follows (as depicted in Fig. 8). Consider an arbitrary angle $A\widehat{O}G$ that needs to be divided in the ratio $a : b$, where a and b are given line segments.

Consider the point H on the cylindrical helix whose projection is G. Find on HG the point J such that GJ is the fourth proportional between a, b, and GH (Euclid VI.12). Draw a circle parallel to the base of the cylinder so that its circumference passes through J. This circumference will intersect the cylindrical helix in a point L. Now let the projection of L onto the base circumference of the cylinder be marked M.

By properties of the cylindrical helix we have

$$AM : ML = AG : GH. \tag{2}$$

Thus,

$$AM : AG = ML : GH. \tag{3}$$

[35] It is interesting to remark that in the first construction of the booklet (pp. 270–272), Viviani proves that one can divide the angle in any given proportion using a cylinder and an ellipse. However, this cannot be the construction Descartes had in mind for dividing the circle into 27 parts, as it is too powerful. Indeed, Viviani's construction will also give a division of the circle into 29 parts, etc. The reason why the construction is too powerful is that Viviani "unwraps" the surface delimited by the ellipse on the cylinder into a plane figure. We also mention that in the third construction Viviani also shows how to obtain the division of the angle into arbitrary parts using the Galilean cycloid (pp. 274–275).

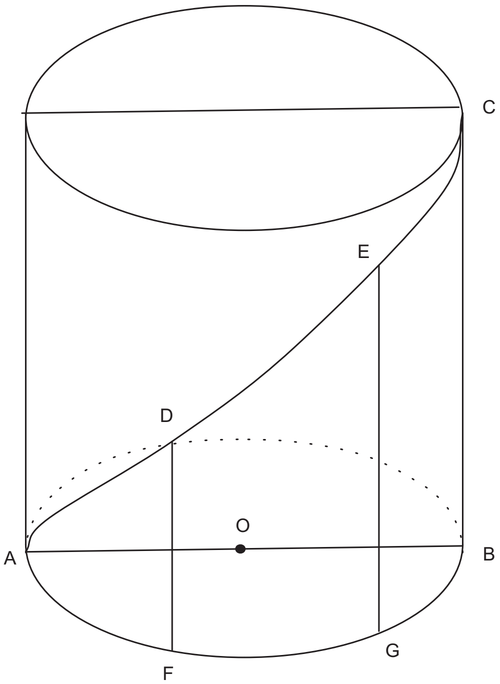

Fig. 7. Viviani's angle multisection by cylindrical helix.

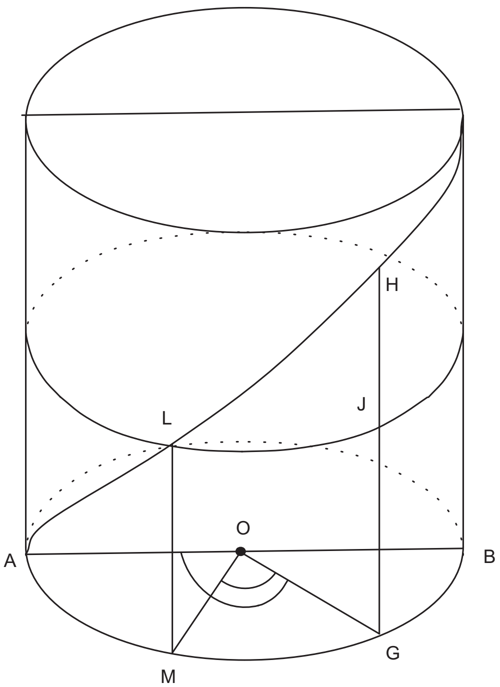

Fig. 8. Viviani's angle multisection by cylindrical helix.

By construction we have that $ML = GJ$. So

$$AM : AG = GJ : GH. \tag{4}$$

Hence $AM : AG = a : b$ since the ratio between GJ and GH is exactly as that of a to b.

And since AM and AG are the arcs subtending the angles $A\widehat{O}M$ and $A\widehat{O}G$ the angles are in the same ratio as the arcs and hence as $a : b$. QEF.

6. Questions and rejoinders

We would now like to consider a few questions regarding our account.

1. Does not the word "invention" as in "l'invention de Gaudey" carry with it the implication that an instrumentlike device was being discussed?

No. While in contemporary usage the word "invention" (both in English and French) might be more often (but not exclusively) used in connection with technological devices, this was not the case in the 17th century. First of all, "invention" can refer to a quality of the mind. Antoine Furetière (1619–1688), in his *Dictionnaire Universel* (Furetière, 1690, p. 1121), defines "invention" as "subtilité d'esprit, certain genie particulier qui donne la facilité de trover quelque chose de nouveaux." But "invention". according to Furetière, "se dit aussi de la chose même inventée." It is obviously this second meaning that applies to "l'invention de Gaudey." Focusing on the latter, a bit of care is required in each individual occurrence as "invention" suffers from the process/product ambiguity (like "explanation"). "Invention" can accordingly mean in certain contexts "determination" or "finding out" (as in "l'invention des tangentes") and in other contexts the thing determined or found (so the "invention" is something that can be the intentional correlate of an act of seeking).[36] But this second meaning in the 17th century is applied not only to technological inventions ("l'invention de la poudre à canon") and to mathematical devices/machines ("l'invention du compass de proportion") but also to constructions that do not appeal to such devices and in addition to methods, solutions of problems, statements, and theorems. Since all we need to establish is that in mathematics the word "invention" was at that time applied to entities other than the machines we discussed in reference to Besson and Huygens, here is an example—but many more could be added—in which a solution of a problem is referred to as "invention":

> Vous commencez par une invention de Monsieur de Roberval, touchant l'espace compris dans la ligne courbe que décrit un point de la circonference d'un cercle, qu'on imagine rouler sur un plan, à laquelle i'avoüe que ie n'ay cy-devant jamais pensé, & que la remarque en est assez belle.... (Descartes to Mersenne, 27 May 1638, AT II, p. 135)

And that the word could be used to refer to all kinds of mathematical "products" is confirmed by the following:

> Or ie vous diray que toutes les autres inventions, tant de M. de Fermat que de ses defenseurs, au moins celles dont i'ay ouy parler iusqu'à present, ne me semblent point d'autre nature. Il faut seulement avoir envie de les trouver & prendre la peine d'en faire le calcul, pour y devenir aussi sçavant qu'eux. (Descartes to Mersenne, 27 May 1638, AT II, p. 137)

"Inventions" includes here, among other things, Fermat's method of tangents, constructions of maxima and minima, and solution of quadrature problems.

[36] "Il y a quelque tems que le Professeur Schooten m'envoya un escrit, que le second fils de Mr de Zuylichem avoit fait, touchant une invention de Mathematique qu'il avoit cherchée; & encore qu'il n'y eust pas tout a fait trouvé son conte (ce qui n'estoit nullement estrange, pource qu'il avoit cherché une chose qui n'a iamais esté trouvée de personne)." (Descartes to Wilhem, 15 June 1646, AT IV, p. 436.)

2. In our first proposed interpretation of the November 1629 letter, Descartes argues that the *helice* is not geometrical, without appealing to his equational criterion of geometricity. Since the cylindrical helix is a spatial curve, traced on the surface of a three-dimensional solid, to apply this criterion he would need an understanding of equations for spatial curves. Without such an understanding, how could Descartes understand the geometrical/mechanical divide for spatial curves?

In response, we first point out that Descartes' other criteria for distinguishing geometrical from ungeometrical curves—construction by arbitrary points, by unregulated motions, or by strings that are sometimes like lines and sometimes like curves—apply straightforwardly to spatial curves (we showed how to apply these in Section 5.1). Thus, it is unproblematic to assume that he understood this. Therefore it would not undermine our interpretation if Descartes did not understand how to apply the equational criterion to spatial curves. But second, we observe that at the end of Book II of the *Géométrie*, Descartes suggests that everything he has said so far in the text, including presumably the algebraic work, could be extended to spatial curves. In particular, he outlines a strategy for projecting spatial curves orthogonally on two planes perpendicular to each other, where each of these planar projections would have equations like any other planar curve (AT VI, pp. 440–441). This is only a hint of how to proceed, but it's noteworthy that Clairaut, in a work noted by scholars as pioneering the development of analytic geometry for spatial curves, cites Descartes as the only person who had previously considered such a development.[37] Thus, we do not think that the issue raised by the second question is a problem for our position.

7. Conclusions

It seems to us that both our interpretations have advantages over the alternatives. To summarize, we find the de Waard–Pintard interpretation problematic on account of the fact that they are committed to interpreting the November 1629 letter as referring to a machine. However, no mention of any machine is found in the letter and, in any case, the machine they propose cannot account for the "*filet*" occurring in Descartes' letter. Bos's ingenious interpretation also seems deficient to us. First of all, it forces a reading of the letter as containing a description of a machine, whereas we do not see the text as forcing that conclusion. Moreover, Huygens's machine has the following drawbacks. First, the cylinder in this case is only a flat disk and paradoxically, there are too many of them, as Huygens's description refers to two small cylinders. Second, it is not clear how we should interpret Descartes' expression "*de biais*" [obliquely] when we apply it to Huygens's machine. Finally, to make sense of the reconstruction, Bos suggests, without evidence, that Huygens could have learned of this machine from Descartes. This assumes that Descartes knew of this machine (and presumably learnt of it from Mersenne's description of Gaudey's invention). But this is exactly the issue at stake, and so this argument strikes us as circular. Finally, both interpretations are committed to reading the reference to the geometricality of the "cylinder" as attributing geometricality to a component of a machine; we have argued that this is absurd.

By contrast, we believe that our interpretations make better sense of the text. In both interpretations, Descartes is simply giving an intuitive account of the construction of the cylindrical helix by strings. This account is so natural that we find it in several 18th- and

[37] Cf. (Clairaut, 1731, preface, first page). In Coolidge [1948], Coolidge discusses the origins of analytic geometry for spatial curves, including the roles of Descartes (p. 77) and Clairaut (pp. 82–84).

19th-century texts, and it is still used nowadays in many textbooks. At the same time, our second interpretation gives an account that requires no appeal to machines but solves the angle multisection problem using only the two elements that were indisputably part of Gaudey's construction, namely the cylinder and the cylindrical helix. While our second reconstruction is speculative, we find it superior to those of de Waard–Pintard and Bos, on account of the arguments we have given.

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