He published two books, *Determinants* (1892), and *Historical Introduction to Mathematical Literature* (1916). He wrote parts of two more, *The Algebraic Equation* in *Monographs on Topics of Modern Mathematics* (1911) and the first part of *Finite Groups* (1916) by Miller, Blichfeldt, and Dickson. He published some 820 papers in the educational, scientific, and mathematical journals of eleven countries. His non-technical papers included expositions of groups for non-specialists, expositions of mathematics for non-mathematicians, and studies on the history of mathematics.

His 450 (approximately) technical papers on groups constitute a permanent addition to the knowledge of finite groups. A knowledge of the substitution groups of low degrees and the abstract groups of low orders has a value in situations far removed from the obvious ones. He was the first to prove many of the things that every student of the subject uses. In some directions he carried his investigations far enough to show that lines of development which looked promising are not feasible.

G. A. Miller was a man of power which he directed to worthy ends; he hastened the development of his subject; and he added largely to the prestige of American mathematics.

THE STORY OF TANGENTS

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1. The Greeks. Whoever has given the least thought to the subject of plane curves has given some consideration to tangents. But what are tangents? To the uninstructed, a plane curve, not a straight line, is the path traced by a moving point whose motion changes its direction at each instant. Had the point at any instant decided not to change the direction of motion, the line through that point in the direction of instantaneous motion would be the tangent. All of this is perfectly clear to any observant and unsophisticated person who has not endeavored to go below the surface, it is only when some one suggests awkward questions like "What do you mean by direction?" "What is meant by instantaneous motion?" that difficulties begin to appear.

Perhaps a beginner starts with a static approach. Here is an arc of a curve. We take a point on the arc and draw through it a line which meets the curve there and nowhere else in the vicinity. This line we may call the tangent and prove, if we can, that it is unique. In the case of the circle it is perpendicular to the radius to that point. But suppose we start with the conic sections and ask whether in the case of a hyperbola a line parallel to an asymptote, which certainly does not meet the curve elsewhere, is really a tangent. There is clearly no suggestion of touching in such a case. It is clear that we can not be too naïve in answering such questions, it is the purpose of the present article to tell the story of how men have attempted to answer them [1].

I can not make out that before Euclid anyone was much concerned with tangents. I turn therefore to Euclid Book III, Definition I [2].

A straight line is said to touch a circle which meeting the circle and, being produced, does not cut the circle.

I am no Greek scholar so I follow what Heath says on the following page of his commentary. He draws a distinction between ἁπτεσθαι which means to meet, and ἐφάπτεσθαι which means to touch. He suggests that this distinction was common with the Greeks, although he cites exceptions one way or the other, one being in the work of Archimedes, though the exact reference to the passage is not given. In the work of Archimedes on spirals, to which I shall presently return, the word used is επιψαυη which Heath also translates "touch" [3]. I think there is no difficulty in seeing what Euclid means. He expands the idea further, for in Euclid III (16) we have

"The straight line drawn at right angles to the diameter of a circle from its extremity will fall outside the circle, and into the space between the straight line and the circumference another straight line can not be interposed." The tangent meets the circle once only and lies wholly outside, and no other line through the point of contact has this property.

When we come to the conic sections, we learn from Apollonius I, (17) and (32) *"If a straight line be drawn through the extremity of the diameter of any conic parallel to the ordinates to that diameter, the straight line will touch the conic, and no other straight line can fall between it and the conic."* This theorem can not, however, have been original with Apollonius, for he speaks of his first four books as having been worked out by his predecessors. Archimedes in the first proposition on the quadrature of the parabola remarks, "These propositions are proved in the elements of the conics," referring, probably, to the works of Euclid and Aristaeus. I think it safe to say that Euclid knew the Apollonian theorems which I have given.

It is time to turn to Archimedes, especially to his work on spirals. An Archimedian spiral is the curve traced by a point advancing uniformly on a line which turns uniformly about a fixed point. Here we run across the very curious fact that although he speaks frequently of tangents to spirals, he does not define them. I think it is evident that he means by a tangent a straight line which meets the spiral at a point but does not cross it there; a tangent meets the curve at one point but nowhere else in the vicinity. He could doubtless have proved that no other line lies between the tangent and the curve.

Archimedes shows next that as the radius vector and angle both increase uniformly, presumably with the time, if a series of angles form an arithmetical progression, the same is true of the corresponding radii vectores. If thus we take the radii vectores to two points of the same turn of the spiral, the radius vector which bisects the angle between them is equal in length to one half their sum.

This leads to the rather curious Proposition 13, *If a straight line touch a spiral, it will touch it once only*. This means at only one point of that turn, as use is made of the preceding theorem about arithmetical series. Let O be the origin, and suppose a certain tangent touches the curve at both P and Q. Let OR be the radius vector which bisects the angle. Then $OP+OQ=2OR$. But we can show, though Archimedes does not do so, the distance to PQ along the bisector is less than this, so there is a point on PQ inside the curve, but this is contrary to the hypothesis that the line PQ was a tangent. As a matter of fact the line PQ meets the curve an infinite number of times, so there are plenty of points on both sides of it but not in the POQ sector. The whole treatment seems to me much looser than we have a right to expect from this wonderful geometer.

Archimedes' great accomplishment is to give an actual construction for the tangent. This is done by giving the polar sub-tangent, that is to say, the distance along the perpendicular at the origin to the radius vector from the origin to the tangent, the distance which we should write $r^2d\theta/dr$. Here he uses the method of exhaustion and the method of verging, which consists in placing a line segment of given length with its ends on given curves while its line passes through given points. His fundamental theorem is that if P be a point of the nth turn and $OP=r$, the polar sub-tangent will be $a=2\pi(n-1)r+\text{arc } AP$ where A is the starting point. The proof I find difficult to follow. It consists in showing that if the theorem were not true we could find a point outside the spiral which should be analytically inside. I can not see where the actual tangency comes in, the proof merely shows that in any other case an impossible construction would result. An analytical expansion of the method will be found in [4].

How did Archimedes discover this result? We do not know, but we can make a shrewd guess. In his *Method*, Archimedes shows how he was first led to theorems about areas and volumes, which he proved rigorously by the method of exhaustion, by cutting his figures into slices and then comparing their turning moments in different positions. We have here the concept of infinitely thin slices. Is it not likely that he also had the idea of a tangent as the line of an infinitely short chord, a perfectly familar concept to the mathematicians of the seventeenth century. If, then, a be the polar sub-tangent corresponding to the infinitely small advances dr, $rd\theta$, we have by similar triangles

$$\frac{a}{r}=\frac{rd\theta}{dr},$$

but in the case of the spiral of Archimedes

$$r = k\theta, \qquad a = r\theta$$

and this is Archimedes rule.

2. Fermat and Descartes. The idea of a tangent as the limiting position of a secant when two intersections with the curve tend to fall together was slow in attaining mathematical acceptance. The fact is that the idea of a limiting position of any sort came to birth slowly and painfully. But the idea of two intersections falling together was well understood before the middle of the seventeenth century. I think that here we should lay great stress on the work of Fermat, who certainly had it clearly in mind even though certain modern writers are very insistent that he did not understand at all the limit idea.

The ideas of Fermat were first set forth in 1629 in a letter to a certain M. Despagnet [5]. The method of finding tangents seems to have been a bi-product of his method of finding maxima and minima. Whether this is really an anticipation of the method of differentiation is disputable, an elaborate discussion by Wieleitner is found in [6]. The best explanation of what is found occurs much later in a letter of 1643 to Brulart de St. Martin [7].

Suppose that we seek a maximum or a minimum for a function f. This will appear on the graph as the top or bottom of an arch at a specific point A. The functions $f(A+E)$ and $f(A-E)$ will both be greater than or less than $f(A)$. Suppose, then, that we write

$$f(A \pm E) = f(A) \pm Ef'(A) + \frac{E^2}{2} f''(A) + \cdots$$

The function f is supposed to be a polynomial, and so the higher powers of E may be neglected. We have then

$$f(A + E) = f(A - E); \qquad f'(A) = 0.$$

The particular example that he takes is $f(A) \equiv BA^2 - A^3$. A similar process is applied to the problem of finding the tangent to a parabola; finding the tangent does not mean finding its equation but finding the sub-tangent [8]. He ends the article with the claim "Nec unquam fallit methodus." A somewhat different explanation is found on page 147 of Vol. I of [5], and Wieleitner in [6] makes much of the difference; neither account strikes me as particularly convincing.

Fermat makes what I think is a much better use of his method on page 158 ff. of Vol. I of [5]. Let the equation of the curve be $F(x, y) = 0$. We seek the subtangent at the point (x, y). A very near point shall be $(x+e)$: The ordinate up to the tangent is found by similar triangles to be $y(1+e/a)$ and this we treat as if it were also on the curve, so that

$$F(x, y) = F\left(x + e, y\left(1 + \frac{e}{a}\right)\right) = 0.$$

He takes the "cissoid" of Diocles and the "conchoid" of Nicomedes, finding a by this limiting process. He points out, incidentally, that to find an inflexion, we must find a maximum or minimum of the angle which a tangent makes with a given direction, and this means finding a maximum or a minimum of its cotangent, and that means to maximize or minimize a/y, where a is the subtangent.

A good example of Fermat's method is found in the case of the "folium" of Descartes, as here we do not have y as an explicit function of x:

$$x^3 + y^3 = nxy,$$

$$(x + e)^3 + y^3\left(1 + \frac{e}{a}\right)^3 - ny(x + e)\left(1 + \frac{e}{a}\right) = 0,$$

$$e\left(3x^2 + \frac{3y^3}{a} - \frac{nxy}{a} - ny\right) + e^2\left(3x + \frac{3y^3}{a^2} - \frac{ny}{a}\right) + e^3\left(1 + \frac{y}{a^2}\right) = 0.$$

This holds for all values of e. We therefore divide e out, and for the tangent assume $e = 0$, then

$$a = -\frac{3y^3 - nxy}{3x^2 - ny}\cdot$$

I will point out that this amounts to putting $a = -yF_y/F_x$ but this general formula did not appear before the work of de Sluse, which we shall see later.

An interesting example is Fermat's oval, which may be written

$$y = K\sqrt{\cos\frac{x}{b}}\cdot$$

Here we introduce an auxiliary circle of radius b, and x becomes the length of an arc on this circle. The process of assuming the identity of two near points is used with regard to this, rather than the original curve. Immediately following the foregoing are examples of finding the tangent to one curve from the tangent to another, a process also followed by Barrow. Corresponding abscissas are equal, and the ordinate to one equals the arc-length to the other. He shows later that the slope of one is equal to the secant of the slope-angle of the other. He solves the problem in the case of the parabola twice, once by his own methods, once by what he calls the "Ancients" method where a tangent is defined as a line meeting a curve but once in a certain region. A preliminary theorem tells us that if we take a point of an arc whose tangent turns continually in one way, and proceed to a nearby ordinate, the distance along the tangent is less than that along the curve, if this be the direction of decreasing ordinates, but greater if it be the direction of increasing ordinates. The proof consists in applying the principle that an arc is longer than its chord, but less than the sum of two tangents from an external point to its ends. This is preliminary to an elaborate study of the lengths of curves.

I can not leave Fermat without expressing admiration for his method, which is essentially that of the infinitesimal calculus, even if he did not see all that was involved. He had, I think, a better grasp of the essential principles than his contemporaries, and was certainly early, perhaps the earliest, in the field.

It is time to pass to Fermat's great rival in this matter, René Descartes. He first attacked the problem of tangents in 1637, which was later than Fermat's letter to Despagnet, but before he had heard directly on the subject from Fermat. It is very curious that Descartes began by seeking to draw a normal to a curve, that is to say, a perpendicular to a tangent at its point of contact [10]. Which straight line cuts a curve at right angles at a given point, or, as he says, cuts the "contingent" at right angles? If (x, y) be the given point, and if $(x, 0)$ be where the normal meets the X-axis, it is the center of a circle, two of whose intersections with the curve fall together. Thus, if we write,

$$(X - x_1)^2 + Y^2 = (x - x_1)^2 + y^2, \qquad f(X, Y) = 0,$$

and eliminate Y, a necessary condition for a normal is that two of the roots of this equation in X should fall together. If we take the case of the parabola,

$$Y^2 = 2mX,$$
$$(X - x_1)^2 + 2mX = (x - x_1)^2 + 2mx,$$
$$X^2 + 2X(m - x_1) = x^2 + 2x(m - x_1).$$

The two roots will be equal if $x_1 = m + x$.

In the examples which Descartes worked out, he did not have available a general method of handling the case of equal roots. He wrote

$$\Phi(X) \equiv (x - x_1)^2(c_0X^m + c_1X^{m-1} + \cdots + c_m)$$

and identified the coefficients on the two sides. The general rule was first worked out by Hudde in 1683 [11].

In 1638, when Descartes first heard of Fermat's method for maxima and minima, he was not a little stirred up, and expressed his disgust in letters to Mersenne, Hardy and others. He disliked especially the quantity e, which was divided out, because it was not equal to 0, and then equated to 0. He attacked Fermat's method of tangents as though it involved considerations of maxima and minima. In this Descartes was wrong. Fermat did not say that finding a tangent was a maximum-minimum problem, but that the methods developed for one case, also fitted the other, and this Descartes finally saw. He put his own method in a letter to Hardy of June 1638 [12].

Let

$$y^3 = lx.$$

Let us find two points (x, y), (x_1, y_1) so that $y_1 = ky$.

$$x_1 = x + e, \qquad y_1 = y\left(1 + \frac{e}{a}\right),$$

$$\frac{y^3}{x} = \frac{y^3\left(1 + \frac{e}{a}\right)^3}{x + e} = \frac{k^3y^3}{x + e},$$

$$a^3 = 3a^2x + 3aex + e^2x.$$

But when $k=1$, $e=0$ and $a=3x$; this does not seem to me essentially different from Fermat's method, even in the reasoning. Nor does it seem to justify the long correspondence involving Descartes, Fermat, Mersenne, Hardy and others. Descartes had, however, a third method, applied at least to the cycloid. The problem of finding a tangent to this curve had occupied various geometers. Fermat had solved it, rather clumsily. Roberval had found a solution which I shall return to later, but Descartes gave an absurdly simple solution which is as follows. Suppose [13] that a polygon rolls along a straight line, a chosen vertex will trace a succession of circular arcs whose centers are successive vertices which lie on the line. The line from the tracing vertex to the corresponding fixed vertex on the line will be perpendicular to the tangent to the arc. Now, if we consider a circle, as presently became popular, as a regular polygon of an infinite number of sides, we see that the instantaneous center is the point of contact with the given line, and the line thence to the tracing point is the normal to the cycloid. The tangent will thus always go from the point of contact to the highest point of the rolling curve.

Descartes puts the matter somewhat differently. He draws through the point of contact a line parallel to the fixed line to meet the rolling circle when it has rolled one half the distance, or when the tracing point has reached its highest position. The line from the intersection to the point of contact of the new circle is parallel to the normal sought. For any rolling curve, the tangent is perpendicular to the line from the point of contact to the instantaneous center.

3. Roberval and Torricelli. In discussing the work of Fermat and Descartes, we have been to a certain degree running around in circles, explaining more or less similar methods of handling infinitesimals. It was the era in which the infinitesimal calculus was struggling towards birth. Let us take a vacation from this sport by looking at a totally different approach to the subject of tangents by two more or less rival geometers whose relative merits I shall certainly not try to evaluate. There is an elaborate weighing of their comparative worth in a recent work by Evelyn Walker [22]. I, therefore, take up first Giles Persone de Roberval. He seems to have first been occupied with the classical question of the tangent to the cycloid, a curve called to his attention by Mersenne, but the best exposition of his work with tangents is found in [23].

Roberval's idea was simplicity itself. A curve is traced by a moving point; the tangent anywhere is the line of instantaneous motion of that point. The real philosophical difficulty, to define what is meant by instantaneous motion, was veiled in the future, to bedevil those of his successors who occupied themselves with the foundations of the calculus. To the unspoiled eye of common sense there was no difficulty. Here is the Axiome or principe d'invention that he gives on pp. 24 and 25c of [23]. *La direction du mouvement d'un point qui décrit une ligne courbe, est la touchante de la ligne courbe en chaque position de ce point là.* This seems to be more or less tautologous, a tangent is a line which touches, but he goes on at once to explain himself. *Par les propriëtez spécifiques, (qui vous seront* *données) examinez les divers mouvements qu'a le point à l'endroit ou vous voulez mener la touchante; de tous ces mouvements composez en un seul et tirez la ligne de direction du mouvement composé, vous aurez la touchante de la courbe.*

The meaning is this. Determine two measurements which connect the moving point with two fixed elements. Determine the vector velocities of the changes of these two. Their vector sum will give the instantaneous velocity. Gomes de Carvalho points out on page 53 of [1] that Roberval is rather cavalier in his reasoning about infinitesimal triangles; the parallelogram whose sides are dx/dt, dy/dt is not the same as that whose sides are dr/dt, $rd\theta/dt$. However, in each case, the diagonal gives the direction of instantaneous motion.

Let us take some examples of Roberval's method. First, we take the parabola. The two motions are away from the focus and away from the directrix. As these two distances are always equal, the instantaneous velocities are equal. Hence the tangent makes equal angles with the axis on the focal radius. He shows carefully that this is the result given by Apollonius.

The central conics are handled in analogous fashion. We have distances from the two foci whose sum or difference is constant. Hence the differences or sums of the instantaneous velocities are constant and so the tangent makes equal angles with the focal radii, or with one radius and the other produced.

Roberval next considers the family of conchoids. Take the conchoid of Nicomedes. Lines radiate from a fixed origin to meet a directrix, a fixed line not through the origin. Each radiating line is produced a fixed distance beyond the directrix. The two motions are a radial one away from the origin and a circular one about the origin. The distance out from the directrix to the curve is independent of the choice of radius vector. If the chosen angle θ gives us ρ for the directrix, and r for the curve, the corresponding rotational velocities are $rd\theta/dt$ and $\rho d\theta/dt$; the stretching velocities are dr/dt, $d\rho/dt$. Hence the tangent of the angle which the tangent to the conchoid makes with the radius vector is ρ/r times the tangent of the angle made with the directrix. The other conchoids come from another choice of directrices.

Roberval gives a simple enough construction for the tangent to the spiral if we assume, as does the Master, that we can draw a straight line equal in length to the circumference of a given circle. He becomes, however, rather deeply involved when he comes to finding the tangents to the quadratrix, or the cissoid. I confess that his analysis of the infinitesimal motions is not convincing to me. He has much better success when he comes to the cycloid. He even allows his rolling wheel to slip a bit, so that the length of the track covered in one complete turn is not necessarily equal to the circumference of the wheel. Assuming that both the sliding and turning motions are uniform, we have merely to draw, through a point on the curve, vectors parallel to the track and tangent to the wheel proportional to the distance slid and the distance turned and find their sum. In reading [23] it is easy to forget that he allows for slipping and one wonders why he does not give the simpler construction of Descartes.

Anything one says about Roberval brings to mind the name of the rival inventor of the method of determining tangents by means of instantaneous velocities, Evangelista Torricelli. In 1644, he published his *Opera Geometrica* where, in the second section of Part 1 entitled "De motu gravium," we find expressed the technique which I will now describe. It will be found in [24]. He starts with some propositions of Galileo about falling bodies. Suppose that we start with a weighted point that falls a certain distance, then is shot off at right angles and thereafter is also allowed to fall naturally. The path will then be a parabola, for the distance slid will be proportional to the time, and the distance fallen to the square of the time. The falling velocity will also be proportional to the time, and, consequently, to the distance slid. If the parabola be $x^2=2my$ and we take the sliding velocity as the constant x, the ratio of dropping to sliding velocity, which will give the direction of the tangent, will be x/m.

We thus get Torricelli's construction for the tangent. We connect the point of contact with the reflection in the vertex of the foot of the ordinate. He adds on page 123, "Haec demonstratio peculiaris est pro parabola, sed universalem habemus pro qualibet sectione conica, consideratis aequalibus velocitatibus unius puncti, quod aequaliter movetur in utraque linea quae ex focis procedit." This is certainly very suggestive of Roberval's procedure for the central conics. He goes on to state, "Eadem ratione Demonstratur Propositio 18 de lineis spiralibus Archimedis unica brevique demonstratione, . . . Immo et hac ratione etiam unico Theoremata tangentes quarundam curvarum, inter quas, omnium linearum Cycloidalum."

Torricelli claims that he has a general method applicable not only to the parabola, but the central conics, the Archimedian spiral, and all cycloids. What is his general method? Presumably it is the composition of velocities, but he carries it out only in the one case. I have an unpleasant feeling that those who have written on the subject have not bothered to think the matter through. Jacobi writes in [25], on page 268, a direct transcript of the original with the statement "progressivi impetus ad lateralem ratio ut *ad* ad *bf* per praecedentem Propositionem," and that's all. Walker [22], page 138, states "The ratio of the progressive impetus to the lateral is as $AD{:}BF$. These are the ordinates of the given point and the focus, with no hint of where he gets this important fact. "Impetus" does not mean acceleration, but instantaneous velocity.

There has arisen a good deal of discussion among historians of mathematics as to which of the geometers, Roberval or Torricelli, first thought of determining tangents by instantaneous velocities. There is an elaborate discussion of the subject with dates in [22], [23], and [24]. I will not go further into the matter, but I should like to insist on the originality of the method. It is a great step forward, to pass from considering a tangent as a line meeting a curve but once, at least in a small region, to that of treating it as the limit of a secant whose intersections fall together. It was equally bold to consider it as the line of instantaneous advance.

4. DeSluse and Barrow. I return to the general line originated by Fermat and Descartes. An admirable extension of this was first put into words by

René François Walter, Baron deSluse. Suppose we have a curve whose equation is $f(x, y)=0$, the function being a polynomial. Reject all constant terms. Let all terms in y be placed on the right with signs changed, and let each term be multiplied by the exponent of y. Let each term in x be placed on the left, and multiplied by the exponent of x, and let one x in each term be replaced by a. If a term involve both x and y, it should appear with the proper sign on both sides and handled appropriately. We then solve for a, the subtangent [14] this will give

$$a = -\frac{yf_y}{f_x}.$$

The question arises at once, where did deSluse get this method? One's first idea is to credit it to Newton who had essentially the same technique, but I think the discovery must have been independent. LePaige, in the carefully written article [15], says that in 1652 deSluse had some sort of a method of drawing tangents. We find him in 1658 writing to Pascal [16], after explaining his method of drawing tangents to certain curves called "perles," "Cette méthode est tirée d'une plus universelle laquelle comprend toutes les ellipsoides, mesme avec peu de changement, les paraboloides et les hyperboloides." Newton did not begin to think about the method of fluxions before 1665. He published nothing on the subject before his paper of 1669, dictated to Collins, "De analysi per aequationes numero terminorum imfinitos." In 1673, he finally gave priority to deSluse [17].

But how did deSluse happen to hit on this technique? LePaige suggests that it comes from the formula for expanding $(x^2-y^2)/(x-y)$. I am afraid I do not see the connection. It seems to me more likely that he reflected on the work of Fermat, and noted the relation of exponents and coefficients in simple differentiation, then stepped from this to partial differentiation. It may be significant that deSluse, like Fermat, used a for the subtangent in [14], though he used η and ω, where we should use x and y.

In connection with deSluse, it is necessary to say something about Barrow. He gives deSluse' method [14] in his [18], stating, modestly enough, that he gives his method at the request of a friend (presumably Newton), "Though I scarcely perceive the use of doing so, considering the several methods of this Nature now in use." His rule, as stated, means very little, but, worked out, it amounts to that of deSluse. In his first example, he commits what seems to me to be the most heinous possible mathematical sin. He uses the same letter to mean two different things in the same problem, writing the equation

$$qq - 2qe + mm - 2ma = BLq,$$

where the right side means BL^2.

Barrow's most interesting example is his fifth where $y=\tan x$. He takes an auxiliary circle of radius r, and finds the point with Cartesian coordinates

($r \cos x$, $r \sin x$). He then gives x a small increment dx and this he puts equal to $\frac{1}{2} \sin 2\, dx$. If then we compare an infinitesimal triangle with a finite one, we get the fundamental formula

$$d \cos x = - \sin x dx.$$

He finally reaches

$$\frac{dy}{dx} = 1 + y^2.$$

J. M. Child, whose admiration for Barrow seems to me a bit excessive, says in [**19**] "If $y = \tan x$, $dy/dx = \sec^2 x$." He must have known (for it is in itself evident) that the same two diagrams can be used for any of the trigonometric ratios. Therefore, Barrow must be credited with the differentiation of the circular functions. This is possible, but not at all certain. Eight years later James Bernoulli gave, as known, the formulae for the derivatives of $\tan x$ and $\sec x$. The credit for differentiating all six functions is usually assigned to Roger Cotes and his *Harmonia Mensurarum* of 1722. Child even assigns to Barrow the credit for inventing the process of differentiation.

I should mention, in connection with Barrow, two other theorems. The first is in his fourth lecture, and consists in a proof of Fermat's sub-tangent formula $a/y = dx/dy$. His proof, like the rest of his work, I find obscurely written. He works, not with infinitesimals, but with velocities. Next, let us take a convex arc and a point M where the slope of the tangent is l. An ordinate shall move at a constant rate, cutting the curve at O and the tangent at K. When K and O are above G, where the moving ordinate cuts the horizontal line through M, if O is further from this horizontal line than is K, it must go further in the same time than does K. Its dropping velocity $dy/dt > l\, dx/dt$, where dx/dt is the constant velocity of the ordinate. But, when the dropping velocity on the tangent is less than that on the curve, we have $dy/dt < l(dx/dt)$. Barrow concludes that at M, $dy/dt = l(dx/dt)$. I have naturally shortened the proof by using modern notation. Fermat's formula will come at once from this. I should mention that on page 61 of [**19**] we have the fundamental statement: *If the arc MN is assumed indefinitely small we may safely substitute instead of it a small bit of the tangent.* This is, of course, the basis of Fermat's process of "adequation" and, in fact, of most seventeenth century work with tangents, except that of the school of Roberval and Torricelli.

5. Newton and Leibniz. We have seen that Newton's method of drawing tangents was the same as that of deSluse. He probably discovered it in 1664 or 1665, when he was first thinking through the methods of the infinitesimal calculus. He published nothing before 1669, and then only in a letter. A complete discussion giving nine different examples appeared under Problem IV of [**26**]. He begins like deSluse. If we take two near points of a curve, and connect them by a straight line, which we treat as if it were the tangent, we have two similar triangles. The one is bounded by the tangent, the sub-tangent and the ordinate; the other by the element of arc and what he calls the "moments" of the two coordinates, which are in the language of Leibniz the differences, and are proportional to their "fluxions." He would write Fermat's first formula

$$\frac{e}{y} = \frac{\dot{x}}{\dot{y}}.$$

The first example is

$$x^3 - ax^2y + axy - y^3 = 0,$$

$$3\dot{x}x^2 - 2a\dot{x}x + a\dot{x}y + ax\dot{y} - 3\dot{y}y^2 = 0,$$

$$l = \frac{3y^3 - axy}{3x^2 - 2ax + ay}.$$

He gives other examples, sometimes introducing other variables, but nothing essentially different. Some of his methods show the influence of Roberval, or a like-thinking geometer. In the third memoir, he assumes that a point is known by r_1 and r_2, its distances from two fixed points. These are connected by a known equation, from which we find $\dot{r}_1/\dot{r}_2$. We draw a perpendicular to the second radius vector at its origin, and extend it till it meets the tangent, and thence drop a perpendicular on the first radius vector. If the distance from this last intersection to the point of contact be S, we have, by similar trapezoids, $s/r_2 = \dot{r}_1/\dot{r}_1$. Hence s is known, and we can find a point on the tangent. In the same way, he handles tangents given in polar coordinates, finding the polar sub-tangent.

When it comes to Leibniz, I move with extreme caution, not wishing to express any opinion on the Newton-Leibniz priority question. Newton wrote in a letter to Oldenburg, October 24, 1676, intended for Leibniz, a long description of his methods, including a reference to deSluse's method of finding tangents. Leibniz answered [27] at once, maintaining that deSluse's methods are not in themselves sufficient, and that he himself had discovered a method. He adds, "Clarissimi Slusii Methodum Tangentium nodum esse absolutam celeberrimo Newtono assentior. Et jam a multo tempore rem Tangentium longe generalius tractavi, scilicet per differentia Ordinaturum."

The essential part is found here in the words, "Longe generalius tractavi." He says that for a long time he had treated tangents by the differences of ordinates, the differences of abscissas were treated as constants. He gives various examples. If we disregard higher infinitesimals, we have

$$dy^2 = 2ydy, \qquad dy^2x = 2xydy + y^2dx,$$

given

$$a + by + cx + dyx + ey^2 + fx^2 + \cdots = 0,$$

$$-\frac{dy}{dx}=\frac{c+dy+2fx+\cdots}{b+dx+2ey+\cdots}.$$

"Quod coincidit cum Regular Slussiana." He writes as a universally accepted principle; "Si sit aliqua potentia aut radix x^z erit $dx^z=zx^{z-1}dx$.

To find $d\sqrt[3]{a+by+cy^2+\cdots}$ he puts

$$z=1/3; \qquad x=a+by+cy^2+\cdots$$

$$dx^z=\frac{dx}{3x^{2/3}}=\frac{bdy+2cydy+\cdots}{3(a+by+cy^2+\cdots)^{2/3}}.$$

Newton had written to Collins that he did not wish to show his method of tangents to Leibniz. The letter ended with the following, "Arbitrior quae celare voluit Newtonus de Tangentibus ducendis, ab his non abludere." I see no point in going further into the famous Newton-Leibniz priority controversy.

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A NEW ABSOLUTE GEOMETRIC CONSTANT?

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In Whitworth, *Choice and Chance*, appears the following problem: If there be n straight lines in one plane, no three of which meet in a point, the number of groups of n of their points of intersection, in each of which no three points lie in one of the straight lines, is $\frac{1}{2}(n-1)!$

We shall assume either that no two lines are parallel, or that the problem refers to the extended plane. To prove that there are just two points on each line, note that since each of the n points lies on two lines, there are $2n$ incidences in the set, which is an average of 2 incidences per line. But since no line may contain as many as 3 incidences, each must contain just 2.

The problem may now be regarded as requiring the number of polygons of n sides formed by a plane network of n lines, no three concurrent. Whitworth's answer is correct only if no polygon is allowed to consist of two or more polygons of fewer sides, *e.g.*, a composite polygon consisting of two polygons of k and $n-k$ sides, respectively. There is, however, nothing in his statement of the problem to rule out such cases.

Let g_n be the number of n-gons formed by a network of n lines. We shall show that g_n becomes infinite like $\sqrt{n}(n-1)!$, which will correct Whitworth's answer. To be more precise, we shall show that g_n becomes infinite like $n^n e^{-n}$, in other words, that

$$\lim_{n\to\infty} \frac{g_n}{n^n e^{-n}} = B,$$

where B is a constant.

Let us proceed by induction. Suppose we raise the number of lines from $(n-1)$ to n by adding one line to $(n-1)$ already given. Each $(n-1)$-gon can be converted into an n-gon by the following operation: Replace any one vertex by the two points where the two sides through it meet the new line. Conversely,