The Story of the Binomial Theorem
Author(s): J. L. Coolidge
Source: *The American Mathematical Monthly*, Vol. 56, No. 3 (Mar., 1949), pp. 147-157
Published by: Mathematical Association of America
Stable URL: http://www.jstor.org/stable/2305028
Accessed: 16-03-2017 22:49 UTC

THE STORY OF THE BINOMIAL THEOREM

J. L. COOLIDGE, Harvard University

1. The early period. The Binomial Theorem, familiar at least in its elementary aspects to every student of algebra, has a long and reasonably plain history. Most people associate it vaguely in their minds with the name of Newton; he either invented it or it was carved on his tomb. In some way or other it was his theorem. Well, as a matter of fact it wasn't, although his work did mark an important advance in the general theory.

We find the first trace of the Binomial Theorem in Euclid II, 4, "If a straight line be cut at random, the square on the whole is equal to the squares on the segments and twice the rectangle of the segments." If the segments are a and b this means in algebraic language

$$(a+b)^2 = a^2 + b^2 + 2ab. \tag{1}$$

The corresponding formula for the square of a difference is found in Euclid II, 7, "If a straight line be cut at random, the square on the whole and that on one of the segments both together, are equal to twice the rectangle contained by the whole and said segment, and the square on the remaining segment."

Here if a represents the whole, and b the first segment, we have

$$a^2 + b^2 = 2ab + (a-b)^2 \tag{2}$$

It would have been perfectly easy for Euclid to go ahead and prove the formula for the cube of a binomial, but that would have broken the thread of the argument. In Books II and X he was prodigiously interested in the squares of binomials, any generalization of these does not seem to have interested him at all. The modern tendency to generalize as far as possible, and stretch each theorem to its most general form, was quite foreign to the thinking of the Greeks in mathematics; clearness and precision were the sovereign qualities which were always sought.

We find a wider mathematical curiosity in Diophantus who cubed various binomials, especially $(n-1)$. Whether he had a general formula, or multiplied out each time is not clear.

It is a curious fact that the first use, beyond Euclid's, for finding binomial power formulae, was to discover the approximate values of roots. We have a significant remark in the commentary of Eutocius on Archimedes' essay on the measurement of the circle:

"Quo modo adpropinquando radix quadratadati numeri invenienda est, dictum est ab Herone in Metricis a Pappo, Theone, compluribus aliis, qui magnum Syntaxin Claudii Ptolemi interpretati sunt" [1].

This suggests a search in Ptolemy's Syntaxis. I have failed to find the passage. Tannery assures us that Pappus followed the general method of Hero of Alexandria [2]. Hero's method is simplicity itself. If we wish to find an approximation to $\sqrt{A}$, and a_1 is a first value, a closer one will be

$$a_2 = \frac{1}{2}\left[a_1 + \frac{A}{a_1}\right]. \tag{3}$$

As a matter of fact, this is merely a special case of a famous method of approximating to a simple root of any function, which we associate with the name of Newton, for if a_1 is an approximation to a root of $f(x) = x^2 - A$ a better approximation is

$$a_2 = a_1 - \frac{f(a_1)}{f'(a_1)}\cdot$$

We find something much closer to our familiar method of finding square roots in the work of Theon of Alexandria, who uses our technique of adding to a_1 our correction $(A - a^2)/2a_1$ [**3**]. Of course it is a question merely of order of procedure, for if we add this correction we have Hero's Formula (3).

We pass to cube roots. Heath says on p. 63, "In no extant Greek writer do we find any description of the method of finding cube roots." If we date Theon about A.D. 390 we have to wait more than 100 years for the Hindu Aryabhata; there are various translations of his Aryabhatiya; I follow that of Datta and Singh, p. 174 [**4**]:

"Divide the second aghana place by thrice the square of the cube root; subtract from the first aghana place the square of the quotient multiplied by thrice the preceding (cube root) and (subtract) the cube (of the quotient) from the aghana place. The quotient put down at the next place (in the line of the root) gives the root."

I think that this shows clearly enough that Aryabhata was familiar with the binomial formula for a cube. Whether the Hindus had the curiosity to raise binomials to higher powers or not I can not say; a power higher than the third may have appeared to them practically useless. Yet someone must have seen the importance of such matters as may be judged from the following quotation, which is highly significant for the whole purpose of this paper. The writer is Omar Khayyam, and in speaking of a work of his own, now most unfortunately lost, he writes [**5**]:

"Les Indiens possèdent des méthodes pour trouver les côtés des carrés et des cubes. J'ai composé un ouvrage sur la démonstration de l'exactitude de ces méthodes, et j'ai prouvé qu'elles conduisent, en effet, à l'objet cherché. J'ai en outre augmenté les espèces, c'est à dire j'ai enseigné à trouver les côtés du carré-carre, du cubo-cube, du quadratro cube, à une étendue quelconque, ce qu'on n'avait pas fait précédément. Les démonstrations que j'ai donné à cette occasion, ne sont que des démonstrations arithmétiques, fondées sur les parties arithmétiques des Eléments d'Euclide."

This is an extremely interesting paragraph. Tropfke expresses his opinion in no uncertain terms:

"Die letzte Bemerkung kan man offenbar nur auf Benutzung der binomschen Entwickelung für beliebig hohe Exponenten deuten, wodurch dann Alkhayammi als Entdecker des Binomialtheorems für ganzzählige Exponenten anzusehen wäre" [6].

This seems to me eminently true and important, provided we take it literally. It all depends on "une étendue quelconque." If he could find any root by arithmetical means, he presumably used the binomial theorem, but the only actual roots he mentions are quartic, sextic, and ninth, each of these could be found by repeating the processes he knew for quadratic and cube roots. When we reflect on how inferior was the mathematical notation of his time, I think there is some doubt whether he could really extract, let us say, a seventh root.

A cautious note is sounded in a very recent discussion:

"Man hat die den modernen Mathematiker naheliegende Vermutung ausgesprochen; das Omar Haiyami den binomschen Entwickelung für beliebig hohe Exponenten etwa in der Weise arbeitete wie wir im 16 Jahrhundert bei Apian, Stifel, und andere Mathematiker der Renaissance beobachten" [7].

Luckey does not definitely pronounce on the point in question, but he seems inclined to the view that Omar could only find roots that were based on the quadratic and cubic. I am puzzled by his writing in connection with the quadrato cubic, "Quadratokubus (x^5)." Personally I can not avoid the sentimental hope that he really found the general formula.

2. The arithmetical triangle. We are safe in saying that by the year 1300 at least one capable mathematician was familiar with the binomial expansion for positive integral exponents. Some two hundred years after Omar, there lived in the Flowery Kingdom Chu-Shih-Chieh to whom we are indebted for the interesting diagram

$$\begin{array}{c} 1 \\ 1 \quad 1 \\ 1 \quad 2 \quad 1 \\ 1 \quad 3 \quad 3 \quad 1 \\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot \\ 1\,8 \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot \cdot\, 8\,1 \end{array}$$

Mikami comments, "This is indicated as an old calculation but not his invention. . . . This may perhaps have been borrowed from the Arabs in some way" [8]. The horizontal arrangement of the figures suggests strongly that these numbers were found by the expansion of binomials, but of course we have nothing suggesting a proof. Various mathematicians have suggested that the Chinese could expand binomials to quite high powers.

The first writer to give us something really solid is Michael Stifel who published in 1544 his *Arithmetica Integra*. Here we find the diagram

1				
2				
3	3			
4	6			
5	10	10		
6	15	20		
7	21	35	35	
8	28	56	70	
9	36	84	126	126
10	45	120	210	252

This, of course, can be extracted at once from Chu Shih-Chieh's table, but it is very unlikely that Stifel ever saw the latter. The two significant facts for us are that he was interested in the approximate extraction of roots, and we should like to know the manner in which he explains the construction of his table. In the first column, we have the integers in natural order. Each subsequent column begins two places lower than the preceding one; it starts with the number immediately on its left, and each subsequent number in the column is the sum of the number immediately above and the number to the left of the latter. Now if we write

$$(a + b)^n = (a + b)(a + b)^{n-1}$$

and, if we know the expansion of $(a+b)^{n-1}$, we find the coefficients of the expansion of $(a+b)^n$ by exactly this process. It would seem that Stifel was showing what we should write in modern notation

$$\binom{n}{r} = \binom{n-1}{r} + \binom{n-1}{r-1}. \tag{4}$$

A third form of this figure we owe to Pascal, whose famous triangle appears in the form

1	1	1	1	1	1	1
1	2	3	4	5	6	
1	3	6	10	15		
1	4	10	20			
1	5	15				
1	6					
1						

Probably Pascal was familiar with Stifel's table; he gives the same rule for the construction of the triangle, as well as some other identities. He points out that the numbers in a N.E. running diagonal are the binomial coefficients, and shows how we find the number of groups of r things taken from n things. Finally in Pascal we have the general rule which we should write [9]

$$\binom{n}{r} = \frac{n(n-1)(n-2)\cdots(n-r+1)}{r(r-1)(r-2)\cdots 1}. \tag{5}$$

It is important to say that a priority in this has been awarded to others, especially Briggs. Netto writes "Die binomsche Formel findet sich zuerst bei H. Briggs Arithmetica Logarithmica 1620," and Tropfke, that it is on p. 21 of Gellibrand's *Trigonometria Britanica*, a posthumous work of Briggs [10]. This may be. I can only say that I have found no trace. Netto gives no page number and I have seen nothing suggestive of it in the French translation, the only thing available to me; as for Tropfke, all I find on the page in question is a non-triangular form of Pascal's triangle, and there is nothing suggesting Formula (5) in the work.

3. Gregory and Newton. The first writer to approach the binomial expansion of a fractional power was James Gregory, who gave the formula in 1670. His method of approach was curiously indirect, his ostensible desire was to find an antilogarithm. Let us start with two numbers b and d with the logarithms $\log b = e$, $\log (b+d) = e+c$.

To find the number whose log is $(e+a)$,

$$\log b + \frac{a}{c}\left[\log (b+d) - \log b\right] = e + a$$

$$e + a = \log\left[b\left(1 + \frac{d}{b}\right)^{a/c}\right].$$

Take the two series,

$$b,\ d,\ \frac{d^2}{b},\ \frac{d^3}{b^2}\cdots$$

$$\frac{a}{c},\ \frac{a-c}{2c},\ \frac{a-2c}{3c},\ \frac{a-3c}{4c}\cdots$$

Combine like this

$$b + \frac{a}{c}d + \frac{a}{c}\cdot\frac{a-c}{2c}\cdot\frac{d^2}{b} + \frac{a}{c}\cdot\frac{a-c}{2c}\cdot\frac{a-2c}{3c}\cdot\frac{d^3}{b^2} = b\left(1 + \frac{d}{b}\right)^{a/c}.$$

There is of course no sign of proof [11].

It is time to turn to Sir Isaac Newton to whom we referred somewhat disparagingly in our opening paragraph. The story of his interest in the subject is told at length in a letter to Oldenburg, dated October 1676, and hence six years after Gregory's letter just mentioned [12]. He tells us that he was interested early in the study of interpolation by Wallis. This admirable mathematician studied curves whose equations were of the type

$$y = (1 - x^2)^{0/2}, \qquad y = (1 - x^2)^{1/2}, \qquad y = (1 - x^2)^{2/2}, \qquad y = (1 - x^2)^{3/2},$$
$$y = (1 - x^2)^{4/2}, \qquad y = (1 - x^2)^{5/2}, \qquad y = (1 - x^2)^{6/2}.$$

If we take the area of the figure bounded by the positive axes, the curve and the ordinate, we have for the cases of the first, third, fifth and seventh curve

$$x, \qquad x - \tfrac{1}{3}x^3, \qquad x - \tfrac{2}{3}x^3 + \tfrac{1}{5}x^5, \qquad x - \tfrac{3}{3}x^3 + \tfrac{3}{5}x^5 - \tfrac{1}{7}x^7.$$

How did Wallis discover these formulae, without the aid of integration? He studied quotients of the form

$$\frac{0^p + 1^p + 2^p \cdots n^p}{n^p + n^p + n^p \cdots n^p}$$

and noticed that the limit as n increased indefinitely was asymptotically $1/(p+1)$, this worked out at least in several instances [13]. We pass easily from this to finding the area under the curve $y = x^p$. We divide into n parts the segment from the origin to $x = n\Delta x$ on the axis, the points of division being 0, Δx, $2\Delta x$, $\cdots$, $n\Delta x$. We erect rectangles on these, one upper vertex being on the curve. The areas will be

$$\Delta x^p \cdot \Delta x, \ (2\Delta x)^p \cdot \Delta x, \ (3\Delta x)^p \cdot \Delta x, \ \cdots, \ (n\Delta x)^p \cdot \Delta x.$$

The sum will be

$$(\Delta x)^{p+1}(1^p + 2^p + 3^p + \cdots + n^p).$$

Now

$$\lim_{n\to\infty} \frac{(0^p + 1^p + \cdots + n^p)}{(n+1)n^p} = \frac{1}{p+1};$$

$$\lim_{n\to\infty} (\Delta x)^{p+1}[0^p + 1^p + 2^p + \cdots + n^p] = \lim_{n\to\infty}\left(\frac{n+1}{n}\right)\frac{(n\Delta x)^{p+1}}{p+1} = \frac{x^{p+1}}{p+1}\cdot$$

This gives the fundamental formula

$$\int x^p dx = \frac{x^{p+1}}{p+1}\cdot \tag{6}$$

Naturally Wallis did not set things up in anything like this form but such is the essence of his reasoning. Moreover, an equivalent formula for quite a number of cases had been proved by Cavalieri and others [14]. Wallis knew also that the integral of a sum is the sum of the corresponding integrals, so there is no real mystery about his discovery of these areas.

Let us return to Newton. He wished to find the areas under the other curves.

beginning with the second which is the circle $y=(1-x^2)^{1/2}$. Newton notes that in all of Wallis' cases the first term is x and the denominators are 1, 3, 4, 7, $\cdots$. They cause no trouble. They come in through the integration, not through the expansion. The second terms are:

$$-\tfrac{0}{3},\ -\tfrac{1}{3},\ -\tfrac{2}{3},\ -\tfrac{3}{3}\cdots.$$

Now $(1-x^2)^{(k+1)/2}$ is a mean proportional between $(1-x^2)^{k/2}$ and $(1-x^2)^{(k+2)/2}$, and this gives the first numerator in cases 1, 3, 5, $\cdots$. He guesses that this averaging process works in every case, and the other expansions should begin

$$x-\frac{\frac{1}{2}x^3}{3},\quad x-\frac{\frac{3}{2}}{3}x^3,\quad x-\frac{\frac{5}{2}}{3}x^3\cdots.$$

Suppose, then, the expansion begins $z-(m/3)x^3$. How shall the subsequent terms be found? Let us follow his own words [15], "Quaerebam itaque quomodo in his seriebus ex datis duobus primus figuris reliquae derivari present. Et inveni quod posita secunda figura, reliquae producerentur per continuarum multiplicationem terminorum hujus serie"

$$\frac{m-0}{}\times\frac{m-1}{}\times\frac{m-2}{}\times\frac{m-3}{}\times\cdots.$$

The essential word here certainly is "inveni." Did Newton work this out for himself or, more likely, did he follow Pascal's and Stifel's formula which holds in the integral case, and guess that it was always correct, and then work out some cases? We shall never know the answer, and on this, largely, I think, depends the amount of credit which he should receive. All that we surely know is that he wrote out

$$(1-x^2)^{1/2}=1-\frac{x^2}{2}-\frac{x^4}{8}-\frac{x^6}{16}\cdots$$

$$(1-x^2)^{3/2}=1-\frac{3x^2}{2}-\frac{3x^4}{8}-\frac{x^6}{16}\cdots$$

$$(1-x^2)^{1/3}=1-\frac{x^2}{3}-\frac{x^4}{9}-\frac{5x^6}{81}\cdots.$$

He squared or cubed the series and reached $(1-x^2)$. In the first case he found the square root by the usual method and reached the series. I cannot see that he did more than this, in which case, brilliant as was his genius in other matters, I do not think he deserves extraordinary credit for his contribution to the binomial theorem.

4. Attempts at proof. What shall we now say about demonstrations of the theorem; have any of these writers really proved it, or have they merely followed the example of all beginners, showing merely that no other solutions are possible? We have seen that Omar Khayyam had doubts on the point, he says "J'ai prouvé qu'elles conduisent en effet à l'objet cherché." Newton's verifications, as far as they go, show that the series are equal to binomials. Pascal in another connection used mathematical induction. He was one of the first users, but he did not prove our theorem in this way. How did anyone know *a priori* that any non-integral power of a binomial was actually equal to a certain convergent series?

In 1742 there appeared an article by Giovanni Salvemini who lived in Castiglione, for which insufficient reason he was frequently referred to as De Castillon. In the *Philosophical Transactions*, vol. 42, 1742–3, we find his article. He points out that everyone knows Newton's formula, but no one, as far as he knows, has proved it. He distinguishes three cases (a) a positive integral exponent, (b) a positive fractional exponent, (c) a negative exponent. The first case he handles by a method still used today. Let us replace $(p+q)^n$ by the product $(p+q_1)(p+q_2)\cdots(p+q_n)$. The numerical coefficient of the term of the rth order in the q's will be the number of combinations of n things taken r at a time, namely $\binom{r}{n}$. Then we set all the q_i's equal to q. When it comes to expanding $(p+q)^{r/n}$, we are safe in taking the first exponent as r/n, for that is the case when $q=0$; thus

$$(p+q)^{r/n} = Ap^{r/n} + Bp^{r/n-1}q + Cp^{r/n-2}q^2 + \cdots$$
$$(p+q)^r = p^r(A + Bp^{-1}q + Cp^{-2}q^2 + \cdots)^n.$$

He knows how to expand a binomial to any positive integral power, and blithely expects that he can do the same thing with a convergent power series, treating it as a binomial. In expanding, the new coefficients kindly come in one at a time, so that we have

$$1 = A^n; \qquad r = nA^{n-1}B; \qquad \frac{r(r-1)}{1\cdot 2} = nA^{n-1}C + \frac{n(n-1)}{1-2}A^{n-2}B^2$$

$$1 = A; \qquad \frac{r}{n} = B; \qquad \frac{\frac{r}{n}\left(\frac{r}{n}-1\right)}{1\cdot 2} = C.$$

The negative expansion comes by taking the reciprocal of the positive one.

A much quicker method was devised by a far abler mathematician, Colin Maclaurin. Here is what he writes on pp. 607–8 of vol. 2 of his *Fluxions*, Edinburgh, 1742:

"Let it be required to find $\overline{1+x}^n$ where n may represent any integer number or fraction, positive or negative. It is evident from what is shown in the common algebra concerning powers and their roots that the first term of any power of $(1+x)$ is 1, and the subsequent terms involve x, x^2, x^3, x^4, $\cdots$ with invariable coefficients. We suppose therefore

$$\overline{1+x}^n = 1 + Ax + Bx^2 + Cx^3 + \cdots$$

represents the general formula. By taking the fluxions on both sides

$$n\dot{x}\,\overline{1+x}^{n-1} = A\dot{x} + 2Bx\dot{x} + 3Cx^2\dot{x}.$$

This is an identity, hence if we take $x=0$ (or because the first term of $\overline{1+x}^n$ must be 1) we must have $A=n$." The other coefficients are quickly found by similar processes and further differentiation.

This demonstration was not essentially new, it appeared five years earlier in the work of Colson [**16**]. The reasoning is less clear as he uses the the same letter to mean two different things; he writes on succeeding lines $y=\overline{a+x}|^m$ and $y=a^m$. However he comes out all right in the end. But he makes on p. 310 an important remark which seems to have escaped Maclaurin:

"Indeed it can hardly be said that this or any other that is developed from the Method of Fluxions is a strict investigation of this Theorem. Because the Method itself is originally derived from the method of raising Powers, at least integral Powers, and presupposes the knowledge of Unciae or numerical coefficients."

Exactly this same difficulty occurred somewhat later to Euler, who gave two other demonstrations, of which I reproduce the second [**17**]. We start with the equations

$$\begin{aligned}(1+x)^n &= 1 + Ax + Bx^2 + Cx^3 + \cdots\\ (1+x)^{n+1} &= 1 + A'x + B'x^2 + C'x^3 + \cdots\\ &= (1+x)(1+x)^n.\end{aligned}$$

Suppose n is an integer. When $n \leqq 0$, all coefficients vanish; when $n \leqq 1$ all after A; when $n \leqq 2$ all after B; and so on. Let us write

$$A = \alpha n, \qquad B = \beta n(n-1), \qquad C = \gamma n(n-1)(n-2), \cdots$$

$$\begin{aligned}(1+x)^{n+1} &= 1 + \alpha(n+1)x + \beta(n+1)nx^2 + \cdots\\ &= (1+x)(1 + \alpha nx + \beta n(n-1)x^2 + \cdots.\end{aligned}$$

Subtracting

$$0 \equiv (\alpha - 1)x + (2\beta n - \alpha n)x^2 + \cdots.$$

Dividing out x, and setting $x=0$, we have

$$\alpha = 1, \qquad \beta = \frac{1}{2}, \qquad \gamma = \frac{1}{2\cdot 3},$$

and so on. Euler concludes, "Prorsus superfluum foret hos casus ulterius prosequi, cum jam luce meridianam clarius apparat pro singulis litteris sequentibus eosdem plane valores necessario prodire debere, quos evolutio Newtonianae docuit, atque haec demonstratio naturae rei tam apprime accomodato videtur, ut illi etiam in primis Analyseos elementis denegeri nequeat. Quin etiam universum ratiocinium qui hic usi sumus, unam vim retinet, etiamso adeo n ut imaginarius spectaretur" [**18**].

I must confess that I am not much impressed by this proof. It has the important advantage of being equally applicable to all values of the exponent but the best reason for the assumption as to the form of the coefficients is that it is correct in the integral case; the statement that something is "luce meridiana clarius" is not the same thing as a mathematical demonstration.

5. Convergence. There remains the important problem of the convergence of the series. The early writers were more or less aware of the existence of this question but were unable to handle it completely. The honor for doing this goes to Niels Hendik Abel. His contribution is much too long to be repeated here [**19**].

LEMMA (1) *Let $\rho_1\rho_2\rho_3 \cdots \rho_n$ be a series of positive quantities such that* $\lim \rho_{m+1}/\rho_m = \alpha > 1$ *and ϵ_m be quantities whose absolute values do not approach* 0 *as a limit as m increases without limit, then the series $\epsilon_0\rho_0+\epsilon_1\rho_1+\epsilon_2\rho_2 \cdots$ is divergent.*

We see in fact that regardless of how great m may be, the set

$$\epsilon_m\rho_m + \epsilon_{m+1}\rho_{m+1} + \cdots + \epsilon_n\rho_n$$

which may contain positive or negative terms, will not approach 0 as a limit.

LEMMA (2) *If in the above series $\alpha < 1$, $|\epsilon_m| < A$, the series is convergent*

We see, in fact that the absolute value of the set is less than $\rho_m A 1/(1-x)$, but $\lim \rho_m = 0$.

Now Abel makes a trigonometric development by the use of De Moivres' theorem.

Let

$$\phi(m) = 1 + \frac{m}{1}x + \frac{m(m-1)}{1\cdot 2}x^2 + \frac{m(m-1)(m-2)}{1\cdot 2\cdot 3}x^3 + \cdots.$$

Let

$$x = a + bi, \qquad m = k + k'i; \qquad \phi(m) = p + qi;$$

$$\sqrt{a^2+b^2} = \alpha; \qquad x = \alpha[\cos\phi + i\sin\phi]$$

$$\frac{m-\nu+1}{\nu} = \delta\nu[\cos\gamma\nu + i\sin\gamma\nu]$$

$$\binom{m}{\mu} = \alpha^\mu\delta_1\delta_2\cdots\delta_\mu[\cos(\mu\phi+\gamma_1+\gamma_2+\cdots+\gamma_\mu) + i\sin(\mu\phi+\gamma_1+\gamma_2+\cdots+\gamma_\mu)].$$

Abel here treats the real and imaginary parts separately, the question of convergence depends, as above, on whether α is greater than or less than unity. The case where it is equal to one, he treats at length, separately. One has the feeling that the last word has been said.

Yet that again is not the case. We have a variety of proofs that, if any power of a binomial can be expressed as a series of positive integral powers, this is the series. But why should such a series exist *a priori?* Omar sensed this difficulty. De Castillon met it, in the case of rational powers of the binomial, and, if his method were strengthened by showing that the algebraic operations with an infinite series were legitimate, and, that an extension by continuity considerations from the rational to the irrational is permissible, we might find the whole here. In a few cases Newton showed that the series which he developed did what they were supposed to do. Perhaps the easiest thing to do would be to find a general proof independent of the binomial theorem that $\dot{x}^n = nx^{n-1}\dot{x}$; then, with the aid of Taylor's Theorem with remainder, give MacLaurin's proof for the binomial case. What a long distance from Euclid II, 4!

References

1. Archimedes, Omnia Opera, Heiberg, Second Ed., vol. 3, Leipzig, 1915, p. 233.

2. Sur un fragment des métriques de Héron d'Alexandrie, Bulletin des Sciences mathématiques, series II, Tome 18, 1894, pp. 18 ff.

3. Heath, History of Greek Mathematics, Oxford, 1921, vol. I, pp. 61, 62. Also Rome, Commentaires de Pappus et de Théon d'Alexandrie, Città del Vaticano, 1936, pp. 470, 471.

4. Bibutibhassan Data and Avadesh Singh, The History of Indian Mathematics, Part 1, Lahore, 1935.

5. Woepcke, L'Algèbre d'Omar Alkhayyami, Paris, 1851, p. 13.

6. Tropfke, Geschichte der Elementarmathematik, 3rd Ed., Breslau, vol. 11, 1933, p. 174.

7. P. Luckey, Die Ausziehung der n-ten Wurzel und der binomsche Lehrsatz in der islamischen Mathematik, Math. Annalen, 120 Band, Heft 2, 1948, p. 216.

8. Mikami, Development of Mathematics in China and Japan, Abhandlungen zur Geschichte der mathematischen Wissenschaften, vol. 30, pp. 90 and 106.

9. Pascal, Traité du triangle arithmétique. Oeuvres, Ed. Brunschwig and Boutroux, Paris, vol. 3, 1923, p. 463.

10. Netto, Enzyklopädie der math Wissenschaften, Section I A 2, p. 24. Tropfke, Elementarmathematik, cit., Berlin, vol. VI, 1924, p. 38.

11. W. H. Turnbull, James Gregory Tercentenary Memorial Volume, London, 1939.

12. Commercium Epistolicum J. P. Collins et aliorum, Biot and Lefort, Paris, 1856.

13. Wallis, Johannes, Operum mathematicorum, pars altera, Oxford, 1656, pp. 31 and 34.

14. Cantor, Geschichte der Mathematik, 2nd Ed., vol. 2, Leipzig, 1900, p. 845.

15. Newton, Commercium Epistolicum, cit., p. 125.

16. John Colson, The Method of Fluxions by the inventor, Sir Isaac Newton, London, 1736, pp. 308 ff.

17. Euler, Nova demonstratio quod evolutio potestatum binomii Newtoniana etiam pro exponentibus fractis valet. Nova Acta Academiae scientiarum Petropolitanae (1787), 1789, pp. 52–58, Opera, Pars prima, vol. 16, pp. 112–121.

19. Abel, Untersuchungen ueber die Reihe . . . , Crelle, vol. I pp. 311 ff. Recherches sur la série . . . , Oeuvres, Christiania, 1881, pp. 219 ff.