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Infinitely Small Quantities in Cauchy's Textbooks

DETLEF LAUGWITZ

Fachbereich Mathematik, Technische Hochschule Darmstadt, D-6100 Darmstadt, Federal Republic of Germany

The fundamental role of infinitely small quantities for his teaching of the calculus was underlined by Cauchy himself in the introduction to his *Cours d'analyse* of 1821 and in the announcements of his later textbooks. First steps toward theories of such quantities which are briefly denoted as variables having zero as their limit were made by Cauchy, who represented them by sequences converging to zero (in the *Cours*) or by functions vanishing at zero (since 1823). It is shown that the famous so-called errors of Cauchy are correct theorems when interpreted with his own concepts. A few gaps in his proofs are explained by the hypothesis that he tacitly assumed continuity. No assumptions on uniformity or on nonstandard numbers are needed. Finally, some possible completions of Cauchy's rudimentary theories of infinitesimals are ventured. © 1987 Academic Press, Inc.

Cauchy betonte selbst die grundlegende Bedeutung unendlich kleiner Größen für seine Art, Analysis zu unterrichten, in der Einführung zum *Cours d'analyse* (1821) und den Selbstanzeigen seiner späteren Lehrbücher. Theoretische Ansätze für solche Größen, welche kurz als Variable mit Limes Null bezeichnet werden, beruhten auf der Darstellung durch Nullfolgen (im *Cours*) oder durch bei Null verschwindende Funktionen (seit 1823). Es wird gezeigt, daß Cauchys oft als falsch bezeichnete Sätze auf der Grundlage seiner eigenen Begriffe korrekt sind. Einige Beweislücken werden damit erklärt, daß er stillschweigend an stetige Funktionen dachte. Historisch unzulässige Annahmen über Gleichmäßigkeit oder über Nichtstandard-Zahlen sind überflüssig. Möglichkeiten zur Vervollständigung von Cauchys Theorieansätzen werden angedeutet. © 1987 Academic Press, Inc.

L'importance fondamentale, pour son enseignement, des quantités infiniment petites a été soulignée par Cauchy lui-même dans son introduction à son "Cours d'analyse" de 1821 et dans les avertissements à ses traités ultérieurs. Le point de départ pour la théorie de ces quantités, qui peuvent être désignées brièvement comme des variables ayant pour limite zéro, repose sur la représentation par les suites qui convergent vers zéro (dans le "Cours") ou par les fonctions qui tendent vers zéro (depuis 1823). On va montrer que les théorèmes soi-disant faux de Cauchy sont exacts en se basant sur ses propres notions. Des lacunes dans les démonstrations seront expliquées par le fait qu'il suppose implicitement qu'il s'agit des fonctions continues. Des hypothèses, historiquement inadmissibles, de l'uniformité ou des nombres non-standard sont superflues. On indiquera comment on peut compléter les théories de base de Cauchy. © 1987 Academic Press, Inc.

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1. INTRODUCTION: CAUCHY'S MOTTO

In the *Avertissements* of [Cauchy 1823, 9] and [Cauchy 1829, 267] we find the motto: "My principal aim has been to reconcile rigor, which I have made a law to myself in my Cours d'analyse, with the simplicity which the direct consideration of infinitely small quantities produces" (translation in [Edwards 1979, 309] of the original: "Mon but principal a été de concilier la rigueur, dont je m'étais fait une loi dans mon Cours d'analyse, avec la simplicité que produit la considération directe des quantités infiniment petites").

Earlier, in the introduction to his *Cours* Cauchy [1821, ii] had stated the importance of infinitely small quantities for the treatment of continuous functions: "En parlant de la continuité des fonctions, je n'ai pu me dispenser de faire connaître les propriétés principales des quantités infiniment petites, propriétés qui servent de base au calcul infinitésimal." ("When speaking of the continuity of functions, I could not dispense with announcing the main properties of infinitely small quantities, properties which serve as foundation of the infinitesimal calculus.")

As a historian of mathematics one cannot but take an author's own intentions and reasons seriously: Infinitely small quantities are fundamental in Cauchy's analysis, they are compatible with rigor, and they produce simplicity. Since Abel [1826] the neglect of the motto has led to difficulties even with the very first theorems on continuity and convergence which I quote for later discussion.

2. TWO CONTROVERSIAL THEOREMS

The first theorem on continuity in [Cauchy 1821, 47] is:

Théorème I.—Si les variables $x, y, z, \ldots$ ont pour limites respectives les quantités fixes et déterminées $X, Y, Z, \ldots$, et que la fonction $f(x, y, z, \ldots)$ soit continue par rapport à chacune des variables $x, y, z, \ldots$ dans le voisinage du système des valeurs particulières

$$x = X, y = Y, z = Z, \ldots,$$

$f(x, y, z, \ldots)$ aura pour limite $f(X, Y, Z, \ldots)$.

(If a function of several variables is continuous in each one separately it is a continuous function of all the variables.)

The following first theorem on convergence of [Cauchy 1821, 120] was repeated without change in [Cauchy 1833, 56]:

Théorème I.—Lorsque les différents terms de la série (1) [i.e., $u_0 + u_1 + u_2 + \ldots$] sont des fonctions d'une même variable x, continues par rapport à cette variable dans le voisinage d'une valeur particulière pour laquelle la série est convergente, la somme s de la série est aussi, dans le voisinage de cette valeur particulière, fonction continue de x.

(The sum $s(x)$ of a convergent series of continuous functions $u_n(x)$ is itself a continuous function.)

Both theorems are incorrect when interpreted in the by now common conceptual framework of analysis (which obviously cannot have been Cauchy's framework). Both theorems become correct as soon as one adds assumptions on uniformity (which, at least in the form by now common, were never used by Cauchy).

The theorems are correct in any of the modern theories of infinitesimals (which, apart from being unknown to Cauchy, lack the "simplicity of infinitesimals," at least in the version of Robinson [1966]).

The three attitudes mentioned (Cauchy erred; Cauchy forgot about essential assumptions; Cauchy was correct, but only when put against a modern background) are unsatisfactory from the point of view of a historian. The first one, shared by a majority, is inadequate even psychologically: Is it believable that Cauchy, the exponent of rigor, should make mistakes at the lowest level of his calculus? Nevertheless: "For instance, it is well known that he asserted the continuity of the sum of a convergent series of continuous functions; Abel gave a counterexample, and *it is clear that Cauchy himself knew scores of them*" [Freudenthal 1971, 137; my italics].

The only satisfactory attitude should be: Try and understand Cauchy's theorems and their proofs from his own concepts. Attempts have been made to do this by eliminating his infinitely small quantities and replacing them by sequences [Giusti 1984]. Apart from a loss in simplicity this is in one more respect against the motto of Cauchy who advocated the *direct* use of infinitely small quantities. Moreover, I shall show that the approach via sequences is mathematically satisfactory only for some concepts and theorems. Attempts toward more comprehensive theories will be sketched in Section 15.

3. CAUCHY'S INFINITESIMALS

Usually, "une quantité infiniment petite" or "un infiniment petit" is translated as "an infinitesimal." For the sake of brevity I shall follow this habit.

Cauchy defines an infinitesimal as a variable having zero as its limit: "Lorsque les valeurs numériques (i.e., absolute values) successives d'une même variable décroissent indéfiniment, de manière à s'abaisser au-dessous de tout nombre donné, cette variable devient ce qu'on nomme un *infiniment petit* ou une quantité *infiniment petite*. Une variable de cette espèce a zéro pour limite" [Cauchy 1821, 19; 1823, 16; 1829, 273]. "On dit qu'une quantité variable devient *infiniment petite,* lorsque sa valeur numérique décroit indéfiniment de manière à converger vers la limite zéro" [Cauchy 1821, 37]. ("One says that a variable quantity becomes *infinitely small,* when its numerical value—i.e., absolute value—decreases indefinitely in such a way as to converge to the value zero"—translation as in [Edwards 1979, 310]).

Unfortunately there is only one example to be found in the *Cours*, the sequence $\frac{1}{4}, \frac{1}{3}, \frac{1}{6}, \frac{1}{5}, \frac{1}{8}, \frac{1}{7}, \ldots$ [Cauchy 1821, 37]. I note in passing that infinitely large quantities are defined in a similar manner, the sequence 1, 2, 3, 4, 5, . . . serving as an example [Cauchy 1821, 38]. In contrast to the *infinitely large* quantities, i.e., variables which in absolute value are increasing indefinitely in such a way as to converge to the limit ∞, there are two *infinite quantities* $\pm\infty$ [Cauchy 1821, 38, 19]. Infinitely large quantities are not unimportant. Occasionally they appear as possible values of subscripts in sequences s_n.

One might conclude from his later texts that Cauchy did not insist on sequences as the only possible representatives of infinitesimals and infinitely large numbers. Beginning with the *Addition* to [Cauchy 1823, 250 ff.] infinitesimals are represented by functions which are continuous in a neighborhood of 0 and vanish at 0. In order to avoid a circulus vitiosus this position is not suitable for [Cauchy 1821] where continuity remains still to be defined. Consequently, I shall postpone this discussion of functions as representatives.

4. CONTINUITY

Briefly, a function is continuous if an infinitesimal change of the variable produces an infinitesimal change of the function itself. This has to be discussed in detail.

Cauchy does not define pointwise continuity. There are definitions of a global continuity (in an interval) and of a kind of local continuity (in the vicinity of a particular value). Concerning the terminology on intervals with or without ends I shall follow Grabiner [1981, footnote on p. 168]: Cauchy's *comprise* is translated as "included," and his *renfermé* as "lying between." If c is included between the limits (i.e., bounds) a and b then $a \leqq c \leqq b$, and c lies between a and b if $a < c < b$.

Global continuity is defined for functions on intervals with ends [Cauchy 1821, 43]; translation following Edwards [1979, 310–311]:

> Si, en partant d'une valeur de x comprise entre ces limites, on attribue à la variable x un accroissement infiniment petit α, la fonction elle-même recevra pour accroissement la différence
>
> $$f(x + \alpha) - f(x),$$
>
> qui dépendra en même temps de la nouvelle variable α et de la valeur de x. Cela posé, la fonction $f(x)$ sera, entre les deux limites assignées à la variable x, fonction *continue* de cette variable, si, pour chaque valeur de x intermédiaire entre ces limites, la valeur numérique de la différence
>
> $$f(x + \alpha) - f(x)$$
>
> décroit indefiniment avec celle de α. En d'autres termes, *la fonction $f(x)$ restera continue par rapport à x entre les limites données, si, entre ces limites, un accroissement infiniment petit de la variable produit toujours un accroissement infiniment petit de la fonction elle-même.*

> If, starting from a value of x included between these limits, one assigns to the variable x an infinitely small increment α, the function itself will take on for an increment the difference $f(x + \alpha) - f(x)$, which will depend at the same time on the new variable α and on the value of x. This granted, the function $f(x)$ will be, between the two limits assigned to the variable x, a *continuous* function of the variable if, for each value of x intermediate between these limits, the numerical value of the difference $f(x + \alpha) - f(x)$ decreases indefinitely with that of α. In other words, *the function $f(x)$ will remain continuous with respect to x between the given limits, if, between these limits, an infinitely small increment of the variable always produces an infinitely small increment of the function itself.* (The italics are by Cauchy.)

The definition of local continuity comes immediately after that: "On dit encore que la fonction $f(x)$ est, dans le voisinage d'une valeur particulière attribuée à la variable x, fonction continue de cette variable, toutes les fois qu'elle est continue entre deux limites de x, même très rapprochées, qui renferment la valeur dont il s'agit." This means that $f(x)$ is continuous in the vicinity of a particular value of x, say $x = x_0$, if a, b exist, $a < x_0 < b$, such that $f(x)$ is continuous between the limits a and b.

It should be noted that Cauchy explicitly speaks of a particular value of x whenever he means it. Otherwise, *valeur* might mean the same as *variable*. Adopting this attitude, Giusti [1984] has given a most interesting interpretation. Consider $f(x + \alpha) - f(x)$, as Cauchy himself does, as a function of two variables, and replace x and α by sequences x_n and a_n, the latter converging to 0, such that $x_n + a_n$ and x_n always belong to the interval under consideration. (The variables vary, as Giusti puts it.) Then $f(x)$ is a continuous function of x in the given interval if $f(x_n + a_n) - f(x_n)$ represents an infinitesimal, i.e., is a sequence converging to 0, whenever the sequences are chosen in the specified way. This interpretation is in agreement with the version which Cauchy put in italics and which is the one he actually uses in the *Cours* and later on, e.g., [Cauchy 1823, 19–20]: "Lorsque, la fonction $f(x)$ admettant une valeur unique et finie pour toutes les valeurs de x comprises entre deux limites données, la difference $f(x + i) - f(x)$ est toujours entre ces limites une quantité infiniment petite, on dit que $f(x)$ est *fonction continue* de la variable x entre les limites dont il s'agit."

If Cauchy had meant it he would certainly have used a less complicated version like the following: If for every particular value of x between the given limits and for each particular infinitesimal i the difference $f(x + i) - f(x)$ is always an infinitesimal, then $f(x)$ is a continuous function of x.

Cauchy proceeds to prove the continuity of elementary functions, and these proofs hold good both in the pointwise and in Giusti's interpretation. The following discussion (of Théorème I) argues against the pointwise concept.

5. CONTINUOUS FUNCTIONS: THÉORÈME I

I return to the theorem restated in Section 2. To simplify notation I shall consider two variables. The proof in [Cauchy 1821, 45–47] is as follows: Let $f(x, y)$ be, in the vicinity of the particular values X, Y, continuous as a function of x and continuous as a function of y, and let α, β be infinitesimals. If x, y attain the values X, Y or values very close to them then, by the assumption, the absolute values of the differences

$$f(x + \alpha, y) - f(x, y)$$

and (A)

$$f(x + \alpha, y + \beta) - f(x + \alpha, y)$$

will decrease indefinitely with those of α, β. The same follows for the sum of the two differences which is $f(x + \alpha, y + \beta) - f(x, y)$. (In other words, this difference is infinitesimal for infinitesimal α and β which means that $f(x, y)$ is a continuous function of two variables.) Cauchy replaces x, y by X, Y and then $x + \alpha$, $y + \beta$ by x, y and obtains Théorème I.

Though Giusti mentions it in his list of the so-called "errors" of Cauchy he does not come back to it. I shall discuss the proof using sequences a_n for α and b_n for β. Until the very last part of the proof, x, y are variables having values in the vicinity of X, Y, which should mean that they can be replaced by sequences converging to X, Y. The differences (A) become

$$f(x_n + a_n, y_n) - f(x_n, y_n)$$

and

$$f(x_n + a_n, y_n + b_n) - f(x_n + a_n, y_n),$$

and the assumption is that both these differences converge to zero. But that is highly unsatisfactory; it means that we assume the formal definition of continuity for $f(x, y)$ as a function of x not only for each fixed y but for all sequences y_n converging to Y, etc. Though this interpretation leads to a correct proof the motto is certainly violated. The assumption is not simple, and there is no direct consideration of infinitesimals.

The simple way out is to consider infinitesimals α, β as mathematical entities of their own specific rank, which can be "considered directly," though it may sometimes be useful to think of them as *represented* by sequences. A variable x may attain particular real values like X, but also values like $X + \alpha$ where α is an infinitesimal. In this interpretation the assumption that $f(x, y)$ be, in the vicinity of X, Y, continuous as a function of x is easily understood: For each fixed value of y which is infinitely close to Y (i.e., $y - Y$ is infinitesimal) the function depends only on x, and the definition of continuity can be applied. I admit that my hypothesis needs further support which will be given in the next sections. Some confirmation may also be found in [Fisher 1978].

6. LIMITS AND CONVERGENCE

The concept of a limit is defined for variables in [Cauchy 1821, 19; 1823, 13; 1829, 269]: "When the successive values attributed to a variable approach indefinitely a fixed value so as to end by differing from it by as little as one wishes, this last fixed value is called the *limit* of all the others. Thus, for example, an irrational number is the limit of diverse fractions which furnish more and more approximate values of it" (translation in [Edwards 1979, 310]). For series with terms u_0, u_1, u_2, . . . the definition is given in [Cauchy 1821, 114]: "Let $s_n = u_0 + u_1 + u_2 + \cdots + u_{n-1}$ be the sum of the first n terms, n being any integer. If, for increasing values of n, the sum s_n approaches a certain limit s, the series will be called convergent, and the limit in question will be called the sum of the series" (translation in [Grabiner 1981, 99]).

Infinitesimals do not appear in the definitions but enter as useful tools, for the first time in an alternative version of the famous convergence criterion [Cauchy 1821, 115]: "in other words, it is necessary and sufficient that, for infinitely large values of the number n, the sums s_n, s_{n+1}, s_{n+2}, . . . differ from the limit s, and consequently among themselves, by infinitely small quantities." Incidentally, that the condition that $s_n - s_m$ be infinitesimal for infinitely large m, n is necessary and sufficient for convergence was stated and used by Euler as early as 1735 [Laugwitz 1986, 50–55].

7. CONVERGENT SERIES: THÉORÈME I

This theorem, which I mentioned in Section 2, has become known as the most famous "false theorem" of Cauchy. Moreover, it is of some interest for the conceptual background of Cauchy that his first theorem on convergence is concerned with series of *functions* and not of *numbers*.

I read the proof as follows. I assume that convergence is postulated for all values x of the variable in the given interval, and moreover for all $x + \alpha$, α infinitesimal; the latter assumption is needed in the proof, but it is not obvious in the statement of the theorem. Let $s(x) = s_n(x) + r_n(x)$ and $s(x + \alpha) = s_n(x + \alpha) + r_n(x + \alpha)$, where $s_n = u_0 + u_1 + \cdots + u_{n-1}$. Then s_n is, for finite n, a continuous function, and $s_n(x + \alpha) - s_n(x)$ is infinitesimal. According to the convergence both $r_n(x)$ and (!) $r_n(x + \alpha)$ will become "imperceptible" for very large (but still finite) numbers n. In other words, for any given finite $\varepsilon > 0$, $|r_n(x)| < \varepsilon$ and $|r_n(x + \alpha)| < \varepsilon$ for some finite n. Hence,

$$|s(x + \alpha) - s(x)| \leqq |s_n(x + \alpha) - s_n(x)| + |r_n(x + a)| + |r_n(x)| < 3\varepsilon$$

for each finite $\varepsilon > 0$, i.e., $s(x + \alpha) - s(x)$ is infinitesimal. It follows that $s(x)$ is a continuous function.

The reader should compare the original:

> Lorsque, les termes de la série (1) renferment une même variable x, cette série est convergente, et ses différents termes fonctions continues de x, dans le voisinage d'une valeur particulière attribuée à cette variable,
>
> $$s_n, \quad r_n, \quad \text{et} \quad s$$
>
> sont encore trois fonctions de la variable x, dont la première est évidemment continue par rapport à x dans le voisinage de la valeur particulière dont il s'agit. Cela posé, considérons les accroissements que reçoivent ces trois fonctions, lorsque'on fait croître x d'une quantité infiniment petite α. L'accroissement de s_n sera, pour toutes les valeurs possibles de n, une quantité infiniment petite; et celui de r_n deviendra insensible en même temps que r_n, si l'on attribue à n une valeur très considérable. Par suite, l'accroissement de la fonction s ne pourra être qu'une quantité infiniment petite. [Cauchy 1821, 120]

My translations are: n finite for "valeur possible de n"; n sufficiently large but finite for "valeur très considérable de n"; r_n becomes imperceptible, i.e., smaller than any given finite $\varepsilon > 0$ for sufficiently large finite n, for "r_n deviendra insensible." Others have given different interpretations. My translations lead to a correct proof which is simple and straightforward. Moreover, they are compatible with Cauchy's own terminology throughout the Cours. For instance, if "très considérable" meant infinitely large Cauchy would have said so; in [Cauchy 1821, 116–117] he states that, for real $|x| < 1$, the values of x^n, $x^n/(1 - x)$ become infinitesimal for infinitely large n.

In my version of the proof for Théorème I only finite values of n enter, as they do in the definition of convergence, and infinitesimals are used as in the definition of continuity.

The text of Théorème I is, to put it mildly, at least misleading. The first to mention that fact was Abel [1826]. In his *proof* Cauchy needs convergence of the series at x as well as at $x + \alpha$ where α is infinitesimal. The assumptions as stated in the *theorem* are weaker: "dans le voisinage d'une valeur particulière pour *laquelle* la série est convergent" (the italics are mine). What he really uses in the proof and in his applications of the theorem amounts to "*lequelle*," referring to the whole infinitesimal neighborhood of the particular value, and he says so quite clearly: "cette série est *convergente,* et ses differents termes fonctions continues de x, *dans le voisinage* d'une valeur particulière." Here, as in the inequalities thereafter, both convergence of the series and continuity of its terms are assumed throughout the neighborhood. Abel failed to mention the correctness of the proof.

Rather late in his life Cauchy [1853] admitted that the statement of his theorem (but not its proof) was incorrect: "Au reste, il est facile de voir comment on doit modifier *l'énoncé* du théorème, pour qu'il n'y plus lieu à aucune exception" [Cauchy 1853, 31–32].

Though not drawing back his old proof he sketches a new one. Suppose that $n' > n$, then the absolute value of

$$s_{n'} - s_n = u_n + u_{n+1} + \cdots + u_{n'-1} \tag{3}$$

is assumed, for sufficiently large n and all x, n', to be smaller than a number ε which one may choose arbitrarily small. This is one of the rare explicit uses of epsilontics that Cauchy ever made. It looks very much like uniform convergence and would lead immediately to a correct proof and theorem. But Cauchy does not stop here; he sticks to his infinitesimals. Moreover, in contrast to the passage in the *Cours*, he even introduces infinitely large numbers: "il est clair qu'il suffira d'attribuer au nombre n une valeur infiniment grande, et à l'accroissement de x une valeur infiniment petite, pour demontrer, entre les limites données, la continuité de la fonction $s = s_n + r_n$" [Cauchy 1853, 32]. He assumes that the expression (3) becomes infinitely small for infinitely large n, which is one version of his convergence criterion. As a consequence the proposition is stated as Théorème I:

> If the different terms of the series (1) . . . are functions of the real variable x, which are continuous with respect to this variable between the given limits; and if, moreover, the sum (3) . . . becomes always [toujours] infinitely small for infinitely large values of the integers n and $n' > n$, then the series (1) will be convergent, and the sum of the series (1) will be, between the given limits, a continuous function of the variable x.

It appears that the proof is more involved than that in the *Cours*. Moreover, the statement of the theorem is correct only if always (toujours) is interpreted as for all $x + \alpha$ from the interval, x real and α infinitely small. This is clarified by a lucid discussion of an example of the type Abel had mentioned [Cauchy 1853, 33–34]: The series $\sum_{k=1}^{\infty} (\sin kx)/k$ does *not* converge for infinitesimal $x \neq 0$. Let $x = 1/n$ for n infinitely large. Then the remainder term

$$r_n(x) = \sum_{k=n}^{\infty} \frac{\sin kx}{k} = \sum_{k=n}^{\infty} \frac{\sin (k/n)}{k/n} \frac{1}{n}$$

differs by an infinitesimal from the value of $\int_1^\infty ((\sin t)/t)\, dt$ which is finite though $r_n(x)$ should be an infinitesimal for a convergent series and n infinitely large. It follows that the series is not a counterexample since the assumption of convergence is violated at $x = 1/n$.

Giusti gives a correct translation of the example into the language of sequences, and he makes a point of it [Giusti 1984, 50]. But he fails to translate the general theorem and its proof. Actually both x (or $x + \alpha$) and n will have to be replaced by sequences which becomes troublesome as soon as they are not connected as in the example where $x = 1/n$. Moreover, the theorem shows the power of Cauchy's "direct consideration of infinitesimals."

Among others, the paper of 1853 is strong evidence against the opinion that Cauchy used his definition of infinitesimals mainly for one purpose: It permitted an easy transfer of the method of limits into the language which was used in the official programs of the École polytechnique [Belhoste 1985, 105].

It is impossible to mention even a part of the literature on Cauchy's "error." A detailed discussion and bibliography can be found in [Spalt 1981].

8. THE BINOMIAL SERIES

The *Cours* contains a beautiful, complete, and correct treatment of the binomial series $\sum_{n=0}^{\infty} \binom{\mu}{n} x^n$ for arbitrary real μ. The series converges for all x, $|x| < 1$, as a consequence of the ratio test [Cauchy 1821, 137]. Then (pp. 141–142) x is a fixed number between -1 and $+1$, and $\phi(\mu) = \sum_{n=0}^{\infty} \binom{\mu}{n} x^n$ is considered as a function of the single variable μ. By his theorem on the multiplication of series and using the relations on binomial coefficients Cauchy obtains the functional equation $\phi(\mu)\phi(\mu') = \phi(\mu + \mu')$. The next step (pp. 146–147) is to show that ϕ is a continuous function of μ. Since $u_n(\mu) = \binom{\mu}{n} x^n$ is, as a function of μ (!), a polynomial of degree n, it is everywhere continuous. By virtue of Théorème I on continuity, the sum $\phi(\mu)$ is a continuous function. For each rational μ, a well-known procedure (pp. 100–102) shows that $\phi(\mu) = (\phi(1))^\mu = (1 + x)^\mu$. Since both $\phi(\mu)$ and $(1 + x)^\mu = \exp(\mu \log(1 + x))$ are continuous functions of μ, we may conclude that

$$(1 + x)^\mu = \sum_{n=0}^{\infty} \binom{\mu}{n} x^n \qquad \text{for all real } \mu \text{ and } |x| < 1.$$

As far as I know this is the first proof of this identity. An earlier attempt was [Euler 1774]. Euler had used the functional equation and succeeded in showing that the series development was correct for *rational* μ. The steps which are essentially new in Cauchy's proof are his concept of a continuous function and his Théorème I on convergence.

Cauchy's extension of the binomial series to complex numbers [Cauchy 1821, Chap. IX] is considered by Fisher [1978]. Fisher points out that in this proof

"Cauchy used infinitely small quantities very much like infinitesimals—i.e., like numbers rather than variables with limit zero," a remark which supports my opinion [Fisher 1978, 323].

Immediate consequences of the binomial series are the exponential and logarithmic series [Cauchy 1821, 147–151]. If $\mu = 1/\alpha$, where α is an infinitesimal, then for $|x\alpha| < 1$

$$(1 + \alpha x)^{1/\alpha} = 1 + \frac{x}{1} + \frac{x^2}{1 \cdot 2}(1 - \alpha) + \frac{x^3}{1 \cdot 2 \cdot 3}(1 - \alpha)(1 - 2\alpha) + \ldots$$

and for $\alpha \to 0$ the series of e^x is obtained. This looks like interchanging limits. But the conclusion can easily be justified by Théorème I. For fixed x, each term of the series is a continuous function of α. The series converges everywhere as $|\alpha| < 1/|x|$, and the sum is continuous by the theorem. Let $\alpha \to 0$.

Similarly, for any fixed x, $|x| < 1$, and $\mu \to 0$ the logarithmic series is a consequence of

$$x - \frac{x^2}{2}(1 - \mu) + \frac{x^3}{3}(1 - \mu)\left(1 - \frac{\mu}{2}\right) - \cdots = \frac{(1 + x)^\mu - 1}{\mu}$$
$$= \frac{e^{\mu \ln(1+x)} - 1}{\mu} = \ln(1 + x) + \frac{\mu}{2}[\ln(1 + x)]^2 + \ldots$$

Further, since $\ln(1 + x)$ is continuous, and $x - x^2/2 + x^3/3 - + \ldots$ converges for $0 \leqq x \leqq 1$, Théorème I gives the series for ln 2.

It is true that Cauchy does not refer the reader to his Théorème I when treating these corollaries. On the other hand, the reasoning is similar to that on the binomial series. Perhaps the fundamental Théorème I was by now considered a matter of course which needs no repeated mention. Reading Cauchy carefully helps to eliminate apparent inconsistencies which some see: "He [Cauchy] proved $\lim(1 + 1/n)^n = \sum 1/n!$ by a popular but unjustified interchange of limit processes . . . , although he was well acquainted with such pitfalls" [Freudenthal 1971, 137].

9. THE DIFFERENTIAL AND THE DERIVED FUNCTION

Since Leibniz the differential had been the true raison d'être of infinitesimals. Cauchy succeeded in eliminating infinitesimals from this concept. His differential $dy = y'dx$ or

$$df(x) = f'(x)dx$$

is nothing but a function of two real variables. Of course, since any variable is permitted to attain infinitesimal values, so is dx. *Cauchy shifted the basic place of infinitesimals from differentials to the concept of continuity.*

The first definition in [Cauchy 1823, 22–23], as translated in [Edwards 1979, 313], states:

> When a function $y = f(x)$ remains continuous between two given limits of the variable x, and when one assigns to such a variable a value enclosed between the two limits at issue, then an

infinitely small increment assigned to the variable produces an infinitely small increment in the function itself. Consequently, if one puts $\Delta x = i$, the two terms of the *ratio of differences*

$$\frac{\Delta y}{\Delta x} = \frac{f(x + i) - f(x)}{i}$$

will be infinitely small quantities. But though these two terms will approach the limit zero indefinitely and simultaneously, the ratio itself can converge towards another limit, be it positive or be it negative. This limit, when it exists, has a definite value for each x. . . . The form of the new function which serves as the limit of the ratio $(f(x + i) - f(x))/i$ will depend on the form of the proposed function $y = f(x)$. In order to indicate this dependence, one gives the new function the name of *derived function* [fonction derivée], and designates it with the aid of an accent by the notation y' or $f'(x)$.

Curiously enough Cauchy never mentions that the existence of $f'(x)$ implies the continuity of $f(x)$ itself, an easy consequence of

$$\frac{f(x + i) - f(x)}{i} - f'(x) = \text{infinitesimal} \qquad \text{and} \qquad f'(x) \text{ finite.}$$

Usually he even states this continuity as an additional assumption, e.g. [Cauchy 1829, 312–313].

It is much more deplorable that he does not mention a fact which is needed in some important proofs: *The derived function as defined by Cauchy is always a continuous function.*

To see this we must realize that differentiability, like continuity, is not a pointwise property. (Compare the definition above!) Hence, not only $f'(x)$ but also $f'(x + i)$ will exist if x, $x + i$ are "between the two limits at issue." Actually, $f'(x + i)$ is needed when $f''(x)$ is defined, etc. Now for $x + i = \bar{x}$ and $-i = k$,

$$-\frac{f(\bar{x} + k) - f(\bar{x})}{k} + f'(\bar{x}) = \text{infinitesimal.}$$

By adding this to

$$\frac{f(x + i) - f(x)}{i} - f'(x) = \text{infinitesimal,}$$

we obtain that $f'(x + i) - f'(x)$ is always an infinitesimal, and $f'(x)$ is a continuous function. Cauchy uses this property liberally, e.g., in the proof of the mean value theorem [Cauchy 1823, 44–46].

10. THE INTEGRAL

It is a common belief that for any proof of the integrability of a continuous function uniform continuity is needed. Consequently, Cauchy's proof is usually declared to be incorrect.

The proof is given in [Cauchy 1823, 122–125]. The crucial step comes when Cauchy passes to the common refinement of two subdivisions of an interval $x_0 \leqq x \leqq X$, $x_0 < x_1 < \cdots < x_{n-1} < X$ and when he has to consider the difference of the approximating sums S belonging to a subdivision and its refinement. This difference will be

$$D = \pm\varepsilon_0(x_1 - x_0) \pm \varepsilon_1(x_2 - x_1) \pm \cdots \pm \varepsilon_{n-1}(X - x_{n-1}).$$

Each ε_k is the difference of values of $f(x)$. As soon as $x_{k+1} - x_x$ becomes infinitesimal, ε_k will be an infinitesimal, by the continuity of the function. Now, by Cauchy's famous theory of means [Cauchy 1821, 27–30] it follows that $D = \varepsilon(X - x_0)$, where ε is a mean (moyenne) of the $\pm\varepsilon_k$. Apparently he uses as a hidden lemma that any mean of infinitesimals is again an infinitesimal and concludes:

> Therefore, when the elements of the difference $X - x_0$ become infinitely small, the mode of division has no more than an imperceptible [insensible] influence on the value of S; and, if one makes the numerical values of these elements decrease indefinitely, by increasing their number, the value of S will end by being perceptibly [sensiblement] constant or, in other words, it will end by attaining a certain limit which will depend solely on the form of the function $f(x)$ and on the extreme values x_0 and X attributed to the variable x. This limit is that which one calls a *definite integral.* (Translation in [Edwards 1979, 320]; for a commented translation, see [Grabiner 1981, 171–174]. Grabiner's translation is "imperceptible" for *insensible,* but "for all practical purposes" for *sensiblement.*)

The hidden lemma is true in any reasonable theory of infinitesimals. It will be difficult to give sense to the statement that each mean of infinitely many sequences with limit zero will again converge to zero. Once more, the direct consideration of infinitesimals is superior to an interpretation in terms of sequences.

Uniform continuity is avoided by the use of the hidden lemma.

11. THE INTEGRATION OF SERIES

Cauchy knew that interchanging limits is not always permitted. In 1814 he had given an example of a discontinuous function of two variables for which the value of the double integral depends on the order of the two integrations. When he needs term-by-term integration of infinite series, he is careful with both the assumptions and the proof [Cauchy 1823, 237–238]. Notations and assumptions are as in Section 7. The proposition is that

$$\sum \int_{x_0}^{X} u_n dx \text{ converges and is equal to } \int_{x_0}^{X} s dx,\ s = \sum_0^{\infty} u_n.$$

The proof relies on his Théorème I on convergence, which is actually mentioned only in his Corollaire II on page 239. Since the series converges everywhere in the interval, the remainder term r_n is infinitely small for infinitely large n and everywhere in the interval. Since $\int r_n dx$ will be a particular value of the product $r_n(X - x_0)$, it is again an infinitesimal. It follows that

$$\int s dx - \sum_0^{n-1} \int u_k dx = \int r_n dx$$

is infinitesimal for infinitely large n and vanishes for $n = \infty$.

Note that Cauchy uses the infinitesimal version of his convergence criterion in both directions.

Again, it is important that $\Sigma\, u_n$ converge everywhere, i.e., for all $x + \alpha$, x real, α infinitesimal, of the interval. Though the theorem and its proof are correct under these assumptions, there is a gap in the proof of the important Corollaire IV of [Cauchy 1823, 239–240], where $u_n = a_n x^n$ and $x_0 = -1/\lambda$, $X = 1/\lambda$, $1/\lambda$ being the radius of convergence of the power series. Cauchy fails to show that the series converges for $X - \alpha$, $x_0 + \alpha$, α infinitesimal, which is actually true by Abel's theorem.

12. TAYLOR'S FORMULA

In [Cauchy 1823] Taylor's formula is derived from the famous remainder integral. Not satisfied with the use of integration Cauchy aimed at a proof needing only the tools of the differential calculus. He succeeded immediately after finishing the *Résumé*, and gave such a proof in the Addition [Cauchy 1823, 243–256]. This is the starting point for his more elaborate theory of infinitesimals which I discuss in Section 14.

The proof itself does not contain any reference to infinitesimals. It rests on his generalized mean value theorem,

$$\frac{f(x_0 + h)}{F(x_0 + h)} = \frac{f^{(n)}(x_0 + \theta h)}{F^{(n)}(x_0 + \theta h)},$$

provided that f, F, and their first $n - 1$ derived functions vanish at $x = x_0$ [Cauchy 1823, 246]. It appears that infinitesimals are not used as a tool for Taylor's theorem but that the new proof of this formula provides an opportunity to establish a theory of infinitesimals. On pages 257–261 immediately following the Addition the editors of the *Oeuvres* reprinted a research paper, "Sur les formules de Taylor and Maclaurin," which contains the proof without any mention of infinitesimals.

I suppose that Cauchy thought of his "direct consideration of infinitesimals" as a means of mathematics teaching. The textbook [Cauchy 1829] gives the theory of infinitesimals as a tool, primarily for the purpose of Taylor's theorem.

13. ORDERS OF CONTACT

In his text on differential geometry Cauchy needs a more general kind of infinitesimals than he has used for Taylor's formula [Cauchy 1826]. He dedicates more than 20%, 7 lines out of 33, of the *Avertissement* of [Cauchy 1826, 9–10] to underlining the importance of infinitesimals:

> On trouvera dans la neuvième, la vingt-unième et la vingt-deuxième Leçon, une nouvelle théorie des contacts des courbes et des surfaces courbes, qui a l'avantage de reposer sur des définitions indépendantes du système de coordonnées que l'on adopte, et de présenter en même temps une idée très-nette du rapprochement plus ou moins considérable de deux courbes ou de deux surfaces qui ont entre elles un contact d'un ordre plus or moins élevé. (In the ninth, twenty-first and twenty-second Lesson, one will find a new theory of the contact of

curves and curved surfaces, which has the advantage to rest on definitions which are independent of the assumed system of coordinates and which, simultaneously, presents a very clear impression of the more or less considerable approach of two curves or two surfaces which have among each other a contact of a more or less high order.)

This is a remarkable idea. Prior to the invention of vector methods a tool is used for an invariant or coordinatefree treatment of geometry, in the shape of infinitesimals.

Let P be the point of contact of two curves, let Q, R be points on the curves with infinitesimal distance i from P, and let $\omega = \omega(i)$ be the angle between the straight lines PQ, PR. The order of contact is defined as the least upper bound of all real numbers r for which ω/i^r is infinitesimal.

14. SYSTEMS OF INFINITESIMALS

Apart from the rather vague "variables converging to zero" Cauchy never says what his infinitesimals *are*; we are told only how infinitesimals can be *represented*. It is the same with real numbers: Mathematicians during the second half of the 19th century developed theories of the real numbers *themselves*; Cauchy was satisfied with *representations*, mainly by decimal numbers. As long as the important properties can be deduced from the representations this makes no difference for the calculus.

After preliminary considerations in [Cauchy 1823, 250 ff. (the *Addition*); 1826, 132 ff.] a final theory is presented in [Cauchy 1829, 281–286; 325–339]. The headline of Chapter 6 is "Sur les derivées des *fonctions qui représentent* des quantités infiniment petites" (my italics).

Without comment Cauchy uses the plural form (systems of infinitesimals) and usually speaks of "un système quelconque." Then i denotes the base of the system: "Soit toujours i la base du système adopté." This letter i is merely a symbol for something which is called an infinitesimal. In general, an infinitesimal is represented by $f(i)$ where $f(x)$ is a function defined in a neighborhood of $x = 0$ and vanishing at $x = 0$. Presumably $f(x)$ should be continuous in the vicinity of $x = 0$. All that is inferred from the texts. Cauchy himself begins by defining the concept of the order of an infinitesimal [Cauchy 1829, 281]:

> Nous terminerons ces Préliminaires en expliquant ce qu'on doit entendre par des quantités infiniment petites de divers ordres. Designons par a un nombre constant, rationnel ou irrationnel; par i une quantité infiniment petite, et par r un nombre variable. Dans le système de quantités infiniment petites dont i sera la *base*, une fonction de i représentée par $f(i)$ sera un infiniment petit de l'*ordre* a, si la limite du rapport $f(i)/i^r$ est nulle pour toutes les valeurs de r plus petite que a, et infinie pour toutes les valeurs de r plus grandes que a.

This means that the order a of the infinitesimal $f(i)$ is the uniquely determined real number (or $+\infty$, as with e^{-1/i^2}) such that $f(i)/i^r$ is infinitesimal for $r < a$ and infinitely large for $r > a$.

Cauchy proceeds to prove some very simple properties of the order. A typical example is: The order of a product equals the sum of the orders of its members.

Obviously, operations on infinitesimals are defined by the corresponding operations on the representing functions. The first theorem is on the relation $<$:

> Théorème I.—Si, dans un système quelconque, l'on considère deux quantités infiniment petites d'ordres différents, pendant que ces deux quantités s'approcheront indéfiniment de zéro, celle qui sera d'un ordre plus élevé finira par obtenir constamment la plus petite valeur numérique. [Cauchy 1829, 282–283; 1826, 133–134] (If, in any system, two infinitesimals of different orders are considered, whilst these quantities approach to zero indefinitely, that which has a higher order will finish by having always the smaller absolute value.)

From the examples considered in the sixth lesson of [Cauchy 1829, 325–339] it will be clear that a function $f(x)$ represents an infinitesimal if it is defined in some *positive* neighborhood of zero, and $\lim f(x) = 0$. Examples are $e^{-1/i}$ and $1/\log i$, with orders ∞ and 0 [Cauchy 1829, 326–327].

For the "prehistory" of Cauchy's orders of the infinitely small, see [Guitard 1986, 12–13].

15. ATTEMPTS TOWARD THEORIES OF INFINITESIMALS

All attempts to understand Cauchy from a "rigorous" theory of real numbers and functions including uniformity concepts have failed. Giusti's attempt based on sequences is only partly successful. One advantage of modern theories like the Nonstandard Analysis of Robinson [1966] is that they provide consistent reconstructions of Cauchy's concepts and results in a language which sounds very much like Cauchy's. I shall briefly sketch two different approaches which follow Cauchy's texts as closely as possible.

Since Cauchy needs expressions like $(f(x + i) - f(x))/i$ I will not restrict myself to functions converging to 0; since infinitesimals appear as denominators, even functions converging to infinity should be admitted. A Cauchy quantity, or C-quantity, will be represented by $f(i)$, where $f(x)$ is defined for $0 < x < \varepsilon$ and some finite ε. I shall say that $f(i)$ and $g(i)$ represent one and the same C-quantity, or $f(i) = g(i)$, if $f(x) = g(x)$ for sufficiently small $x > 0$. This kind of an equivalence class has sometimes been called a function germ. The algebraical operations and other relations like $<$ are defined via the representing functions in the obvious way, as is $F(f(i))$. The properties of a partially ordered ring are easily verified. If $\lim f(x) = a$ exists as a real number, then Cauchy writes $\lim f(i) = a$. The modern term is standard part. If $\lim f(i) = 0$, the $f(i)$ represents an infinitesimal C-quantity.

Though a complete theory of such C-quantities is still missing we are in possession of a partial theory which covers at least the concepts and results of the *Cours* and the *Résumé* [Cauchy 1821, 1823]. If the independent variable of a function representing a C-quantity is restricted to $x = 1/n$, n natural numbers, then only sequences $a_n = f(1/n)$ enter, and two sequences a_n, b_n represent one and the same quantity if $a_n = b_n$ for $n \geqq n_0$ and some natural number n_0. The theory was initiated in [Schmieden & Laugwitz 1958] and developed in [Laugwitz 1978, 1980a]. While Cauchy represents his base i by $f(x) = x$, we choose ω represented by $a_n = 1/n$ (or rather its reciprocal, Ω, represented by the sequence of natural numbers). The key concept is that of normal sequences and normal functions of quantities. (The term "normal" was replaced by Robinson's "internal" in more recent publications.) It turns out that Cauchy's concepts and theorems (including the hidden lemma mentioned in Section 10) fit in with this theory.

A different theory has been developed in [Laugwitz 1980b, 1983, 1986]. Its starting point is a generalization of the method of field extension. A symbol Ω (corresponding to Cauchy's $1/i$) is adjoined to the real numbers. If a formula $F(n)$ containing the variable n (for natural numbers) is true for all sufficiently large n, then $F(\Omega)$ is defined to be true in the extended theory. Again, Cauchy's concepts and results are reconstructed. This theory had been modeled after ideas of Leibniz and Euler. It can easily be modified in a manner which may be close to Cauchy's approach of the Addition through to [Cauchy 1829]: If a formula $F(x)$ is true for $0 < x < \varepsilon$, ε some positive real number, then $F(i)$ is true in the extended theory.

Even if we try, as present-day mathematicians we can hardly free ourselves from the influence of set-theoretical thinking which is certainly uncauchyous (sit venia verbo). As I have pointed out, neither single numbers nor sets of single numbers nor sets of pairs of numbers are basic in the theory of real functions, but variables and functions themselves are the primary objects (compare Section 7). Cauchy's intervals are not sets but loci where a variable can move freely. Even a nonstandard set (or overdense continuum, as Lakatos called it) does not hit the concept. Apparently continuity (at least piecewise) of any function appearing in the calculus is a deep-rooted conviction with Cauchy. That may explain the fact that he does not bother to show the continuity of differentiable functions.

For these reasons the attempts mentioned here should be taken with reluctance and reserve.

16. CONCLUDING REMARKS

It has been impossible to include comments or even bibliographical hints on the extensive work concerned with Cauchy's so-called errors; neither was it suitable to mention all of the purported justifications of Cauchy by nonstandard analysts, by Lakatos, and by others. Further references can be found in the books and papers mentioned in the bibliography, in particular the books of Robinson, Spalt, and myself.

Information on the origins of Cauchy's concepts and methods can be found in [Grabiner 1981] and [Guitard 1986]. The influence of Euler should not be neglected, with regard both to the organization of Cauchy's texts and, in particular, to the fundamental role of infinitesimals.

A preliminary version of this paper was dedicated to Curt Schmieden on the occasion of his 80th birthday (D. Laugwitz, "Cauchy and Infinitesimals," Preprint 911, Fachbereich Mathematik, Technische Hochschule Darmstadt, June 1985).

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[1] Page numbers for Cauchy refer to *Oeuvres.*