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Another theorem of Cauchy which 'admits exceptions'

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Abstract

Several exceptions are provided for a theorem in Cauchy's *Cours d'Analyse* in the proof of which the need for uniform convergence has been ignored. A reconstruction of this theorem is offered.

Résumé

Quelques exceptions sont donnés d'un théorème du *Cours d'Analyse* de Cauchy dans lequel il n'a pas été tenu compte de la nécessité de considérer la notion de la convergence uniforme. Une refection de ce théorème est proposé.

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1. Abel's example

Chapter VI of Cauchy's *Cours d'Analyse* [Cauchy, 1821, 123–172] is about series, but in what today would be, for the most part, a masterly first course on the convergence of series, Cauchy claimed that the limit of a convergent sequence of continuous functions must be continuous [Cauchy, 1821, 131–132]. In 1826 Abel pointed out in a letter to his teacher Holmboe in Oslo (added in [Abel, 1839, 71] as a note to his paper on the binomial series) that this theorem 'admitted exceptions' by citing the example:

$\sin x - \frac{1}{2}\sin 2x + \frac{1}{3}\sin 3x - \cdots = \frac{1}{2}x$ on $[0,\pi)$, but which is equal to 0 at $x = \pi$ and so discontinuous there.

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This example fruitfully stimulated Seidel [1847], Stokes [1847], and Cauchy himself [1853], in the developments which led to Weierstrass' account of uniform convergence.

2. Cauchy's theorems

Chapter II of Cauchy's *Cours d'Analyse* [Cauchy, 1821, 26–69] is divided into three sections. The third section [Cauchy, 1821, 45–69] is about singularities, points where the value of a function may be problematic (seemingly 0/0, or ∞/∞ for example, or infinite). In this third section, Cauchy proposed four related theorems.

1. If $f(x + 1) - f(x) \to k$ as $x \to \infty$, then $\frac{f(x)}{x} \to k$ as $x \to \infty$.
2. If $\frac{f(x+1)}{f(x)} \to k$ as $x \to \infty$, then $\sqrt[x]{f(x)} \to k$ as $x \to \infty$, provided that $f(x) > 0$ for large x.
3. If $A_{n+1} - A_n \to k$ as $n \to \infty$, then $A_n/n \to k$ as $n \to \infty$.
4. If $A_{n+1}/A_n \to k$ as $n \to \infty$, then $\sqrt[n]{A_n} \to k$ as $n \to \infty$, provided A_n is positive.

By taking logarithms, Theorem 2 may be deduced from Theorem 1, and similarly Theorem 4 may be deduced from Theorem 3. Theorem 3 will turn out to be a special case of Theorem 1, so that apparently the edifice is built on Theorem 1.

Consider the function defined by $f(x) = \frac{1}{1-x+[x]}$, previously proposed in Burn [2004], where $[x]$ denotes the integral part of x, $x \leqslant [x] < x + 1$.

For this function, $f(x + 1) - f(x) = 0$ for all values of x, while, for $x > 1$, $\frac{f(x)}{x}$ takes every value in the range $[1,\infty)$, on every interval of unit length.

Thus despite Theorem 1, there is no limit for $\frac{f(x)}{x}$ as $x \to \infty$, and we have an exception to the theorem. It is easy to construct further exceptions; for example, $f(x) + kx$, where k is a real number and f is the function just defined, and also $\tan \pi x$ may be adjusted for this purpose. Cauchy also claimed Theorem 1 for $k = \infty$, but this too is contradicted by the example $g(x) = 2^x - \frac{1}{1-x+[x]}$. In this case $g(x + 1) - g(x) = 2^x$ for all x, and for $x > 1, \frac{g(x)}{x}$ takes every value in the range $(-\infty,0]$, on every interval of unit length. It seems that we must search for some unstated assumption in Cauchy's argument.

3. Cauchy's proof

We examine Cauchy's proof of Theorem 1, for finite k [Cauchy, 1821, 48–50].

Cauchy's argument started by claiming that, given $\varepsilon > 0$, for sufficiently large h, $k - \varepsilon < f(x + 1)–f(x) < k + \varepsilon$, for $x \geqslant h$, where k is the limit.

Cauchy then claimed that the arithmetic mean of the numbers

$$\begin{aligned}&f(h+1) - f(h)\\&f(h+2) - f(h+1)\\&\&c\ldots\\&f(h+n) - f(h+n-1),\end{aligned}$$

where n is a positive integer, namely $\frac{f(h+n)-f(h)}{n}$, also lies between $k - \varepsilon$ and $k + \varepsilon$.

So,

$$k - \varepsilon < \frac{f(h+n) - f(h)}{n} < k + \varepsilon.$$

At this point Cauchy put $x = h + n$, and obtained $f(x) = f(h) + (x - h)(k + \alpha)$, where $-\varepsilon < \alpha < \varepsilon$. He then divided through by x and let $x \to \infty$, which seemed to show that $\frac{f(x)}{x} \to k$. But in shifting from $h + n$ to x there was a covert increase in generality, and what Cauchy had in fact proved is that $\frac{f(h+n)}{h+n} \to k$, as $n \to \infty$, that is to say, a theorem about the convergence of a sequence. This result holds for every possible choice of h, and therefore applies to an infinite family of sequences, but it does not expose what may happen when x increases continuously.

It is now clear that Cauchy's argument, as it stands, may be used to construct an excellent proof of Theorem 3, and as we have noted, Theorem 4 may be deduced.

Theorem 2 is subject to the same kind of exception as Theorem 1, using the function $F(x) = e^{\frac{1}{1-x+[x]}}$, for which $\frac{F(x+1)}{F(x)} = 1$ for all x, and for which $(F(x))^{1/x}$ has no limit as $x \to \infty$. Although we have cast some doubt on Theorems 1 and 2, Cauchy followed each of these theorems with three corollaries, applications of these theorems, which are manifestly sound and interesting.

4. The corollaries

Theorem 1.

Corollary 1 [Cauchy, 1821, 52]: $\frac{\log x}{x} \to 0$ *as* $x \to \infty$.
Corollary 2 [Cauchy, 1821, 52]: $\frac{A^x}{x} \to \infty$ *as* $x \to \infty$, *provided* $A > 1$.
Corollary 3 [Cauchy, 1821, 53]: *It is not necessary to use Theorem* 1 *to find the value of the ratio* $\frac{f(x)}{x}$ *when* $x = \infty$, *except when* $f(x)$ *becomes infinite with* x. *If the function is finite* [*i.e. bounded*] *for* $x = \infty$, *the ratio* $\frac{f(x)}{x}$ *clearly has* 0 *as limit.*

Theorem 2.

Corollary 1 [Cauchy, 1821, 57]: $x^{1/x} \to 1$, *as* $x \to \infty$.
Corollary 2 [Cauchy, 1821, 57]: *For a polynomial* P *in* x, $P^{1/x} \to 1$, *as* $x \to \infty$.
Corollary 3 [Cauchy, 1821, 58]: $[\log x]^{1/x} \to 1$, *as* $x \to \infty$.

With such corollaries, the theorem must clearly be salvaged. What then is the unstated assumption in the theorem whose validity enables these corollaries?

5. A reconstructed Theorem 1

Given. f is a real function such that $f(x+1) - f(x) \to k$, *as* $x \to \infty$, *and f is bounded on closed bounded intervals*:

To prove. $\frac{f(x)}{x} \to k$, as $x \to \infty$.

Let $g(x) = f(x) - kx$, then $g(x + 1) - g(x) \to 0$, as $x \to \infty$, and g is bounded on closed bounded intervals.

Given $\varepsilon > 0$, there exists an $L > 1$ such that, $-\varepsilon < g(x + 1) - g(x) < \varepsilon$, for $x \geqslant L - 1$.

$$\begin{aligned} g(x) = g(x) &- g(x-1) \\ &+ g(x-1) - g(x-2) \\ &+ g(x-2) - g(x-3) \\ &\dots \\ &+ g(x-N+1) - g(x-N) \\ &+ g(x-N), \\ &\text{for any positive integer } N. \end{aligned}$$

Now, suppose $x > L$. Choose N so that $L > x - N \geqslant L - 1$. Then $-N\varepsilon < g(x) - g(x - N) < N\varepsilon < x\varepsilon$, and so $| \frac{g(x)}{x} - \frac{g(x-N)}{x} | < \frac{N}{x}\varepsilon < \varepsilon$. But g is bounded on $[0, L]$, so for some M, $-M < g(x) < M$, on this interval, and so $-M < g(x - N) < M$, for all $x > L$. Thus $\frac{-M}{x} < \frac{g(x-N)}{x} < \frac{M}{x}$ for $x > L$, and $-2 < \frac{g(x)}{x} < 2\varepsilon$ for $x > \max\{L, M/\varepsilon\}$, and so $\frac{g(x)}{x}$ becomes arbitrarily small as $x \to \infty$. Thus $\frac{f(x)-kx}{x} \to 0$ and $\frac{f(x)}{x} \to k$, as $x \to \infty$. The same adjustment to the wording of Theorem 1 enables a proof in the case $k = \infty$.

The property of being bounded on closed bounded intervals, which we have added to the statement of Cauchy's theorem, holds for every continuous function, and so this version of the theorem establishes all of Cauchy's corollaries. This property of continuous functions was first established by Weierstrass in the late 1860s [Dugac, 2003, 131]. However the language of boundedness was only introduced by Pasch [1882] following the use of the concept by Weierstrass. Previous to that, a 'bound' was called a 'limit' (using the common-sense meaning of that term) and the 'boundedness' of a set of values described by calling the values 'finite', as in Corollary 1.3 above. But the term 'finite' is ambiguous. The function f which we have used for our counter-example is in fact finite for every value of x, though not bounded. In Smithies [1986], the distinction between the analysis of Cauchy and Weierstrass is drawn. As in some other errors of Cauchy, the lacuna in the present theorem is the absence of the notion of uniform convergence; for if from Cauchy's original proof we construct a sequence of functions $h_n(x) = f(x + n)/(x + n)$ for $x > 0$ and n a positive integer, then for each sufficiently large x, Cauchy's argument proves that $\lim_{n\to\infty} h_n(x) = k$, that is, pointwise convergence of (h_n), and for Cauchy, this argument established the theorem, but an attempt to show uniform convergence would have exposed the mistake.

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