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On the Relations between Georg Cantor and Richard Dedekind

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This paper gives a detailed analysis of the scientific interaction between Cantor and Dedekind, which was a very important aspect in the history of set theory during the 19th century. A factor that hindered their relationship turns out to be the tension which arose in 1874, due to Cantor's publication of a paper based in part on letters from his colleague. In addition, we review their two most important meetings (1872, 1882) in order to establish the possible exchange of ideas connected with set theory. The one-week meeting in Harzburg (September 1882) was particularly rich in consequences, among other things Dedekind's proof of the Cantor–Bernstein equivalence theorem. But the analysis of this episode will corroborate the lack of collaboration between both mathematicians. © 1993 Academic Press, Inc.

Este artículo presenta un análisis detallado de la interacción científica entre Cantor y Dedekind, que fue muy importante para el desarrollo de la teoría de conjuntos en el siglo XIX. Un factor que obstaculizó sus relaciones fue la tensión provocada hacia 1874 por la publicación de un artículo de Cantor, parcialmente basado en cartas de su colega. Aparte de esto, se repasan las principales ocasiones de contacto directo entre los dos matemáticos (1872, 1882) con el fin de aclarar el posible intercambio de ideas conjuntistas. El encuentro de una semana en Harzburg (septiembre de 1882) resulta especialmente rico en consecuencias, entre otras la prueba del teorema de equivalencia Cantor–Bernstein dada por Dedekind. El análisis de este episodio mostrará sin embargo que no hubo una verdadera colaboración entre ambos matemáticos. © 1993 Academic Press, Inc.

In der vorliegenden Arbeit wird die wissenschaftliche Wechselwirkung zwischen Cantor und Dedekind, die für die Entwicklung der Mengenlehre von grosser Bedeutung gewesen ist, ausführlich untersucht. Ein Faktor, der das Verhältnis verhinderte, war die Spannung, die 1874 entsprang, infolge der Veröffentlichung einer Abhandlung von Cantor, die zum Teil auf Dedekinds Briefen beruhte. Weiterhin sollen die beiden wichtigsten persönlichen Begegnungen (1872, 1882) betrachtet werden, um zu klären, inwieweit mengentheoretische Ideen ausgetauscht worden sind. Das einwöchige Treffen in Harzburg (1882) war besonders folgenreich: unter anderem stand es in einer unmittelbaren Beziehung zum Dedekindschen Beweis des Cantor–Bernsteinschen Equivalenzsatzes. Die Untersuchung dieser Episode aber wird den Mangel an Zusammenarbeit unterstreichen. © 1993 Academic Press, Inc.

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> Es ist für mich die Lücke [in the relations] umso empfindlicher, als ich seit einer Reihe von Jahren mich daran gewöhnt habe, meine inneren mathematischen Erlebnisse Ihrem gereiften Urtheil zu unterbreiten [Cantor to Dedekind, November 5, 1882. In Cantor & Dedekind 1937, 55]

1. INTRODUCTION

In 1908, Zermelo presented his famous axiomatization of set theory as a way of recovering "the entire theory created by Cantor and Dedekind" from the difficulties posed by the antinomies [Zermelo 1908, 200]. Zermelo's opinion regarding the origins of set theory, as expressed in that sentence, is somewhat unorthodox, for it has usually been said that Cantor was, alone, the only creator [1]; nevertheless, in my view that opinion is quite right if properly construed (see Section 2 below). Now, if we accept it, an interesting and still unsolved question appears when we consider that Dedekind and Cantor knew each other personally early on, starting in 1872, when neither of them had yet published on set-theoretical matters. The question is, of course, that of relevant influences which could have come through their personal meetings.

Not only is their interesting correspondence well known, but the traditional view is that Cantor and Dedekind were good friends who met each other frequently. Fraenkel (in his [1930] and in the brief version of it included in [Cantor 1932]) spoke about frequent personal meetings which occurred mainly in Harzburg [Cantor 1932, 456]. Of course, this opens the possibility of an intense interchange of ideas, and Fraenkel himself commented on the influence of Dedekind's writings and style on Cantor [456–457] [2]. This article is the result of an analysis of the available material in order to investigate the possible influences and their effects on the development of set theory.

The Cantor–Dedekind correspondence does not suffice, alone, to answer that question: almost all of it was written at Cantor's initiative and in order to discuss his new ideas while, in most of the correspondence, Dedekind played only the role of a critic. Nevertheless, the letters offer some information on the development of the relations between both mathematicians and especially on their personal meetings. These portions of the correspondence have not been completely exploited in the past, and I have used them, together with other published and unpublished sources, in order to discuss the exchange of set-theoretical ideas between both mathematicians.

First, I will present a brief account of the state of Cantor's and Dedekind's views on sets up to 1872, that is, prior to knowing each other personally. We will see that Dedekind's conceptions were more advanced, and so it seems more probable that a substantial influence would come from him.

But, second, a close study of the early correspondence brings unexpected results, and we will see that the image suggested by Fraenkel is to a large extent wrong. Relations between both mathematicians were difficult after 1874, when they underwent an interruption, studied in Section 3. To some extent, this implies a negative answer to the question stated above; the difficulties originating in 1874 had as a consequence a certain lack of collaboration and mutual reinforcement between both mathematicians.

In fact, the correspondence follows a peculiar rhythm, with intense periods of contact, followed by characteristic gaps which demand some explanation (see Appendix I). This pattern contrasts sharply with both Cantor's correspondence with Mittag-Leffler, which involved hundreds of letters over a short period, and Dedekind's correspondence with Heinrich Weber, which was sustained and constant. Yet in spite of the accessibility of this material, no historian seems to have analyzed such questions in full detail, and the lack of such a detailed study explains why the interruption studied here has been overlooked by most historians. It is still generally believed, following [Grattan-Guinness 1971, 1974], that the only interruption of the relations occurred in 1882/83 (this view is supported by [Purkert & Ilgauds 1987, 73–75]); and most experts on Cantor (Grattan-Guinness, Dauben) simply avoid the uncomfortable events related to [Cantor 1874]. The only exception seems to be [Dugac 1976, 116–118], who stressed the relevant facts, but without supporting them with further analysis. I hope the analysis offered here, in Section 3 and Appendix II, is balanced enough to settle the question.

Likewise, the total number of meetings that we can trace is *six* (see Appendix I), a surprisingly small number if judged from Fraenkel's viewpoint. Given the common interests that both mathematicians had, the proximity of their cities, and even the fact that both spent their vacations in the Harz Mountains, the small amount of contact is surprising.

As a result, we should not expect many unnoticed influences, with the exception of two particular meetings. The first was their initial encounter in 1872, considered in Section 2; little is known about it. The second important meeting was in 1882, when, for reasons that we will see in Section 4, their relations bettered. The material available throws some light on the content of the 1882 conversations, and here we will consider the Cantor–Bernstein equivalence theorem, which was discussed on this occasion and which played a role in the subsequent work of both mathematicians.

A careful judgement of the events discussed here is impossible without reference to the Cantor–Dedekind correspondence, which has been published in its entirety, although in scattered locations. The mathematical portions of the correspondence were published by Emmy Noether and Jean Cavaillès [Cantor & Dedekind 1937], except for letters from 1899 that had already been edited by Ernst Zermelo in [Cantor 1932]. A French translation of both sets of letters can be found in [Cavaillès 1962, 177–251]. The remainder of the correspondence seemed lost after World War II, but it was rediscovered in the United States some 30 years ago [Grattan-Guinness 1974] and published in [Dugac 1976].

In order to avoid misunderstandings, I should indicate that the letters contained in [Meschkowski & Nilson 1991] are insufficient to judge the events studied here: the editors did not have in mind, as an aim in the selection of passages, the events we are going to discuss, and so they did not include some texts which are essential for our purposes here.

2. CANTOR AND DEDEKIND IN 1872

Dedekind was 14 years older than Cantor, and it is possible to show that, by 1872, his strong interest in foundational matters had led him to what I would term a *general program* for the set-theoretical foundation of classical mathematics. On this point, I will give here a brief presentation of my argument, a detailed discussion of which will appear in a forthcoming article (many of the relevant facts can be found in [Dugac 1976] and especially in [Dugac 1981]).

In 1871, as a result of an idiosyncratic effort, Dedekind published his ideal theory, where he proposed notions stated in terms of sets for dealing with algebraic questions: fields for algebra in general, ideals for the factorization of integers [Dedekind 1871] [3]. In 1872 he presented his well-known theory of real numbers on the basis of the so-called Dedekind cuts, i.e., partitions of the set of rational numbers. But this theory was just part of a set-construction of the whole number system, as is apparent from older manuscripts in which Dedekind develops the usual definitions of integers by means of pairs of natural numbers, and rationals by means of pairs of integers [Dedekind *Nachlass*, III, 2 and III, 4]. And it was also in 1872 that he began writing the draft for his monograph *Was sind und was sollen die Zahlen*? [Dedekind 1888], in which he established a theory of sets and mappings as the basis for a rigorous development of the theory of natural numbers and all of arithmetic. This draft can be found in [Dugac 1976, 293–309] and its dating comes from Dedekind himself (see [Dedekind 1888, 336], where he says that it was written from 1872 to 1878). The fact that it had been finished by 1878 is further corroborated by two letters to H. Weber written in November 1878 [Dugac 1976, 272–273].

Moreover, arithmetic was in Dedekind's view a very broad discipline which subsumed both algebra and analysis [Dedekind 1888, 335]. His expositions of ideal theory in 1871 and 1879 show the way in which sets and mappings formed a foundation for algebra [Dedekind 1871, 1879] (see also the constant references to the theory of sets and maps [Dedekind 1888] in the last version of ideal theory [Dedekind 1893]). In fact, the 1871 presentation of ideal theory in terms of sets, so strange for contemporary authors, can be partially explained by the fact that it was very natural within the context of Dedekind's general program. As he states in [Dedekind 1877, 268–269], Dedekind regarded ideals as arithmetical objects analogous to real numbers, and thus their definition was similar to that of real numbers [Dedekind 1872] and obeyed analogous methodological principles. As regards analysis, Dedekind never explained its arithmetical foundations, but given the set-theoretical definitions of $\mathbb{R}$ and $\mathbb{C}$, real- and complex-valued functions could obviously be defined by means of mappings [4].

In short, for Dedekind the theory of sets and maps was the foundation of all pure mathematics, and he seems to have developed this view already around 1872. To stress the relevance of Dedekind's conceptions for the development of set theory, the fact should be mentioned that the draft begun in 1872 already contained an extensional notion of set, the general notion of mapping, and Dedekind's famous definition of infinite sets. All of this appears at the very beginning of the draft (cf.

[Dugac 1976, 293–294]) and therefore seems to belong to Dedekind's 1872 view. We see that his concept of sets and their place in mathematics was already a very sophisticated one, and it is natural to wonder whether he informed Cantor of his views, thereby helping to shape the future orientation of Cantor's research.

Meanwhile, Cantor was also advancing in the direction of set theory. He began his work on sets while developing a line of research which was central at this time: the analytical study of functions with infinitely many singularities, which also attracted the attention of mathematicians like Hankel, du Bois-Reymond, and Dini [5]. Precisely in 1872, Cantor introduced a notion that was crucial for the development of his set theory and the theory of point-sets generally: the notion of a derived set [Cantor 1872]. (Given a point-set P, its derived sets are defined iteratively: P′ is the set of all limit points of P, and similarly for P″ with respect to P′, etc.; see [Hawkins 1975] or [Dauben 1979].) In the same paper, Cantor presented his theory of real numbers, based on Cauchy sequences. This theory did not build upon a broader set-based conception of the number system, as was the case with Dedekind's, but it was related to the study of derived sets and introduced a subject which was to guide Cantor in his exploration of infinity.

In connection with this, it is interesting to note that, according to Fraenkel, Cantor had already considered matters of cardinality prior to 1872. He had proved the denumerability of $\mathbb{Q}$ [Fraenkel 1930, 199] and possibly that of n-tuples (cf. [Cantor & Dedekind 1937, 13]). This shows his early speculative interest in the infinite, although these early results were far less significant than his crucial discovery of 1873. In stating the question of the denumerability of $\mathbb{R}$ and answering it negatively, Cantor opened the "paradise" of the transfinite; he proved the existence of two different infinite cardinalities, thereby revealing that an unexpectedly rich structure underlies infinite sets.

Therefore, it is clear that Cantor had made important advances prior to knowing Dedekind personally. In fact, one can argue that the notion of a derived set and that of power or cardinality of an infinite set were the origins of transfinite set theory, and neither of these ideas was suggested to Cantor by Dedekind. Thus, transfinite set theory has to be considered basically a creation of Cantor, even if it had important areas of overlap with Dedekind's theory of infinite sets—an instance will be shown in Section 4.2.

Other authors of the time did not develop a set-theoretical view of mathematics, nor an abstract set theory, and so the contributions of Dedekind and Cantor seem to have been crucial for the process of development of a theory of sets. Cantor can be described as the "creator" of transfinite set theory, even though the stress which is normally placed on him as the first mathematician who took seriously the infinite overlooks the important contributions of Dedekind. At the same time, Dedekind has to be taken into account centrally in order to discuss the rise of the set-theoretical view of mathematics, and also in any general account of the development of abstract set theory. It is in this sense that Zermelo's opinion, quoted earlier, seems justified.

Having reviewed the main motives for Dedekind's and Cantor's involvement with the study of sets and the state of their views around 1872, we turn to discuss the possible relevance of their first, accidental encounter in Switzerland, during the year 1872. Although Cantor had already taken important steps toward a theory of infinite sets, his ideas seem to have been much less systematic than those of Dedekind. Thus, the latter's views could have played an important role in convincing Cantor of the interest of an autonomous study of sets, of their important role in mathematics, or even of the acceptability of actually infinite sets.

It is therefore very tempting to speculate on possible topics that came up in their conversations. For instance, since they had both just published their well-known theories of the real numbers, the foundations of the number system were almost certainly a subject dealt with. Through incidental comments, we know this was not all (cf. [Dugac 1976, 225], where we read that an article on probability by Dedekind was mentioned). Since both had wide-ranging interests, we might consider many other possibilities including Riemann's work on the integral—published by Dedekind in 1868 and directly relevant to Cantor's research—and Dedekind's algebraic number theory—Cantor was a student of Kronecker, the other great expert on algebraic number theory, and had written his theses on number-theoretical subjects.

As regards this last possibility, there is in fact some evidence suggesting that Dedekind's number-theoretical writings influenced Cantor. The terminology used by Dedekind [1871, 224] for set operations in the context of fields reappears in the crucial series of papers published by Cantor from 1879 to 1884 (cf. [1880b, 145–146] and the following issues). Dedekind had named the set operations after basic notions of number theory; thus, subsets were "divisors," the union of sets was their "least common multiple," etc. It is, indeed, surprising to see how Cantor employed this very same terminology in the context of point-sets, where it has little meaning [6].

Coming back to the 1872 meeting, the fact that Dedekind began his draft [Dedekind 1872/78] precisely in that year could well have been a consequence of his conversations with Cantor. But unfortunately, at present we do not know of any document containing substantial information about the 1872 conversations. Instead of further speculating about the meeting, I will go on to the next episode in the relationship: the beginning of the scientific correspondence and the early personal problems.

3. THE INTERRUPTION IN 1874

In 1874, Cantor published his famous paper [1874] containing the proof that $\mathbb{R}$ is non-denumerable; the paper was entitled "Über eine Eigenschaft des Inbegriffes aller reellen algebraischen Zahlen" ("On a Property of the Collection of All Real Algebraic Numbers"). Although the property mentioned in the title was the denumerability of algebraic numbers, demonstrated in Section 1, the paper included a Section 2 devoted to Cantor's crucial result showing the non-denumerability of $\mathbb{R}$. The content of the article had been discussed with Dedekind in letters of November and December 1873, and it seems to have led to tensions between them.

The letters written between November 29 and December 27, 1873 were the only portion of their correspondence on which Dedekind felt the need to take notes. These notes were published as a complement to Cantor's letters in [Cantor & Dedekind 1937, 17–20]. Exactly when they were written is unknown, although it seems probable that it was at a somewhat later time.

Despite the careful and objective tone which characterizes Dedekind's notes, they reveal a feeling of surprise about Cantor's behavior. According to Dedekind, Cantor's paper [1874] was based on a transcription of two of his letters, one written on November 30 or December 1, the other on December 8, 1873, both of which are now lost [Cantor & Dedekind 1937, 18–19]. Dedekind's remark concerns the redaction of the paper and does not mean that the crucial non-denumerability theorem was due to him. Yet, although this theorem was Cantor's own, the paper also contained another which stemmed from Dedekind.

When Cantor posed the problem of the denumerability of $\mathbb{R}$, on November 29, Dedekind answered that he was unable to solve it, but at the same time he stated and proved the theorem on the denumerability of the set of algebraic numbers [Cantor & Dedekind 1937, 18]. Although Dedekind's letter is no longer extant, the point is confirmed by Cantor's next letter, acknowledging receipt of the proof on December 2 [Cantor & Dedekind 1937, 13]. Now, as Dedekind wrote, "after a short time, this theorem and its proof were reproduced almost literally, including the use of the technical term 'height' [Höhe], in Cantor's article" [Cantor & Dedekind 1937, 18] (cf. [Cantor 1874, 116]). After having confirmed the difficulty of the original problem he posed to Dedekind, Cantor was soon able to show the non-denumerability of $\mathbb{R}$. He wrote a letter to Dedekind on December 7 containing his new result, and the following day Dedekind sent a new version of Cantor's proof, making its core simpler and more precise. Again, "this presentation was transcribed, almost word for word, in Cantor's article" [Cantor & Dedekind 1937, 19] (cf. [Cantor 1874, 117]).

Above all, it had been Dedekind who stated and proved the theorem of denumerability of the algebraic numbers; and according to the title, this was the main result in Cantor's paper! [7]. According to Dedekind's notes, Cantor used his theorem without asking for permission, and in fact he is never mentioned in the paper. Moreover, Cantor did not claim simultaneous discovery of the proof in any of his letters. He did tell Dedekind on December 25, after having sent the paper, that his comments and formulations had been "very useful" for his publication [Cantor & Dedekind 1937, 16]. This episode became a source of some tension between them.

This is not essentially new: the information mentioned above has been available since the publication of the Cantor–Dedekind correspondence in 1937, and the relevant passages have been stressed by Dugac [1976, 116–118]. Nevertheless, some historians are still not convinced of the accuracy of Dedekind's comments or their implications. Meschkowski [1983, 235–236] has even responded to Dugac by trying to show that the former's interpretation was untenable. Nevertheless, an analysis of the correspondence and Cantor's paper, based on rather technical details of the proofs, seems to support Dedekind's claim; the interested reader can find the details in Appendix II.

Moreover, the existence of an interruption in the correspondence and of a certain amount of tension is also suggested by other sources. It appears that from March 1874 onward Cantor's letters remained unanswered, and in a letter of May 5, 1874 he feels the need to stress his interest in maintaining communications [Dugac 1976, 228]. It could be that some of Dedekind's letters are missing, but another letter by Cantor, written October 10, 1876 [Dugac 1976, 229], confirms that there had been no exchanges of letters or personal contacts during the preceding two years. In addition, many years later, Cantor himself recalled these tensions. In a recently published letter to Hilbert, dated November 15, 1899, he speaks about his old desire to communicate with Dedekind on the problem of the antinomies:

> *Only this fall* did I have an opportunity to discuss it with him, because for *reasons unknown to me*, he had been angry with me [mir . . . gezürnt] for years, and *from 1874* he *had almost broken off* [abgebrochen] the earlier correspondence [which began in] 1871. [quoted in Purkert & Ilgauds 1987, 154; italics are Cantor's]

This letter was written during a period in which Cantor began, or was about to begin, to suffer serious attacks of mental illness [Grattan-Guinness 1971, 365–368]. This could explain some of its peculiarities and forces us to treat its contents carefully. Still, as regards a break in the correspondence, it fits perfectly well with the rest of the evidence.

To be sure, the core of the non-denumerability theorem, and the problem itself, were completely original achievements of Cantor's. This theorem alone would have sufficed to secure him a place in the history of mathematics. Nevertheless, Dedekind had reason enough to take offense: his contribution had gone unmentioned, and, according to the title and presentation, the main content of Cantor's paper was in fact Dedekind's theorem.

Cantor and Dedekind met once again in Switzerland around October 1874. Although no details from this meeting are known, Dedekind's notes, and the fact that another two years would pass before they heard from each other (Cantor to Dedekind, October 10, 1876, in [Dugac 1976, 228–229]), indicate that the meeting was probably less than cordial. Taking this into account, it would even appear justified to speak about a rupture in the relationship.

Still, I must add that we have no reason to think that Dedekind ever discussed the source of these tensions openly with Cantor; on the contrary, at the end of his notes he wrote that they had never talked about it [Cantor & Dedekind 1937, 20]. These circumstances reveal a characteristic trait of Dedekind's personality; being very ceremonious and formal, he probably was astounded by Cantor's behavior and took it for granted that he, Cantor, would make the first move and explain himself. On the other hand, these events show the worst side of Cantor's passionate nature.

Nevertheless, the strains seem to have slowly relaxed over the years, and the next contact occurred in 1876, at Dedekind's initiative. At that time, he sent some just published works to Cantor [Dugac 1976, 229], thereby showing that he was now open to reconciliation. This paved the way for a new interchange of letters in May 1877, and during June and July they discussed the subject of Cantor's next treatise: one-to-one correspondences between continua with different dimensions.

Surprisingly, some remarks which Dedekind had expressed regarding Cantor's new ideas appeared once again in a paper by Cantor and, again, without any mention of the source. Dedekind found a weakness in the first proof communicated to him by Cantor, which is explained in Section 7 of the paper [Cantor 1878, 130–131] (cf. Cantor's original proof of June 20, 1877 in [Cantor & Dedekind 1937, 25–26] and Dedekind's objection of June 22 in [Cantor & Dedekind 1937, 27–28]). Beyond this, Dedekind convinced Cantor that he should not be too critical of prior conceptions of dimension, and he made very interesting remarks concerning the interpretation of the theorem, including a precise statement of the theorem on the invariance of dimension under bicontinuous mappings (letter of July 2 in [Cantor & Dedekind 1937, 37–38]; cf. also [Johnson 1979/81, Chap. 2]) [8].

By the end of 1877, the correspondence showed signs of warmth, especially from Dedekind [Dugac 1976, 230–231], but this seems to have ceased after the appearance of Cantor's paper in 1878. During all of 1878, Cantor and Dedekind remained out of contact (letter from Cantor, December 29, 1878, in [Dugac 1976, 232]) and throughout the whole period from 1879 to 1881 there was very little communication between them (cf. letter from Cantor, January 18, 1880 [Dugac 1976, 233], where Cantor considers the possibility that he had unintentionally angered his colleague). Thus the possibility of a friendship, which was glimpsed at in 1877, disappeared, and distance characterized the relationship.

Why did Cantor act as he did? One possible reason could have been some kind of rivalry or jealousy regarding the paternity of set theory [9]. Yet this seems forced; if such a feeling existed, it should have arisen later; after all, there was still no published set theory. On other occasions, Cantor was honest and appreciative; for instance, his 1870/72 papers on trigonometrical series acknowledge debts to Heine, Schwarz, Weierstrass, and Kronecker [Cantor 1932, 71, 82, 84]. So why did he behave so differently in his dealings with Dedekind? It is noteworthy that all of the names mentioned above were mathematicians closely tied with the Berlin school. It thus appears plausible that Cantor's behavior was related to concerns regarding his academic career and his relations with Berlin mathematicians.

Although his later confrontation with Leopold Kronecker has been told repeatedly (cf. [Fraenkel 1930; Grattan-Guinness 1974; Dauben 1979; Purkert & Ilgauds 1987]), Cantor's relations with him and with Ernst Eduard Kummer seem to have been reasonably good throughout the 1870s. Cantor worked under them while preparing his two theses (doctorate in 1867, habilitation in 1869), and he received some ideas from Kronecker for a short paper of 1871 [Cantor 1932, 84]. It is well known that toward the end of 1877 Cantor was nervous about a presumed delay in the type-setting of his [1878], a delay which he only attributed to Kronecker [Schoenflies 1927, 5–6] some seven years later. Nevertheless, in the same letter of November 10, 1877, where he talks about the delay, he reports the positive reaction of the Berlin masters—i.e., Weierstrass, Kronecker, and Kummer—to his new result [10]. And even a letter from October 15, 1879 reports on a visit to Kronecker which occurred on good terms [Dugac 1976, 233]. Taking all of this into account, we can safely say that Cantor faced no more opposition from Kronecker than did Heine or Weierstrass himself [11], and that, although it seems probable that from 1877 onward he felt suspicious of Kronecker, there was no real confrontation between them until well into the 1880s (see his letters, from 1882 to the first nervous breakdown in 1884, in [Meschkowski & Nilson 1991, 59–61, 65, 127–129, 162–173, 192–201]).

As a young mathematician, Cantor hoped for a brilliant academic career, and most probably he thought that his relationship with the Berlin school would help secure an attractive position in the future (cf. [Purkert & Ilgauds 1987, 53–55, 76, 93]; see also [Dauben 1979, 162–164] and the letter to Hermite, January 22, 1894 in [Meschkowski 1965, 514]). But there are reasons to believe that Kronecker and Kummer were somewhat unhappy with Dedekind. Both Dedekind and Kronecker worked on a subject initially investigated by Kummer, the problem of unique factorization of algebraic integers, and both were able to provide completely general solutions (cf. [Edwards 1980]). Kronecker was Kummer's closest friend and colleague, and both wrote that Kronecker had obtained a general theory in 1858 [Edwards 1980, 329]. Nevertheless, the latter only came to publish a version of this in 1882, 11 years after the appearance of Dedekind's ideal theory. In connection with this, Frobenius once wrote to Dedekind about his surprise to see a letter in which

> Kronecker acknowledges without reservation your priority in publishing ideal theory, something that he had never done before, orally or in writing. This acknowledgement astonishes [frappirt] me all the more, since I am quite certain that he has never forgiven you for that publication [Dugac 1976, 280].

It should be noted that Frobenius knew Kronecker very well; for instance, it was he who held the "Gedächtnisrede" upon his death before the Berlin Akademie der Wissenschaften. Moreover, in a letter to Mittag-Leffler written on November 9, 1883, Cantor commented that in his [1883, 183] he had payed Kronecker, "together with Dedekind (which angers him much), a compliment" in connection with their algebraic and number-theoretical works [Meschkowski & Nilson 1991, 144] [12]. Thus, it seems quite plausible that in the 1870s Cantor was aware of these feelings and that he feared the enmity of Kronecker and Kummer if he called attention to Dedekind's collaborative role in his own published work [13].

4. HARZBURG, SEPTEMBER 1882

Toward the end of 1881, a professorship in mathematics became open at the University of Halle, and Cantor immediately offered it to Dedekind [14]. Yet despite all his efforts and the many letters he sent, it was impossible to convince Dedekind to move from Braunschweig to Halle. The episode is well documented in [Grattan-Guinness 1974, 116–123] and [Dugac 1976, 126–128], and it is generally assumed that Dedekind's refusal to accept the position in Halle was the reason their correspondence ended in 1882. Nevertheless, a more detailed analysis sheds a different light on the events.

Cantor's interest in having Dedekind at Halle seems to have engendered once again good feelings in Dedekind, and in his letters from late 1881 and January 1882, the latter spoke very positively about the idea of working together with Cantor [Dugac 1976, 239–240, 246]. Moreover, after Dedekind declined the offer in January, their correspondence continued, addressing interesting themes, such as continuous motion in discontinuous spaces (April 7 and 15 [Cantor & Dedekind 1937]), the definition of a continuum (September 15 and 30, October 2 [Cantor & Dedekind 1937]), and the introduction of transfinite numbers (November 5 and 6 [Cantor & Dedekind 1937]). Thus, an improvement in their relations took place during 1882, and they actually met on two separate occasions in September. The first meeting took place in Harzburg September 7–12, as Dedekind mentioned in the second draft for his [1888] [Dedekind *Nachlass,* III, 1, II, p. 40]. The second took place in Eisenach, where both attended the *Naturforscherversammlung* [Dugac 1976, 255–256].

Regarding the Eisenach meeting, we know that the question of the general definition of a continuum was discussed, since Cantor had sent related materials in preparation for such a discussion [Cantor & Dedekind 1937, 53–54]. But it is the Harzburg meeting which is highly interesting for our purposes. Above all, we know that Dedekind's draft [Dedekind 1872/78] for *Was sind und was sollen die Zahlen?* was discussed by both mathematicians during the Harzburg meeting [Dedekind *Nachlass*, III, 1, II, p. 40] (cf. also [Dedekind 1888, 336, 356 footnote]). Taking this fact into account, it is possible to find several interesting similarities between this draft and Cantor's subsequent writings.

In his draft Dedekind presented his foundational viewpoint based on a theory of sets and maps, and so it is natural to think that they should have discussed the role of set theory in mathematics. Here their positions differed, for up until this time Cantor had never, either in published writings or in letters, considered set theory as a *foundation* for mathematics, whereas this had been Dedekind's position for some years past. Significantly, in his unpublished paper of 1884/85, Cantor spoke about set theory as the basis of mathematics [Cantor 1970, 84], so that here the influence of Dedekind's vision seems to have been decisive.

Similarly, it was only after the 1882 meeting that Cantor began publishing ideas about the set-theoretical foundations of natural numbers (see [Dauben 1979, 176–179; Purkert & Ilgauds 1987, 131–132]). Therefore, it is difficult to escape the conclusion that he had been led to this by his conversations with Dedekind (contrary to the reading of Purkert & Ilgauds).

The Harzburg meeting also gives some clues which shed new light on the emergence of Cantor's transfinite numbers, although a satisfactory account of this would take us too far afield. Instead, I wish to discuss another surprising episode connected with the 1882 meeting which again underscores the lack of collaboration between the two mathematicians.

The fact that Cantor and Dedekind discussed problems in general set theory at

Harzburg is further corroborated by Cantor's letter of November 5, 1882 [Cantor & Dedekind 1937, 55], where he affirms having told Dedekind about his difficulties in proving a theorem, namely the crucial lemma in the proof of the Cantor–Bernstein theorem on the comparability of transfinite cardinals.

CANTOR–BERNSTEIN THEOREM. *If set A is equipollent with a proper subset of set B, and B is equipollent with a proper subset of A, then A and B are themselves equipollent.*

To prove this, it suffices to solve the following "problem" posed by Cantor: given $M_2 \subset M_1 \subset M$, where M_2 and M are equipollent, to show that M_1 and M are equipollent [Cantor & Dedekind 1937, 59]; this is the lemma mentioned above [15]. In fact, suppose that Φ maps A bijectively onto a subset of B, and Θ maps B bijectively onto a subset of A; then, using Cantor's lemma with $M = A$, $M_1 = \Theta(B)$, and $M_2 = \Theta\Phi(A)$, obviously M and M_2 are equipollent, and so $\Theta(B)$ and B are equipollent to A.

Dedekind was perfectly aware of the importance Cantor attached to this lemma: they talked about it in Harzburg, and Cantor mentioned it twice in his letter of November 5 [Cantor & Dedekind 1937, 55, 59] and once in [Cantor 1883, 201]. In 1887, while working on a second draft of [Dedekind 1888], Dedekind obtained a proof of both the lemma and the Cantor–Bernstein theorem; this proof was found by Cavaillès in Dedekind's *Nachlass* and published in [Dedekind 1930/32, Vol. 3, 447–448]. The tool for demonstrating these results was Dedekind's chain theory, the most original part of [Dedekind 1888], which was essential for his abstract characterization of the set of natural numbers.

Given a mapping $\Phi: K \rightarrow K$, Dedekind defines *the chain of set $A \subseteq K$*, denoted A_0, to be the intersection of all sets $C \subseteq K$ such that $\Phi(C) \subseteq C$ and $A \subseteq C$ [Dedekind 1888, 353]. The chain of A is just a subset of K, namely the closure of A under Φ in K; the reason for calling it a chain is the structure imposed by Φ on it. Since A_0 depends on the base-mapping Φ, Dedekind also proposed to denote it more explicitly as $\Phi_0(A)$. Since Dedekind used chains to define numbers, it was essential that he avoid any implicit use of the natural numbers (for instance, he could have said that A_0 contains A and all elements $k \in K$ such that, for any $a \in A$ and $n \in \mathbb{N}$, $\Phi^n(a) = k$). For this reason, Dedekind came to employ a typically impredicative definition in Poincaré's sense [Poincaré 1908].

Asuming an injective mapping $\Phi: K \rightarrow K$, Dedekind characterized the set of natural numbers as any set $N \subseteq K$ with a distinguished element (that we call 1), for which the following conditions hold: (a) $\Phi(N) \subseteq N$; (b) $1 \notin \Phi(N)$; (c) $N = \{1\}_0$ [Dedekind 1888, 359]. Here, condition (c) is essential: the fact that N is the chain of the unitary set $\{1\}$ ensures that induction will hold, being a condition equivalent to Peano's axiom of induction. It also ensures, as Dedekind noted, that N does not contain "extra" elements of K, i.e., what we would call non-standard elements (cf. Dedekind's 1890 letter to Keferstein in [Sinaceur 1974, 251–278], or the English translation in [van Heijenoort 1967]).

But the application of chain theory in Dedekind's proof of the Cantor–Bernstein theorem showed that it was more than a mere instrument for dealing with natural numbers and that it was, in fact, a general tool for set theory [16]. Oddly enough, Dedekind decided not to include this theorem in [Dedekind 1888], thereby missing an opportunity to show the importance of his chain theory. In fact, the matter is even more mysterious: Dedekind's exposition of chain theory concluded with an obscure proposition which, as he himself remarked, was not used in the remainder of the book and whose proof was left to the reader. The proposition is an obscurely formulated version of Cantor's lemma.

Dedekind's general theory of chains in Section 4 of [Dedekind 1888] is characterized by the fact that the mappings are not required to be injective. Proposition 63, the last in Section 4, is also stated for non-injective mappings, and this is what obscures its content. Given a mapping Φ, and using the notation K' for images $\Phi(K)$, Dedekind [1888, 356] starts by assuming that $K' \subset L \subset K$, which means that K (and also L, since $L' \subseteq K' \subseteq L$) is a chain. This starting supposition is the one that characterizes Cantor's lemma. Then, Dedekind asserts that if one takes $U = K \backslash L$ (the complement of L in K) and $V = K \backslash U_0$ (where U_0 is the chain of set U), one can establish the following decomposition of L and K: $K = \bigcup(U_0, V)$ and $L = \bigcup(U_0', V)$. (Here, U_0' is what Dedekind calls the image-chain of U_0, i.e., $\Phi(U_0) = \{\Phi(U)\}_0$.)

As I said above, the proof of this theorem, which rests on the theory of chains developed in [Dedekind 1888], was left to the reader, and Dedekind made no further comment on its meaning. Nevertheless, it is easy to see that if the mapping Φ were bijective, U_0 would then be equipollent to U_0', and so, according to the decompositions established above, L and K would be equipollent. This proves Cantor's lemma, and it was precisely in this way that Dedekind proved it, together with the Cantor–Bernstein theorem, in 1887 [Dedekind 1930/32 III, 447–448].

Why did Dedekind include this obscure proposition, whose purpose was unclear and which played no role in his deductive theory? And given the fact that he was aware of Cantor's interest in this matter, why did he not just communicate the 1887 proof to him? It seems likely that he wanted to test the alertness of his colleague by proposing the lemma in a deliberately obscure form, in the manner of many mathematicians of the 17th century. If so, Cantor apparently failed the test, for in 1895 he still considered the theorem unproven [Cantor 1895/97, 285]. When, in a letter of August 29, 1899, Dedekind finally communicated the result to him, Cantor accepted the proof as new [Cantor 1932, 449–450]. This incident shows plainly the lack of collaboration and mutual reinforcement between them. It also confirms that Cantor paid little attention to Dedekind's theory, which he regarded as merely a contribution to the most elementary, and sometimes trivial, propositions of arithmetic (see his letter to Vivanti of April 2, 1888 in [Meschkowski & Nilson 1991, 302]).

4.3. Coda

Toward the end of 1882, Cantor made his crucial "discovery" of transfinite numbers, although as far as we know Dedekind showed no interest in it. By then, Cantor was already in correspondence with Gösta Mittag-Leffler, the editor of the new journal *Acta Mathematica* (cf. [Meschkowski & Nilson 1991, 68ff]), and thus he found a more congenial correspondent. Probably he was also offended by Dedekind's refusal to accept the position in Halle. All of these circumstances help to explain why Cantor did not write to Dedekind again, and so the relationship ended. It was only in 1899 that a new, but short-lived exchange of letters took place.

This last exchange occurred in July and August, and it concerned the antinomies of set theory and the well-ordering theorem. Cantor had already tried to discuss this matter with Dedekind in 1897, through the intervention of Felix Bernstein. According to Bernstein's recollections (quoted in [Dedekind 1930/32, III, 449]), the news of the antinomies led Dedekind to ponder whether human reasoning was not completely rational. Nevertheless, a more direct exchange on the matter had to wait until Dedekind "reopened" the correspondence in July 1899 (see Cantor's reply in [Dugac 1976, 259] and also Cantor's letter to Hilbert, quoted in Section 3.1). On this occasion, we find Cantor well ahead of Dedekind in set-theoretical insights. In his letters of August, he presented the antinomies of the classes of all transfinite ordinals and all alephs. Furthermore, he tried to establish a distinction between consistent and inconsistent "multiplicities" [17], and he used this material in an attempt to prove the well-ordering theorem [Cantor 1932, 441–448]. Had Cantor published his new ideas, they would have made available his most important new concepts of the 1890s, and they could have promoted the next important episode in the history of set theory, the discussion of the antinomies and the well-ordering theorem which led to Zermelo's axiomatization. Nevertheless, it seems that Cantor was never completely satisfied with them and that their discussion with Dedekind did not lead to the clarifications that he had hoped might result (cf. [Purkert & Ilgauds 1987, 154]). To this end, Cantor also arranged what was to be the last meeting between them, which took place in September 1899 [Dugac 1976, 262; Landau 1917, 54].

To summarize, both Cantor and Dedekind made important contributions to the development of set theory and to the set-theoretical view of mathematical objects, but, contrary to the commonly accepted view, they did not enjoy a strong collaboration. The initial cause for this was the tension created by the publication of [Cantor 1874]. Dedekind kept playing the role of critic upon Cantor's request, but they enjoyed good relations only around 1877 and again in 1882. The Harzburg episode shows that both mathematicians had many insights and common interests to share and that collaboration between them could have been of great advantage to the development of set theory. A closer contact might have convinced Dedekind to devote some effort to transfinite set theory, while Cantor's creativity would have profited from Dedekind's analytic powers and sense of rigor [18]. But as events happened, the 19th century ended with two rather different set theories, diverse in aims and tools (cf. [Medvedev 1984]).

APPENDIX I: CHRONOLOGY OF THE RELATIONS

The Cantor–Dedekind correspondence follows a pattern of short but intense outbursts, followed by long periods without contact. The main exchanges of letters occurred in:

1873 (related to [Cantor 1874]),
1877 (connected with [Dedekind 1872; Cantor 1878]),
1882 (connected with [Cantor 1882; 1883, cf. Section 4]), and
1899 (related to the question of the antinomies).

We can see that most of the exchanges were motivated by Cantor's desire to discuss his current work and upcoming publications. The same is true for short interchanges in January 1879 (related to [Cantor 1879]) and January 1880 (in which he proposed to Dedekind that they collaborate on a translation and where he discussed the subject of [Cantor 1880a]).

It is important to stress that, for most of the intermediate periods, we have confirmation of the lack of contact. In October 1876 Cantor remarked that no contact had occurred since 1874 [Dugac 1976, 228–229] and later that there had been no interchanges during all of 1878 [Dugac 1976, 232] and almost all of the period 1879/81 [Dugac 1976, 232–239, especially 233]. For the period between 1883 and 1899 we have only two short, formal letters [Dugac 1976, 259], so it also seems to have been almost completely devoid of contacts. Moreover, on several occasions Cantor expressed regret about this state of affairs (cf. [Dugac 1976, 233, 258]), and he showed signs of relief when contact was renewed (cf. [Dugac 1976, 232, 259]).

As regards personal meetings, these tended to occur during periods when relations were relatively smooth. As I indicated above, we only know of six meetings during the whole period 1872/1899. The first two seem to have taken place by chance, due to the fact that both mathematicians were spending their vacations in the Swiss mountains. Following their first encounter in 1872, they met again in 1874, in an atmosphere probably marked by the tensions surrounding the publication of [Cantor 1874]. The third meeting occurred in May 1877, when Cantor visited Dedekind on his way back to Halle from the *Gaussfeier* in Göttingen (see [Lipschitz 1986, 88]). Then there were the two meetings in September 1882, discussed in Section 4 [Dugac 1976, 255–256]. The last meeting took place in 1899 [Dugac 1976, 260–262], after which Cantor and Dedekind seem never to have been in touch with each other again.

APPENDIX II: AN ANALYSIS OF CANTOR'S [1874]

The purpose of this appendix is to show how an analysis of the 1873 correspondence, together with [Cantor 1874], seems to support Dedekind's claims concerning the use of his letters. As we have seen, Cantor's [1874] contains two theorems: the first showing the denumerability of the set of algebraic numbers; the second proving the non-denumerability of $\mathbb{R}$. The original proofs and comments may be found in [Cantor & Dedekind 1937, 12–20; Cantor 1874], while [Dauben 1979, Chap. 3] offers detailed expositions of them.

We have seen that, according to the letters, Dedekind had been the one who formulated and proved the first theorem. Nevertheless, Meschkowski [1983, 235–236] has raised the possibility of a simultaneous discovery of this theorem by both mathematicians. First of all, Cantor did not claim this in any letter of the time, and so, even in case of a simultaneous discovery, Dedekind had reasons enough for taking offense upon seeing the publication.

It is true, nevertheless, that when posing the problem of the denumerability of $\mathbb{R}$, in his letter of November 29, 1873, Cantor commented also that one could prove the denumerability of $\mathbb{Q}$ and that of the set of n-tuples of integers [Cantor & Dedekind 1937, 13]. After receiving Dedekind's theorem on the denumerability of the set of algebraic numbers, which I will denote $\mathbb{A}$, Cantor wrote that it was similar to his own proof that the set of n-tuples is denumerable [Cantor & Dedekind 1937, 13]. And, interestingly, much later he wrote that the denumerability of $\mathbb{A}$ was a consequence of the denumerability of n-tuples [Cantor 1882, 152]. In fact, if the set of all finite n-tuples is denumerable, it is easy to prove that the set of polynomials with integer coefficients is also denumerable, thereby showing by means of the fundamental theorem of algebra the desired property of $\mathbb{A}$.

But surprisingly, a comparison of Cantor's proof on n-tuples, as presented in his letter of December 2, with Dedekind's proof for algebraic numbers (that is, the published proof) shows that Cantor's proof contained an error. To order the n-tuples $(n_1, n_2, \ldots, n_v)$, Cantor considered the number $N = n_1^2 + n_2^2 + \ldots + n_v^2$ [Cantor & Dedekind 1937, 13], while in order to enumerate the polynomials with integer coefficients Dedekind used the "height" $v - 1 + |n_1| + |n_2| + \ldots + |n_v|$ (see [Cantor 1874, 116; Cantor & Dedekind 1937, 18]). The difference between these two methods becomes clear in case some of the n_i are zero, for then Cantor's method fails to indicate how many zeros there are. This means that to each N these exist infinitely many n-tuples, and so Cantor's method of enumeration fails.

The flaw was certainly easy to overcome, but at the same time Dedekind's proof was more straightforward, since it yielded directly an enumeration of the polynomials. This could have been the reason why Cantor preferred it for his publication.

Let me now come to the second theorem, that is, Cantor's first non-denumerability proof, which was different from the one commonly given in textbooks (the latter goes through a diagonal argument that originated in 1891 [Cantor 1932, 278–280]). The original proof used tools of analysis: departing from a supposed enumeration of $\mathbb{R}$, Cantor constructed a sequence of nested intervals such that the number(s) included in all the intervals fell outside the enumeration scheme.

Now, a comparison of his original proof (letter of December 7) with the published version does show also some traits which make Dedekind's claim highly probable. Apart from a simplification which was apparently seen by both mathematicians, the latter part of the proof was carried out in different ways. As a member of the Berlin School, Cantor was familiar with the Bolzano–Weierstrass *Schlussweise* [method of proof], namely that which proceeds by constructing a sequence of (closed) nested intervals and concludes by showing the existence of a real number included in the intersection of those intervals. Weierstrass used it constantly (cf. [Ullrich 1989, 156]), and Cantor's original proof makes a straightforward use of that *Schlussweise* in order to draw his conclusion [Cantor & Dedekind 1937, 15].

Even in 1884 Cantor was still using that method, and he affirmed that it was hardly replaceable by an essentially different one [Cantor 1932, 212].

Meanwhile, Dedekind had chosen another proposition as the cornerstone of real analysis: the theorem which affirms the existence of a limit to each monotonically increasing and bounded sequence of real numbers [Dedekind 1872, 315–316, 332–334]. In his [1872] Dedekind stated a "principle of continuity" and defined the real numbers in terms of cuts; at the end of the paper, he used those definitions to prove rigorously the truth of the above-mentioned "cornerstone theorem." It is then most notable that the published version of Cantor's theorem does not employ directly the Bolzano–Weierstrass *Schlussweise*, but *contains a proof of it* from Dedekind's preferred proposition [Cantor 1874, 117]. The clear difference between both proofs gives further support to Dedekind's claim that Cantor had based the article on his letters. Dedekind also affirmed—with an exclamation mark—that Cantor had avoided the phrase "according to the principle of continuity" which appeared in his letter [Cantor & Dedekind 1937, 19] [19]; obviously, he had mentioned it in the letter in order to justify the use of the "cornerstone theorem," in reference to [Dedekind 1872].

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NOTES

1. Zermelo himself supported this view later, in his introduction to [Cantor 1932].

2. Later scholars (Grattan-Guinness, Dauben, etc.) have not gone beyond Fraenkel's analysis of the relations between Cantor and Dedekind, and they usually refer to his exposition.

3. On Dedekind's algebraic work, see [Dugac 1976, 1981]; [Mehrtens 1979]; [Edwards 1980, 1983].

4. This use of mappings as a means of defining functions is the natural way of formulating Dirichlet's "abstract" notion of function within Dedekind's framework.

5. That line of research did not lead directly to abstract set theory; its direct results were the development of a topology of point-sets, and above all the notion of content, a forerunner of measure theory which appeared in the 1880s. Regarding these developments in real analysis, the reader should consult [Hawkins 1975] or the short exposition of Cantor's ideas in [Grattan-Guinness 1980, Chap. 5]; [Dauben 1979, Chaps. 1 and 2] is also helpful.

6. I should point out a mistake in [Dauben 1979, 80], where he discusses this matter: in Cantor's notation $\mathbf{D}(P', P'', \ldots)$, the **D** comes from the word "Divisor" which he uses, and not from "Durchschnitt," which he did not use at this time (in [Cantor 1880b, 145], we can find this latter word, but introduced as an editorial comment by Zermelo). A later instance of the influence of Dedekind's writings seems to be Cantor's effort to structure in a deductive way his last presentation of the theory of transfinite sets [Cantor 1895/97]; his presentation probably follows the model of [Dedekind 1888]. On the other hand, as Greg Moore has brought to my notice, the notation introduced by Cantor in his [1880b], which considered unions and intersections of an infinite family of sets, was followed by Dedekind in his [1888].

7. Cantor's strange choice of the title originated with Weierstrass, who recommended that he publish his discoveries, but only so long as they concerned the denumerability of algebraic numbers, that is, Dedekind's result (cf. [Cantor & Dedekind 1937, 17]). Weierstrass even advised Cantor to

avoid references to essential differences between infinite sets [Dugac 1976, 226]. In fact, there are indications that he never accepted the idea of transfinite distinctions, for example, a transcription of his summer course 1874 (after knowing Cantor's result) which contains the following words: "Two infinitely large magnitudes are not comparable, they can always be considered as equal" [Dugac 1973, 126] (cf. also texts from 1878 [ibid. 103] and 1885 [ibid. 141]). Given this situation, Cantor arranged the matter very wisely: he presented the non-denumerability theorem as a mere addition, in order to apply the "main" theorem to prove the existence of transcendental numbers in any given interval [Cantor 1874, 115–116]. Instead of the non-denumerability proof, which interested him most, Cantor emphasized a new demonstration of Liouville's theorem.

8. Although Dedekind's conjecture was not discussed at length in the article, Cantor indicated it as something natural in the context of geometrical investigations [Cantor 1878, 121]. The conjecture caused an avalanche of contributions by Lüroth, Thomae, Jürgens, Netto, and Cantor himself (cf. [Johnson 1979/81, Chap. 3] and also [Dauben 1979, 70–76]), but even in 1932 Zermelo attributed it exclusively to Cantor [Cantor 1932, 133].

9. Although I will not attempt it here, a correct interpretation of this episode should take into account the more or less abrupt endings of Cantor's friendships with Schwarz and Mittag-Leffler (cf. [Grattan-Guinness 1974, 125–126]), as well as his controversy with du Bois-Reymond.

10. This letter is not contained in [Dugac 1976], but can be found in [Grattan-Guinness 1974, 112]. Concerning the 1877 delay, this is, curiously enough, not apparent when one looks at the submission dates of the papers in the 84th volume of the *Journal für die reine und angewandte Mathematik* [Johnson 1979/81, 144].

11. The conflict with Weierstrass only reached important proportions during the 1880s (cf. [Biermann 1988, 137–139]), but already in 1870 Kronecker tried to convince E. Heine not to publish a paper [Dauben 1979, 308–309] and began to make criticisms that affected Weierstrass's analysis (cf. [Meschkowski 1983]).

12. A similar attitude is shown in an event reported by Bernstein, after talking with Dedekind in 1911. Dedekind told him that during a stay in Berlin he visited Kummer, who received him in an unfriendly manner; while opening the door, his salute was: "so you are coming to see whether I will pass away [abgehen] soon" [Dedekind 1930/32 III, 481].

13. Despite Cantor's [Dugac 1976, 252–253] and Frobenius's comments, Dedekind had a good opinion of Kronecker [Dugac 1976, 253–254], which is today maintained by Edwards [1980, 368–372; 1981].

14. Taking into account the explanation of Cantor's behavior given in Section 3.3, this might seem contradictory. But the fact is that during the 1870s Cantor was not offered a position in a German university, and in the 1880s he felt more and more isolated: his work was not receiving the recognition he expected, and he increasingly charged Kronecker and Schwarz with having intrigued against him (cf. his letters to Mittag-Leffler and Thomé in [Purkert & Ilgauds 1987, 76, 217]). As a natural consequence, he began to search for different connections and to think of Halle as the place where he might spend the rest of his life. Cantor seems to have perceived Dedekind as the only outstanding German mathematician who had shown a lively interest in his research, and so he took him immediately into consideration.

15. In Cantor's letter of November 5, 1882, he mentions the theorem twice. The first time, he seems to think that his new transfinite numbers will enable him to prove the theorem, but at the end of the letter he again poses it as an open problem. It is clear that, in the meantime, he had seen that he could only prove a restricted version of the theorem for sets of numbers of the first or second number classes, which is the one we find in [Cantor 1883]. Of course, this version is enough to cover subsets of $\mathbb{R}$, *if* the continuum hypothesis is true [Moore 1982, 42–43].

16. In 1908, Zermelo used a transfinite generalization of Dedekind's chains, which he called θ-chains, to prove the well-ordering theorem [Zermelo 1908a, 184–185, 190]. This is a graphic example of how the theories of Cantor and Dedekind could have been profitably united, had their relations been better.

17. When he spoke about "multiplicities," Cantor was equating these with Dedekind's "systems" and trying to convince him that the theory developed in [Dedekind 1888] led to inconsistencies (cf.

his letters in [Cantor 1932, 443, 447–448]). At this point, Cantor shows a knowledge of Dedekind's ideas that could only have come from their conversations, probably from 1882. He was aware of the fact, unnoticeable in [Dedekind 1888], that Dedekind's concept of sets was still based on the naive idea that any well-defined property determines a set—in a word, the principle of comprehension (see Cantor to Hilbert in [Purkert & Ilgauds 1987, 154]).

18. Particular examples of possible improvements are the following. Dedekind's notion of mapping could have systematized various Cantorian ideas which in his work appear as independent from each other: those of equipollence, "Abbildung" in the sense of mapping preserving linear order, and "Belegung" or covering, used for defining exponentiation of alephs. Dedekind's theorem of induction, generalized through an appropriate notion such as Zermelo's θ-chain, could also have rigorized Cantor's theory of transfinite numbers.

19. In the notes he was even so careful as to determine the precise position where the phrase should have been, corresponding to [Cantor 1874, 117, first five lines of the last paragraph].

REFERENCES

Unpublished Sources

Dedekind, R. *Nachlass*. Niedersächsische Staats- und Universitätsbibliothek Göttingen (Germany). Handschriftenabteilung.

Published sources

Biermann, K. R. 1988. *Die Mathematik und ihre Dozenten an der Berliner Universität 1810–1933*, Second, expanded edition. Berlin: Akademie Verlag.

Cantor, G. 1872. Über die Ausdehnung eines Satzes aus der Theorie der trigonometrischen Reihen. *Mathematische Annalen* **5**, 123–132. References to the reprint in [Cantor 1932, 92–101].

——— 1874. Über eine Eigenschaft des Inbegriffes aller reellen algebraischen Zahlen. *Journal für die reine und angewandte Mathematik* **77**, 258–262. References to the reprint in [Cantor 1932, 115–118].

——— 1878. Ein Beitrag zur Mannigfaltigkeitslehre. *Journal für die reine und angewandte Mathematik* **84**, 242–258. References to the reprint in [Cantor 1932, 119–133].

——— 1879. Über einen Satz aus der Theorie der stetigen Mannigfaltigkeiten. *Nachrichten der Königlichen Gesellschaft der Wissenschaften zu Göttingen* (1879), 127–135. References to the reprint in [Cantor 1932, 134–138].

——— 1880a. Zur Theorie der zahlentheoretischen Funktionen. *Mathematische Annalen* **16**, 583–588. References to the reprint in [Cantor 1932, 65–70].

——— 1880b. Über unendliche, lineare Punktmannichfaltigkeiten II. *Mathematische Annalen* **17**, 355–358. References to the reprint in [Cantor 1932, 145–148].

——— 1882. Über unendliche, lineare Punkmannichfaltigkeiten III. *Mathematische Annalen* **20**, 113–121. References to the reprint in [Cantor 1932, 149–157].

——— 1883. Über unendliche, lineare Punktmannichfaltigkeiten V. *Mathematische Annalen* **21**, 545–586. References to the reprint in [Cantor 1932, 165–208]. Also published separately as *Grundlagen einer allgemeinen Mannigfaltigkeitslehre*.

——— 1895/97. Beiträge zur Begründung der transfiniten Mengenlehre. *Mathematische Annalen* **46**, 481–512; **49**, 207–246. References to the reprint in [Cantor 1932, 282–351].

——— 1932. *Gesammelte Abhandlungen mathematischen und philosophischen Inhalts*, E. Zermelo, Ed. Berlin: Springer-Verlag. Reprinted in Hildesheim: Olms, 1962, and Berlin: Springer, 1980.

——— 1970. Principien einer Theorie der Ordnungstypen. Erste Mitteilung. Paper written in 1884/85, first published with an extensive introduction by I. Grattan-Guinness. *Acta Mathematica* **124**, 65–107.

Cantor, G., & Dedekind, R. 1937. *Cantor–Dedekind Briefwechsel.* E. Noether and J. Cavaillès, Eds. Paris: Hermann. French translation in [Cavaillès 1962, 179–251].

Cavaillès, J. 1962. *Philosophie mathématique.* Paris: Hermann.

Dauben, J. W. 1979. *Georg Cantor. His mathematics and philosophy of the infinite.* Cambridge, MA: Harvard University Press.

Dedekind, R. 1871. Über die Komposition der binären quadratischen Formen. Supplement X to the 2nd edition of P. G. Lejeune Dirichlet, *Vorlesungen über Zahlentheorie.* R. Dedekind, Ed. Braunschweig: Vieweg. References to the reprint in [Dedekind 1930/32, III, 223–261].

——— 1872. *Stetigkeit und irrationale Zahlen.* Braunschweig: Vieweg. References to the reprint in [Dedekind 1930/32, III, 315–334].

——— 1872/78. *Gedanken über die Zahlen.* First draft version of [Dedekind 1888], published in [Dugac 1976, 293–309].

——— 1877. Sur la théorie des nombres entiers algébriques. *Bulletin des sciences mathématiques et astronomiques* ser. 1 **11** (1876); ser. 2 **1** (1877). Published separately in Paris: Gauthier–Villars, 1877. References to the partial reprint in [Dedekind 1930/32, III, 262–296].

——— 1879. Über die Theorie der ganzen algebraischen Zahlen. Supplement XI to the 3rd edition of P. G. Lejeune Dirichlet, *Vorlesungen über Zahlentheorie.* R. Dedekind, Ed. Braunschweig: Vieweg.

——— 1888. *Was sind und was sollen die Zahlen?* Braunschweig: Vieweg. References to the reprint in [Dedekind 1930/32, III, 335–390].

——— 1893. Über die Theorie der ganzen algebraischen Zahlen. Supplement XI to the 4th edition of P. G. Lejeune Dirichlet, *Vorlesungen über Zahlentheorie.* R. Dedekind, Ed. Braunschweig: Vieweg. Reprinted in New York: Chelsea, 1968.

——— 1930/32. *Gesammelte mathematische Werke.* 3 vols. R. Fricke, E. Noether, O. Ore, Eds. Braunschweig: Vieweg. Reprinted in New York: Chelsea, 1969, 3 vols. in 2.

Dugac, P. 1973. Eléments d'analyse de Karl Weierstrass. *Archive for History of Exact Sciences* **10,** 41–176.

——— 1976. *Richard Dedekind et les fondements des mathématiques.* Paris: Vrin.

——— 1981. Richard Dedekind et l'application comme fondement des mathématiques. In [Scharlau 1981, 134–144].

Edwards, H. M. 1980. The genesis of ideal theory. *Archive for History of Exact Sciences* **23,** 321–378.

——— 1983. Dedekind's invention of ideals. *Bulletin of the London Mathematical Society* **15,** 8–17. Also in *Studies in the History of Mathematics.* Studies in Mathematics, Vol. 26 (1987), E. Philips, ed., pp. 8–20. Washington, DC: The Mathematical Association of America.

Fraenkel, A. 1930. Georg Cantor. *Jahresbericht der Deutschen Mathematiker-Vereinigung* **39,** 189–266.

Grattan-Guinness, I. 1971. Towards a biography of Georg Cantor. *Annals of Science* **27,** 345–391.

——— 1974. The rediscovery of the Cantor–Dedekind correspondence. *Jahresbericht der Deutschen Mathematiker-Vereinigung* **76,** 104–139.

——— (Ed.) 1980. *From the Calculus to Set Theory.* London: Duckworth.

Hawkins, T. W. 1975. *Lebesgue's Theory of Integration. Its Origins and Development.* New York: Chelsea.

Johnson, D. M. 1979/81. The problem of the invariance of dimension in the growth of modern topology. *Archive for History of Exact Sciences* **20,** 97–188; **25,** 85–266.

Landau, E. 1917. Richard Dedekind—Gedächtnisrede. *Nachrichten der Königlichen Gesellschaft der Wissenschaften zu Göttingen,* 50–70.

Lipschitz, R. 1986. *Briefwechsel mit Cantor, Dedekind, Helmholtz, Kronecker, Weierstrass und anderen.* W. Scharlau, Ed. Braunschweig/Wiesbaden: Vieweg.

Medvedev, F. A. 1984. Über die abstrakten Mengenlehren von Cantor und Dedekind. *Berichte zur Wissenschaftsgeschichte* **7,** 195–200.

Mehrtens, H. 1979. Das Skelett der modernen Algebra. Zur Bildung mathematischer Begriffe bei Richard Dedekind. In *Disciplinae Novae*, C. J. Scriba, Ed., pp. 25–43. Göttingen: Vandenhoeck & Ruprecht.

Meschkowski, H. 1965. Aus den Briefbüchern Georg Cantors. *Archive for History of Exact Sciences* **2,** 503–519.

——— 1983. *Georg Cantor. Leben, Werk und Wirkung.* Mannheim/Wien/Zürich: Bibliographisches Institut. Reprint, with some additions, of *Probleme des Unendlichen* (1967) Braunschweig: Vieweg.

Meschkowski, H., & Nilson, W. 1991. *Georg Cantor: Briefe.* Berlin: Springer-Verlag.

Moore, G. H. 1982. *Zermelo's Axiom of Choice. Its Origins, Development and Influence.* New York/Heidelberg/Berlin: Springer-Verlag.

Poincaré, H. 1908. *Science et méthode.* Paris: Flammarion.

Purkert, W., & Ilgauds, H. J. 1987. *Georg Cantor 1845–1918.* Basel/Boston/Stuttgart: Birkhäuser.

Scharlau, W. (Ed.) 1981. *Richard Dedekind 1831/1981.* Braunschweig/Wiesbaden: Vieweg.

Sinaceur, M. A. 1971. Appartenance et inclusion: Un inédit de Richard Dedekind. *Revue d'histoire des sciences et de leurs applications* **24,** 247–254.

——— 1974. L'infini et les nombres. Commentaires de R. Dedekind a "Zahlen." La correspondence avec Keferstein. *Revue d'histoire des sciences* **27,** 251–278.

Ullrich, P. 1989. Weierstrass', Vorlesung zur "Einleitung in die Theorie der analytischen Funktionen." *Archive for History of Exact Sciences* **40,** 143–172.

Van Heijenoort, J. (Ed.) 1967. *From Frege to Gödel.* Cambridge, MA/London: Harvard University Press.

Zermelo, E. 1908a. Neuer Beweis für die Möglichkeit einer Wohlordnung. *Mathematische Annalen* **65,** 107–128.

——— 1908b. Untersuchungen über die Grundlagen der Mengenlehre, I. *Mathematische Annalen* **65,** 261–281.