Fractional Indices, Exponents, and Powers

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The word *power* is defined by Webster as "The product arising from the continued multiplication of a number into itself." The word exponent is taken to mean "A symbol written above another symbol and on the right, denoting how many times the latter is repeated as a factor."[1] However, so loosely have these terms come to be used that statements inconsistent with the above definitions can be found not only in Webster but also in almost any good elementary textbook or history of mathematics. For example, the word "logarithm" is defined as "The exponent of that power of a fixed number (called the *base*) which equals a given number (called the *antilogarithm*)."[2] In terms of the previous definitions this is sheer nonsense. Only very exceptionally do the logarithms of numbers turn out to be positive and integral, or even rational. Hence the exponent in this case usually designates not a repeated multiplication but a sequence of operations: raising to powers, finding principal roots, determining a limit, and then perhaps taking a reciprocal. Meticulous authors point out that in this case the meaning of the phrase "raising to a power" is extended to include this set of operations, and even the function theory involved in the use of imaginary exponents. There can be, of course, no objection on the grounds of logic to such an extension of meaning through appropriate definition, but there remains an objection on the basis of the fitness of things: the same phrase is used to denote both the series of operations and a single component of this series. Thus "raising four to the minus three-halves power" includes as a necessary operation that of "raising four [or $\frac{1}{4}$ or ± 2 or $\pm\frac{1}{2}$, depending on the order of operations indicated in the definition] to the third power." Here the phrase "raising to a power" is used in two entirely different senses. Moreover, the one sense is not a generalization of the other. One might equally well speak of $4^{-3/2}$ as "finding a minus-three-halves

[1] These are substantially the definitions given also in the *Mathematics Dictionary* (Ed. by Glenn James, Van Nuys, Cal., 1942), the *Mathematisches Wörterbuch*, (by G. S. Klügel, 5 vols., Leipzig, 1803-1831), and the *Encyclopedie des Sciences Mathématiques*, (Ed. by Jules Molk and Franz Meyer, vol. I (1), Paris, 1904, pp. 53-56.

[2] Cf., for example, W. L. Hart, *Plane Trigonometry* (New York, 1933), p. 21; J. B. Rosenbach, E. A. Whitman, David Moskovitz, *Plane Trigonometry* (New York, 1937), p. 139.

root" of four, or even as "taking a minus-three-halves reciprocal of four." The simplest procedure would appear to be to retain Webster's meaning of the word "power" and to substitute some other expression for the extended sense. The word "exponent" might well be reserved for this use, but there is a slight practical objection here also, as a short excursion into the past will show.

The close association in thought of the concept of power and the notation of exponents has led inadvertently to some confusion in the history of mathematics. Standard works on the subject[3] state that Nicole Oresme in the fourteenth century first used fractional exponents. In substantiation of this assertion they indicate that in the *Algorismus proportionum* one finds such expressions as

$$\frac{p1}{1\cdot 2}$$

to denote what Oresme expressed as the three-halves "proportion" (i. e., the cube of the principal square root), so that $(\sqrt{4})^3$ might appear[4] as

$$\frac{1p}{1\cdot 2\cdot 4} \text{ or } \frac{3p}{2\cdot 4}.$$

Here are clear-cut examples of a not inconvenient notation for fractional "powers" as described above; but they do not illustrate the use of *exponents* in the ordinary sense as given by Webster. Oresme gave rules equivalent to such expressions as

$$(a^m)^{p/q} = (a^{mp})^{1/q} \text{ or } a^m \cdot a^{1/n} = a^{m+(1/n)};$$

but the statements of these are largely verbal rather than symbolic, and in no case does the indicator of the power or root appear in the position of a modern exponent.[5] The *idea* of fractional "powers"

[3] See, for example, Moritz Cantor, *Vorlesungen über Geschichte der Mathematik*, vol. II (Leipzig, 1892), p. 121ff; Kark Fink, *A Brief History of Mathematics* (transl. by W. W. Beman and D. E. Smith, Chicago, 1910), p. 102; D. E. Smith, *History of Mathematics*, vol. I (Boston, 1923), p. 239; Johannes Tropfke, *Geschichte der ElementarMathematik*, vol. I (Leipzig, 1902), p. 200. Cf. also Maximilian Curtze, *Der algorismus proportionum des Nicolaus Oresme* (Berlin, 1868), p. 9ff; H. G. Funkhouser, "Historical development of the graphical representation of statistical data" [*Osiris*, III (1937), 269-404), p. 274; Hermann Hankel, *Zur Geschichte der Mathematik im Alterthum und Mittelalter* (Leipzig, 1874), p. 350; *Encyclopédie des sciences mathématiques*, I(1), 56; Florian Cajori, *A History of Mathematical Notations*, (2 vols., Chicago, 1928-1929), I, 343, 354.

[4] Heinrich Wieleitner, "Zur Geschichte der gebrochenen Exponenten", *Isis*, VI (1924), 509-520. Cf. references to Cantor, Curtze, Fink, Hankel, Smith, and Tropfke above; also Florian Cajori, *A History of Mathematics* (2nd ed., New York, 1931), p. 127.

[5] See Wieleitner, *op. cit.*, Curtze, *op. cit.*

or "proportions" quite possibly goes back long before the time of Oresme, but the *notation* of fractional exponents did not appear until several hundred years later. It is not unlikely that the Scholastic doctrine of fractional proportions may some time be traced through Arabic treatises and Greek works on arithmetic back to Pythagorean musical theory.[6] At any rate, Oresme was not the earliest medieval scholar to deal with fractional "powers", for Thomas Bradwardine in his *Liber de proportionibus* of 1328 had referred to "medietas duplæ proportionis" and "medietas sesquioctavæ proportionis"[7] (i. e., $\sqrt{2}$ and $\sqrt{9/8}$). Nevertheless it may be that Oresme first discussed proportions made up of both powers and roots. Moreover, he appears to have been the first one to represent such proportions symbolically. There is at hand a convenient word, *index*, which might well be used to denote all such symbolisms, for Webster defines it as "The figure, letter, or expression showing the power or root of a quantity." This would correctly characterize the notation of Oresme without giving the false impression that here one finds *exponents* in the strict sense.

The use of exponents as indicators of positive integral powers is widely ascribed to Descartes, but in reality this goes back a century and a half before his time. Nicolas Chuquet in 1484 composed a *Triparty en la science des nombres* which was probably inspired by the work of Oresme of about a century before. In the *Triparty* there are expressions such as $.5.^1$ and $.6.^2$ and $.10.^3$ to designate what now would appear as $5x$ and $6x^2$ and $10x^3$. Here powers are clearly indicated by exponents, although Chuquet used the word *denominacion* instead of *potence* and his form differs slightly from the modern Cartesian notation.[8] In this remarkable work negative integers and zero also are used as exponents: 9 (or $9x^0$) is written as $.9.^0$ and one reads correctly that $.72.^1$ divided by $.8.^3$ is $.9.^{2\cdot m}$ (i. e., $72x \div 8x^3 = 9x^{-2}$). Chuquet possessed also a brief notation for roots,—such as ℞^{2} .7. for the square root of 7 and ℞^{4} .10. for the fourth root of 10,— but this corresponded to our form $\sqrt[2]{7}$ and $\sqrt[4]{10}$ rather than to the fractional-exponent type in $7^{1/2}$ and $10^{1/4}$.

Tradition has attributed the earliest use of fractional exponents to John Wallis, but this should be interpreted cautiously. In 1585 in the *Arithmétique* of Simon Stevin powers of one-tenth and powers of

[6] See Heinrich Wieleitner, "Geschichte der gebrochenen Exponenten", *Isis*, VII (1925), 490-491.

[7] See Wieleitner, *op. cit.* (1924), p. 515.

[8] For Chuquet's work see Aristide Marre, "Notice sur Nicolas Chuquet et son Triparty en la science des nombres," *Bulletino di Bibliografia e di Storia delle Scienze Matematiche e Fisiche*, XIII (1880), 555-659, 693-814, especially pp. 737ff; and Ch. Lambo, "Une algèbre française de 1484. Nicolas Chuquet," *Revue des Questions Scientifiques* (3), II (1902), 442-472, especially pp. 459-463.

unknowns were denoted by figures, frequently encircled, placed either over or after the digit or the coefficient. Thus the number 6.789 might appear as $\overset{0}{6}\,\overset{1}{7}\,\overset{2}{8}\,\overset{3}{9}$, and the polynomial $1+2x+3x^2+4x^3$ could be expressed by 1 ⓪+2 ①+3 ②+4 ③. Stevin indicated clearly that this notation could be extended readily to include all roots:

> Toutefois le 1/2 en circle seroit le charactere de racine de ①, & par consequent 2/3 en un circle seroit le charactere de racine quarrée de ③, par ce que telle 3/2 en circle multipleé en foi donne produict ③, & ainsi des austres; de sorte que par tel moyen on pourroit de toutes simples quantitez extraire especes de racines quelconques, comme racine cubique de ② seroit 2/3 en circle, etc.[9]

Such notations are clearly equivalent to fractional exponents, but literal-minded readers will notice that the indices were placed over *or* on the right, rather than above *and* on the right, as common usage and the definition of Webster require. For fractional exponents in the strict sense one waits almost another century.

The analytic geometry of Descartes in 1637 popularized the use of positive integral exponents in the modern manner.[10] This work exerted a strong influence upon John Wallis, who applied the ideas and notations in his *Arithmetica infinitorum* of 1655. Here Wallis proposed his well-known principle of interpolation or of (incomplete) induction, asserting that inasmuch as the area under the curves $y=x^n$ was given by the expression

$$\frac{x^{n+1}}{n+1}$$

for all integral values of n, this formula was seen, by analogy, to hold also for fractional and even irrational values of the exponent![11] The period of mathematical rigor was yet a century and a half removed. Throughout the treatise he uses integral exponents in the strict modern sense and speaks freely of fractional and irrational indices. These latter apparently first actually appeared, however, in Newton's famous

[9] Simon Stevin, *Les Oeuvres mathématiques* (ed. by Albert Girard, Leiden 1634), I, 6. See also Eugène Prouhet, "Sur l'invention des exponants fractionnaires ou incommensurables," *Nouvelles Annales de Mathématiques*, vol. XVIII (1859), *Bulletin de Bibliographie*, pp. 42-46. Cf. also Tropfke, *op. cit.*, p. 102. Joost Bürgi made use of a similar form, writing indices of positive integral powers of the unknown as Roman numerals placed over the corresponding coefficients. It should perhaps be remarked also that Stevin's notation for polynomials resembles somewhat that used earlier by Bombelli and later by Girard. See Cajori, *A History of Mathematical Notations*, I, 343-360.

[10] Roman numerals had been used as exponents the year before by James Hume who wrote A^3 as A^{iii}. See Cajori, *A History of Mathematical Notations*, I, 345f.

[11] See John Wallis, *Opera mathematica* (2 vols., Oxonii, 1656-1657), II, 52-53.

letter to Oldenburg of June 13, 1676.[12] In Wallis' *Algebra* of 1685 fractional exponents, both positive and negative, appear frequently.[13]

The use of fractional exponents quickly became common practice. Newton stated the binomial theorem so as to include all (rational) exponents, and numerous other formal expressions involving positive integral powers were found to be satisfied also when these integers were replaced by corresponding negative or fractional or even irrational values. Leibniz, in a letter to Wallis, suggested the possibility of fractional derivatives and integrals. The operation of the principle of the permanence of form tended to obscure the fact that while the type of *expression* remained essentially the same, the entire *meaning* had been radically altered. The exponent in x^n indicates a continued multiplication if n is a positive integer, but not otherwise. In the days of Bradwardine and Oresme the very same word had been used for powers and roots: *proportio dupla* meant varying as the square, *proportio subdupla* signified varying as the square root. Hence the term *proportio* was naturally carried over to all fractional indices, and the index $1\frac{1}{2}$ denoted *proportio sesquialtera.* As the Greek and Latin emphasis upon the idea of proportion gradually gave way to the development in terms of the arithmetic operations as now defined, powers and roots were more clearly dissociated. Whereas Oresme regarded the index 2/3 as designating *one proportion,* Stevin explicitly stated that it denoted *two distinct operations*, involution and evolution. However, the success of logarithms, Wallis' principle of induction, and the excessive formalism in the calculus of Leibniz, tended to obscure this distinction. No essential difference was seen in the expression x^3 and $x^{\sqrt{3}}$, for algorithmic rules applied to them indifferently. This tendency remains to the present day in ever so many textbooks which prove the laws of exponents for positive integral powers and then, with no warning or apology to the reader, treat these laws as adequately justified for entirely different situations indicated by real indices, integral or fractional, rational or irrational. Such specious procedures are encouraged by the careless use of the phrase, "raising to a power". Having raised four to the second power, a beginner experiences a comfortable feeling of understanding when he is told that similarly ten "raised to the .30103...th power" is equal to two. A clear-cut distinction between the terms index, exponent, and power would go far toward ameliorating the bliss of such ignorance. Incidentally it would serve also to render less obscure certain historical situations.

[12] See Isaac Newton, *Opera quæ exstant omnia* (ed. by Samuel Horsley, Londini, 1779-1785), IV, 215.

[13] See John Wallis, *A Treatise on Algebra* (London, 1685), p. 332; cf. pp. 310, 319, and *passim.*

One then would be able to state unequivocally, so far as extant evidence permits, that the idea of fractional "proportions" was referred to by Bradwardine and probably was of much earlier origin; that these appear first to have been represented symbolically by indices by Oresme; that exponents were adopted for integral indices by Chuquet and popularized in modern form by Descartes; that fractional exponents were adumbrated by Stevin and effectively established by Wallis and Newton; and that fractional "powers" are no longer *comme il faut.*