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Bolzano and uniform continuity

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Abstract

It has often been thought that the distinction between pointwise and uniform continuity was a relatively late arrival to real analysis, due to the mathematicians associated with Weierstrass. In this note, it is argued that Bolzano, in his work on real function theory dating from the 1830s, had grasped the distinction and stated two key theorems concerning uniform continuity.

Résumé

La distinction entre la continuité dans un point et la continuité uniforme est souvent représentée comme une retardataire à l'analyse, due aux mathématiciens autour de Weierstrass. Dans cette note, nous soutenons que Bolzano, dans ses travaux sur l'analyse des années 1830, a bien compris cette distinction, et qu'il a formulé deux théorèmes clés sur la continuité uniforme.

MSC: 01-26

Keywords: Uniform continuity; Bolzano; Heine's theorem; Heine; Dirichlet

1. Introduction

The Bohemian philosopher Bernard Bolzano (1781–1848) has long been recognized for his early and decisive contributions to the foundations of real analysis. Among the best parts of his work are those

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concerned with continuous functions. In his "Purely Analytic Proof" [Bolzano, 1817], for example, he provided a proof of the Intermediate Value Theorem which set out for the first time, as Pierre Dugac has remarked, significant parts of the foundations of real analysis [Dugac, 1980, 92]. Bolzano's *Function Theory* [Bolzano, 1930], written in the 1830s, but only published some 100 years later, confirms his mastery of the concept of continuity and its role in analysis. There, he constructed a continuous, nowhere differentiable function [Bolzano, 1930, I, §75; II, §19][1] and gave nice proofs of two other central results which are usually associated with later mathematicians, Weierstrass in particular. These are

> that a function continuous on a closed interval is bounded there [Bolzano, 1930, I, §§20–21]; and

> that a function continuous on a closed interval assumes global maximum and minimum values on the interval [Bolzano, 1930, I, §§22, 24].

Bolzano's statements are general and precise (that these propositions were even recognized as theorems requiring proof is remarkable for that time), and his proofs are strikingly modern, both involving applications of what is now known as the Bolzano–Weierstrass theorem. (Bolzano used this in the following form: an infinite point-set contained in a closed interval has a limit point in the interval. He alluded to a proof of this theorem within his "Theory of Measurable Numbers" [Bolzano, 1930, 28n].[2] There is no compelling reason to doubt that he had a proof, but so far it has not been found in his papers.)

Another important proposition concerning continuous functions is the following:

> Theorem 1. A function which is continuous on a closed interval is also uniformly continuous there.

On the other hand, we have

> Theorem 2. A function can be continuous on an open interval without being uniformly continuous there.

In his *Function Theory* [Bolzano, 1930, I, §13], and in a manuscript containing corrections to this work [van Rootselaar, 1969, 8–9], Bolzano stated results which bear an uncanny resemblance to these theorems. Previous commentators on Bolzano's mathematics, however, have consistently denied that Bolzano grasped the concept of uniform continuity [Bolzano, 1930, Editor's Notes, p. 4; van Rootselaar, 1969, 1–2; van Rootselaar, 1970, 275–276; Sebestik, 1992, 402n23, 431].[3] They have thus given indirect support to the received view that the definition of uniform continuity and the proof of Theorem 1 were due to Weierstrass and his students, in particular to Eduard Heine (see, e.g., [Bourbaki, 1969, 182; Kline, 1970, 953; Edwards, 1979, 325; Grattan-Guinness, 1980, 135; Laugwitz, 1994, 321]).

Heine was indeed the first to publish a definition of uniform continuity [Heine, 1870, 361] and a proof of Theorem 1 [Heine, 1872, 188]. He claimed no originality in these papers, however, and as it turns out

[1] Bolzano actually only claimed (and proved) that his function had no derivative on a set of points dense in the interval on which it is defined.

[2] For Bolzano's theory of measurable numbers see [Bolzano, 1962, 1976; Laugwitz, 1965; Sebestik, 1992; van Rootselaar, 1963].

[3] Since this article was accepted for publication, van Rootselaar [Bolzano, 2000, 10] has also argued that Bolzano had grasped the concept of uniform continuity. Rusnock [1999; 2000; 2004] also discusses related issues.

his proof is an almost verbatim transcription of one given by Dirichlet in his lectures on definite integrals in 1854 [Lejeune-Dirichlet, 1904, §2]. (The transmission of this result is discussed in [Dugac, 1989].)

The concern of this note, however, is to establish that Bolzano has a legitimate claim to priority. We intend to show, in particular, that he not only grasped the notion of uniform continuity but also gave an adequate characterization of the concept, stated and proved Theorem 2, and stated Theorem 1 in addition to providing a useful fragment of its proof.

2. Bolzano on continuity

In 1817, Bolzano published his best known paper in analysis, his "Purely Analytic Proof" of the Intermediate Value Theorem [Bolzano, 1817]. The definition of continuity he gives there is well-known and close to those in current usage today:

> According to a correct definition, the expression that a function fx varies according to the law of continuity for all values of x inside or outside certain limits means just that: if x is some such value, the difference $f(x+\omega) - fx$ can be made smaller than any given quantity provided ω can be taken as small as we please. With the notation that I introduced in §14 of *Der binomische Lehrsatz* ..., this is $f(x+\omega) = f(x) + \Omega$ [Bolzano, 1817, Preface].[4]

It is clear that the concept which is here defined is what would later be called pointwise continuity on a domain. Bolzano spoke quite explicitly of a function which varies continuously *for all values of a certain domain*, and the definition displays the quantificational structure quite plainly: f is said to be continuous on a domain if and only if, given any point of the domain, a certain condition is satisfied. The condition in question, namely, continuity *at a point*, is thus present inside Bolzano's definition and can be readily detached from the reference to a domain (e.g., an interval). Bolzano, as discussed below, later did just this.

Bolzano's formulation differs from modern ones in two respects. First—a minor point—he made no use of absolute values in his statement, although they are tacitly understood. Second, and potentially more misleading, is the use Bolzano made of the symbol ω. In the language of the "Binomial Theorem," ω is a *variable* quantity "which can become as small as desired" [Bolzano, 1816, v]. It should not be confused with a *constant* or *fixed* quantity. If we were to make the assumption, natural enough for a modern reader, that ω refers to a constant quantity (i.e., that it is a logical variable ranging over fixed real numbers), then Bolzano's definition would turn out to be defective. To take one example, the function

$$f(x) = \begin{cases} 1 & \text{if } x \text{ is of the form } \frac{1}{2^n} \text{ for some } n \in \mathbb{N} \\ 2x & \text{otherwise} \end{cases} \tag{1}$$

would then have to be said to be continuous at the point $x = 0$: the difference $f(0+\omega) - f(0)$ can be made smaller than any given quantity by taking ω sufficiently small, provided that the ω chosen is not of the form $\frac{1}{2^n}$.[5]

[4] Quoted after the translation of Stephen B. Russ [1980, 162]. The same definition may be found in [Bolzano, 1816, §29].

[5] This example is taken from Bolzano's later work *Function Theory* [Bolzano, 1930, I, §9].

We would want to say, instead, that there is a value of ω such that for it, *and* for all the values of ω', where $|\omega'| < \omega$, we have $f(x+\omega') - fx$ smaller than a given quantity.[6] That this was Bolzano's understanding is confirmed by his usage of it elsewhere in the "Purely Analytic Proof." In §15, for example, he considered functions f and ϕ, both continuous on an interval $[a, b]$, with $f(a) < \phi(a)$. From the continuity of the two functions, he inferred that $f(a+i) < \phi(a+i)$ for *all* i less than a certain value.

A natural interpretation of Bolzano's ω, then, would be as a range of values (or a neighborhood) of the form $\{x \mid -\omega_0 \leqslant x \leqslant \omega_0\}$ for some fixed ω_0, but this is nowhere clearly spelled out in either the "Purely Analytic Proof" or the "Binomial Theorem." Variable quantities which can become as small as desired were commonly used in the mathematical literature of that time,[7] and perhaps Bolzano thought that there was no need for him to give a detailed explanation of them in these papers. Most historians have—either wittingly or not—extended the benefit of the doubt to Bolzano on this point and credited him with formulating the first adequate definition of continuity. One could, however (as Bolzano himself later recognized), be more precise concerning the points just mentioned.

In his *Function Theory*, written in the 1830s, Bolzano dealt with these problems directly, and there we find a precise, thoroughly modern definition of pointwise continuity. (The concept was also sharpened, as Bolzano defined left and right continuity.) Here is his definition:

> If a uniform function Fx of one or more variables is so constituted that the variation it undergoes when one of its variables passes from a determinate value x to the different value $x + \Delta x$ diminishes *ad infinitum* as Δx diminishes *ad infinitum*—if, that is, Fx and $F(x + \Delta x)$ (the latter of these at least from a certain value of the increment Δx and for all smaller values) are measurable [i.e., roughly speaking, real and finite], and the absolute value of the difference $F(x + \Delta x) - Fx$ becomes and remains less than any given fraction $\frac{1}{N}$ if one takes Δx small enough (and however smaller one may let it become): then I say that *the function Fx is continuous for the value x*, and this *for a positive increment* or *in the positive direction*, when that which has just been said occurs for a positive value of Δx; *for a negative increment* or *in the negative direction*, on the other hand, when that which has been said holds for a negative value of Δx; if, finally, the stated condition holds for a positive as well as a negative increment of x, I say, simply, that Fx is *continuous* at the value x. [Bolzano, 1930, I, §2][8]

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6 Thus interpreted, Bolzano's definition differs from (although it is equivalent to) the usual ones in confining the values of the variable $x + \omega$ to a *closed* rather than an open interval about x.

7 In his well-known work on the metaphysics of the calculus, for instance, Lazare Carnot had used them to define infinitely small quantities as follows: "I call *infinitely small quantity*, one which is considered as continually decreasing, so that it can be made as small as desired ..." [J'appelle *quantité infiniment petite*, toute quantité qui est considerée comme continuellement décroissante, tellement qu'elle puisse être rendu aussi petite qu'on le veut ...] [Carnot, 1970, ch. 1, §14]. This "clarification" of infinitely small quantities would later become standard usage thanks to Cauchy, who defined continuity as follows: a function f is continuous at x if and only if an infinitely small increase of the variable produces an infinitely small increase in the function, i.e., $f(x+\alpha) - fx$ is infinitely small whenever α is [Cauchy, 1821, 34ff.]. Infinitely small quantities, he had explained earlier, are not fixed quantities at all, but rather *variable* quantities which have zero as their limit [Cauchy, 1821, 4, 26].

8 "Wenn eine einförmige Function Fx von einer oder auch mehreren Veränderlichen so beschaffen ist, daß die Veränderung, die sie erfährt, indem eine ihrer Veränderlichen x aus dem bestimmten Werthe x in den Veränderten $x + \Delta x$ übergehet, in das Unendliche abnimmt, wenn Δx in das Unendliche abnimmt, wenn also der Werth Fx sowohl als auch der Werth $F(x + \Delta x)$, der letztere wenigstens anzufangen von einem gewissen Werthe der Differenz Δx für alle kleineren abermahls meßbar ist, der Unterschied $F(x + \Delta x) - Fx$ aber seinem absoluten Werthe nach kleiner als jeder gegebene Bruch $\frac{1}{N}$ wird und verbleibt, wenn man nur Δx klein genug nimmt, und so klein man es dann auch noch ferner werden läßt: so sage ich, daß die Function

In this later definition, a certain value of the increment Δx is distinguished, one, namely, which is small enough so that for it, and for all values of Δx smaller than it in absolute value, we have $|F(x+\Delta x) - F(x)| < \frac{1}{N}$ ("if one takes Δx small enough (and however smaller one may let it become)"). According to today's conventions, two different symbols would be used here, rather than two occurrences of Δx. However, Bolzano's intentions are clear and perfectly correct; he took a certain fixed value of Δx, but he allowed as well, in his inequalities, all nonzero values of Δx smaller in absolute value than the fixed value. In short, Δx was used with the same intention that ω had been used in the 1817 paper, only here the meaning was explicitly set out.

In order to simplify discussion, let us call the distinguished value of Δx in Bolzano's definition a *modulus of continuity* for F. That is, given x and $\frac{1}{N}$, a modulus of continuity for F is a positive number $\omega_x = \omega(x, \frac{1}{N})$ such that, for all values of Δx with $|\Delta x| \leqslant \omega_x$, we have $|F(x+\Delta x) - F(x)| < \frac{1}{N}$. (Bolzano did not introduce a special symbol for the modulus of continuity, letting the phrase "a small enough value of Δx" serve to designate the fixed value of the increment.) We can then paraphrase Bolzano's definition as follows: a function F is continuous at a value x if and only if for any $\frac{1}{N}$ there exists a modulus of continuity ω_x for F at x.

A function is said to be continuous on an interval, in the *Function Theory* as in the *Purely analytic proof* of 1817, iff it is continuous at every point in the interval: that is, if and only if, for each value x in the interval and given $\frac{1}{N}$, there exists a modulus of continuity ω_x for F at x. One can now ask the following question: if F is continuous for each x in an interval, can we take ω_x the same size for every x? In the case of functions continuous on an open interval, Bolzano answered: not necessarily. Shortly after giving the definition of continuity, he made this observation:

> Theorem. Merely because a function $F(x)$ is continuous for all values of its variable x lying between a and b, it does not follow that for all x between these values there is a fixed number e which is small enough so that one can claim that Δx never has to be taken smaller in absolute value than e in order to ensure that the difference $F(x+\Delta x) - F(x)$ will turn out to be smaller than $\frac{1}{N}$.[9] [Bolzano, 1930, I, §13]

He proved this as follows:

> It is neither contradictory in itself, nor contradictory to the given concept of continuity to assume that for any x there is always another (e.g., for the x approaching a certain limit C) for which it is necessary to take a smaller Δx in order to fulfill the condition that the difference $F(x+\Delta x) - Fx$ becomes less than $\frac{1}{N}$ and remains so, as one makes Δx smaller and smaller. We have such an example in the function $Fx = \frac{1}{1-x}$ for values of x approaching 1. Let us write for the sake of brevity $x = 1 - i$. Then $F(x+\Delta x) - Fx = \frac{\Delta x}{i(i-\Delta x)}$; if this is to be $< \frac{1}{N}$, then Δx must be $< \frac{i^2}{N+i}$. Thus as i becomes smaller, one must take Δx smaller; and

Fx für den Werth x stetig verändere, und zwar bey einem positiven Zuwachse oder im positiver Richtung, wenn das nur eben gesagte bey einem positiven Werthe von Δx eintritt: und daß sie dagegen sich stetig verändere bey einem negativen Zuwachse oder in negativer Richtung, wenn das Gesagte bey einem negativen Werthe von Δx Statt hat: wenn endlich das Gesagte bey einem positiven sowohl als negativen Zuwachs von Δx gilt: so sage ich schlechtweg nur, daß Fx stetig sey für den Werth x."

[9] "Blos daraus, daß eine Function Fx für alle innerhalb a und b gelegenen Werthe ihrer Veränderlichen x stetig sey, folgt nicht, daß es für alle innerhalb dieser Grenze gelegenen Werthe von x eine und eben dieselbe Zahl e geben müsse, klein genug, um behaupten zu können, daß man Δx nach seinem absoluten Werthe nie $< e$ zu machen brauche, damit der Unterschied $F(x+\Delta x) - Fx < \frac{1}{N}$ ausfalle."

> when i diminishes *ad infinitum*, i.e., when x approaches 1 *ad infinitum*, Δx must be taken smaller than any given number, in order to ensure that the difference ΔFx turns out $< \frac{1}{N}$.[10] [Bolzano, 1930, I, §13]

This is Bolzano's claim and proof that a function continuous on an open interval need not be uniformly continuous there. A reading of this proof makes it clear that what he had in mind is exactly today's notion of uniform continuity. The definition of uniform continuity which one can extract from his statement, however, has given some readers pause:

> [T]here is a fixed number e which is small enough so that one can claim that Δx never has to be taken smaller in absolute value than e in order to ensure that the difference $F(x + \Delta x) - F(x)$ will turn out to be smaller than a [given] number $\frac{1}{N}$.

The phrase "never has to be taken smaller" seems incongruous here, and has led some to think that Bolzano was confused. Karel Rychlik, the editor of the *Function Theory*, for example, paraphrased this condition as follows:

> Let x and $x + \Delta x$ be points in a set M; given $\varepsilon > 0$, there exists $e > 0$ independent of x in M such that it is not necessary for $|\Delta x|$ to be less than e in order to ensure $|F(x + \Delta x) - F(x)| < \varepsilon$. [Bolzano, 1930, Editor's notes, 4][11]

He then constructed an example which showed that this property (in conjunction with pointwise continuity) was not equivalent to uniform continuity on a set M. From his remarks, it is apparent that Bob van Rootselaar [1969, 1] agreed with Rychlik's interpretation. Both evidently assumed that by Δx, Bolzano referred simply to a fixed value of the increment.

Bolzano's language in the proof, it seems to us, indicates that this interpretation is not justified. For he says quite clearly that the Δx must be chosen smaller and smaller in order to ensure that $F(x+\Delta x) - Fx$ is less than $\frac{1}{N}$ not just for that particular value, but also for all smaller values of the increment Δx ("in order to fulfill the condition that the difference $F(x + \Delta x) - Fx$ becomes less than $\frac{1}{N}$, *and remains so, as one makes* Δx *smaller and smaller*"). What is being chosen here, in other words, is not a single fixed value of Δx, but rather a modulus of continuity. Considered by itself, apart from its context, Bolzano's formulation is not a definition of uniform continuity. However, the proof makes clear that it is incomplete not because Bolzano had no idea of uniform continuity, but rather because his formulation is elliptical,

[10] "Es ist weder an sich, noch dem gegebenen Begriffe der Stetigkeit widersprechend anzunehmen, daß für jedes andere x ein anderes, z. B. nahmentlich für jedes x, das einer gewissen Grenze C sich nähert, ein kleineres Δx nothwendig sey, um die Bedingung zu erfüllen, daß der Unterschied $F(x + \Delta x) - Fx < \frac{1}{N}$ wird und verbleibt, sofern man Δx noch immer verkleinert. Ein Beyspiel haben wir an der Function $Fx = \frac{1}{1-x}$ für solche Werthe von x, die sich dem Werthe von 1 in das Unendliche nahen. Schreiben wir nämlich zur Abkürzung $x = 1 - i$, so ist $F(x + \Delta x) - Fx = \frac{\Delta x}{i(i-\Delta x)}$: soll dieß $< \frac{1}{N}$ werden; so muß $\Delta x < \frac{i^2}{N+i}$ seyn. Also je kleiner i wird, um desto kleiner muß man auch Δx machen, und wenn i ins Unendliche abnimmt, d.h., wenn x sich der Grenze 1 in das Unendliche nahet, so muß Δx nach und nach kleiner als jeder gegebene Zahl werden, bloß damit der Unterschied $\Delta Fx < \frac{1}{N}$ ausfalle."

[11] "Sind x und $x + \Delta x$ Punkte aus M, so kann zu jedem $\varepsilon > 0$ ein $e > 0$ unabhängig von x aus M auf solche Art bestimmt werden, daß es nicht nötig ist, $|\Delta x|$ kleiner als e zu wählen, wenn $|\Delta F(x)| < \varepsilon$ sein soll."

and requires us to refer back to the definition of continuity given in §2.[12] The faults of his statement can be explained, if perhaps not excused, by noting that §13 is a remark concerning the definition of pointwise continuity and was meant to be read with that definition before one's eyes.

Things become somewhat clearer if we incorporate the definition of continuity directly into the statement of the theorem. Then we have

> Theorem. Merely because a function Fx is so constituted that for all values of its variable x lying between a and b, the absolute value of the difference $F(x + \Delta x) - Fx$ becomes and remains less than any given fraction $\frac{1}{N}$ if one takes Δx small enough (and however smaller one may let it become) it does not follow that for all x between these values there is a fixed number e which is small enough so that one can claim that Δx never has to be taken smaller in absolute value than e in order to ensure that the difference $F(x + \Delta x) - F(x)$ will turn out to be smaller than $\frac{1}{N}$.

With this in mind, Bolzano's text can be summarized as follows. First, a function F is said to be continuous at a point x if and only if for any $\frac{1}{N}$ there exists a modulus of continuity for F at x. Bolzano then pointed out in §13 that a function may be continuous on an open interval without it being the case that for any $\frac{1}{N}$, there exists $e > 0$ such that, for all x in the interval, the modulus of continuity ω_x never has to be taken less than e. And this was shown through his example, where the size of the moduli of continuity required to ensure $|F(x + \Delta x) - Fx| < \frac{1}{N}$ for $x \in (a, 1)$ are not in fact bounded away from zero.

The property that Bolzano denied of this particular function can be paraphrased as follows:

> Given $\frac{1}{N}$, there exists $e > 0$ such that, for all x in the interval, there exists a modulus of continuity for F at x which never has to be taken $< e$.

This differs slightly from now-standard definitions of uniform continuity, which have

> Given $\frac{1}{N}$, there exists $e > 0$ such that, for all x in the interval, there exists a modulus of continuity for F which is $= e$.

But it is easily seen that these two formulations are equivalent; for if there exists a modulus of continuity equal to e, then there trivially exists one which is $\geqslant e$. On the other hand, if there exists a modulus of continuity which is $\geqslant e$, then the modulus of continuity can always be taken equal to e.

Thus it seems clear to us that Bolzano here characterized the property of uniform continuity and proved, with the help of his example, that pointwise continuity on an open interval does not imply uniform continuity there.

He was not done, however, for in a manuscript published by van Rootselaar containing additions and emendations to the *Function Theory*, Bolzano stated that pointwise continuity on a closed interval *is* sufficient to ensure uniform continuity. He wrote:

[12] A further point supporting this reading is that the number $\frac{1}{N}$ appears out of nowhere in §13. If we read the passage as a comment on the definition of pointwise continuity, however, it becomes clear that what is being talked about is the previously given number $\frac{1}{N}$ mentioned there.

> Theorem. If a function Fx is continuous for all values between and including $x = a$ and $x = b$, then there is a certain number e which is sufficiently small so that for all x which do not lie outside of a and b, the increment Δx does not have to be taken smaller than e in order for the difference $F(x + \Delta x) - Fx$ to turn out to be less than a given number $\frac{1}{N}$. [van Rootselaar, 1969, 8–9][13]

He did not, however, produce a satisfactory proof of this result. What we find in the manuscript is some rough notes towards a proof. These contain—although Bolzano did not seem to have recognized this—a useful fragment of a correct proof. Here is a sketch of Bolzano's attempted proof [van Rootselaar, 1969, 9ff.].

Suppose that $F(x)$ is continuous on $[a, b]$. Suppose further that given a number $\frac{1}{N}$ there are $x_1, x_2, x_3, \ldots \in [a, b]$ such that the allowable increment Δx_i must be taken smaller and smaller in order to ensure that $|F(x_i + \Delta x_i) - F(x_i)|$ remains smaller than $\frac{1}{N}$. If the set of such x_i is only finite, then the Δx_i will have a minimum, which will therefore serve for the whole interval. The only remaining case of interest is where the Δx_i are infinite in number and tend to zero with increasing i (i.e., a case where the stated condition fails). In this case, applying the Bolzano–Weierstrass theorem, there is a limit point of the x_i, say c, which will lie in $[a, b]$. By hypothesis, f will be continuous at $x = c$.

At this point, it is relatively straightforward to obtain a contradiction to complete the proof. Instead of this, Bolzano attempted, without success, to prove the result directly. A direct proof is possible from the beginning sketched by Bolzano, but is considerably more complicated than an indirect one.[14] Thus, although he stated the key theorem linking pointwise and uniform continuity, Bolzano did not manage to produce a satisfactory proof.

3. Conclusion

The distinction between pointwise and uniform continuity is often cited as a typical advance of later 19th-century, in particular Weierstrassian, analysis and as a sign of the increasingly sophisticated use of quantificational concepts in mathematics. While there is no doubt much truth in the general picture of the development of analysis this example has been used to support, the results of this paper indicate that such distinctions were within the reach of careful mathematicians of an earlier generation like Dirichlet or, still earlier, mathematicians like Bolzano who made a special point of attending to the fine points of conceptual and logical structure.

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[13] "Lehrsatz: Wenn eine Function Fx für alle Werthe der Veränderlichen x von $x = a$ bis $x = b$ einschließlich stetig ist: so gibt es eine gewisse Zahl e klein genug, daß für alle Werthe der x nicht außerhalb a und b liegen, der Zuwachs Δx nicht $< e$ zu werden braucht, damit der Unterschied $F(x + \Delta x) - Fx <$ als eine gegebene Zahl $\frac{1}{N}$ ausfalle."

[14] See Rusnock [2004, Appendix] for details.

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