HISTORIA MATHEMATICA 20 (1993), 265–284

Algebraic Analysis in Germany, 1780–1840: Some Mathematical and Philosophical Issues

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Algebraic analysis is the algebraic treatment of functions and of infinitesimal calculus originating from Euler's *Introductio in analysin infinitorum*. This approach, as developed by the "Combinatorial School," was influential in Germany at the turn of the 19th century and became the basis for the mathematical syllabus of the Prussian gymnasium in the Humboldt educational reforms. The present paper discusses algebraic analysis under the viewpoint of two problems of legitimation. On the one hand, there was the conceptual and technical problem of calculating with infinite formal (divergent) series. How can equality between terms involving formal series be interpreted, and how is this "formal equality" related to numerical equality? The second problem regards the fact that algebraic analysis was not legitimized by referring to geometry or applications, but as an autonomous theory which is established by its internal coherence and harmony. Regarding this point the paper argues that the hermeneutic view of the sciences, which was widespread in Germany, led to a sometimes anti-Kantian attitude that sought to overcome the link of mathematics to the intuition of space and time. © 1993 Academic Press, Inc.

Die Algebraische Analysis ist die von L. Euler's *Introductio in analysin infinitorum* ausgehende algebraische Auffassung der Funktionenlehre und des Infinitesimalkalküls. An der Wende von 18. zum 19. Jahrhundert war sie in Deutschland in ihrer Ausformulierung durch die "Kombinatorische Schule" von Einfluß und wurde in der Humboldtschen Bildungsreform zur Grundlage des mathematischen Lehrplans der Preußischen Gymnasien. Das vorliegende Papier diskutiert die Algebraische Analysis unter der Perspektive eines zweifachen Begründungsproblems. Zum einen gab es das begrifflich-technische Problem des Rechnens mit unendlichen formalen (divergenten) Reihen. Wie kann die Gleichheit zwischen Termen, die formale Reihen beinhalten, verstanden werden, und wie verhält sich diese "formale Gleichheit" zu numerischer Gleichheit? Das zweite Problem betrifft die Tatsache, daß die Algebraische Analysis nicht durch Bezug auf Geometrie oder Anwendungen begründet wurde, sondern als eine autonome, selbstgenügsame Theorie, die durch ihre innere Kohärenz und Harmonie gerechtfertigt ist. Dazu wird argumentiert, daß die in Deutschland verbreitete hermeneutische Sicht der Wissenschaften zu einer manchmal auch gegen Kant gerichteten Einstellung führte, die Mathematik von ihrer Bindung an die Anschauung von Raum und Zeit zu lösen. © 1993 Academic Press, Inc.

L'analyse algébrique est la conception algébrique de la théorie des fonctions et du calcul infinitésimal dérivée de l'Introductio de L. Euler. Au tournant du XVIIIème au XIXème siècle, elle était influentielle en Allemagne dans sa formulation par "l'école combinatoire," devenant la base du curriculum mathématique des lycées prussiennes sous la réforme de Humboldt. La contribution discute l'analyse algébrique sous la perspective d'un double problème de fondation. Le premier était le problème conceptuel–technique de calculer avec des séries formelles infinies (divergentes). Comment concevoir l'égalité entre termes contenant des séries formelles, et quelle est la relation de cette "égalité formelle" avec l'égalité numérique? Le second problème concerne le fait que l'analyse algébrique ne fut pas fondée par référence à la géometrie ou aux applications, mais comme théorie autonome

0315-0860/93 $5.00

et autosatisfaisante justifiée par sa cohérence et harmonie internes. La contribution dit que la vue herméneutique des sciences répandue en Allemagne a mené a une attitude quelquefois anti-Kantienne qui veut dissocier la mathématique de son rapport à l'intuition de l'espace et du temps.

AMS 1991 subject classifications: 01A50, 01A55, 01A72, 01A74, 05-03.
KEY WORDS: hermeneutic culture, Humboldt educational reforms, combinatorial school, formal/divergent series, binomial and polynomial theorem, reversion of series.

1. THE CULTURAL FOUNDATION OF SCIENCE

A remarkable nexus of political and cultural events took place in Germany and France around the turn of the 19th century. Indeed, the dates of Napoleon's reign, 1799 to 1814, coincide very nearly with the first and most productive period of the romantic movement in Germany. The year 1794 saw the appearance of the first edition of Johann Gottlieb Fichte's (1762–1814) *Doctrine of Science* (*Grundlage der gesamten Wissenschaftslehre als Handschrift für seine Zuhörer*), a philosophical work which heavily influenced the romantic movement and the general culture climate of the time. Fichte's views represented a characteristic German reaction to events in France. He began his career with the voluminous *Defense of the French Revolution* of 1793 (*Beitrag zur Berichtigung der Urteile des Publikums über die französische Revolution*), and in 1808 he reached the zenith of his fame with *Speeches to the German Nation* (*Reden an die deutsche Nation*), a work destined to excite national resistance against the Napoleonic occupation of Germany. The core of both works was a *philosophy of education* that constituted the inner identity of the German reaction to the revolutionary events.

It was precisely during this Napoleonic era that Germany developed a cultural identity that it would maintain throughout much of the 19th century and which created a *cognitive framework* that shaped the development of science in Germany. In what follows we sketch some features of this framework and then analyze how it was related to mathematics, taking as our principal example the so-called algebraic analysis, a theory that derived from Leonhard Euler's famous *Introductio in analysin infinitorum*, Vol. I of 1748, which later became important as a mathematical background for mathematics teaching at the German Gymnasium.

The first and most important feature of this cognitive framework concerns the *relationship between science and practice* [Jahnke 1990a, part A]. The Humboldtian reforms in Germany created the modern role of a university professor as both teacher and researcher, thereby establishing a model for the institutionalization of science which, above all, furthered fundamental research and pure science far removed from practical applications. This was possible only because for a certain time there was, both within scientific circles and outside them, a broad consensus about the nature of the relationship between science and practice. This consensus, characteristic of the German cultural climate, shall be referred to as the *cultural foundation of science*.

This term is intended to convey the idea that science was pursued not for the sake of technical or commercial applications but because it can contribute to the development of a certain awareness of life, to the discussion of notions by which society may gain a better understanding of itself and its aims, and to the cultivation (*Bildung*) of the individual. Therefore science was seen as part of a comprehensive effort to produce a *purely spiritual world,* and, under certain circumstances, such an effort was considered more important for society than technical innovations. In short, science should foster *understanding* of nature and of culture. In this sense, we can thus speak of the formation of a *hermeneutic culture* in Germany. Wilhelm Dilthey has described the spirit of these times by the apt phrase that the interpretation of the world out of itself became the watchword of all free minds [Dilthey 1905, 211]. Just as philosophy and art generate interpretations of reality, so science was now seen as part of an interpretive effort.

This hermeneutic culture changed the aim of classical philology from the study of language to the production of a holistic interpretation of the classical world. A similar shift took place in the natural sciences; the primary task was no longer to gain new empirical knowledge but to develop a coherent synthetical view of nature. In the words of Wilhelm von Humboldt: "Genuine science has to be imbued and animated with a presentiment of a fundamental power whose essence is reflected in an original idea like in a mirror and must connect the totality of phenomena to it" [Humboldt 1814, 557]. The embedding of mathematics into this hermeneutic culture also had far-reaching consequences for the view of its methodology. Rather than considering pure mathematics in terms of algorithmic procedures for calculating certain magnitudes, the emphasis fell on *understanding* certain relations from their own presuppositions in a purely conceptual way. To understand given relations in and of themselves one must generalize them and see them abstractly. Recurring to the theory of art, we can speak of an act of *alienation.* In fact, since this time the analogy between art and mathematics has held a special attraction for many pure mathematicians.

That such a hermeneutic attitude to the world need not lead to a contemplative stance can be seen from the views of Johann Gottlieb Fichte, who favored an approach to education involving pure mathematics because the ability to think abstractly was, in his opinion, the decisive precondition for thinking in terms of alternatives, and of being able to develop a new design for the future, thereby creating an ethically acceptable world. According to Fichte: "That ability to independently design images which are by no means copies of reality but suitable to become ideals for it would be the first principle from which the cultivation of the species by means of the new education would have to proceed." [Fichte 1808, 31/2]

Immanent in this general cognitive framework was an *epistemological* motive related particularly to mathematics. In his *Kritik der reinen Vernunft* Immanuel Kant had defined mathematics as a science that constructed its concepts through the pure intuition of space and time. Therefore, these concepts are synthetic a priori. Without doubt this conception reflected the views of many mathematicians. Yet it stood in curious contrast to the actual trends in mathematical research at that time. It was precisely in the second half of the 18th century that the *analytical calculus* achieved remarkable results in many fields of mathematics and mathematical physics, and often these results depended on the use of analytical constructs which had no interpretation in the intuition of space and time. This objection was raised in the critical discussions of Kant's philosophy. For instance, Johann Gottfried Herder (1744–1803), the eminent philosopher of language, spoke of Kant's "radical misconception that visible construction should exhaust the essence of mathematics" [Herder 1799, 265]. By partly mathematical, partly philosophical arguments, Bernard Bolzano, in an appendix (*On Kant's theory of the construction of concepts through intuition*) to his *Contributions to a Better Founded Representation of Mathematics*, criticized Kant's views as too narrow and not really covering arithmetic and algebra [Bolzano 1810, 135 ff]. Consequently, he arrived at a definition of mathematics as a "science treating the general laws (forms) to which things must obey in their existence" [Bolzano l.c., 11]. A definition of mathematics as a "theory of forms" (*Formenlehre*) became common at that time. Around 1800, there arose a trend among mathematicians, philosophers, and educators to overcome the linkage of mathematics to the intuition of space and time.

This process was reinforced by the basic tenets of hermeneutic culture. Since the main intellectual interest at the time was to produce a purely spiritual world, or as Humboldt put it, to look for the invisible within the visible [Humboldt 1814, 560], abstract speculation going beyond the empirically obvious was an essential element of this worldview. Thus, the notion became more and more common that the objects of mathematics are purely mental constructions produced by man's faculty of productive imagination (*produktive Einbildungskraft*) [Fries 1822, 58]. For Jakob Friedrich Fries (1773–1843) this productive imagination did not operate in the pure intuition of space and time, but was a faculty of man's reason (*Vernunft*). This deviation from Kant was the more remarkable as its author was a strict follower of Kant's philosophy and sharply opposed the romantic trends of his time.

These assumptions had a decisive importance for algebraic analysis, a mathematical theory which was separated from its applications to geometry and the natural sciences and which was conceived as an autonomous theory justified by its own inner coherence. This point of view has been most aptly expressed by Julius Plücker: "I would like to advocate myself the view that analysis is a science which, independent of any application, exists autonomously by itself, and that geometry as well as in another regard mechanics, appears merely as a visual interpretation of the sublime whole" [Plücker 1831, IX].

Before analyzing these matters further, we should ask ourselves what insights we can expect to gain by viewing algebraic analysis within the cultural climate of Germany. In what sense can our notion of the *cultural foundation of science* explain what was going on? Surely, the separation of analysis from geometry and intuition was a general trend of the time. It can be found in France and Great Britain as well as in Germany. Moreover, the cultural context can only partially explain the scientific work and even the philosophy of a single author. This is true not only because science cannot be reduced to philosophical or metascientific motives but also because modern science is international. A scientist may live in a certain cultural context and may even be actively involved in a philosophical discourse there, and yet his true orientation may come from a totally different context. Thus, an analysis of the cultural and philosophical climate only contributes to our understanding in a, so to speak, statistical sense. It cannot replace the analysis of the individual work of a scientist. In my opinion, the main power of explanation provided by the philosophical/cultural context concerns the language in which scientists speak, and the types of explanations and legitimations which are considered acceptable or unacceptable. In our case, the separation of analysis from geometry and intuition may have been an imperative based on the internal state of mathematics, but the manner in which this was discussed and legitimized can hardly be understood without considering the cultural context in which mathematics was then practiced.

2. BASIC PROBLEMS OF ALGEBRAIC ANALYSIS

At the beginning of the 19th century, algebraic analysis meant both a subject matter field and the algebraic treatment of functions. In Germany, the field was also called *Analysis des Endlichen.* The great model of the theory was the first volume of Leonhard Euler's *Introductio in Analysin Infinitorum* [Euler 1748], where Euler presented an elementary theory of functions which was to provide the methods for differential and integral calculus but which did not contain this subject matter itself. Euler treated functions as purely algebraic objects, and therefore his methods consisted of calculating with infinite expressions: infinite series, products, continued fractions. Around the turn of the century, there were several attempts to systematize the field. The most important approach was that of Joseph Louis Lagrange [1797], while in Germany the so-called *Combinatorial School* led by the Leipzig mathematician Carl Friedrich Hindenburg (1739–1808) attempted to give a combinatorial treatment of the subject.

The key mathematical problem of algebraic analysis concerned the meaning and importance of infinite formal series. Whereas at the turn of the 19th century formal series played a considerable role, they were mostly considered illegitimate mathematical entities after 1850. Only at the end of the 19th century did divergent series again come up when several authors studied asymptotic series expansions and notions of generalized limits ("summability") [Kline 1972, Chap. 47]. Yet the viewpoint relevant in our context is that of a formal equality between a function and its series expansion in the sense of generating functions. This aspect has been expounded most lucidly by Andreas Speiser (1885–1970) in his preface to Euler's *Introductio* [Speiser 1945].

Speiser's argument is interesting for several reasons and I discuss it in detail. It should be noted that Speiser's main field of research was algebra, especially the theory of groups, and that he was among those mathematicians of our century who have tried to promote a cultural view of mathematics. Indeed, Speiser has intensively studied the period dealt with herein.

Concerning Euler's treatment of series, Speiser argues that with regard to series given by a law, and therefore having a general term, one can distinguish an *arithmetical* and an *algebraic* conception. According to the arithmetical conception, the terms of a series are interpreted as numbers, and therefore convergence is required. Following the algebraic view, the + or − signs are just symbols of combination, so that convergence plays no real role. This view is analogous, for example, to that in the theory of groups, where abstract elements are combined. Euler was mainly interested in this algebraic conception. As to its fruitfulness, Speiser wrote:

> By expansion in a series the law of a function is mapped onto a law for the members of the series, which brings to light deep properties of the function. Thus, for example, the number π bears a hidden relation to the odd natural numbers, log 2 to all natural numbers, e^x, sin x, cos x, and the integral logarithm to the factorials. In his immense expeditions into the empire of series, Euler made the overwhelming discovery that divergent series provide the strongest means for detecting unexpected facts. The passage through the divergent is even more fruitful than the passage through the complex within function theory. . . . That only convergent series should have a meaning is an unmathematical assertion—rather, the powerful Eulerian problem should be taken up again. Our time, which like the eighteenth century has turned to higher philosophical questions, should have the strength to do that. [Speiser 1945, IX/X]

This quotation is remarkable for several reasons. First, it contains an original and historically adequate legitimation of formal series. Second, there are some other important aspects hinted at by the key words *deep properties, hidden relations,* and *higher philosophical questions*. Here Speiser indicates certain connections that fit very well with the historical relation of mathematics and hermeneutic culture we have sketched. Above all, there is the spirit of abstract speculation. The passage through the divergent proves fruitful, although one cannot really explain why. The decimal expansion for π displays extreme irregularity (it has passed every statistical test for randomness so far devised), and yet it has a series representation displaying the utmost regularity. Thus something needs to be *understood* which mathematicians had not yet been able to understand. For this, one cannot be content with obvious approaches—one requires bold, abstract, speculative ideas analogous to philosophical theories and to Euler's approach in treating formal series even when one is unable to bring them into a coherent and fully controllable calculus.

Thus Speiser not only gives a conclusive argument for the legitimation of formal series but also shows a sensitive historical judgement. The algebraic conception allowing equalities between series which may not have a numerical interpretation aims at more abstract relations between laws of functions and the formal series in which they can be expanded. This was of decisive importance around 1800, but there were only limited attempts to develop a comprehensive conception embracing and combining both the algebraic and the arithmetic notion of series. (For Euler's attempts to relate formal and numerical equality, see [Barbeau & Leah 1976]; some later German approaches are discussed in the next chapter of this paper.) Thus, the entire formal approach to analysis eventually vanished from mathematics, and it was not before our century that a higher appreciation of algebraic methods within analysis was revived.

Without doubt there was a link between the spirit of abstract speculation in the early 19th century and the abstract approach to mathematics. This can be seen very well if one looks at an attempt to distinguish between formal and numerical equality made by Christian Gudermann (1798–1852), later the teacher of Karl Friedrich Weierstrass. In 1825 he wrote in a paper on the polynomial formula:

> One may be allowed to presume here, as generally known, the more general meaning of the concepts of equality, equation and of the corresponding sign = when applied even to infinite series. . . . The sign = then refers to a necessary relation between a primary (original) function and its expansion in a series such that this expansion is a mere representation of the primary function in a necessary (not arbitrary) manner, and serving for its determination. The infinite expansion in fact offers in each of its terms a special feature for the determination of the nature of the primary function. Thus it comprises innumerable features of the primary function, and only magnitude is not contained in these, because of the infinite nature of the expansion. Therefore, in those cases where the expansion is finite the sign = retains its usual restricted meaning." [Gudermann 1825, 21/22].

Surely, this is not a mathematically explicit conception. Yet the idea of postulating an abstract relation between the primary function and its infinite expansion is, in fact, a plausible mathematical intuition which may well be made precise some day. Speiser seems to have had a similar notion in mind when he drew an analogy between Euler's treatment of formal series and the modern theory of Fourier series. In the theory of Fourier series every continuous function is uniquely determined by its Fourier coefficients and if the Fourier series converges uniformly it represents the function in a numerical sense. But in the general case this is not true, and indeed the series need not even converge. In this case, there is a necessary relation between the primary function and its expansion which does not comprise the special feature of magnitude [Speiser 1945, X].

The notion that there can be equations between power series which may not have a numerical interpretation was commonplace in the 18th and early 19th century. As a consequence, a terminological distinction between numerical and formal equality arose between 1800 and 1820. Still, there were only a few attempts to make this distinction rigorous, and most authors relied on vague ideas like the one quoted by Gudermann.

3. THE WORK OF HINDENBURG AND OHM

Beginning with Cauchy's *Analyse Algébrique* [Cauchy 1821] a general trend set in that sought more and more to eliminate the formal-algebraic approach, although in some German textbooks it proved to be remarkably resistant. As examples of how equations involving formal series were considered and treated we discuss some works on the binomial and polynomial formula.

In 1779 Carl Friedrich Hindenburg gave a proof of the general polynomial formula [Hindenburg 1779]. Hindenburg was by education a philologist as well as a physicist and a mathematician, and he held a chair in experimental physics rather than in mathematics. His mathematical papers were not really deep. However, he was obviously a good scientific organizer and attracted some people with a certain mathematical stature. His treatment of the polynomial formula was in his time considered as a real achievement. The theorem says that the *m*th power of a general polynomial or power series is again a power series,

$$(1 + a_1x + a_2x^2 + a_3x^3 + \cdots)^m = 1 + A_1x + A_2x^2 + A_3x^3 + \cdots,$$

whose coefficients can be calculated by

$$A_r = \sum_{h=1}^{r} \binom{m}{h} p^{r^h}C.$$

Here the symbol ${}^{r^h}C$ is understood as designating the sum over all products which can be formed by taking h factors out of the coefficients a_1 to a_r (irrespective of their order and with repetitions allowed) and where the sum of the indices equals r. The symbol p represents a numerical operator indicating how many times the respective summands have to be taken. For instance, ${}^{5^3}C$ signifies the sum $a_1a_2^2 + a_1^2a_3$, whereas $p^{5^3}C$ is equal to

$$\frac{3!}{1!2!}a_1a_2^2 + \frac{3!}{2!1!}a_1^2a_3.$$

This notation is not exactly Hindenburg's, as he did not use numerical indices. Hindenburg's formula appeared for the first time in the above notation in a textbook by the Göttingen mathematician Bernhard Friedrich Thibaut [1809]. Apart from notation, however, Hindenburg's version of the polynomial formula had the merit that, in contrast to earlier formulations, it immediately allowed for a generalization to fractional and negative exponents, since the quantity m appears only in the binomial coefficients.

Hindenburg had arrived at this formula by simply setting

$$(1 + a_1x + a_2x^2 + a_3x^3 + \cdots)^m = (1 + y)^m$$

and expanding $(1 + y)^m$ according to the binomial formula. The powers

$$y^k = (a_1x + a_2x^2 + a_3x^3 + \cdots)^k$$

with *natural numbers k*, which appeared in this expansion, could then be calculated by using the polynomial formula for natural numbers.

This natural number version of the polynomial formula had already been found by Leibniz [1695] and Johann Bernoulli [1695]. A proof of it had been published by Abraham de Moivre in 1697, but he gave only a recursive rule for the calculation of the coefficients [cf. Schneider 1968, 1969, 252 pp]. Although Leibniz and de Moivre hinted at the possibility of extending the formula to rational and negative exponents [cf. Knobloch 1973, 98], this generalization would have required deriving a new form of the theorem, a step they never took. In 1730, de Moivre gave the special expansion for $m = -1$ [Pensivy 1986/1987, 65]. Later, Euler derived the polynomial theorem from a differential equation, valid also for rational and negative exponents with a recursive rule for the calculation of the series [Euler 1755, pars posterior, sect. 202]. Hindenburg's version of the polynomial theorem was, thus, the first *independent* formula which explicitly allowed for rational and negative exponents m. To write down this formula, however, he had to develop a good deal of combinatorial terminology and symbolism, and with this he was not very successful. Nevertheless, in a letter to Hindenburg, Lagrange acknowledged the former's efforts to develop a coherent combinatorial symbolism and wrote in regard to his polynomial formula: "La règle générale que vous y donnez pour former les puissances d'un polinome quelquonque ne me paroit rien laisser à desirer sur cet objet" [quotation by Hindenburg in Archiv der reinen und angewandten Mathematik II (1798), 370].

The polynomial theorem was of central importance for all calculations with power series since the operations of division, of exponentiation, and of extraction of roots may be reduced to this formula by taking appropriate negative, positive, and fractional exponents, respectively. Moreover, in 1793 Heinrich August Rothe (1773–1842), a disciple of Hindenburg, showed that even the solution of an arbitrary algebraic or transcendental equation may be effected by applying the polynomial formula to obtain the so-called *reversion of series*. Newton had been the first to design a recursive algorithm for this problem [Newton 1676], and afterward it had been treated by de Moivre [1698; cf. Pensivy 1986/1987, 66 pp.]. If an equation between two formal power series in x and y is given by

$$a_1 y^{\alpha_1} + b_1 y^{\alpha_1+\delta_1} + c_1 y^{\alpha_1+2\delta_1} + \cdots = ax^{\alpha} + bx^{\alpha+\delta} + cx^{\alpha+2\delta} + \cdots$$

with $a, \alpha > 0$, then an arbitrary power x^{γ}, and thus x itself, can be represented as a power series in y whose coefficients are calculated with the help of the polynomial formula [Rothe 1793]. This theorem, although forgotten today, represents something like an *implicit function theorem for formal series*.

It is important to note that Rothe's theorem is a purely combinatorial relation. Some twenty years earlier, Lagrange had proved another theorem solving the same problem of reverting series, but by use of the differential calculus [Lagrange 1770]. Given an equation

$$x = y + zf(x)$$

with arbitrary, i.e., analytic f (z being a parameter), then every function $\varphi(x)$, and in particular x itself, can be expanded into the following power series in z:

$$\varphi(x) = \varphi(y) + zf(y)\varphi'(y) + \frac{z^2 \cdot d[f(y)^2 \cdot \varphi'(y)]}{1 \cdot 2 \cdot dy} + \frac{z^3 \cdot d^2[f(y)^3 \cdot \varphi'(y)]}{1 \cdot 2 \cdot 3 \cdot dy^2} + \cdots.$$

In 1795 Rothe and Johann Friedrich Pfaff could show that, presupposing Taylor's theorem, Lagrange's and Rothe's formulae are equivalent [Pfaff 1795a,b; Rothe 1795]. For this reason, Rothe's theorem was seen as a remarkable success. For

Hindenburg and his adherents the polynomial formula was the most important theorem of analysis, because it seemed to be "a pinnacle from which one can survey the regions of analysis." [Klügel 1796, 51]

Hindenburg did not explicitly discuss how he conceived of the relation between the algebraic and arithmetic view of power series. Although he had derived a version of the polynomial theorem which, on the right hand side, allowed an algebraic as well as an arithmetic reading, he did not explain the meaning of equality and exponentiation in this theorem. Only later did a working definition come about which made possible a purely algebraic interpretation of the polynomial theorem and other power series formulae [cf. for instance Gudermann 1825]: a power series Q may be raised to a fractional exponent m/n by means of the definition

$$Q^{m/n} = R \Leftrightarrow Q^m = R^n.$$

Generally speaking, mathematicians of the 18th century believed that substituting numerical quantities into formal equations would lead to correct numerical results. Examples where this assumption obviously failed were treated by introducing new parameters and thus *generalizing* the respective formula. However, by the end of the 18th century the so-called *numerical factorials*

$$a^{n|d} := a(a + d)(a + 2d)(a + 3d) \cdots (a + (n - 1)d)$$

introduced by Euler, Alexandre Théophile Vandermonde, and the German mathematician and member of the combinatorial school Christian Kramp (1760–1826) (see [Kramp 1798]), provided an example where these strategies definitely failed. This function $a^{n|d}$ was one of Euler's *functiones inexplicabiles* and was closely related to his Γ-function. It motivated Gauß' work on hypergeometric series and later was extensively treated by Karl Weierstrass who used it to discuss his view of 18th-century methods of analysis [Weierstrass 1842/1843 and 1856].

One of the most debated paradoxes, however, involved the following two equations:

$$(2\cos x)^m = \sum_{k=0}^{\infty} \binom{m}{k} \cos(m - 2k)x$$

$$0 = \sum_{k=0}^{\infty} \binom{m}{k} \sin(m - 2k)x.$$

These formulae are correct for natural numbers m, as can be easily verified, and proofs were given in the 18th century that they remain correct if arbitrary rational or negative exponents m are admitted. Lagrange, in particular, had treated the problem as a paradigm for the power of his methods [Lagrange 1806, 138–141]. (For a full discussion of the history of this problem cf. [Jahnke 1987]). In 1811, however, Siméon Denis Poisson showed that the simple substitution of the numerical values $m = \frac{1}{3}$ and $x = \pi$ leads to a contradiction. This problem remained unsolved until 1823 when Louis Poinsot (1777–1859) in Paris and Martin Ohm in Berlin provided the correct analytical sums for the two trigonometrical series [Ohm 1823, Poinsot 1825]. Moreover, as can be seen from his correspondence [Abel 1902, 16/7], Niels Henrik Abel's work on the summation of the binomial series of 1826, which was later considered so important because it presented the first complete and exhaustive summation of the binomial series (including problems of convergence and complex exponents) had been motivated by these same paradoxes [Abel 1826].

The German mathematician Martin Ohm (1792–1872) [cf. Bekemeier 1987], brother of the famous physicist Georg Simon Ohm, was one of the few mathematicians in the early nineteenth century who systematically treated the relation between numerical and formal equations. Ohm had been a student of Rothe and, therefore, was trained within the broader context of the combinatorial school, although he did not follow the combinatorial approach. He viewed all analytical formulae as being either *numerical* or *symbolical*, the latter comprising only *meaningless symbols* as means to express relations between algebraic operations. Only the natural numbers were accepted as numbers in the strict sense, whereas negative, rational, and complex numbers were adjoined to the calculus as symbolic forms of the type $a - b$, a/b, $a + ib$. Ohm clearly recognized the necessity of defining equality for the new symbolic objects and of proving the consistency of the resulting system. His system of mathematics represented real progress for the rigorous foundation of elementary algebra. The whole procedure was similar to the approach of George Peacock (1791–1858) and Augustus de Morgan (1806–1871) in England with their principle of permanence of algebraic operations. In this regard, Ohm can even claim priority over them, and he was certainly of equal mathematical rank with them.

A particular problem which Ohm solved was the treatment of multivalued algebraic expressions. It is well known that in general the expression a^x is infinitely valued for complex a and x. To handle this problem, Ohm systematically distinguished between *perfect* and *imperfect* equations. An equation was called perfect if the sets of values on both sides were equal. If, however, the set of values on one side was only a subset of the set of values on the other side, it was called imperfect. Thus, for instance, the equation

$$a^x \cdot a^y = a^{x+y}$$

is imperfect because, in general, the left hand comprises values which are not represented by the right side. This equation becomes perfect by introducing a multiplier providing the missing values:

$$a^x \cdot a^y = a^{x+y} \cdot e^{2\pi i(kx+ly)}, \qquad k, l \in Z.$$

Note that even in the case of perfect equations the exact correspondence between the values on both sides need not necessarily be known. To derive this correspondence from an equation remained a problem which frequently proved rather subtle. At the end of the nineteenth century the distinction between perfect and imperfect equations was again used by Otto Stolz (1842–1905), who introduced it, with a reference to Ohm, in his *Theoretische Arithmetik* [Stolz & Gmeiner 1915, 322]. Florian Cajori commented on Ohm's treatment of the power function: "It must be granted that Ohm surpassed all his predecessors in the generality and fullness of discussion of the expression a^x, and that he is the first writer to successfully base the general theory of logarithms (having a complex number as a base) fully and unreservedly upon the theory of the general power a^x" [Cajori 1913, 177].

By employing this general framework Ohm could successfully treat formal series. He adjoined such series as ideal elements to the universe of finite algebraic expressions, defining equality and the basic arithmetical operations of series by means of the respective coefficient representations. Passing, then, from formal equations to numerical ones meant two things: to determine the *domain of convergence* and to derive from the possibly multivalued expressions a single-valued one.

If we take as an example the *binomial formula,* we see that the usual notation

$$(1 + x)^m = \sum_{k=0}^{\infty} \binom{m}{k} x^k$$

is an imperfect equation because for rational or irrational exponents m the left side is multivalued whereas the right side is single-valued. This equation becomes perfect by multiplying the right-hand side by the factor $e^{2\pi i m s}$, $s \in Z$, providing the missing values. For the sake of clarity, we modify Ohm's manner of notation and introduce also an analog factor on the left side, the underlined expression designating a single value of the mth power,

$$(\cos 2rm\pi x + i \cdot \sin 2rm\pi x) \cdot \underline{(1 + x)^m} = \left(\sum_{k=0}^{\infty} \binom{m}{k} x^k \right) \cdot (\cos 2sm\pi x + i \cdot \sin 2sm\pi x)$$

with r, $s \in Z$. Both sides now represent equally many values. This equation is still a purely formal equation, and passing to a numerical equation requires determining the domain of convergence of the binomial series and then deriving a single-valued relation, i.e., investigating which r on the left side belongs with which s on the right. By means of this technique. Ohm was able to resolve the difficulty with the above-mentioned trigonometric series thereby showing that it was the multivalued behavior of these functions and not the divergence of their series representations that was responsible for the paradox. This confirms Heinrich Burkhardt's conclusion that most of the mistaken results which occurred in dealing with divergent series were, in fact, the consequences of other procedures of doubtful validity [Burkhardt 1910/1911, 205].

All in all, Ohm's conception was coherent, and it must be emphasized that in regard to the multiplicity of algebraic expressions, it represented a decisive step beyond Lagrange, who, in his theory of analytic functions had used the multiplicity of algebraic expressions to argue that in the power series expansion of an arbitrary function rational exponents cannot occur [cf. Fraser 1987, 40].

It is interesting to compare Ohm's interpretation of the binomial formula with that of the "*Newer Analysis,*" as the approach of Cauchy and Abel was called in Germany. In the historiography of 19th-century analysis Cauchy's and Abel's treatments of the binomial theorem are usually mentioned because of their convergence proofs and their programmatic rejection of divergent series (see for example [Grattan-Guinness 1970, 80/1]). This is correct, but it does not comprise the whole truth. In fact, the conceptual change went much deeper. Cauchy and Abel were well aware that the rejection of divergent series was not compelling and they were very cautious in their published remarks on this subject. For Abel the rejection of divergent series was a *logical consequence* of his decision to abandon the notion of formal equality and to confine mathematical analysis completely to numerical equations. In the beginning of his paper on the binomial series Abel explicitly discussed how the binomial formula could be interpreted in the case of exponents which were not natural numbers. As a result of this discussion he rejected the notion of formal equality and took the position that only numerical equality should be accepted in mathematics [Abel 1826, 4/5].

In general, 18th-century analysts were basically algebraic thinkers. They thought in terms of formulae and, consequently, of variables and indeterminates. Therefore, for them, power series had a cognitive and psychological primacy over numerical series. The latter appeared only as derived from power series by substitution of numerical quantities. The distinction between formal and numerical equality reflected this 18th-century preoccupation with variables/indeterminates and power series. Abel's rejection of formal equality was thus a conscious break with the 18th-century manner of thinking algebraically, and his new approach to calculus is not adequately described as a mere introduction of precise and rigorous methods of calculation which the 18th century lacked. In recent books on the history of calculus, as for example [Grattan-Guinness] and [Grabiner 1981], the authors do more justice to the 18th-century approach to divergent series than is done in older studies such as, for example, [Boyer 1959], and they rightly point out that Cauchy's rejection of divergent series was an overreaction [Grabiner, 1981, 100]. In this context, Grabiner refers to work on generalized limits and asymptotic expansions from the end of the 19th century. However, notwithstanding the work done by 18th-century mathematicians on these topics, it seems to me that their general cognitive views are more adequately reflected, although not fully covered, by modern notions of formal power series and formal equality.

In reflecting these fundamental problems Abel's paper on the binomial series had a very programmatic flavor, so much that Abel even avoided writing down the binomial formula in the usual notation. Not once can one find the formula

$$(1 + x)^m = \sum_{k=0}^{\infty} \binom{m}{k} x^k$$

in this paper. Instead, Abel wrote down his results completely in the language of single-valued real functions, and in this language the binomial formula reads like this:

$$\begin{aligned}
&1 + \frac{m+ni}{1}(a+bi) + \frac{(m+ni)(m-1+ni)}{1\cdot 2}(a+bi)^2 + \cdots \\
&\quad + \frac{(m+ni)(m-1+ni)\cdots(m-u+1+ni)}{1\cdot 2\cdot\cdots\cdot u}(a+bi)^u + \cdots \\
&= ((1+a)^2+b^2)^{m/2}\cdot e^{-n\arctan(b/(1+b)} \\
&\quad \cdot\left[\cos\left(m\arctan\left(\frac{b}{1+b}\right) + \frac{n}{2}\log((1+a)^2+b^2)\right)\right. \\
&\quad \left. + i\cdot\sin\left(m\arctan\left(\frac{b}{1+b}\right) + \frac{n}{2}\log((1+a)^2+b^2)\right)\right].
\end{aligned}$$

This formula is rather telling. On the one hand, it shows the real complexity of the binomial theorem. On the other hand, however, it also becomes clear how the rigorous realization of the program of calculating only with single-valued numerical functions had, in a certain sense, rather negative effects. It takes some labor to verify that the right hand above is only the explicit form of the expression $(1 + z)^w$ (with $z = a + ib$ and $w = m + in$), not to mention the difficulties that arise when working with such a formula. The algebraic transparence and heuristic power of the binomial theorem had thus vanished completely and, from this perspective, Abel's point of view had nothing attractive to offer his contemporaries. In retrospect, it is clear that only complex function theory and Riemann's treatment of multivalued functions could really overcome this difficulty since in that theory complex functions are treated in a way where the algebraic structure of the formulae is respected.

Abel's version of the binomial theorem provides a good example illustrating why Cauchy's approach to algebraic analysis met with certain reservations in Germany. A number of mathematicians judged Cauchy's treatment as lacking inner lucidity and sound structure. Like present day criticisms of Weierstrass's (ε–δ) approach to limits as being too complicated for beginners while obscuring the clear and fundamental ideas of the founders of the infinitesimal calculus, Cauchy was criticized for conjuring up tricks rather than presenting really well-organized mathematics. This opinion was even held by the mathematician who did the most to make Cauchy's approach the norm for mathematical reasoning in Germany: Oskar Schlömilch (1823–1901). Indeed, Schlömilch conceded that Cauchy's *Cours d'analyse* lacked a solid architecture (Schlömilch 1845, V/VI], and, for this reason, he decided to write a treatise on algebraic analysis in which he hoped to combine Cauchy's rigorous treatment with a lucid presentation. Unfortunately, Schlömilch overlooked some essential mathematical points in Cauchy's approach. He did not realize that in deriving the cornerstone of the theory, the binomial formula, it is necessary to prove the continuity of the binomial series as a function of its exponent, and thus he fell back on older approaches [Schlömilch

1845, 62]. In fact, no correct proof of the binomial theorem can be found in a German textbook before the 1860s, and the formula remained even later a problematical case. This was especially irksome, since the binomial theorem was considered a culmination point for school mathematics which was supposed to provide inner coherence to the area of algebraic analysis, whereas its proof went beyond the scope of school teaching [cf. Jahnke 1990a, part C)].

At the same time, some mathematicians felt that in some way the restriction to convergent series stood in contradiction to the spirit of algebraic analysis since it severely disturbed the intuitive analogy between calculations with finite and infinite expressions. As an example of this line of thinking, consider a book on algebraic analysis that appeared in 1860, written by Moritz Abraham Stern (1807–1894), a well-known number theorist and the first doctoral student of Gauss. In this work, Stern turned back to the original views of the Combinatorial School and introduced two different signs for equality, one for numerical ($=$) and one for formal equality ($\neq$) [Stern 1860]. This time, however, the climate had changed and Stern's book failed to exert any influence on the teaching of algebraic analysis.

4. THE COGNITIVE UNIVERSE OF ALGEBRAIC ANALYSIS

In the wake of Humboldt's educational reforms, algebraic analysis became the scientific background for the mathematical syllabus of the Gymnasium. In contemporary pedagogical discussions this was not so much jsutified by its fertility in applications to analytic geometry and physics, but, rather, by the inner harmony and coherence of the whole theory. Although in the syllabus of 1812, the so-called Süvern syllabus, applications played a considerable role, they were more and more repressed with the course of time and in 1834 analytic geometry was even officially removed from the syllabus. Algebraic analysis became the core of school teaching because it was seen as an *elementary model of pure mathematics*. (For details cf. [Jahnke 1990a and b].)

To explain the spirit behind this conception, consider the following two quotations from authorities outside the realm of mathematics. The romantic poet Friedrich von Hardenberg (Novalis) (1772–1801) considered combinatorial analysis as a *model of mathematical genius,* since "the genius makes the impossible possible," since it is "perfect calculation," "without modifications" [Novalis 1983, 167 and Jahnke 1990a, 105 pp.]. These phrases alluded to the idea that every limitation of a calculation can be symbolically overcome by introducing ideal elements making operations possible which had been impossible before. Thus one finds here the clear anticipation of a technique which later was frequently used in mathematics to formally introduce number forms of ever higher order, thereby overcoming the limitations of the earlier systems. These *ideal elements* do not have an empirical meaning but are products of our imagination.

In his pedagogical writings, W.v. Humboldt attributed a high value to mathematics teaching, not for the training of logical thinking, as formal education was later interpreted, but to provide an *allgemeingültige Anschauung*. He saw a deep-rooted analogy between mathematical and aesthetic intuition and, therefore, it was the inner harmony and coherence of mathematics which constituted in his eyes its educational value. One of the consequences of this view was the radical refusal to introduce everyday applications into school teaching [Humboldt 1810, 261]. Mathematics had meaning not because of its applications, but out of itself, its inner relations and coherence. The systematic and purely formal nature of algebraic analysis made it ideally suited for this educational purpose. Consider this passage from a textbook by Enno Heeren Dirksen (1792–1850), who wrote: "The science, usually called mathematics, has the peculiarity that its objects as well as their determinations exist only insofar as they are produced by a free activity of the intellect; and this is the reason why in this field of knowledge nothing is recognized from the outside, but only from the way it is constructed" [Dirksen 1845, III]. This free activity of the mind results in a mental progress. "Being conscious of this mental progress and of the resulting relations, viewed in their necessary coherence, that is what makes up the science of analysis" [Dirksen 1845, IV]. According to this view, analysis is a purely mental construction starting with free constructive acts of thinking, followed by studying the inner relations by which these constructions are connected. Although this is a truly remarkable definition of analysis, it was by no means unique at the time as the quotation by Plücker given in the first section above shows.

Such general views corresponded very well with the systematic classification of mathematics found in certain textbooks which reflected the primacy of pure analysis. The following example is instructive because it shows that even in the second half of the 19th century the combinatorial approach had its adherents. It is taken from a textbook by Carl Anton Bretschneider (1808–1878) who was in his time a leading teacher and attained some international fame with his historical study, *Die Geometrie und die Geometer vor Euklides, ein historischer Versuch.* Bretschneider's classification reads as follows [Bretschneider 1856–1857, 11]:

Scheme of Mathematics:
I. Combinatorics.
1) Theory of Permutations.
2) Theory of Variations.
3) Theory of Combinations.
II. Theory of Quantities.
A. Theory of Numbers.
1) Theory of Discrete Numbers, Arithmetic.
2) Theory of Continuous Numbers, Analysis.
B. Theory of Forms. ("Gebindelehre")
1) Theory of Discrete Forms, Syntactics.
2) Theory of Continuous Forms, Theory of Extension. (Hermann Graßmann's "Ausdehnungslehre")

Views like these, which one can find in the German textbook literature, suggest the following description of the *cognitive universe* (*Denkwelt*) of algebraic analysis: in regard to its mathematical substance, algebraic analysis was derived from Euler's *Introductio;* ontologically, the theory was a universe of mentally produced symbolic forms; and in its theoretical elaboration it followed an ideal of organic and systematic coherence. Behind this was a spirit stressing an interest in "hidden relations" that have no practical value but are part of a purely spiritual world. This included the expectation that speculative ideas would succeed where routine methods had failed and an effort to free oneself from conventional viewpoints and to look within the visible for the invisible.

Algebraic analysis, as a universe of symbolic forms, determined the reality of mathematics education at the gymnasium. Since calculations in an arbitrary place value system can be interpreted as operations with polynomials, even elementary arithmetic can be seen under a combinatorial viewpoint. Thus, from elementary arithmetic up to the binomial theorem, the whole field comprised a remarkable unity. Later, the step-by-step extensions of number domains according to the principle of permanence of algebraic operations became the core of the whole arithmetic–algebraic syllabus. It was seen as giving inner coherence to arithmetic and algebra and as a perfect realization of the idea of an organic whole. In a well-known schoolbook we can read: "Equally important for the success of instruction seems to the author the distinct emphasis on the repetition of the same laws at the three different levels of arithmetical operations which shows arithmetic to the student as an organic whole . . ." [Müller 1838, V].

Beginning with the 1840's, Cauchy's views on the necessity of convergence proofs began to enter into school teaching. Yet this merely led to some compromises with the older view. Textbook authors added some convergence proofs to their power series expansions. The truly revolutionary step of Cauchy in grounding analysis in the theory of continuous functions instead of relying on arithmetic or algebraic expressions was not introduced at the gymnasium before the end of the century.

ACKNOWLEDGMENTS

Versions of this paper were delivered at the Symposium on "Mathematics and the French Revolution: Decades of Change" within the XVIIIth International Congress of History of Science, August 1989, and at the conference on "European Mathematics during the Napoleonic Period" held in Cortona in October 1989 under the auspices of the Istituto Nazionale di Alta Matematica Francesco Severi. All German quotations have been translated into English by myself. I would like to thank E. Knobloch, D. Rowe, F. Seeger, A. Shenitzer, and an anonymous referee for comments on the written version and for help with the English language.

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