Notes on isotropic convex bodies
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Chapter 1

Central limit theorem

1.1 Concentration property for $p$-balls

Let $1 \leq p \leq \infty$ and let $r_{p,n} > 0$ be a constant such that $|r_{p,n}B_p^n| = 1$. We write $L_{p,n}$ for the isotropic constant of $B_p^n$ and $\mu_{p,n}$ for the Lebesgue measure on $r_{p,n}B_p^n$.

As the next Theorem shows, most of the volume of the normalized $\ell_p^n$-ball lies in a very thin spherical shell around the radius $\sqrt{n}L_{p,n}$:

**Theorem 1.1.1.** For every $t > 0$,

$$\mu_{p,n}\left(\frac{|x|^2}{n} - L_{p,n}^2 \geq t\right) \leq CL_{p,n}^4 nt^2,$$

where $C > 0$ is an absolute constant.

The proof is based on the fact that normalized $\ell_p^n$-balls have the following subindependence property.

**Theorem 1.1.2 (Subindependence).** Let $K := r_{p,n}B_p^n$ and $P := \mu_{p,n}$. If $t_1, \ldots, t_n$ are non-negative numbers, then

$$\Pr\left(\bigcap_{i=1}^n \{|x_i| \geq t_i\}\right) \leq \prod_{i=1}^n \Pr(|x_i| \geq t_i).$$

**Proof.** The Theorem will follow by induction if we show that

$$\Pr\left(\bigcap_{i=1}^n \{|x_i| \geq t_i\}\right) \leq \Pr(|x| \geq t_1) \Pr\left(\bigcap_{i=2}^n \{|x_i| \geq t_i\}\right).$$
Set
\[(1.1.4) \quad S = \bigcap_{i=2}^{n} \{ |x_i| \geq t_i \}. \]

Then, we need to prove that
\[(1.1.5) \quad \frac{|K \cap S \cap \{ |x_1| \geq t_1 \}|}{|K|} \leq \frac{|K \cap \{ |x_1| \geq t_1 \}|}{|K|} \cdot \frac{|K \cap S|}{|K|}. \]

We will apply the following simple fact: if \( \mu \) is a positive measure on \([0, 1] \) and \( f : [0, 1] \to \mathbb{R} \) is increasing, then
\[(1.1.6) \quad \mu([0, 1]) \int_0^s f d\mu \leq \mu([0, s]) \int_0^1 f d\mu \]
for all \( s \in [0, 1] \). If
\[(1.1.7) \quad f(u) = \frac{|K \cap S \cap \{ |x_1| = 1-u \}|}{|K \cap \{ |x_1| = 1-u \}|}, \]
it is not hard to check that \( f \) is increasing. Let \( \mu \) be the probability measure with density
\[(1.1.8) \quad g(u) = \frac{|K \cap \{ |x_1| = 1-u \}|}{|K|}. \]

Then,
\[(1.1.9) \quad \int_0^1 f(u) d\mu = \int_0^1 \frac{|K \cap S \cap \{ |x_1| = 1-u \}|}{|K \cap \{ |x_1| = 1-u \}|} \frac{|K \cap \{ |x_1| = 1-u \}|}{|K|} du = \frac{|K \cap S|}{|K|} \]
and
\[(1.1.10) \quad \int_0^{1-t} f(u) d\mu = \frac{|K \cap S \cap \{ |x_1| \geq t \}|}{|K|}. \]

Applying (1.1.6) for \( s = 1-t_1 \) we get the result. \( \square \)

Theorem 1.1.2 immediately implies an anti-correlation inequality for the coordinate functions.

**Corollary 1.1.3.** Let \( K := r_{p,n}B_p^n \). Then,
\[(1.1.11) \quad \int_K x_i^2 x_j^2 dx \leq \int_K x_i^2 dx \cdot \int_K x_j^2 dx \]
for all \( i \neq j \) in \( \{1, \ldots, n\} \).
Proof. We write
\[
\int_K x_i^2 x_j^2 \, dx = 4 \int_{K \cap \{x_i \geq 0, x_j \geq 0\}} x_i^2 x_j^2 \, dx
\]
\[
= 4 \int_0^\infty \int_0^\infty 4t_i t_j P(x_i \geq t_i, x_j \geq t_j) dt_i dt_j
\]
\[
\leq 4 \int_0^\infty \int_0^\infty 4t_i t_j P(x_i \geq t_i) P(x_j \geq t_j) dt_i dt_j
\]
\[
\leq 4 \left( \int_0^\infty 2t_i P(x_i \geq t_i) dt_i \right) \left( \int_0^\infty 2t_j P(x_j \geq t_j) dt_j \right)
\]
\[
= 4 \int_{K \cap \{x_i \geq 0\}} x_i^2 \, dx \int_{K \cap \{x_j \geq 0\}} x_j^2 \, dx
\]
\[
= \int_K x_i^2 \, dx \int_K x_j^2 \, dx.
\]

Proof of Theorem 1.1.1: From the Cauchy-Schwarz inequality we have

\[(1.1.12)\]
\[n^2 L_{p,n}^4 = \left( \int_K |x|^2 \, dx \right)^2 \leq \int_K |x|^4 \, dx.\]

On the other hand, using Corollary 1.1.3 we have

\[
\int_K |x|^4 \, dx = \int_K \left( \sum_{i=1}^n x_i^2 \right)^2 \, dx = \sum_{i=1}^n \int_K x_i^4 \, dx + \sum_{i \neq j} \int_K x_i^2 x_j^2 \, dx
\]
\[
\leq n \int_K x_i^4 \, dx + \sum_{i \neq j} \int_K x_i^2 \, dx \int_K x_j^2 \, dx
\]
\[
= n \int_K x_i^4 \, dx + n(n-1) L_{p,n}^4.
\]

Since

\[(1.1.13)\]
\[\int_K x_i^4 \, dx \leq C \left( \int_K x_1^2 \, dx \right)^2 = CL_{p,n}^4\]

for some absolute constant \(C > 0\), we get

\[(1.1.14)\]
\[L_{p,n}^4 \leq \frac{1}{n^2} \int_K |x|^4 \, dx \leq \left( 1 + \frac{C}{n} \right) L_{p,n}^4.\]

This implies that

\[(1.1.15)\]
\[\int_K \left( \frac{|x|^2}{n} - L_{p,n}^2 \right)^2 \, dx = \frac{1}{n^2} \int_K |x|^4 \, dx - L_{p,n}^4 \leq \frac{C}{n} L_{p,n}^4.\]
Then, Chebyshev’s inequality gives

$$\left(1.1.16\right) \quad \frac{t^2}{n} P_p,n \left( \left| \frac{|x|^2}{n} - L_{p,n}^2 \right| \geq t \right) \leq \int_K \left( \frac{|x|^2}{n} - L_{p,n}^2 \right)^2 \, dx \leq \frac{CL_{p,n}^4}{nL_{p,n}^2}$$

for every $t > 0$, which is exactly the assertion of the Theorem.

**Corollary 1.1.4.** For every $t > 0$,

$$\mu_p,n \left( \left| \frac{|x|^2}{\sqrt{n}} - L_{p,n} \right| \geq t \sqrt{n} \right) \leq \frac{C L_{p,n}^4}{nt^2}.$$  

**Proof.** Let $t > 0$. We have

\[
\mu_p,n \left( \left| \frac{|x|^2}{\sqrt{n}} - L_{p,n} \right| \geq t \sqrt{n} \right) \leq \mu_p,n \left( \left| \frac{|x|^2}{n} - nL_{p,n}^2 \right| \geq tnL_{p,n} \right) \leq \frac{CL_{p,n}^4}{t^2nL_{p,n}^2} = \frac{C L_{p,n}^4}{nt^2},
\]

by Theorem 1.1.1.

### 1.2 Busemann’s inequality

Let $K$ be a symmetric convex body in $\mathbb{R}^n$. Busemann’s inequality states that the function

$$\left(1.2.1\right) \quad x \mapsto \frac{|x|}{|K \cap x^+|}$$

is a norm. This fact is a special case (take $k = 2$) of the following Theorem.

**Theorem 1.2.1.** Let $K$ be a symmetric convex body in $\mathbb{R}^n$. Let $2 \leq k \leq n-1$ and let $E$ be a $k$-codimensional subspace of $\mathbb{R}^n$. Let $F = E^\perp$ and, for every $z \in F$ define $E(z) = \{x + tz : x \in E, t > 0\}$. Then, the function

$$\left(1.2.2\right) \quad z \mapsto \frac{|z|}{|K \cap E(z)|}$$

is a norm on $F$.

**Proof.** Let $z_1$ and $z_2$ be linearly independent vectors in $F$. Set $z_3 = z_1 + z_2$ and define

$$f_i(t) = |K \cap (tz_i/|z_i| + E)| \quad \text{for all } t > 0, \; i = 1, 2, 3.$$  

Then,

$$F_i = |K \cap E(z_i)| = \int_0^\infty f_i(t) \, dt.$$
We will prove that

\[(1.2.5) \quad \frac{|z_3|}{F_3} \leq \frac{|z_1|}{F_1} + \frac{|z_2|}{F_2}.\]

Let \(t_1, t_2 > 0\) and let \(y_i = t_i z_i/|z_i|, i = 1, 2.\) The segment \([y_1, y_2]\) intersects the ray in the direction of \(z_3\) at the point \(y_3 = t_3 z_3/|z_3|\). Writing \(y_3 = \alpha y_1 + (1 - \alpha)y_2\), we see that

\[(1.2.6) \quad \alpha = \frac{t_2/|z_2|}{t_1/|z_1| + t_2/|z_2|}.\]

Then, from the equation

\[(1.2.7) \quad \frac{\alpha t_1}{|z_1|} z_1 + \frac{(1 - \alpha)t_2}{|z_2|} z_2 = \frac{t_3}{|z_3|} z_3\]

we get

\[(1.2.8) \quad \frac{|z_3|}{t_3} = \frac{|z_1|}{t_1} + \frac{|z_2|}{t_2}.\]

For every \(s \in [0, 1]\) we define \(t_1(s)\) and \(t_2(s)\) by the equations

\[(1.2.9) \quad s = \frac{1}{F_1} \int_0^{t_1(s)} f_1(u) du = \frac{1}{F_2} \int_0^{t_2(s)} f_2(u) du.\]

We have

\[(1.2.10) \quad \frac{dt_1}{ds} = \frac{F_1}{f_1(t_1(s))},\]

and differentiating (1.2.8) we see that

\[(1.2.11) \quad \frac{|z_3|}{t_3(s)} \frac{dt_3}{ds} = \frac{|z_1|}{t_1(s)} \frac{F_1}{f_1(t_1(s))} + \frac{|z_2|}{t_2(s)} \frac{F_2}{f_2(t_2(s))}.\]

Applying the Brunn-Minkowski inequality (in log-concave form) we see that

\[(1.2.12) \quad f_3(t_3(s)) \geq f_1(t_1(s))^\alpha f_2(t_2(s))^{1-\alpha}.\]

We write

\[(1.2.13) \quad \frac{F_3}{|z_3|} \geq \int_0^1 \frac{f_3(t_3(s))}{|z_3|} \frac{dt_3}{ds} ds.\]

Now, the integrand is greater than or equal to

\[(1.2.14) \quad \frac{t_3^2(s)}{|z_3|^2} \left( \frac{|z_1|}{t_1^2(s)} \frac{F_1}{f_1(t_1(s))} + \frac{|z_2|}{t_2^2(s)} \frac{F_2}{f_2(t_2(s))} \right) f_1(t_1(s))^\alpha f_2(t_2(s))^{1-\alpha}.\]
If we set \(a = |z_1|/t_1\) and \(b = |z_2|/t_2\), from (1.2.6) and (1.2.8) we may write the last expression in the form

\[
1/(a + b)^2 \left( a^2 F_1 \left| z_1 \right| f_1(t_1(s)) + b^2 F_2 \left| z_2 \right| f_2(t_2(s)) \right) f_1(t_1(s))^{a/b} f_2(t_2(s))^{b/a}.
\]

By the arithmetic-geometric means inequality,

\[
a \left( a F_1 \left| z_1 \right| f_1(t_1(s)) \right) + b \left( b F_2 \left| z_2 \right| f_2(t_2(s)) \right) \geq (a + b) \left( a F_1 \left| z_1 \right| f_1(t_1(s)) \right)^{a/(a+b)} \left( b F_2 \left| z_2 \right| f_2(t_2(s)) \right)^{b/(a+b)},
\]

so the integrand in (1.2.13) is greater than

\[
1/(a + b)^2 \left( a F_1 \left| z_1 \right| f_1(t_1(s)) + b F_2 \left| z_2 \right| f_2(t_2(s)) \right) f_1(t_1(s))^{a/b} f_2(t_2(s))^{b/a}.
\]

Applying once again the arithmetic-geometric means inequality, we see that

\[
(a F_1 \left| z_1 \right| f_1(t_1(s)))^{a/(a+b)} (b F_2 \left| z_2 \right| f_2(t_2(s)))^{b/(a+b)} \geq \left( a + b \right)^{-1} \left( a F_1 \left| z_1 \right| f_1(t_1(s)) + b F_2 \left| z_2 \right| f_2(t_2(s)) \right)^{-1}.
\]

This shows that the integrand in (1.2.13) is greater than \(\left( |z_1| F_1 + |z_2| F_2 \right)^{-1}\), which proves (1.2.5).

\[\square\]

### 1.3 Average section function

Let \(K\) be a convex body in \(\mathbb{R}^n\) with volume 1 and center of mass at the origin. Recall that

\[
f_{K,\theta}(t) = |K \cap (\theta^\perp + t\theta)|
\]

for every \(\theta \in S^{n-1}\) and \(t \geq 0\). We define a function \(f_K\), which measures the average decay of the volume of hyperplane sections of \(K\), by

\[
f_K(t) = \int_{S^{n-1}} f_{K,\theta}(t) \sigma(d\theta).
\]

The next Proposition gives an integral formula for \(f_K\).

**Proposition 1.3.1.** Let \(K\) be a convex body in \(\mathbb{R}^n\) with volume 1 and center of mass at the origin. For every \(t \geq 0\),

\[
f_K(t) = c_n \int_{U_K(t)} \frac{1}{|x|^2} \left( 1 - \frac{t^2}{|x|^2} \right)^{n-3} dx,
\]
where \( U_K(t) = \{ x \in K : |x| \geq t \} \) and

\[
c_n = \frac{\Gamma\left(\frac{n}{2}\right)}{\sqrt{\pi} \Gamma\left(\frac{n+1}{2}\right)}.
\]

**Proof.** We denote by \( \lambda_{\theta,t} \) the Lebesgue measure on the hyperplane \( N_{\theta}(t) = \{ x : \langle x, \theta \rangle = t \} \). Consider the measure

\[
\lambda_t = \int_{S^{n-1}} \lambda_{\theta,t} \sigma(d\theta).
\]

Then, \( \lambda_t \) is a positive measure on \( \mathbb{R}^n \) and

\[
f_K(t) = \int_{S^{n-1}} \int_{N_{\theta}(t)} \chi_K(x) d\lambda_{\theta,t}(x) \sigma(d\theta) = \lambda_t(K).
\]

The density of \( \lambda_t \) is invariant under orthogonal transformations, therefore

\[
d\lambda_t = p_t(|x|)
\]

where \( p_t : [t, +\infty) \to [0, +\infty) \). In order to find \( p_t \), for every \( r > t \) we compute

\[
\lambda_t(B(0, r)) = \int_{B(0, r)} p_t(|x|) dx = n \omega_n \int_t^r p_t(s) s^{n-1} ds.
\]

On the other hand, since the intersection of \( B(0, r) \) with the hyperplane \( N_{\theta}(t) \) is an \((n-1)\)-dimensional ball of radius \( \sqrt{r^2 - t^2} \) for every \( \theta \in S^{n-1} \), we get

\[
\lambda_t(B(0, r)) = \int_{S^{n-1}} \lambda_{\theta,t}(B(0, r)) \sigma(d\theta) = \omega_{n-1}(r^2 - t^2)^{\frac{n-1}{2}}.
\]

Differentiating with respect to \( r \geq t \), we see that

\[
\omega_{n-1} \frac{1}{2} (r^2 - t^2)^{\frac{n-3}{2}} 2r = n \omega_n p_t(r) r^{n-1}.
\]

This shows that

\[
p_t(r) = \frac{(n-1) \omega_{n-1}}{n \omega_n} \frac{(r^2 - t^2)^{\frac{n-3}{2}}}{r^{n-2}}.
\]

Therefore,

\[
f_K(t) = \int_{U_K(t)} p_t(|x|) dx = \frac{(n-1) \omega_{n-1}}{n \omega_n} \int_{U_K(t)} \frac{(|x|^2 - t^2)^{\frac{n-3}{2}}}{|x|^{n-2}} dx,
\]

and the result follows. \( \square \)

**Corollary 1.3.2.** \( f_K \) is a decreasing function. \( \square \)
1.4 The \( \varepsilon \)-concentration hypothesis

Let \( K \) be an isotropic convex body in \( \mathbb{R}^n \). We view \( K \) as a probability space (with the Lebesgue measure \( \mu_K \) on \( K \)) and for every \( \theta \in S^{n-1} \) we consider the random variable \( X_\theta(x) = \langle x, \theta \rangle \). Since \( K \) is isotropic, we have

\[
\mathbb{E} X_\theta = 0 \quad \text{and} \quad \text{Var}(X_\theta) = L_K^2
\]

for every \( \theta \in S^{n-1} \).

It is conjectured that most of these random variables have to be very close to a Gaussian random variable \( \gamma \) with mean 0 and variance \( L_K^2 \). In this Section we will see that this is true, at least for isotropic symmetric convex bodies, under the following general hypothesis which states that the Euclidean norm concentrates near the value \( \sqrt{n}L_K \) as a function on \( K \).

**Concentration hypothesis:** Let \( 0 < \varepsilon < \frac{1}{2} \). We say that \( K \) satisfies the \( \varepsilon \)-concentration hypothesis if

\[
\mu_K \left( \left| \frac{|x|}{\sqrt{n}} - L_K \right| \geq \varepsilon L_K \right) \leq \varepsilon.
\]

**Remark 1:** The results of §1.1 (see Corollary 1.1.4) show that the class of \( l_1^n \) balls satisfies the \( \varepsilon \)-concentration hypothesis with \( \varepsilon \approx \frac{1}{n^{1/3}} \).

Before stating the Theorem, we need to introduce some notation. We denote by \( g(s) \) the density of the Gaussian random variable \( \gamma \) with variance \( L_K^2 \) and for simplicity we write \( g_\theta(s) \) for the density of \( X_\theta \). Note that

\[
g_\theta(s) = f_{K,\theta}(s) = |K \cap (\theta^\perp + s\theta)|
\]

and

\[
g(s) = \frac{1}{\sqrt{2\pi}L_K} \exp \left( -\frac{s^2}{2L_K^2} \right).
\]

**Theorem 1.4.1.** Let \( K \) be an isotropic symmetric convex body in \( \mathbb{R}^n \) which satisfies the \( \varepsilon \)-concentration hypothesis for some \( 0 < \varepsilon < \frac{1}{2} \). Then, for every \( \delta > 0 \)

\[
\sigma \left( \left\{ \theta : \left| \int_{-t}^{t} g_\theta(s) \, ds - \int_{-t}^{t} g(s) \, ds \right| \leq \delta + 4\varepsilon + \frac{c_1}{\sqrt{n}} \quad \text{for every} \ t \in \mathbb{R} \right\} \right) 
\geq 1 - ne^{-c_22^n},
\]

where \( c_1, c_2 > 0 \) are absolute constants.
The proof is divided into three steps. We first consider the average function

\[ A(t) = \int_{S^{n-1}} \int_{-t}^{t} g(s) \, ds \, \sigma(d\theta) \]

and show that, under the \( \varepsilon \)-concentration hypothesis,

\[ |A(t) - \int_{-t}^{t} g(s) \, ds| \leq 4\varepsilon + \frac{c_1}{\sqrt{n}} \]

for every \( t > 0 \).

**Lemma 1.4.2.** Let \( K \) be an isotropic convex body in \( \mathbb{R}^n \). For every \( t > 0 \),

\[ |A(t) - \frac{2}{\sqrt{2\pi}} \int_{K} \int_{0}^{\frac{1}{\sqrt{n}}} e^{-\frac{s^2}{2}} \, du \, dx| \leq \frac{c_1}{\sqrt{n}}, \]

where \( c_1 > 0 \) is an absolute constant.

**Proof.** From Proposition 1.3.1 we have

\[ f_K(s) = \int_{S^{n-1}} g_K(s) \sigma(d\theta) = c_n \int_{\{|x| \geq s\}} \frac{1}{|x|} \left( 1 - \frac{s^2}{|x|^2} \right)^{\frac{n-3}{2}} \, dx, \]

where \( c_n = \Gamma\left( \frac{n}{2} \right) / \sqrt{\pi} \Gamma\left( \frac{n-1}{2} \right) \). Note that

\[ \frac{1}{2c_n} = \int_{0}^{1} (1 - u^2)^{\frac{n-3}{2}} \, du. \]

Now, Fubini’s theorem gives

\[ A(t) = 2 \int_{0}^{t} f_K(s) \, ds = 2c_n \int_{0}^{1} \int_{\{|x| \geq t\}} \frac{1}{|x|} \left( 1 - \frac{s^2}{|x|^2} \right)^{\frac{n-3}{2}} \, dx \, ds = 2c_n \int_{K} \int_{0}^{\min\{1, \frac{1}{\sqrt{n}}\}} (1 - u^2)^{\frac{n-3}{2}} \, du \, dx. \]

We will prove that

\[ 2c_n \int_{0}^{\min\{1, \frac{1}{\sqrt{n}}\}} (1 - u^2)^{\frac{n-3}{2}} \, du = \frac{2}{\sqrt{2\pi}} \int_{0}^{\frac{1}{\sqrt{n}}} e^{-\frac{u^2}{2}} \, du \leq \frac{c_1}{\sqrt{n}} \]

for every \( t > 0 \) and \( x \in K \), where \( c_1 > 0 \) is an absolute constant. This will prove the Lemma since the volume of \( K \) is equal to 1.
Case 1: If \( |x| \leq t \), then by (1.4.10) we have to show that
\[
1 - \frac{2}{\sqrt{2\pi}} \int_0^{t\sqrt{n}} e^{-\frac{x^2}{2}} \, dx \leq \frac{c_1}{\sqrt{n}}.
\]
But the left hand side is equal to
\[
\frac{2}{\sqrt{2\pi}} \int_0^{t\sqrt{n}} e^{-\frac{x^2}{2}} \, dx \leq \frac{2}{\sqrt{2\pi}} \int_0^{\infty} e^{-\frac{x^2}{2}} \, dx \leq \frac{2}{\sqrt{2\pi}} e^{-\frac{t^2}{2}} \leq \frac{2}{\sqrt{2\pi}}.
\]

Case 2: If \( |x| \geq t \), we write
\[
2c_n \int_0^{t\sqrt{n}} (1 - u^2) \frac{n-3}{n} \, du = \frac{2c_n}{\sqrt{n}} \int_0^{t\sqrt{n}} \left(1 - \frac{u^2}{n}\right)^{n-3} \, du.
\]
Then, the left hand side of (1.4.11) is bounded by
\[
\frac{2c_n}{\sqrt{n}} - 2 \frac{2}{\sqrt{2\pi}} \int_0^{t\sqrt{n}} e^{-\frac{x^2}{2}} \, dx + 2c_n \int_0^{t\sqrt{n}} \left|\left(1 - \frac{u^2}{n}\right)^{n-3} - e^{-\frac{u^2}{2}}\right| \, du.
\]
Using the asymptotic formula \( \Gamma(x) = x^{x-1}e^{-x}\sqrt{2\pi x} \left(1 + \frac{1}{12x} + O(x^{-2})\right) \) as \( x \to +\infty \), we see that
\[
\frac{2c_n}{\sqrt{n}} = 2 \frac{2}{\sqrt{2\pi}} \left(\frac{n}{n-1}\right)^{\frac{2}{n-1}} - \frac{1}{\sqrt{\pi}} \left(1 + O\left(\frac{1}{n}\right)\right).
\]
It follows that the first term in (1.4.15) is bounded by \( c/n \) as \( n \to \infty \).

For the second term, consider the function \( h(u) = e^{-\frac{u^2}{2}} - \left(1 - \frac{u^2}{n}\right)^{n-3}\frac{u^2}{n} \) on \([0, \sqrt{n}]\). Note that \( h(0) = 0 \) and \( h(\sqrt{n}) = \exp(-n/2) \). If there is a point \( v \in [0, \sqrt{n}] \) such that \( h'(v) = 0 \), then \( \left(1 - \frac{v^2}{n}\right)^{n-3} = \frac{n}{n-3} e^{-\frac{v^2}{2}} \), and hence, \( h(v) = \frac{v^2}{n-3} e^{-\frac{v^2}{2}} \). Since this last expression is \( O(1/n) \) on \([0, \sqrt{n}]\), we have
\[
\int_0^{t\sqrt{n}} \left|\left(1 - \frac{u^2}{n}\right)^{n-3} - e^{-\frac{u^2}{2}}\right| \, du \leq \frac{c\sqrt{n}}{\sqrt{n}}.
\]
On the other hand we have \( c_n \simeq \sqrt{n} \), which shows that the second term in (1.4.15) is bounded by \( c/\sqrt{n} \) as \( n \to \infty \).

\[\text{Theorem 1.4.3.}\] Let \( K \) be an isotropic convex body in \( \mathbb{R}^n \). If \( K \) satisfies the \( \varepsilon \)-concentration hypothesis, then
\[
\left|A(t) - \int_{-t}^{t} g(s) \, ds\right| \leq 4\varepsilon + \frac{c_1}{\sqrt{n}}
\]
for every \( t > 0 \).
Proof. Let $t > 0$ and set

$$F_t(s) = \frac{2}{\sqrt{2\pi}} \int_0^t e^{-\frac{u^2}{2}} du.$$  

In this notation, Lemma 1.4.2 states that

$$\left| A(t) - \int_K F_t \left( \frac{|x|}{\sqrt{n}} \right) dx \right| \leq \frac{c_1}{\sqrt{n}}.$$

Note that

$$F_t(L_K) = \frac{2}{\sqrt{2\pi}} \int_0^t e^{-\frac{s^2}{2}} ds = \frac{1}{\sqrt{2\pi} L_K} \int_{-t}^t e^{-\frac{s^2}{2 L_K^2}} ds = \int_{-t}^t g(s) ds.$$

So, the Theorem states that

$$\left| A(t) - F_t(L_K) \right| \leq 4\varepsilon + \frac{c_1}{\sqrt{n}}.$$

Because of (1.4.20), the Theorem will be proved if we check that

$$\left| \int_K F_t \left( \frac{|x|}{\sqrt{n}} \right) dx - F_t(L_K) \right| \leq 4\varepsilon.$$

We divide $K$ into two subsets:

$$K_1 = K \cap \left\{ \left| \frac{|x|}{\sqrt{n}} - L_K \right| \leq \varepsilon L_K \right\}$$

and

$$K_2 = K \cap \left\{ \left| \frac{|x|}{\sqrt{n}} - L_K \right| \geq \varepsilon L_K \right\}.$$

Then,

$$\left| \int_K F_t \left( \frac{|x|}{\sqrt{n}} \right) dx - F_t(L_K) \right| \leq \sum_{i=1}^2 \int_{K_i} \left| F_t \left( \frac{|x|}{\sqrt{n}} \right) - F_t(L_K) \right| dx.$$  

To estimate the integral on $K_2$, we just use the fact that $F_t \left( \frac{|x|}{\sqrt{n}} \right)$ and $F_t(L_K)$ are bounded by 1. Since $K$ satisfies the $\varepsilon$-concentration hypothesis,

$$\int_{K_2} \left| F_t \left( \frac{|x|}{\sqrt{n}} \right) - F_t(L_K) \right| dx \leq 2|K_2| \leq 2\varepsilon.$$
For the integral on $K_1$ we shall use a Lipschitz estimate for $F_t$. Observe that
\[(1.4.28) \quad |F'(s)| = \frac{2}{\sqrt{2\pi} s^2} e^{-\frac{s^2}{2}} \leq \frac{1}{s},\]
since $x \exp(-x^2/2) \leq 1/\sqrt{\pi}$ on $[0, +\infty)$. Also, since $\varepsilon < 1/2$, for every $x \in K_1$ we have
\[(1.4.29) \quad \frac{|x|}{\sqrt{n}} > L_K \frac{2}{\varepsilon}.\]
It follows that
\[(1.4.30) \quad \int_{K_1} \left| F_t \left( \frac{|x|}{\sqrt{n}} \right) - F_t(L_K) \right| \, dx \leq \int_{K_1} \frac{2}{L_K} \left| \frac{|x|}{\sqrt{n}} - L_K \right| \, dx \leq \int_{K_1} 2\varepsilon \, dx \leq 2\varepsilon.
\]
Combining the above we get (1.4.23), and taking into account (1.4.20) we conclude the proof.

In the second step we use the estimate of Theorem 1.4.3 for the average $A(t)$ to obtain a similar estimate for “most directions” $\theta \in S^{n-1}$. The idea is to show that
\[\int_{-t}^{t} g_\theta(s)ds\]
is the (restriction on $S^{n-1}$ of the) radial function of a symmetric convex body in $\mathbb{R}^n$ and then use the spherical isoperimetric inequality in the context of Lipschitz continuous functions on the sphere.

Here, we make use of Busemann’s inequality (see §1.4).

**Proposition 1.4.4.** Let $K$ be a symmetric convex body in $\mathbb{R}^n$. Fix $t > 0$ and define
\[(1.4.31) \quad \|x\|_t = \int_{-t}^{t} \frac{|x|}{\sqrt{n}} \, g_\theta(s)ds.\]
Then, $\| \cdot \|_t$ is a norm on $\mathbb{R}^n$.

**Proof.** Recall that $g_\theta(s) = |K \cap (\theta^\perp + s\theta)|$ for every $\theta \in S^{n-1}$. We define
\[(1.4.32) \quad v(x, t) = \int_{-t}^{t} g_\frac{x}{\sqrt{n}}(s)ds\]
and prove that for all $x, y \in \mathbb{R}^n$,
\[(1.4.33) \quad \frac{1}{2} \left( \frac{|x|}{v(x, t)} + \frac{|y|}{v(y, t)} \right) \geq \frac{|x + y|}{v \left( \frac{x + y}{2}, t \right)}.\]
We may clearly assume that $x$ and $y$ are linearly independent. Consider the convex body $K' = K \times [-1, 1]$ in $\mathbb{R}^{n+1}$. Lemma 1.4.4 shows that $\frac{|\theta|}{|K' \cap \theta^\perp|}$ defines a norm on $\mathbb{R}^{n+1}/\{0\}$. That is,
\[(1.4.34) \quad \frac{1}{2} \left( \frac{|\theta|}{|K' \cap \theta^\perp|} + \frac{|\phi|}{|K' \cap \phi^\perp|} \right) \geq \frac{|\theta + \phi|}{|K' \cap (\frac{\theta + \phi}{2})^\perp|}.\]
for all linearly independent $\theta, \phi \in \mathbb{R}^{n+1}$.

Let $r \in (0, 1)$ be defined by the equation $tr = \sqrt{1 - r^2}$. We observe that if $z \in \mathbb{R}^n \setminus \{0\}$ and

$$u(z) = \left( r \frac{z}{|z|}, \sqrt{1 - r^2} \right),$$

then the projection of $K' \cap u(z) \perp$ onto the first $n$ coordinates is $\{ w \in K : |\langle w, z \rangle| \leq t|z| \}$. It follows that

$$v(z, t) = \sqrt{1 - r^2} |K' \cap u(z) \perp|.$$

We define $\eta(z) = |z| u(z)$. Then, $|K' \cap u(z) \perp| = |K' \cap \eta(z) \perp|$ and $|\eta(z)| = |z|$. If we set $\theta = \eta(x)$ and $\phi = \eta(y)$, Busemann's inequality shows that

$$1 \geq \frac{1}{2} \left( \frac{|x|}{v(x, t)} + \frac{|y|}{v(y, t)} \right) \geq \frac{1}{\sqrt{1 - r^2}} \frac{|\frac{\theta + \phi}{2}|}{|K' \cap \left( \frac{\theta + \phi}{2} \right) \perp|}.$$

Observe that

$$\frac{\theta + \phi}{2} = \left( r \frac{x + y}{2}, \sqrt{1 - r^2} \frac{|x| + |y|}{2} \right) = \frac{|x + y|}{2} \left( r \frac{x + y}{|x| + |y|}, \sqrt{1 - r^2} \frac{|x| + |y|}{|x + y|} \right).$$

Then, the projection of $K' \cap \left( \frac{\theta + \phi}{2} \right) \perp$ onto the first $n$ coordinates is a strip perpendicular to $\frac{x + y}{2}$, with width

$$s = \frac{|x| + |y|}{|x + y|} t,$$

and this gives

$$\frac{v \left( \frac{x + y}{2}, s \right)}{|K' \cap \left( \frac{\theta + \phi}{2} \right) \perp|} = \sqrt{1 - r^2} \frac{|x| + |y|}{\frac{\theta + \phi}{2}}.$$

Then, (1.4.37) takes the form

$$1 \geq \frac{1}{2} \left( \frac{|x|}{v(x, t)} + \frac{|y|}{v(y, t)} \right) \geq \frac{1}{2} \frac{|x| + |y|}{v \left( \frac{x + y}{2}, s \right)}.$$

Observe that if $a \geq 1$, then for every $z \in \mathbb{R}^n$ we have

$$v(z, at) \leq a v(z, t).$$
Central limit theorem

(this follows from the fact that \( g_\theta(as) \leq g_\theta(s) \) for all \( \theta \in S^{n-1} \) and \( s > 0 \). So,

\[
\frac{|x| + |y|}{2v\left(\frac{x+y}{2}, s\right)} = \frac{|x| + |y|}{2v\left(\frac{x+y}{2}, \frac{|x+y|}{|x+y|} t\right)} \geq \frac{|x| + |y|}{2\frac{|x+y|}{|x+y|} v\left(\frac{x+y}{2}, t\right)} = \frac{|x+y|}{2}.
\]

Going back to (1.4.40) we conclude the proof.

**Lemma 1.4.5.** Let \( K \) be a symmetric convex body in \( \mathbb{R}^n \) with volume 1. For every \( \theta \in S^{n-1} \) and every \( t > 0 \),

\[
\int_1^\infty g_\theta(s)ds \leq \frac{1}{2} e^{-2g_\theta(0)t}.
\]

**Proof.** Consider the function

\[
H(t) = \int_1^\infty g_\theta(s)ds = \int_0^\infty \chi_{[t, \infty)}(s) g_\theta(s)ds.
\]

Using the fact that \( g_\theta \) is log-concave and applying the Prékopa-Leindler inequality, we may easily check that \( H \) is log-concave. It follows that

\[
(\log H)(t) - (\log H)(0) \leq (\log H)'(0)t
\]

for every \( t > 0 \). Observe that \( H(0) = 1/2 \) by the symmetry of \( K \), and

\[
(\log H)'(0) = -\frac{g_\theta(0)}{H(0)} = -2g_\theta(0).
\]

It follows that

\[
H(t) \leq H(0) \exp\left((\log H)'(0)t\right) = \frac{1}{2} \exp(-2g_\theta(0)t),
\]

as stated in the Lemma.

**Lemma 1.4.6.** Let \( K \) be an isotropic symmetric convex body in \( \mathbb{R}^n \). For every \( t > 0 \), the norm

\[
\|x\|_t = \frac{|x|}{\int_{-t}^t g_{\frac{|x|}{t}}(s)ds}
\]

satisfies

\[
a|x| \leq \|x\|_t \leq b|x|
\]

for every \( x \in \mathbb{R}^n \), where \( a, b \) are two positive constants such that \( a \geq 1 \) and \( b/a \leq c \) for some absolute constant \( c > 0 \).
1.4 The \( \varepsilon \)-concentration hypothesis

Proof. Since \( K \) is isotropic, we know that \( g_\theta(0) \simeq L_K^{-1} \) for every \( \theta \in S^{n-1} \). Then, by the symmetry of \( K \) we have

\[
\int_{-t}^{t} g_\theta(s) \, ds \leq \min \left\{ 2t \, g_\theta(0), 1 \right\} \leq \min \left\{ \frac{c_1 t}{L_K}, 1 \right\}.
\]

Also, Lemma 1.4.6 shows that

\[
\int_{-t}^{t} g_\theta(s) \, ds = 1 - 2 \int_{t}^{\infty} g_\theta(s) \, ds \geq 1 - e^{-2g_\theta(0)t} \geq 1 - e^{-\frac{c_2}{K}}.
\]

We easily check that

\[
1 - e^{-\frac{c_2}{K}} \geq \frac{c_2 t}{2L_K}
\]

if \( c_2 t \leq L_K \). In any case,

\[
\int_{-t}^{t} g_\theta(s) \, ds \geq \min \left\{ \frac{c_3 t}{L_K}, 1 - e^{-1} \right\}.
\]

In other words

\[
a := \max \left\{ \frac{L_K}{c_4 t}, 1 \right\} \leq ||\theta||_t \leq b := \max \left\{ \frac{L_K}{c_3 t}, \frac{e}{e - 1} \right\}
\]

for every \( \theta \in S^{n-1} \). Note that \( a \geq 1 \) and \( b/a \) is bounded independently of \( t \) and \( L_K \).

\[\square\]

Theorem 1.4.7. Let \( K \) be an isotropic symmetric convex body in \( \mathbb{R}^n \). If \( K \) satisfies the \( \varepsilon \)-concentration hypothesis, then for every \( t > 0 \) and \( \delta > 0 \),

\[
\sigma \left\{ \theta : \left| \int_{-t}^{t} g_\theta(s) \, ds - \int_{-t}^{t} g(s) \, ds \right| \geq \delta + 4\varepsilon + \frac{c_3}{\sqrt{n}} \right\} \leq 2e^{-c_4 \delta^2 n},
\]

where \( c_3, c_4 > 0 \) are absolute constants.

Proof. Let \( t > 0 \) and \( \delta > 0 \) be fixed. We will use the spherical isoperimetric inequality through the following fact: If \( f : S^{n-1} \to \mathbb{R} \) is \( d \)-Lipschitz and \( M(f) \) is its mean, then

\[
\sigma \left\{ \theta : |f - M(f)| \geq \delta + \frac{c_2}{\sqrt{n}} \right\} \leq 2e^{-\frac{\delta^2}{2n}}.
\]

We shall apply this to the function \( f(\theta) = \int_{-t}^{t} g_\theta(s) \, ds \). Observe that

\[
\left| \int_{-t}^{t} g_\theta(s) \, ds - \int_{-t}^{t} g_\phi(s) \, ds \right| = \left| \frac{1}{||\theta||} - \frac{1}{||\phi||} \right| \leq \frac{||\theta - \phi||}{||\theta|| \, ||\phi||} \leq \frac{b}{\sqrt{n}} ||\theta - \phi|| \leq c ||\theta - \phi||.
\]
where \( c \) is the absolute constant in Lemma 1.4.7. Also, note that \( M(f) = A(t) \). It follows that

\[
\sigma \left( \left\{ \theta : \left| \int_{-t}^{t} g_{\theta}(s) \, ds - A(t) \right| \geq \delta + \frac{c_2}{\sqrt{n}} \right\} \right) \leq 2 \exp \left( -\frac{\delta^2 n}{2c^2} \right).
\]

Combining this with Theorem 1.4.3, we get

\[
\sigma \left( \left\{ \theta : \left| \int_{-t}^{t} g_{\theta}(s) \, ds - \int_{-t}^{t} g(s) \, ds \right| \geq \delta + 4\varepsilon + \frac{c_3}{\sqrt{n}} \right\} \right) \leq 2 \exp \left( -c_4\delta^2 n \right),
\]

where \( c_3 = c_1 + c_2 \) (\( c_1 \) is the constant in Theorem 1.4.3) and \( c_4 = 1/(2c^2) \).

**Proof of Theorem 1.4.1:** First, fix some \( \theta \in S^{n-1} \). Since

\[
g_{\theta}(s) \leq g_{\theta}(0) \leq \frac{c_1}{L_K}
\]

and

\[
g(s) = \frac{1}{\sqrt{2\pi L_K}} \exp\left(-s^2/(2L_K^2)\right) \leq \frac{1}{\sqrt{2\pi L_K}}
\]

for every \( s > 0 \), the function

\[
H(t) = \left| \int_{-t}^{t} g_{\theta}(s) \, ds - \int_{-t}^{t} g(s) \, ds \right|
\]

is Lipschitz continuous with constant \( d \leq \frac{c_2}{L_K} \), where \( c_2 \) is an absolute constant.

Also, there is an absolute constant \( c_3 > 0 \) such that \( H(t) \leq 1/\sqrt{n} \) for every \( t \geq c_3 L_K \log n \). This is a consequence of the equality

\[
H(t) = 2 \left| \int_{t}^{\infty} g_{\theta}(s) \, ds - \int_{t}^{\infty} g(s) \, ds \right|
\]

and of Lemma 1.4.6: if \( c_3 > 0 \) is chosen large enough, when \( t \geq c_3 L_K \log n \) we have

\[
\max \left\{ \int_{t}^{\infty} g_{\theta}(s) \, ds, \int_{t}^{\infty} g(s) \, ds \right\} < \frac{1}{2\sqrt{n}}.
\]

Define \( t_k = k\alpha \), where \( \alpha = L_K/\sqrt{n} \) and \( k = 1, \ldots, k_0 = [c_3\sqrt{n} \log n] + 1 \). From Theorem 1.4.8, for every \( \delta > 0 \) we have

\[
\sigma (A) \leq 2c_3\sqrt{n}(\log n)e^{-c_6\delta^2 n},
\]
where
\[
A = \left\{ \theta : \exists k \leq k_0 \text{ s.t. } \left| \int_{-t_k}^{t_k} g_0(s)ds - \int_{-t_k}^{t_k} g(s)ds \right| \geq \delta + 4\varepsilon + \frac{c_5}{\sqrt{n}} \right\}
\]
and \(c_5, c_6 > 0\) are absolute constants. If \(\theta\) is not in \(A\), then
\[
H(t_k) \leq \delta + 4\varepsilon + \frac{c_5}{\sqrt{n}}
\]
for all \(k = 1, \ldots, k_0\). Since \(H\) is \(\frac{c_5}{L_K}\)-Lipschitz, we get a similar estimate for \(H(t)\), \(t \in [0, c_3 L_K \log n]\). Finally, if \(t > c_3 L_K \log n\), we know that \(H(t) < 1/\sqrt{n}\).

\section{The variance hypothesis}

Let \(K\) be an isotropic convex body in \(\mathbb{R}^n\). In this Section we study the parameter \(\sigma_K\) of \(K\) which is defined by

\[
\sigma^2_K = \frac{\text{Var}(|x|^2)}{nL^4_K}
\]

and its connections with the problems we discussed in this Chapter as well as in Chapter 6.

It is useful to write \(\sigma_K\) in the form

\[
\sigma^2_K = \frac{n\text{Var}(|x|^2)}{(E|x|^2)^2}.
\]

In this way the quantity becomes invariant under homotheties, and hence, easier to compute.

A simple computation shows that if \(K = B^n_2\) then

\[
E|x|^4 = \frac{n}{n + 4} \quad \text{and} \quad E|x|^2 = \frac{n}{n + 2}.
\]

Therefore,

\[
\sigma^2_{B^n_2} = n \left( \frac{E|x|^4}{(E|x|^2)^2} - 1 \right) = \frac{4}{n + 4}.
\]

Actually, Euclidean balls have minimal \(\sigma_K\) as the next Theorem shows.

\textbf{Theorem 1.5.1.} Let \(K\) be an isotropic convex body in \(\mathbb{R}^n\). Then,

\[
\sigma_K \geq \sigma_{B^n_2}.
\]
Proof. Let $x$ be uniformly distributed in $K$. The distribution function $F(r) = |\{x \in K : |x| \leq r\}|$ has density

$F'(r) = n\omega_n r^{n-1} \sigma \left( \frac{1}{r} K \right)$

for $r > 0$. We define $q(r) = n\omega_n \sigma \left( \frac{1}{r} K \right)$. Observe that $q$ is increasing and can be assumed absolutely continuous. Therefore, we can write $q$ in the form

$q(r) = n \int_r^{\infty} \frac{p(s)}{s^n} ds,$

where $p : (0, +\infty) \to \mathbb{R}$ is a non-negative measurable function. Then, Fubini’s theorem shows that

$\int_0^{\infty} p(s) ds = n \int_0^{\infty} \frac{s}{s^n} \left( \int_0^s r^{n-1} dr \right) ds = \int_0^{\infty} r^{n-1} q(r) dr = 1,$

which means that $p$ is the density of some positive random variable $\xi$.

Also, for every $\alpha > -n$,

$E |x|^\alpha = \int_0^{\infty} r^{\alpha+n-1} q(r) dr = \frac{n}{n+\alpha} \int_0^{\infty} s^{\alpha} p(s) ds = \frac{n}{n+\alpha} E \xi^\alpha.$

We can now compute

$\text{Var}(|x|^2) = \frac{n}{n+4} E \xi^4 - \left( \frac{n}{n+2} E \xi^2 \right)^2$

$= \frac{4n}{(n+4)(n+2)^2} (E \xi^2)^2 + \frac{n}{n+4} \text{Var}(\xi^2)$

$\geq \frac{4n}{(n+4)(n+2)^2} (E \xi^2)^2.$

It follows that

$\sigma_K^2 = n \frac{\text{Var}(|x|^2)}{(E |x|^2)^2} \geq \frac{4n}{(n+4)(n+2)^2} \left( \frac{n}{n+2} E \xi^2 \right)^2 = \frac{4}{n+4},$

and the Theorem follows from (1.5.4). □

Remark 1: Simple computations show that

$\sigma_{\hat{p}_n}^2 = 1 - \frac{2(n+1)}{(n+3)(n+4)} \to 1$ as $n \to \infty$
and
\begin{equation}
\sigma_{B^n} = \frac{4}{5} \quad \text{for every } n.
\end{equation}

In the next Section we briefly discuss various consequences of the following hypothesis.

**Variance hypothesis:** There exists an absolute constant $C > 0$ such that $\sigma^2_K \leq C$ for every isotropic convex body.

Let us first note that $\sigma^2_K$ is uniformly bounded for all $\ell^n_p$. This follows by the subindependence theorem of §1.1. Actually, the argument is inside the proof of Theorem 1.1.1.

**Proposition 1.5.2.** There exists an absolute constant $C > 0$ such that $\sigma^2_{B^n_p} \leq C$ for every $n$ and every $p \in [1, \infty]$.

**Proof.** In the proof of Theorem 1.1.1 we saw that
\begin{equation}
n^2 L_{p,n}^4 \leq \int_K |x|^4 dx \leq (n^2 + Cn)L_{p,n}^4
\end{equation}
for some absolute constant $C > 0$. Then,
\begin{equation}
\sigma^2_{B^n_p} = n \left( \frac{\mathbb{E}|x|^4}{n^2 L_{p,n}^4} - 1 \right) \leq C
\end{equation}
for all $p$ and $n$. \qed