On the isotropic constant of marginals

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Abstract

Let $1 \leqslant p < \infty$ and $n \geqslant 1$. We write $\mu_{B,p,n}$ for the probability measure in $\mathbb{R}^n$ with density $\mathbf{1}_{B_p^n}$, where $B_p^n := \{x \in \mathbb{R}^n : \|x\|_p \leqslant b_{p,n}\}$ and $|B_p^n| = 1$. Let $\mu_1, \cdots, \mu_{k_1}$ be one-dimensional log-concave measures, let $p_j \in [1, \infty]$ and let n_j, $1 \leqslant j \leqslant k_2$ be positive integers with $k_1 + \sum_{j=1}^{k_2} n_j = N > n$. We show that, for every $F \in G_{N,n}$,

$$L_{\pi_F\left(\left(\otimes_{i=1}^{k_1}\mu_i\right)\otimes\left(\otimes_{j=1}^{k_2}\mu_{B,p_j,n_j}\right)\right)} \leqslant C,$$

where $C > 0$ is an absolute constant, L_μ stands for the isotropic constant of μ and $\pi_F(\mu)$ denotes the marginal of μ on F.

1 Introduction

A famous open problem in convex geometry is the hyperplane conjecture (**HC**) asking if there exists a constant $c > 0$ such that for every $n \geqslant 1$ and any symmetric convex body K of volume 1 in $\mathbb{R}^n$ there exists $\theta \in S^{n-1}$ such that

$$|K \cap \theta^\perp| \geqslant c. \tag{1.1}$$

The question was posed in this form by J. Bourgain in [6]. A classical reference on the subject is the paper of V. D. Milman and A. Pajor [23] (see also [12]). It this paper we will consider an equivalent formulation of the hyperplane conjecture, given by K. Ball [1]. Let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$ (i.e. the density f_μ of μ is of the form $f_\mu(x) = e^{-V(x)}$, where $V : \mathbb{R}^n \to [0, \infty]$ is a convex function). Then, the question is whether

$$L_\mu := f_\mu(0)^{\frac{1}{n}} \leqslant C, \tag{1.2}$$

where $C > 0$ is an absolute constant. The best known bound is due to B. Klartag [17] who proved that $L_\mu \leqslant C n^{\frac{1}{4}}$ (see also [7] and [18]).

Another famous conjecture – which at first sight seems unrelated to the hyperplane conjecture – was proposed by Kannan, Lovász and Simonovits [15]. We will

*The author is partially supported by an NSF grant (DMS-0906150).

use the abbreviation (**KLS**). In equivalent form, (**KLS**) asks if for any isotropic log-concave probability measure μ on $\mathbb{R}^n$ and any smooth function $g : \mathbb{R}^n \to \mathbb{R}$,

$$\mathrm{var}_\mu(g) := \mathbb{E}|g - \mathbb{E}(g)|^2 \leqslant C\mathbb{E}\|\nabla g\|_2^2, \tag{1.3}$$

where $C > 0$ is an absolute constant (see [22] for other equivalent formulations of the question). Recently, Eldan and Klartag ([10]) showed that if (**KLS**) has a positive answer (for all isotropic log-concave measures) then (**HC**) is also true (for all isotropic log-concave measures). More precisely, they showed that a weaker version of the (**KLS**) conjecture (the so-called variance conjecture) is sufficient. We refer to [14] for the best known bound and more information related to the latter problem. The validity of (**HC**) has been verified in many cases (see e.g. the references in [27]); on the contrary, (**KLS**) has been established in some very special cases only (1-dimensional log-concave probability measures [4] and indicators of B_p^n [30]). However, it is known that if μ_1, μ_2 are two probability measures satisfying (1.3) with the same constant, then so does their product $\mu_1 \otimes \mu_2$ (see e.g. [19, pp. 98]). Moreover, if μ satisfies (1.3) with some constant D, then any marginal $\pi_F(\mu)$ of μ also satisfies (1.3) with the same constant. So, combining these two operators one can construct a rich family of isotropic log-concave probability measures which satisfy (**KLS**). For example, if $\mu_1, \cdots, \mu_k$ are 1-dimensional log-concave probability measures and $\mu_{B,p_j,n_j} := \mathbf{1}_{a_{p_j,n_j} B_{p_j}^{n_j}}$, $j = 1, \cdots, m$, (where $p_j \in [1, \infty]$ and a_{p_j,n_j} is chosen so that μ_{B,p_j,n_j} is isotropic) then, for any $F \in G_{N,n}$, where $n < N := k + \sum_{j=1}^{m} n_j$, any isotropic log-concave probability measure of the form

$$\mu := \pi_F \left(\left(\otimes_{i=1}^k \mu_i\right) \otimes \left(\otimes_{j=1}^m \mu_{B,p_j,n_j}\right)\right) \tag{1.4}$$

satisfies (1.3) for some absolute constant $C > 0$.

Our aim is to investigate the isotropic constant of measures of the form (1.4). It is well known that the isotropic constant of the product of two measures is bounded by the maximum of the corresponding isotropic constants (see, for example, [12, Lemma 1.6.6]). So, the difficulty arises on the marginal operator. It is not known if given an isotropic log-concave probability measure μ on $\mathbb{R}^N$ and a subspace $F \in G_{N,n}$ one has

$$L_{\pi_F(\mu)} \leqslant C L_\mu. \tag{1.5}$$

Actually (1.5) is another equivalent formulation of (**HC**) – see the end of §4 for the details.

Our main result states that probability measures of the form (1.4) satisfy the (**HC**).

Theorem 1.1. *There exists an absolute constant $C > 0$ such that for any log-concave probability measure μ of the form* (1.4)*,*

$$L_\mu \leqslant C. \tag{1.6}$$

Acknowledgments. I would like to thank Apostolos Giannopoulos and Emanuel Milman for many interesting discussions.

2 Preliminaries

2.1 Basic notation. We work in $\mathbb{R}^n$, which is equipped with a Euclidean structure $\langle\cdot,\cdot\rangle$. We denote by $\|\cdot\|_2$ the corresponding Euclidean norm, and write B_2^n for the Euclidean unit ball, and S^{n-1} for the unit sphere. Volume is denoted by $|\cdot|$. We write D_n for the Euclidean ball of volume 1 and σ for the rotationally invariant probability measure on S^{n-1}. The Grassmann manifold $G_{n,k}$ of k-dimensional subspaces of $\mathbb{R}^n$ is equipped with the Haar probability measure $\mu_{n,k}$. Let $1\leqslant k\leqslant n$ and $F\in G_{n,k}$. We will denote by P_F the orthogonal projection from $\mathbb{R}^n$ onto F.

The letters c,c',c_1,c_2 etc. denote absolute positive constants which may change from line to line. Whenever we write $a\simeq b$, we mean that there exist absolute constants $c_1,c_2>0$ such that $c_1a\leqslant b\leqslant c_2a$. If $A\subseteq\mathbb{R}^n$ with $|A|>0$, we write $\widetilde{A}:=|A|^{-\frac{1}{n}}A$.

2.2 Probability measures. We denote by $\mathcal{P}_{[n]}$ the class of all probability measures in $\mathbb{R}^n$ which are absolutely continuous with respect to the Lebesgue measure. We write $\mathcal{A}_n$ for the Borel σ-algebra in $\mathbb{R}^n$. The density of $\mu\in\mathcal{P}_{[n]}$ is denoted by f_μ.

The subclass $\mathcal{SP}_{[n]}$ consists of all symmetric measures $\mu\in\mathcal{P}_{[n]}$; μ is called symmetric if f_μ is an even function on $\mathbb{R}^n$.

The subclass $\mathcal{CP}_{[n]}$ consists of all $\mu\in\mathcal{P}_{[n]}$ that have center of mass at the origin; so, $\mu\in\mathcal{CP}_{[n]}$ if

$$\int_{\mathbb{R}^n}\langle x,\theta\rangle d\mu(x)=0 \tag{2.1}$$

for all $\theta\in S^{n-1}$.

Let $\mu\in\mathcal{P}_{[n]}$. For every $1\leqslant k\leqslant n-1$ and $F\in G_{n,k}$, we define the F-marginal $\pi_F(\mu)$ of μ as follows: for every $A\in\mathcal{A}_F$,

$$\pi_F(\mu)(A):=\mu(P_F^{-1}(A)). \tag{2.2}$$

It is clear that $\pi_F(\mu)\in\mathcal{P}_{[\dim F]}$. Note that, by the definition, for every Borel measurable function $f:\mathbb{R}^n\to[0,\infty)$ we have

$$\int_F f(x)\,d\pi_F(\mu)(x)=\int_{\mathbb{R}^n}f(P_F(x))\,d\mu(x). \tag{2.3}$$

The density of $\pi_F(\mu)$ is the function

$$f_{\pi_F(\mu)}(x)=\pi_F(f_\mu)(x)=\int_{x+F^\perp}f_\mu(y)\,dy. \tag{2.4}$$

Let $\mu_1\in\mathcal{P}_{[n_1]}$ and $\mu_2\in\mathcal{P}_{[n_2]}$. We will write $\mu_1\otimes\mu_2$ for the measure in $\mathcal{P}_{[n_1+n_2]}$ which satisfies

$$(\mu_1\otimes\mu_2)(A_1\times A_2)=\mu_1(A_1)\mu_2(A_2) \tag{2.5}$$

for all $A_1\in\mathcal{A}_{n_1}$ and $A_2\in\mathcal{A}_{n_2}$. It is easily checked that $f_{\mu_1\otimes\mu_2}=f_{\mu_1}f_{\mu_2}$.

Moreover the marginal operator and the product operator "commute": Let $\mu_1 \in \mathcal{P}_{N_1}$, $\mu_2 \in \mathcal{P}_{N_2}$, $F_1 := P_{\mathbb{R}^{N_1}}(F)$ and $F_2 := P_{\mathbb{R}^{N_2}}(F)$. Then.

$$\pi_{F_1}(\mu_1) \otimes \pi_{F_2}(\mu_2) = \pi_F(\mu_1 \otimes \mu_2), \tag{2.6}$$

where $F := F_1 \oplus F_2$.

Let $\mu \in \mathcal{P}_{[n]}$ and $\lambda > 0$. We define $\mu_{(\lambda)} \in \mathcal{P}_{[n]}$ as the measure that has density $f_{\mu_{(\lambda)}}(x) := \lambda^n f_\mu(\lambda x)$. Moreover if $T \in SL(n)$ we define $\mu \circ T \in \mathcal{P}_{[n]}$ as the measure with density $f_{\mu\circ T}(x) := f_\mu(T^{-1}x)$.

If $\mu_i \in \mathcal{P}$ we write $\mu_i \Rightarrow \mu$ for the weak convergence of μ_i to μ.

2.3 Log-concave measures. We denote by $\mathcal{L}_{[n]}$ the class of all log-concave probability measures on $\mathbb{R}^n$. A measure μ on $\mathbb{R}^n$ is called log-concave if for any $A, B \in \mathcal{A}_n$ and any $\lambda \in (0,1)$,

$$\mu(\lambda A + (1-\lambda)B) \geqslant \mu(A)^\lambda \mu(B)^{1-\lambda}.$$

A function $f : \mathbb{R}^n \to [0,\infty)$ is called log-concave if $\log f$ is concave.

It is known that if $\mu \in \mathcal{L}_{[n]}$ and $\mu(H) < 1$ for every hyperplane H, then $\mu \in \mathcal{P}_{[n]}$ and its density f_μ is log-concave (see [5]). As an application of the Prékopa-Leindler inequality one can check that if f is log-concave then, for every $k \leqslant n-1$ and $F \in G_{n,k}$, $\pi_F(f)$ is also log-concave. As before, we write $\mathcal{CL}_{[n]}$ or $\mathcal{SL}_{[n]}$ for the classes of centered or symmetric non degenerate $\mu \in \mathcal{L}_{[n]}$ respectively.

If $\mu_1, \mu_2 \in \mathcal{L}_{[n]}$ we define their convolution $\mu_1 * \mu_2$ as the measure with density $f_{\mu_1*\mu_2}(x) := \int_{\mathbb{R}^n} f_{\mu_1}(y) f_{\mu_2}(x-y)dy$. It follows from the Prékopa-Leindler inequality that $\mu_1 * \mu_2$ is well defined and belongs to $\mathcal{L}_{[n]}$. In the notation given above, one can check that

$$(\mu_1 * \mu_2)_{(\sqrt{2})} = \pi_F(\mu_1 \otimes \mu_2), \tag{2.7}$$

where $F := \{(x,y) \in \mathbb{R}^{2n} : x = y\}$.

2.4 Convex bodies. A convex body in $\mathbb{R}^n$ is a compact convex subset C of $\mathbb{R}^n$ with non-empty interior. We say that C is symmetric if $x \in C$ implies that $-x \in C$. We say that C is centered if $\int_C \langle x, \theta \rangle \, dx = 0$ for every $\theta \in S^{n-1}$. The support function $h_C : \mathbb{R}^n \to \mathbb{R}$ of C is defined by $h_C(x) = \max\{\langle x, y \rangle : y \in C\}$. Note that if K is a convex body in $\mathbb{R}^n$ then the Brunn-Minkowski inequality implies that $\mathbf{1}_{\widetilde{K}} \in \mathcal{L}_{[n]}$.

We denote by $\mathcal{K}_{[n]}$ the class of convex bodies in $\mathbb{R}^n$ and by $\widetilde{\mathcal{K}}_{[n]}$ the subclass of bodies of volume 1. Also, $\mathcal{CK}_{[n]}$ is the class of centered convex bodies (bodies with center of mass at the origin) and $\mathcal{SK}_{[n]}$ is the class of origin symmetric convex bodies in $\mathbb{R}^n$.

2.5 L_q–centroid bodies. Let $\mu \in \mathcal{P}_{[n]}$. For every $q \geqslant 1$ and $\theta \in S^{n-1}$ we define

$$h_{Z_q(\mu)}(\theta) := \left(\int_{\mathbb{R}^n} |\langle x, \theta \rangle|^q f_\mu(x) \, dx \right)^{1/q},$$

where f_μ is the density of μ. If μ is log-concave then $h_{Z_q(\mu)}(\theta) < \infty$ for every $q \geqslant 1$ and every $\theta \in S^{n-1}$. We define the L_q-centroid body $Z_q(\mu)$ of μ to be the centrally symmetric convex set with support function $h_{Z_q(\mu)}$. One can check that for any $T \in SL(n)$ and $\lambda > 0$,

$$Z_p((\mu \circ T)_{(\lambda)}) = \frac{1}{\lambda} T(Z_p(\mu)). \tag{2.8}$$

Note that (2.3) implies that

$$P_F(Z_p(\mu)) = Z_p(\pi_F(\mu)). \tag{2.9}$$

L_q–centroid bodies were introduced, with a different normalization, in [20] (see also [21] where an L_q affine isoperimetric inequality was proved). Here we follow the normalization (and notation) that appeared in [25]. The original definition concerned the class of densities $\mathbf{1}_K$ where K is a convex body of volume 1. In this case, we also write $Z_q(K)$ instead of $Z_q(\mathbf{1}_K)$.

If K is a compact set in $\mathbb{R}^n$ and $|K| = 1$, it is easy to check that $Z_1(K) \subseteq Z_p(K) \subseteq Z_q(K) \subseteq Z_\infty(K)$ for all $1 \leqslant p \leqslant q \leqslant \infty$, where $Z_\infty(K) = \mathrm{conv}(\{K, -K\})$. Note that if $T \in SL(n)$ then $Z_p(T(K)) = T(Z_p(K))$. Moreover it was proved in [26, Theorem 4.4] that, for all $1 \leqslant n < N$, $K \in \mathcal{CK}_{[N]}$ and $F \in G_{N,n}$,

$$|P_F(Z_n(K))|^{\frac{1}{n}} |K \cap F^\perp|^{\frac{1}{n}} \simeq 1. \tag{2.10}$$

For additional information on L_q–centroid bodies, we refer to [25] and [26].

2.5 Isotropic probability measures. Let μ be a centered measure in $\mathcal{P}_{[n]}$. We say that μ is isotropic if $Z_2(\mu) = B_2^n$. Note that if $\mu \in \mathcal{CL}_{[n]}$, then there exist $T \in SL(n)$ and $\lambda > 0$ such that $(\mu \circ T)_{(\lambda)}$ is isotropic. We write μ_{iso} for an "isotropic image" of μ. Note that μ_{iso} is unique up to orthogonal transformations. If $\mu \in \mathcal{CL}_{[n]}$ then we define the isotropic constant of μ by $L_\mu := f_{\mu_{iso}}(0)^{\frac{1}{n}}$. We denote by $\mathcal{IL}$ the class of isotropic log-concave measures.

It is known (see [26, Proposition 3.7]) that if $\mu \in \mathcal{IL}_{[n]}$, then

$$\frac{1}{L_\mu} \simeq |Z_n(\mu)|^{\frac{1}{n}}. \tag{2.11}$$

We say that a centered convex body K is isotropic if $Z_2(K)$ is a multiple of the Euclidean ball. We define the isotropic constant of K by

$$L_K := \left(\frac{|Z_2(K)|}{|B_2^n|} \right)^{1/n}. \tag{2.12}$$

So, K is isotropic if and only if $Z_2(K) = L_K B_2^n$. Let $K \in \mathcal{CK}_{[n]}$ and $a > 0$. We write $\mu_{K,a} := a^n \mathbf{1}_{\frac{K}{a}}$. Note that K is isotropic if and only if $\mu_{K,L_K} = L_K^n \mathbf{1}_{\frac{K}{L_K}}$ is isotropic as a measure.

We refer to [23], [12] for additional information on isotropic convex bodies and to the books [29], [24] and [28] for basic facts from the Brunn-Minkowski theory and the asymptotic theory of finite dimensional normed spaces.

3 Coherent classes of measures

We start with the definition of coherent classes of measures – see [9].

Definition 3.1. Let $\mathcal{C} \subseteq \mathcal{P}$ be a class of probability measures. Then, $\mathcal{C}$ is called coherent if

1. For every n_1, n_2 and $\mu_1 \in \mathcal{C}_{[n_1]}, \mu_2 \in \mathcal{C}_{[n_2]}$ one has that $\mu_1 \otimes \mu_2 \in \mathcal{C}_{[n_1+n_2]}$.
2. For every n, $1 \leqslant k \leqslant n-1$, $F \in G_{n,k}$ and $\mu \in \mathcal{C}_{[n]}$ one has that $\pi_F(\mu) \in \mathcal{C}_{[k]}$.
3. If $\mu_i \in \mathcal{C}_{[n]}$, $i = 1, 2, \cdots$ and $\mu_i \Rightarrow \mu$, then $\mu \in \mathcal{C}$.

   We will say that $\mathcal{C}$ is τ-coherent if instead of 3. it satisfies the following
4. If $\mu \in \mathcal{C}$, $\lambda > 0$ and $T \in SL(n)$, then $(\mu \circ T)_{(\lambda)} \in \mathcal{C}$.

We also agree that the null class is coherent. Note that if $\mathcal{U}_1$ and $\mathcal{U}_2$ are τ-coherent then $\mathcal{U}_1 \cap \mathcal{U}_2$ is also τ-coherent. Known results show that the classes $S\mathcal{P}$, $C\mathcal{P}$ and $\mathcal{L}$ are τ-coherent. Also, $\mathcal{I}$ is coherent – see [9].

Let $\mathcal{A} \subseteq \mathcal{P}$ be a family of probability measures. We define

$$\overline{\mathcal{A}} := \bigcap\{\mathcal{U} \subseteq \mathcal{P} : \mathcal{U} \text{ coherent and } \mathcal{A} \subseteq \mathcal{U}\}. \tag{3.1}$$

It is clear that if $\mathcal{A}_1 \subseteq \mathcal{A}_2$ then $\overline{\mathcal{A}_1} \subseteq \overline{\mathcal{A}_2}$.

Note that the class $\mathcal{K} := \bigcup_{n=1}^{\infty}\{\mu \in \mathcal{P}_{[n]} : \mu = \mathbf{1}_{\widetilde{K}}\ ;\ K \in \mathcal{K}_{[n]}\}$ is *not* coherent. However, it is not difficult to check that $\overline{\mathcal{K}} = \mathcal{L}$.

Let $\mathcal{A} \subseteq C\mathcal{L}$. Then, we define

$$L_{\mathcal{A}} := \sup\{L_\mu : \mu \in \mathcal{A}\}. \tag{3.2}$$

We will need the following fact (the proof is based on [31, Proposition 2.11]).

Proposition 3.2. *Let $\mathcal{A} \subseteq \mathcal{IL}$ be a family of probability measures and set $\mathcal{C} := \overline{\mathcal{A}}$. Then, for every $n \geqslant 1$, for every $\mu \in \mathcal{C}_{[n]}$ and $\varepsilon > 0$, there exist $k \in \mathbb{N}$, $\mu_i \in \mathcal{A}_{n_i}$, $i \leqslant k$ with $\sum_{i=1}^k n_i = N$ and $F \in G_{N,n}$ such that*

$$|L_\mu - L_\nu| \leqslant \varepsilon, \text{ where } \nu := \pi_F\left(\otimes_{i=1}^k \mu_i\right). \tag{3.3}$$

Proof. Let $\mathcal{U} \subseteq \mathcal{IL}$ be the smallest class which is closed under products and marginals and contains $\mathcal{A}$. Then, it is proved in [31, Proposition 2.11] that $\mathcal{C}$ is the closure of $\mathcal{U}$ with respect to the Lévy metric. So, in order to finish the proof, it is enough to observe that L_μ is continuous with respect to the Lévy metric and use (2.6). $\square$

The next proposition follows from the definition of a τ-coherent class and (2.6).

Proposition 3.3. *Let $\mathcal{C} \subseteq C\mathcal{L}$ be a τ-coherent class and let $\mu_1, \mu_2 \in \mathcal{C}_{[n]}$. Then, $\mu_1 * \mu_2 \in \mathcal{C}_{[n]}$.*

The next proposition allows us to work only with symmetric isotropic log-concave measures. The proof follows an argument of B. Klartag [16].

Proposition 3.4. *Let $\mathcal{C} \subseteq \mathcal{CL}$ be a τ-coherent class of measures and set $\mathcal{SC} := \mathcal{SL} \cap \mathcal{C}$. Then,*

$$L_{\mathcal{C}} \leqslant e\sqrt{2} L_{\mathcal{SC}}.$$

Proof. Let $\mu \in \mathcal{C}_{[n]}$ and let $\bar{\mu}$ the measure with density $f_{\bar{\mu}}(x) = f_\mu(-x)$. Since $\mathcal{C}$ is τ-coherent $\bar{\mu} \in \mathcal{C}$. Also, by Proposition 3.3, $\mu^s := \mu * \bar{\mu} \in \mathcal{C}$ and it is straightforward to check that μ^s is also symmetric. Note that $L_\mu = L_{\bar{\mu}}$. In order to finish the proof it is enough to show that for every $\mu_1, \mu_2 \in \mathcal{CL}$,

$$L_{\mu_1 * \mu_2} \leqslant e\sqrt{2} \min\{L_{\mu_1}, L_{\mu_2}\}. \tag{3.4}$$

We may assume that μ_1, μ_2 are isotropic. Then, one can check that

$$(\mu_1 * \mu_2)_{(\sqrt{2})} = (\mu_1)_{(\sqrt{2})} * (\mu_1)_{(\sqrt{2})}$$

is also isotropic. So,

$$\begin{aligned} L^n_{\mu_1 * \mu_2} &:= f_{(\mu_1 * \mu_2)_{(\sqrt{2})}}(0) = \int_{\mathbb{R}^n} (f_{\mu_1})_{(\sqrt{2})}(y)(f_{\mu_2})_{(\sqrt{2})}(-y)dy \\ &\leqslant \|(f_{\mu_2})_{(\sqrt{2})}\|_\infty \leqslant e^n (f_{\mu_2})_{(\sqrt{2})}(0) \\ &= \left(e\sqrt{2}\right)^n f_{\mu_2}(0) \leqslant \left(e\sqrt{2}L_{\mu_2}\right)^n, \end{aligned}$$

where we have also used a theorem of M. Fradelizi [11] stating that, for any centered log-concave density f in $\mathbb{R}^n$ one has $\|f\|_\infty \leqslant e^n f(0)$. We work in the same way to get that $L^n_{\mu_1 * \mu_2} \leqslant \left(e\sqrt{2}L_{\mu_1}\right)^n$. This proves (3.4). □

Let $\mu_i \in \mathcal{IL}_{[n_1]}$, $i \leqslant m$ and $N := \sum_{i=1}^m n_i$. Then

$$L_{\mu_1 \otimes \cdots \otimes \mu_m} = \prod_{i=1}^m L_{\mu_i}^{\frac{n_i}{N}} \leqslant \max\{L_{\mu_i}, i \leqslant m\}. \tag{3.5}$$

Indeed, it follows from the definition that

$$L^N_{\mu_1 \otimes \cdots \otimes \mu_m} = f_{\mu_1 \otimes \cdots \otimes \mu_m}(0) = \prod_{i=1}^m f_{\mu_i}(0) = \prod_{i=1}^m L_{\mu_i}^{n_i}. \tag{3.6}$$

We have the following:

Proposition 3.5. *Let $\mathcal{C}_i \subseteq \mathcal{IL}$, $i \in I$ a family of coherent classes. Then*

$$L_{\overline{\{\mathcal{C}_i, i \in I\}}} \leqslant \sup\{L_{\mathcal{C}_i}, i \in I\}. \tag{3.7}$$

Proof. We may assume that I is finite. Let $m := \text{card} I$. Let $\mathcal{C} := \overline{\{\mathcal{C}_i, i \in I\}}$. In the proof of Proposition 3.2 we have shown that the class $\mathcal{C}_0$ which is closed under products and marginals and contains $\{\mathcal{C}_i, i \in I\}$ is dense in $\mathcal{C}$ in the Lévy metric. So it is enough to bound $L_{\mathcal{C}_0}$. Let $\mu \in \mathcal{C}_0$. Then since the marginal operator commutes with the product operator and any marginal of a measure in $\mathcal{C}_i$ belongs also to $\mathcal{C}_i$ by definition, we have that μ is of the form

$$\mu := \mu_1 \otimes \cdots \otimes \mu_m,$$

where $\mu_i \in \mathcal{C}_i$. The result follows from (3.5). $\square$

In the introduction, for every $p \in [1, \infty]$ and $n \geqslant 1$, we defined the measure $\mu_{B,p,n} := \mu_{B_p^n, a_{p,n}}$, where $B_p^n := \{x \in \mathbb{R}^n : \|x\|_p \leqslant 1\}$ and $a_{p,n}$ is chosen so that $\mu_{B,p,n}$ is isotropic. For all $p_1 \leqslant p_2 \in [1, \infty]$ we define

$$\mathcal{L}_{[p_1,p_2]} := \overline{\{\mu_{B,p,n};\ p \in [p_1, p_2], n \geqslant 1\}}. \tag{3.8}$$

Let $\mathcal{G}$ be the class of standard Gaussian measures. By the central limit theorem and the definition of a coherent class, for any non-empty $\mathcal{C} \subseteq \mathcal{IP}$ which is coherent we have $\mathcal{G} \subseteq \mathcal{C}$.

We also define, for every $k \geqslant 1$,

$$\mathcal{IL}^{[k]} := \overline{\{\mu \in \mathcal{CL} : \mu \text{ is } k - \text{dimensional}\}}. \tag{3.9}$$

J. Wojtaszczyk showed in [31] that if $\mu = \mathbf{1}_{K,a} \in \mathcal{IL}^{[1]}$ then, necessarily, $K = B_\infty^n$. This shows that $\mathcal{IL}^{[1]}$ is strictly contained in $\mathcal{IL}$.

With the notation introduced above, Theorem 1.1 would follow from the following.

Theorem 3.6. *There exists $C > 0$ such that*

$$L_{\overline{\{\mathcal{L}_{[1,\infty]}, \mathcal{IL}^{[1]}\}}} \leqslant C. \tag{3.10}$$

4 Supergaussian and Subgaussian measures

Let $\mu \in \mathcal{IL}_{[n]}$ and $\theta \in S^{n-1}$. The subgaussian constant of μ in the direction of θ is defined by

$$\widetilde{\psi}_{2,\mu}(\theta) := \sup_{\lambda > 0} \left(\log \int_K e^{\lambda \langle x, \theta \rangle} d\mu(x) \right)^{\frac{1}{2}}. \tag{4.1}$$

We define the subgaussian constant of μ by

$$\beta_{2,\mu} := \sup_{\theta \in S^{n-1}} \widetilde{\psi}_{2,\mu}(\theta). \tag{4.2}$$

The usual definition of the subgaussian constant is different:

$$\psi_{2,\mu}(\theta) := \inf \left\{ \lambda > 0 : \int_{\mathbb{R}^n} e^{\frac{|\langle x, \theta \rangle|^2}{\lambda^2}} d\mu(x) \leqslant 2 \right\}. \tag{4.3}$$

Our modification is justified by the next proposition (see [9, Propositions 4.5 and 4.9]).

Proposition 4.1. *Let $\mu \in \mathcal{SIL}_{[n]}$. Then, for every $\theta \in S^{n-1}$,*

$$\psi_{2,\mu}(\theta) \simeq \widetilde{\psi}_{2,\mu}(\theta). \tag{4.4}$$

Moreover, if for some $b > 0$ and for all $n \geqslant 1$ we define

$$\mathcal{SBG}(b)_{[n]} := \{\mu \in \mathcal{IL}_{[n]} : \beta_{2,\mu} \leqslant b\} \text{ and } \mathcal{SBG}(b) := \bigcup_{n=1}^{\infty} \mathcal{SBG}(b)_{[n]}, \tag{4.5}$$

then $\mathcal{SBG}(b)$ is a coherent class.

Let γ_n denote the standard Gaussian distribution. Then there exists a universal constant c_γ such that $\gamma_n \in \mathcal{SBG}(c_\gamma)_{[n]}$.

The fact that if μ is subgaussian with constant b then L_μ is bounded by a constant $c(b)$ depending only on b was first established by J. Bourgain in [8]. His estimate $c(b) \leqslant cb \log b$ has been slightly improved in [9]. The best known estimate is due to B. Klartag and E. Milman [18]:

Theorem 4.2. *There exists $c > 0$ such that for any $b \geqslant c_\gamma$,*

$$L_{\mathcal{SBG}(b)} \leqslant cb. \tag{4.6}$$

The assumption that $b \geqslant c_\gamma$ is only to guarantee that the class $\mathcal{SBG}(b)$ is not empty. We will need the following consequence of Theorem 4.2.

Proposition 4.3. *Let K be an isotropic convex body in $\mathbb{R}^N$ which is subgaussian with constant b. Then, for any $F \in G_{N,n}$,*

$$|K \cap F^\perp|^{\frac{1}{n}} \leqslant cb, \tag{4.7}$$

where $c > 0$ is an absolute constant.

Proof. Note that μ_{K,L_K} is isotropic (as a measure) and $\pi_F(\mu_{K,L_K})$ is also subgaussian with constant b. So, using Theorem 4.2, we have that

$$(cb)^n \geqslant \pi_{\mu_F(K,L_K)}(0) = L_K^N \left| \frac{K}{L_K} \cap F^\perp \right| = L_K^n |K \cap F^\perp| \geqslant c_0^n |K \cap F^\perp|.$$

This completes the proof. □

The next result has been proved by F. Barthe and A. Koldobsky in [2, §6.2] (see also [3]).

Theorem 4.4. *There exists $c > 0$ such that*

$$\mathcal{L}_{[2,\infty]} \subseteq \mathcal{SBG}(c).$$

Let $\mu \in \mathcal{IL}_{[n]}$. We say that μ is supergaussian with constant $a \geqslant L_{D_1} = \frac{1}{2\sqrt{3}}$ if, for all $p \geqslant 1$,

$$Z_p(\mu) \supseteq Z_p(\mathbf{1}_{D_n,a}). \tag{4.8}$$

An equivalent way to describe (4.8) is to say that, for every $1 \leqslant p \leqslant n$ and $\theta \in S^{n-1}$,

$$h_{Z_p(\mu)}(\theta) \geqslant \frac{\sqrt{p}}{ca}.$$

It is not difficult to show the following (see [27, Proposition 5.1]).

Proposition 4.5. *Let $\mu \in \mathcal{IL}$ be supergaussian with constant $a \geqslant \frac{1}{2\sqrt{3}}$. Then,*

$$L_\mu \leqslant ca,$$

where $c > 0$ is an absolute constant.

It follows from the definition and (2.8) that if μ is supergaussian with constant a, then $\pi_F(\mu)$ is also supergaussian with constant a. However, the class of supergaussian measures (with constant lass than a) is not a coherent class, because the product of two supergaussian measures fails (in general) to be supergaussian. Assuming that $a \geqslant L_{D_1}$ we have that the class of supergaussian measures with constant less than a is non-empty. So, for $a \geqslant L_{D_1}$ we define

$$\mathcal{SPG}(a)_{[n]} := \overline{\{\mu \in \mathcal{IL}_{[n]} : \mu \text{ supergaussian with constant } a\}}$$

and

$$\mathcal{SPG}(a) := \bigcup_{n=1}^{\infty} \mathcal{SPG}(a)_{[n]}.$$

Let us emphasize that the class $\mathcal{SPG}$ contains probability measures that are not necessarily "supergaussian".

We will prove the following.

Theorem 4.6. *There exists $c > 0$ such that for all $a \geqslant L_{D_1}$,*

$$L_{\mathcal{SPG}(a)} \leqslant ca.$$

Proof. We write $\mu_{a,n} := \mathbf{1}_{D_n,a}$. Let $\mu \in \mathcal{IL}_{[n]}$ be symmetric and supergaussian with constant a. Then, for every $t \in \mathbb{R}$, for every even integer $p \geqslant 2$ and for all $y \in \mathbb{R}^n$,

$$\int_{\mathbb{R}^n} |\langle x, y\rangle + t|^p d\mu(x) \geqslant \int_{\mathbb{R}^n} |\langle x, y\rangle + t|^p d\mu_{a,n}(x). \tag{4.9}$$

Indeed, since $\mu, \mu_{a,n}$ are symmetric,

$$\begin{aligned}
\int_{\mathbb{R}^n} |\langle x, y\rangle + t|^p d\mu(x) &= \sum_{i=0}^{p} \binom{p}{i} \int_{\mathbb{R}^n} t^i \langle x, y\rangle^{p-i} d\mu(x) \\
&= \sum_{i=0, i \text{ even}}^{p} \binom{p}{i} \int_{\mathbb{R}^n} t^i \langle x, y\rangle^{p-i} d\mu(x) \\
&= \sum_{k=0}^{\frac{p}{2}} \binom{p}{2k} |t|^{2k} \int_{\mathbb{R}^n} |\langle x, y\rangle|^{p-2k} d\mu(x) \\
&\geqslant \sum_{k=0}^{\frac{p}{2}} \binom{p}{2k} |t|^{2k} \int_{\mathbb{R}^n} |\langle x, y\rangle|^{p-2k} d\mu_{a,n}(x) \\
&= \int_{\mathbb{R}^n} |\langle x, y\rangle + t|^p d\mu_{a,n}(x).
\end{aligned}$$

Let $n_1, \cdots, n_k \in \mathbb{N}$ and let $N := \sum_{i=1}^{k} n_i$. Let $\mu_1 \in \mathcal{SIL}_{n_1}, \cdots, \mu_k \in \mathcal{SIL}_{n_k}$ be supergaussian measures with constant a. Let $\mu_N := \mu_1 \otimes \cdots \otimes \mu_k$ and $\bar{\mu}_a := \mu_{n_1,a} \otimes \cdots \otimes \mu_{n_k,a}$. Then, for every even integer $p \geqslant 2$ and every $\bar{y} := (y_1, \cdots, y_k) \in \mathbb{R}^N$, applying k times (4.9) and using Fubini's theorem, we have that

$$\begin{aligned}
\int_{\mathbb{R}^N} |\langle \bar{x}, \bar{y}\rangle|^p d\mu_N(\bar{x}) &= \int_{\mathbb{R}^{n_1}} \cdots \int_{\mathbb{R}^{n_k}} |\sum_{i=1}^{k} \langle x_i, y_i\rangle|^p d\mu_k(x_k) \cdots d\mu_1(x_1) \\
&\geqslant \int_{\mathbb{R}^{n_1}} \cdots \int_{\mathbb{R}^{n_k}} |\sum_{i=1}^{k} \langle x_i, y_i\rangle|^p d\mu_{n_k,a}(x_k) \cdots d\mu_{n_1,a}(x_1) \\
&= \int_{\mathbb{R}^N} |\langle \bar{x}, \bar{y}\rangle|^p d\bar{\mu}_a(\bar{x}).
\end{aligned}$$

So, we have shown that

$$Z_p(\mu_N) \supseteq Z_p(\bar{\mu}_a). \tag{4.10}$$

Let $n \geqslant 1$ and $F \in G_{N,n}$. Let $D := D_{n_1} \times \cdots \times D_{n_k}$. Then D is isotropic (in the convex body sense) and subgaussian with some absolute constant $c > 0$. Then, by Proposition 4.3, we have that

$$|D \cap F^{\perp}|^{\frac{1}{n}} \simeq 1. \tag{4.11}$$

Moreover, for any $p > 0$,

$$Z_p(\bar{\mu}_a) = \frac{1}{a} Z_p(D). \tag{4.12}$$

Let $n \geqslant 1$ and $F \in G_{N,n}$. Then, using (2.9), (4.10) and (4.12) we get

$$Z_p(\pi_F(\mu_N)) = P_F(Z_p(\mu_N)) \supseteq P_F(Z_p(\bar{\mu}_a)) = \frac{1}{a} P_F(Z_p(D)). \tag{4.13}$$

So, for $p = n$, using (2.11), (4.13), (2.10) and (4.11) we see that

$$\frac{1}{L_{\pi_F(\mu_N)}} \simeq |Z_n(\pi_F(\mu_N))|^{\frac{1}{n}} \geqslant \frac{c}{a}|P_F(Z_n(D))|^{\frac{1}{n}} \geqslant \frac{c'}{a}\frac{1}{|D\cap F^{\perp}|^{\frac{1}{n}}} \geqslant \frac{c''}{a}.$$

In other words,

$$L_{\pi_F(\mu_N)} \leqslant c'''a. \tag{4.14}$$

The result follows from Propositions 3.2 and 3.4. □

The class $\mathcal{SPG}$ is quite rich as the following two propositions show.

Proposition 4.7. *There exists $c > 0$ such that*

$$\mathcal{IL}_{[1,2]} \subseteq \mathcal{SPG}(c).$$

Proof. Given $a > 0$, let $\gamma_{n,\frac{1}{a}}$ be the centered Gaussian measure in $\mathbb{R}^n$ with variance $\frac{1}{a}$. Let $\mathcal{B}(a)_{[n]}$ the class of all measures satisfying $Z_p(\mu) \supseteq Z_p(\gamma_{n,\frac{1}{a}})$ for all $p \geqslant 1$. Then, if $\mu_i \in \mathcal{B}_{n_i}$, $1 \leqslant i \leqslant k$, and if $N := \sum_{i=1}^k n_i$, working as in the proof of (4.10) we have that

$$Z_p\left(\otimes_{i=1}^k \mu_i\right) \supseteq Z_p(\gamma_{N,\frac{1}{a}}) \supseteq Z_p(\mathbf{1}_{D_n,ca}), \tag{4.15}$$

for all $p \geqslant 1$, where $c > 0$ is an absolute constant. The first inclusion combined with (2.8) shows that $\mathcal{B}(a) := \bigcup_{n=1}^{\infty} \mathcal{B}(a)_{[n]}$ is a coherent class and the second inclusion implies that

$$\mathcal{B}(a) \subseteq \mathcal{SPG}(ca), \tag{4.16}$$

for some $c > 0$. Let $p \in [1, 2]$ and write $\mu_{p,n}$ for the isotropic log-concave probability measure with density $f_{\mu_{p,n}}(x) = a_{p,n}e^{-\|x\|_p^p}$. It is a straightforward computation to check that $\mu_{p,1} \in \mathcal{B}(c_1)$ for some $c_1 > 0$. So (4.15) implies that $\mu_{n,p} \in \mathcal{B}(c_1)$. Let K be a symmetric convex body in $\mathbb{R}^n$, let $\|\cdot\|_K$ the corresponding norm to K and let $r > 0$. We define a probability density $g_{K,r}$ on $\mathbb{R}^n$ by

$$g_{K,r}(x) := \frac{1}{|K|\Gamma(\frac{n+r}{r})}e^{-\|x\|_K^r}.$$

Then (see [13, Lemma 4.3]), for any $q > 0$,

$$Z_q(g_{K,r}) = \left(\frac{\Gamma(\frac{n+q+r}{r})}{\Gamma(\frac{n+r}{r})}\right)^{\frac{1}{q}} Z_q(\widetilde{K}). \tag{4.17}$$

Since $g_{B_p^n,p} = \mu_{p,n}$, it is not hard to check that, for all $q \leqslant n$,

$$Z_q(\widetilde{B}_p^n) \simeq Z_q(\mu_{p,n}). \tag{4.18}$$

This shows that, for all $q \leqslant n$,

$$Z_q(\mu_{B,p,n}) \supseteq c' Z_q(\widetilde{B_p}) \supseteq c'' Z_q(\mu_{p,n}) \supseteq c''' \sqrt{q} B_2^n,$$

and the proof is complete. □

Note that, if μ is a k-dimensional isotropic log-concave probability measure, then it is "supergaussian" with constant $c\sqrt{k}$ (see [27] Proposition 3.2). This means that $\mathcal{IL}^{[k]} \subseteq \mathcal{SPG}(c\sqrt{k})$. In particular, we have the following:

Proposition 4.8. *There exists $c > 0$ such that*

$$\mathcal{IL}^{[1]} \subseteq \mathcal{SPG}(c).$$

Proposition 3.5 and Theorems 4.2 and 4.6 imply the following.

Corollary 4.9. *There exists $c > 0$ such that, for any $a, b \geqslant c_\gamma$,*

$$L_{\overline{\{\mathcal{SPG}(a), \mathcal{SBG}(b)\}}} \leqslant c \max\{a, b\}.$$

Proof of Theorem 3.6 (and 1.1). Theorem 4.4 and Propositions 4.7 and 4.8 imply that $\overline{\{\mathcal{L}_{[1,\infty]}, \mathcal{L}^{[1]}\}} \subseteq \overline{\{\mathcal{SPG}(c_1), \mathcal{SBG}(c_2)\}}$ for some universal constants $c_1, c_2 > 0$. The result follows from Corollary 4.9. □

It was mentioned in the introduction that the main difficulty we had to overpass in this work is that is not known if there exists an absolute constant $a > 0$ such that

$$L_{\pi_F(\mu)} \leqslant a L_\mu \tag{4.10}$$

for all $\mu \in \mathcal{IL}_{[n]}$ and $F \in G_{n,k}$. In fact, (4.19) is just another equivalent formulation of (**HC**). Indeed, if (**HC**) is true then clearly (4.19) is also true. The other direction follows from the next Proposition.

Proposition 4.10. *Let $\mathcal{C} \subseteq \mathcal{IL}$ be a non-empty coherent class. Assume that there exists $a > 0$ such that, for any $\mu \in \mathcal{C}$, (4.10) holds. Then,*

$$L_{\mathcal{C}} \leqslant a. \tag{4.11}$$

Proof. Let $\mu \in \mathcal{C}_{[n]}$ satisfy $L_\mu = L_{\mathcal{C}}$. Since $\mathcal{C}$ is non-empty, we have $\gamma_N \in \mathcal{C}$ for all $N \geqslant 1$. We define $\mu_1 := \mu \otimes \gamma_N$. Note that, if $F = \mathbb{R}^n$ then $\pi_F(\mu_1) = \mu$. Moreover, if N is large enough we have that

$$L_{\mu_1} = f_{\mu_1}(0)^{\frac{1}{n+N}} = (f_\mu(0) f_{\gamma_N}(0))^{\frac{1}{n+N}} \leqslant (\sqrt{n})^{\frac{n}{n+N}} \left(\frac{1}{\sqrt{2\pi}}\right)^{\frac{N}{n+N}} \leqslant 1. \tag{4.12}$$

Applying (4.10) for $\pi_F(\mu_1) = \mu$, we get (4.11). □

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