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Functional Analysis

Concentration of mass on isotropic convex bodies

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Abstract

We establish sharp concentration of mass for isotropic convex bodies: there exists an absolute constant $c > 0$ such that if K is an isotropic convex body in $\mathbb{R}^n$, then

$$\mathrm{Prob}\big(\{x \in K\colon \|x\|_2 \geqslant c\sqrt{n}L_K t\}\big) \leqslant \exp\big(-\sqrt{n}\,t\big)$$

for every $t \geqslant 1$, where L_K denotes the isotropic constant. ***To cite this article: G. Paouris, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

Résumé

Concentration de masse pour les corps convexes isotropes. Nous démontrons qu'il existe une constante absolue $c > 0$, telle que, si K est un corps convexe isotrope, alors

$$\mathrm{Prob}\big(\{x \in K\colon \|x\|_2 \geqslant c\sqrt{n}\,L_K t\}\big) \leqslant \exp\big(-\sqrt{n}\,t\big)$$

pour tout $t \geqslant 1$, où L_K désigne la constante d'isotropie. ***Pour citer cet article : G. Paouris, C. R. Acad. Sci. Paris, Ser. I 342 (2006).***

1. Introduction

A convex body K in $\mathbb{R}^n$, with volume equal to 1 and center of mass at the origin, is called isotropic if its inertia matrix is a multiple of the identity. Equivalently, if there exists a positive constant L_K such that $\int_K \langle x, \theta\rangle^2\,\mathrm{d}x = L_K^2$ for every $\theta \in S^{n-1}$. The starting point of this paper is the following concentration estimate of Alesker [1]: if K is an isotropic convex body in $\mathbb{R}^n$ then, for every $t \geqslant 1$ we have

$$\mathrm{Prob}\big(\{x \in K\colon \|x\|_2 \geqslant c\sqrt{n}\,L_K t\}\big) \leqslant 2\exp\big(-t^2\big).$$

Throughout this note, we write B_2^n for the Euclidean unit ball and $\|\cdot\|_2$ for the Euclidean norm; c, c_1, c_2 etc. will denote absolute positive constants.

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Bobkov and Nazarov (see [2,3]) have obtained a striking strengthening of Alesker's estimate for the class of 1-unconditional isotropic convex bodies: in this case,

$$\mathrm{Prob}\big(\{x \in K\colon\ \|x\|_2 \geqslant c\sqrt{n}L_K t\}\big) \leqslant \exp\big(-\sqrt{n}t\big)$$

for every $t \geqslant 1$. Strong dimension dependent volume concentration was recently confirmed in [5] for the unit balls of the Schatten trace classes as well.

The purpose of this Note is to establish the fact that the 'Bobkov–Nazarov estimate' holds true in full generality.

Theorem 1.1. *If K is an isotropic convex body in $\mathbb{R}^n$ then, for every $t \geqslant 1$ we have that*

$$\mathrm{Prob}\big(\{x \in K\colon\ \|x\|_2 \geqslant c\sqrt{n}L_K t\}\big) \leqslant \exp\big(-\sqrt{n}\,t\big).$$

2. Sketch of the proof

Let K be an isotropic convex body in $\mathbb{R}^n$. For any $q \geqslant 1$, we define $I_q(K) := (\int_K \|x\|_2^q\,\mathrm{d}x)^{1/q}$. As observed in [11], in order to prove Theorem 1.1 it is enough to show that

$$I_{c_1\sqrt{n}}(K) \leqslant c_2 I_2(K). \tag{1}$$

For every $q \geqslant 1$ we define the L_q-centroid body $Z_q(K)$ of K by its support function $h_{Z_q(K)}(y) := (\int_K |\langle x, y\rangle|^q\,\mathrm{d}x)^{1/q}$. Under a different normalization these bodies were introduced in [8]. The family $\{Z_q(K)\colon\ q \geqslant 1\}$ increases to the body $Z_\infty(K) = \mathrm{conv}\{K, -K\}$. Since K is isotropic, we have $Z_2(K) = L_K B_2^n$. We will use the following facts:

(i) Let C be a symmetric convex body in $\mathbb{R}^n$ and let $w_q(C) := (\int_{S^{n-1}} h_C^q(\phi)\,\mathrm{d}\sigma(\phi))^{1/q}$ be the q-th mean width of C. It is easily checked that

$$w_q\big(Z_q(K)\big) \simeq \sqrt{\frac{q}{q+n}}\, I_q(K). \tag{2}$$

(ii) In [6] it is proved that if $k_*(C)$ is the largest positive integer for which $\mu_{n,k}\{F \in G_{n,k}\colon\ \frac{1}{2}w_1(C)\|x\|_2 \leqslant h_C(x) \leqslant 2w_1(C)\|x\|_2$ for all $x \in F\} \geqslant 1 - \frac{k}{n+k}$, then

$$w_1(C) \simeq w_q(C) \tag{3}$$

for every $q \leqslant k_*(C)$. Here $G_{n,k}$ is the Grassmann manifold of k-dimensional subspaces of $\mathbb{R}^n$ equipped with the Haar probability measure $\mu_{n,k}$. Recall that the critical dimension k_* is completely determined by the mean width $w_1(C)$ and the circumradius $R(C)$ of C; in [10] it is shown that $k_*(C) \simeq n\frac{w_1(C)^2}{R(C)^2}$.

(iii) **Definition.** Let K be a convex body of volume 1 in $\mathbb{R}^n$. We define

$$q_* := q_*(K) := \max\big\{q \in \mathbb{N}\colon\ k_*\big(Z_q^\circ(K)\big) \geqslant q\big\},$$

where $Z_q^\circ(K)$ is the polar body of $Z_q(K)$. A related parameter was introduced in [12], where the following lower bound for $q_*(K)$ was also proved.

Proposition 2.1. *If K is an isotropic convex body in $\mathbb{R}^n$ then*

$$q_*(K) \geqslant c\sqrt{n}. \tag{4}$$

From the above discussion it becomes clear that (1), and hence Theorem 1.1, will follow if we show that

$$w_{q_*}\big(Z_{q_*}(K)\big) \leqslant c\sqrt{q_*}\,L_K \quad \text{or, equivalently,} \quad w_1\big(Z_{q_*}(K)\big) \leqslant c\sqrt{q_*}\,L_K. \tag{5}$$

By the definition of $q_*(K)$ there exists $F \in G_{n,q_*}$ such that $\frac{1}{2}w_1(Z_{q_*}(K)) \leqslant h_{Z_{q_*}(K)}(\theta) \leqslant 2w_1(Z_{q_*}(K))$ for all $\theta \in S_F := S^{n-1} \cap F$. The following proposition completes the proof:

Proposition 2.2. *Let K be an isotropic convex body in $\mathbb{R}^n$. For every integer $q \geqslant 1$ and every $F \in G_{n,q}$ there exists $\theta \in S_F$ such that*

$$h_{Z_q(K)}(\theta) \leqslant c\sqrt{q}\,L_K. \tag{6}$$

Sketch of the proof. Fix $F \in G_{n,q}$ and write E for the orthogonal subspace of F and P_F for the orthogonal projection onto F. For every $\phi \in S_F$ we define $E^+(\phi) = \{x \in \operatorname{span}\{E, \phi\}\colon \langle x, \phi\rangle \geqslant 0\}$.

Let $q \geqslant 0$ and write $B_q(K, F)$ for the convex body in F defined by the gauge function

$$\phi \mapsto \|\phi\|_2^{1+q/(q+1)} \left(\int_{K\cap E^+(\phi)} \left|\langle x, \phi\rangle\right|^q \,\mathrm{d}x \right)^{-1/(q+1)}$$

(see [9] for details and references). Integration in polar and cylindrical coordinates shows that

$$P_F\big(Z_q(K)\big) = (2q)^{1/q}\left|B_{2q-1}(K, F)\right|^{2/q} Z_q\big(\overline{B}_{2q-1}(K, F)\big), \tag{7}$$

where $\bar{A}$ denotes the homothet $A/|A|^{1/n}$ of volume 1 of a convex body A. Using well known Khintchine type inequalities for log-concave functions (see [9] for details and references) we get

$$\left|B_{2q-1}(K, F)\right|^{2/q} \leqslant cL_K. \tag{8}$$

Therefore,

$$P_F\big(Z_q(K)\big) \subseteq c_1 L_K Z_q\big(\overline{B}_{2q-1}(K, F)\big) \subseteq c_2 L_K Z_\infty\big(\overline{B}_{2q-1}(K, F)\big). \tag{9}$$

Taking volumes in (9) and estimating the volume of $Z_\infty(\overline{B}_{2q-1}(K, F))$ by a standard use of the Rogers–Shephard inequality, we complete the proof. □

It is interesting to note that the estimate of Theorem 1.1 is sharp in both n and t; the ℓ_1^n-ball B_1^n is the extremal isotropic convex body in the following sense: For every isotropic convex body K in $\mathbb{R}^n$ and for every $2 \leqslant q \leqslant \infty$,

$$\frac{I_q(K)}{I_2(K)} \leqslant c\,\frac{I_q(\overline{B_1^n})}{I_2(\overline{B_1^n})}.$$

3. Further results

3.1. Reverse L_q-affine isoperimetric inequality

Lutwak, Yang and Zhang proved in [7] that if K is a convex body of volume 1 in $\mathbb{R}^n$, then

$$\left|Z_q(K)\right|^{1/n} \geqslant \left|Z_q\big(\overline{B_2^n}\big)\right|^{1/n} \geqslant c\sqrt{q/n}$$

for every $1 \leqslant q \leqslant n$, where $c > 0$ is an absolute constant. Our analysis of the L_q-centroid bodies leads to the following reverse inequality.

Theorem 3.1. *Let K be a convex body in $\mathbb{R}^n$, with volume 1 and center of mass at the origin. For every $1 \leqslant q \leqslant n$ we have that*

$$\left|Z_q(K)\right|^{1/n} \leqslant c\sqrt{q/n}\,L_K,$$

where $c > 0$ is a universal constant.

3.2. Random points in isotropic convex bodies

Let $\varepsilon \in (0, 1)$ and consider N independent random points $x_1, \ldots, x_N$ uniformly distributed in an isotropic convex body K in $\mathbb{R}^n$. A question of Kannan, Lovász and Simonovits is to find N_0, as small as possible, for which the following holds true: if $N \geqslant N_0$ then with probability greater than $1 - \varepsilon$ one has $\|\mathrm{Id} - \frac{1}{NL_K^2}\sum_{i=1}^N x_i \otimes x_i\| \leqslant \varepsilon$. Bourgain in [4] proved that one can choose $N_0 \simeq c(\varepsilon)n(\log n)^3$; this was improved to $N_0 \simeq c(\varepsilon)n(\log n)^2$ by Rudelson [13]. Theorem 1.1 allows us to remove one more logarithmic term.

Theorem 3.2. *Let $\varepsilon \in (0,1)$ and let K be an isotropic convex body in $\mathbb{R}^n$. If $N \geqslant c(\varepsilon) n \log n$, and if $x_1, \ldots, x_N$ are independent random points uniformly distributed in K, then with probability greater than $1-\varepsilon$ we have*

$$(1-\varepsilon)L_K^2 \leqslant \frac{1}{N}\sum_{i=1}^{N}\langle x_i, \theta\rangle^2 \leqslant (1+\varepsilon)L_K^2$$

for every $\theta \in S^{n-1}$.

3.3. Concluding remark

All the results of this note remain valid if we replace Lebesgue measure on an isotropic convex body by an arbitrary isotropic log-concave measure. Detailed references, proofs and various extensions of the results of this note will appear elsewhere.

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