Neighborhoods on the Grassmannian of marginals with bounded isotropic constant

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Abstract

We show that for any isotropic log-concave probability measure μ on $\mathbb{R}^n$, for every $\varepsilon > 0$, every $1 \leqslant k \leqslant \sqrt{n}$ and any $E \in G_{n,k}$ there exists $F \in G_{n,k}$ with $d(E,F) < \varepsilon$ and $L_{\pi_F \mu} < C/\varepsilon$.

1 Introduction

Let K be a symmetric convex body in $\mathbb{R}^n$ of volume 1, $|K| = 1$. The Hyperplane Conjecture (posed by J. Bourgain in [2]) claims that there exist a universal constant $c > 0$ and a unit vector θ ($\theta \in S^{n-1}$) such that

$$|K \cap \theta^{\perp}| \geqslant c,$$

where $|\cdot|$ stands for the n-dimensional volume. K. Ball in [1] showed that the question has an equivalent formulation in the more general setting of log-concave measures. A Borel probability measure μ on $\mathbb{R}^n$ is called log-concave if for any compact sets A, B in $\mathbb{R}^n$ we have

$$\mu((1-\lambda)A + \lambda B) \geqslant \mu(A)^{1-\lambda}\mu(B)^{\lambda}$$

for all $\lambda \in (0,1)$. The measure μ is called centered if $\int \langle x, y\rangle \, d\mu(x) = 0$ for all $y \in \mathbb{R}^n$. The covariance matrix for a centered measure μ is defined as:

$$\mathrm{Cov}(\mu)_{ij} = \int_{\mathbb{R}^n} x_i x_j \, d\mu(x), \quad i, j = 1, \ldots, n.$$

A log-concave probability measure μ on $\mathbb{R}^n$ is called isotropic if it is centered and its covariance matrix is the identity.
The isotropic constant of a centered log-concave probability measure on $\mathbb{R}^n$ is defined as:

$$L_\mu := \|\mu\|_\infty^{1/n} [\det \mathrm{Cov}(\mu)]^{\frac{1}{2n}}, \tag{1.1}$$

where $\|\mu\|_\infty = \|f_\mu\|_\infty$ and f_μ is the density function of μ. So if μ is isotropic then $L_\mu = \|\mu\|_\infty^{1/n}$. The Hyperplane Conjecture can be formulated equivalently as follows: There exists a constant $C > 0$ such that for all n and any μ log-concave isotropic probability measure in $\mathbb{R}^n$, $L_\mu < C$. A classical reference on the problem is [21]. For a more detailed exposition on recent developments see [4].

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The first non-trivial bound on this question was given by Bourgain in [2] proving that $L_\mu \leqslant c_1 \sqrt[4]{n}\log n$ for all isotropic measures on $\mathbb{R}^n$. In [12] Klartag removed the logarithmic term (see also [15] for an alternative approach and [35] for further refinements). Throughout this note, all constants $c, C, C', \ldots$ denote positive, dimension-independent numerical constants, whose value may change from line to line. We write $A \simeq B$ to denote $c \leqslant A/B \leqslant C$ for some numerical constants $c, C > 0$.

Actually Klartag's approach developed in [12] gave an affirmative answer to the "isomorphic version" of the Hyperplane Conjecture in the setting of convex bodies:

Theorem (Klartag) *For any $\varepsilon > 0$ and any convex body K on $\mathbb{R}^n$ there exists a convex body T on $\mathbb{R}^n$ which satisfies:*

- *$d_G(K,T) < 1+\varepsilon$ and*
- *$L_T < C/\sqrt{\varepsilon}$.*

In this note we show that a result of the same flavor holds true for marginals of a log-concave isotropic probability measure. Marginals of an isotropic log-concave measure are also log-concave from Prékopa-Leindler inequality [29] and isotropic. However, if μ has bounded isotropic constant, it is not known whether its marginals have also bounded isotropic constant. It's not hard to show that this is an equivalent formulation of the problem (see [28], Proposition 5.3). Our main result states that even if for a given marginal one cannot decide if the isotropic constant is bounded, there exists another one "close" to it which has bounded isotropic constant. To formulate the statement precisely let us fix the distance d on $G_{n,k}$ as:

$$E, F \in G_{n,k}, \quad d(E,F) := \inf\{\|I-U\|_{\mathrm{op}} : U \in O(n), U(E) = F\}. \tag{1.2}$$

Under this notation our result reads as follows:

Theorem 1.1. *Let μ be isotropic log-concave probability measure on $\mathbb{R}^n$ and let $1 \leqslant k \leqslant \sqrt{n}$. For every $\varepsilon > 0$ and any $E \in G_{n,k}$ there exists $F \in G_{n,k}$ with $d(E,F) < \varepsilon$ such that*

$$L_{\pi_F \mu} \leqslant C/\varepsilon, \tag{1.3}$$

where $C > 0$ is an absolute constant. Additionally, if L_μ is bounded then we can take $1 \leqslant k \leqslant n-1$.

Apart from the obvious similarities to Klartag's result, there are some significant differences that we would like to point out. First, the way of measuring distance is different. Second, both measures we consider here, the given and the perturbed one, are isotropic while in Klartag's result are not. Third, the dependence on ε is weaker than Klartag's result. However, in the last section of this note we show that if one can prove the above statement with a dependence $o(1/\varepsilon)$ then the Hyperplane Conjecture will follow. In other words, if the Hyperplane Conjecture turns out to be false, the dependence on ε in Theorem 1.1 is optimal.

Let us also comment on the restriction on the dimension ($k \leqslant \sqrt{n}$) in the main theorem. It follows from the work of [9] and [26] that for every $k \leqslant \sqrt{n}$ that a randomly chosen marginal of dimension k has bounded isotropic constant. Moreover if L_μ is bounded (for some specific measure μ) then for every $k \leqslant n-1$ any randomly chosen marginal has bounded isotropic constant. In general, it is not known if one can find a marginal of dimension $k \gg \sqrt{n}$ with bounded isotropic constant. It follows from [6] that if one could prove that a random marginal of dimension $k \gg \sqrt{n}$ has bounded isotropic constant this would improve the best known result for the isotropic constant. Hence, the restriction on the dimension in the theorem is because the best known bound for the isotropic constant is $O(n^{1/4})$. For a more detailed analysis on the range of dimension that Theorem 1.1 holds see Theorem 3.4 and the discussion after.

The paper is organized as follows: In Section 2 we give the necessary background for the Grassmann manifold and some facts from convexity that we are going to use. In Section 3 we briefly refer to some standard facts about log-concave measures and we proceed to the proof of the main result. Finally, in Section 4 we conclude with the optimality on the theorem and some further remarks.

2 Grassmann manifold and convexity

§2.1. The Grassmann manifold. The Grassmann manifold consisting of all k-dimensional subspaces F of $\mathbb{R}^n$ is denoted by $G_{n,k}$. We work in $G_{n,k}$ equipped with a metric ρ which is induced by some unitarily invariant ideal norm on $L(\ell_2^n)$ (see [34] for details). A typical example of such a metric is $\sigma_\infty(E_1,E_2) := \|P_{E_1} - P_{E_2}\|_{\mathrm{op}}$ induced by the operator norm under the embedding $F \mapsto P_F$ of the Grassmann manifold into $L(\ell_2^n)$, where P_F stands for the orthogonal projection onto F. Another example of a unitarily invariant metric on $G_{n,k}$ is: $d(E,F) = \inf\{\|I-U\| : U \in O(n), U(E) = F\}$. Note that σ_∞ and d are Lipschitz equivalent metrics, i.e. $\sigma_\infty(E,F) \leqslant d(E,F) \leqslant \sqrt{2}\sigma_\infty(E,F)$, for all $E,F \in G_{n,k}$. Under the orthogonal group action over $G_{n,k}$ we get that $G_{n,k}$ becomes a homogeneous space, thus there exists a unique probability measure $\nu_{n,k}$ which is invariant under this action – the so-called Haar measure.

We will need an entropy estimate for $G_{n,k}$ with respect to metrics described above. For a compact set A in a metric space (X,d) and any $\varepsilon > 0$ the ε-entropy of A denoted by $N(A,d,\varepsilon)$ is the minimum number of balls of radius ε required to cover A. In [33] (see also [34]) Szarek proved the following result:

Theorem 2.1. *Let $1 \leqslant k \leqslant n-1$ and let ρ be a metric on $G_{n,k}$ induced by some unitarily invariant ideal norm on $L(\ell_2^n)$ normalized such that* $\mathrm{diam}(G_{n,k},\rho) = 1$. *Then, for any $0 < \varepsilon < 1$ we have:*

$$(c_1/\varepsilon)^{k(n-k)} \leqslant N(G_{n,k},\rho,\varepsilon) \leqslant (c_2/\varepsilon)^{k(n-k)}, \tag{2.1}$$

where $c_1, c_2 > 0$ are absolute constants.

See also [22] for an alternative proof of this result. Using Theorem (2.1), the invariance of the Haar measure and the sub-additivity of the measure we can immediately conclude the following:

Corollary 2.2. *Let $1 \leqslant k \leqslant n-1$, ρ be a metric as above on $G_{n,k}$ and let $\delta \in (0,1)$. Then, for any $F \in G_{n,k}$ we have:*

$$(\delta/c_2)^{k(n-k)} \leqslant \nu_{n,k}(B_\rho(F,\delta)) \leqslant (\delta/c_1)^{k(n-k)}, \tag{2.2}$$

where $B_\rho(F,\delta) = \{E \in G_{n,k} : \rho(F,E) < \delta\}$.

We are going to use only the left-hand side inequality in (2.2).

§2.2. Affine and dual affine quermassintegrals. The next affine invariants of any body K were initially introduced by Lutwak (under different normalization) in [17]. The definition we expose here follows [6]. Let K be a convex body in $\mathbb{R}^n$. For any $1 \leqslant k \leqslant n-1$ we define the *k-th dual affine quermassintegral* of K as:

$$\tilde{\Phi}_{[k]}(K) := \left(\int_{G_{n,k}} |K \cap F^\perp|^n \, d\nu_{n,k}(F) \right)^{\frac{1}{kn}}, \tag{2.3}$$

where $|\cdot|$ stands for the volume.

Grinberg proved in [11] that these quantities are invariant under volume preserving linear transformations, as conjectured by Lutwak. The following inequality was proved by Busemann and Strauss [5] and independently by Grinberg in [11].

Theorem 2.3. *For any $1 \leqslant k \leqslant n-1$ and any convex body K in $\mathbb{R}^n$ of volume 1, we have:*

$$\int_{G_{n,k}} |K \cap E|^n \, d\nu_{n,k}(E) \leqslant \int_{G_{n,k}} |D_n \cap E|^n \, d\nu_{n,k}(E). \tag{2.4}$$

Here D_n stands for the Euclidean ball of volume 1. Note that according to the above notation this inequality can be equivalently rewritten as $\tilde{\Phi}_{[k]}(K) \leqslant \tilde{\Phi}_{[k]}(D_n)$ for all convex bodies of volume 1. We also have the following asymptotic estimate:

$$\tilde{\Phi}_{[k]}(D_n) = \omega_n^{1/n} \left(\frac{\omega_{n-k}}{\omega_n} \right)^{1/k} \simeq 1,$$

where ω_m denotes the volume of the Euclidean ball on $\mathbb{R}^m$ of radius 1.

Next, we give the definition of the affine quermassintegrals: Let K be a convex body in $\mathbb{R}^n$. For any $1 \leqslant k \leqslant n-1$ we define the *k-th affine quermassintegral* of K as:

$$\Phi_{[k]}(K) := \left(\int_{G_{n,k}} |P_F K|^{-n} \, d\nu_{n,k}(F) \right)^{-\frac{1}{kn}}. \tag{2.5}$$

We have that $K \mapsto \Phi_{[k]}(K)$ is homogeneous of order 1, that is if $\lambda > 0$ then $\Phi_{[k]}(\lambda K) = \lambda \Phi_{[k]}(K)$.

These quantities were introduced by Lutwak in [18] under different normalization. The definition given here follows again [6]. Grinberg also proved in [11] that these quantities are invariant under volume preserving affine transformations. The quantity $\Phi_{[k]}(\cdot)$ is an affine version of the classic quermassintegral. We refer to the book [32] for additional basic facts from Convex Geometry.

Lutwak conjectured that for any convex body K on $\mathbb{R}^n$ of volume 1 one must have $\Phi_{[k]}(K) \geqslant \Phi_{[k]}(D_n)$. An isomorphic version of this estimate was verified by P. Pivovarov and the first named author in [23, Theorem 5.1]. This estimate is crucial for our argument. Therefore, we give a brief sketch of proof of this result for the reader's convenience. The argument makes use of Theorem 2.3, Blaschke-Santaló's inequality [31] and reverse Santaló inequality due to Bourgain and V. Milman [3].

Theorem 2.4. *[23] Let K be a centrally symmetric convex body on $\mathbb{R}^n$. Then, for all $1 \leqslant k \leqslant n-1$ one has:*

$$\Phi_{[k]}(K) \geqslant c_1 |K|^{1/n} \sqrt{\frac{n}{k}}, \tag{2.6}$$

where $c_1 > 0$ is an absolute constant.

Proof (Sketch). Using reverse Santaló inequality we may write:

$$\Phi_{[k]}(K) \geqslant c\omega_k^{2/k} \left(\int_{G_{n,k}} |K^\circ \cap F|^n \, d\nu_{n,k}(F) \right)^{-\frac{1}{kn}}.$$

On the other hand by Theorem 2.3 we have:

$$\int_{G_{n,k}} |K^\circ \cap F|^n \, d\nu_{n,k}(F) \leqslant |K^\circ|^k \int_{G_{n,k}} |D_n \cap F|^n \, d\nu_{n,k}(F) = |K^\circ|^k \omega_n^{-k} \omega_k^n.$$

Hence, by Santaló's inequality we obtain:

$$\left(\int_{G_{n,k}} |K^\circ \cap F|^n \, d\nu_{n,k}(F) \right)^{\frac{1}{kn}} \leqslant |K^\circ|^{1/n} \omega_n^{-1/n} \omega_k^{1/k} \leqslant |K|^{-1/n} \omega_n^{1/n} \omega_k^{1/k}.$$

The result follows. □

3 Proof of the main result

We first collect some known results from the theory of log-concave probability measures that we will need for the proof. For any log-concave probability measure μ and every $q \geqslant 1$ we define the *L_q-centroid body of μ*, denoted by $Z_q(\mu)$ through its support function:

$$h_{Z_q(\mu)}(y) := \left(\int_{\mathbb{R}^n} |\langle x, y\rangle|^q \, d\mu(x) \right)^{1/q}, \quad y \in \mathbb{R}^n. \tag{3.1}$$

The marginal $\pi_F \mu$ onto subspaces F of a Borel probability measure μ is defined as the push forward of μ under the orthogonal projection P_F, that is:

$$(\pi_F \mu)(A) := \mu(P_F^{-1}(A)) = \mu(A + F^{\perp}),$$

for any Borel subset A in F. By the definition of the marginal we have that for every subspace F of $\mathbb{R}^n$,

$$P_F(Z_q(\mu)) = Z_q(\pi_F \mu). \tag{3.2}$$

Note that the isotropicity of a centered μ can be equivalently described by the condition $Z_2(\mu) = B_2^n$. The following estimate for the volume of L_n-centroid body has been proved in [26, Proposition 3.7]:

$$|Z_n(\mu)|^{1/n} \simeq \frac{1}{f_\mu(0)^{1/n}}. \tag{3.3}$$

Moreover, for any centered log-concave measure by Fradelizi's theorem [8] we know that:

$$f_\mu(0) \leqslant \|f_\mu\|_\infty \leqslant e^n f_\mu(0) \tag{3.4}$$

and in view of the definition of the isotropic constant we may write:

$$|Z_n(\mu)|^{1/n} \simeq \frac{[\det \mathrm{Cov}(\mu)]^{\frac{1}{2n}}}{L_\mu} \leqslant c_1 [\det \mathrm{Cov}(\mu)]^{\frac{1}{2n}}, \tag{3.5}$$

where in the last inequality we have used the fact that $L_\mu \geqslant c > 0$ for all probability measures μ (see [21]). Moreover, for the full range of $1 \leqslant q \leqslant n$ we have that [25]:

$$|Z_q(\mu)|^{1/n} \leqslant c \sqrt{\frac{q}{n}} [\det \mathrm{Cov}(\mu)]^{\frac{1}{2n}}. \tag{3.6}$$

For our purpose we will like to know the reverse inequality. Of course if the reverse inequality was known for all $q \leqslant n$ then (just apply for $q = n$ and use (3.3) and (3.4)) the Hyperplane Conjecture would follow. We introduce (for $\beta \geqslant 1$) the auxiliary parameter $q_v(\mu, \beta)$ as follows:

$$q_v(\mu, \beta) := \max\left\{ q \leqslant n : |Z_q(\mu)|^{1/n} \geqslant \frac{1}{\beta} \sqrt{\frac{q}{n}} [\det \mathrm{Cov}(\mu)]^{\frac{1}{2n}} \right\}. \tag{3.7}$$

The results of Klartag and E. Milman in [15] imply that for any log-concave measure μ on $\mathbb{R}^n$ inequality can be reversed (up to absolute constants) for $q \leqslant \sqrt{n}$ (see also [35] for subsequent refinements). In our notations their results give the following:

Theorem 3.1. *[15] There exists an absolute constant $c > 0$ such that for every centered* log-*concave probability measure μ,*

$$q_v(\mu, c) \geqslant \sqrt{n}.$$

The Lutwak-Yang-Zhang inequalities [19] (see also [24] for the measure theoretic version we use here) say that for all $1 \leqslant q \leqslant n$,

$$|Z_q(\mu)|^{1/n} \geqslant c\sqrt{\frac{q}{n}}\frac{[\det \mathrm{Cov}(\mu)]^{\frac{1}{2n}}}{L_\mu}. \tag{3.8}$$

In our notation (3.8) implies that

$$q_v(\mu, cL_\mu) \geqslant n. \tag{3.9}$$

Also, since $p \mapsto |Z_p(\mu)|^{\frac{1}{n}}$ is increasing by Hölder's inequality, we have that for every $t, \beta \geqslant 1$,

$$q_v(\mu, t\beta) \geqslant c q_v(\mu, \beta) t^2. \tag{3.10}$$

For any log-concave probability measure μ on $\mathbb{R}^n$ we introduce the averages:

$$\mathcal{A}_{[k]}(\mu) := \left(\int_{G_{n,k}} f_{\pi_F \mu}(0)^n \, d\nu_{n,k}(F)\right)^{\frac{1}{kn}}. \tag{3.11}$$

Similar quantities have appeared previously in the literature: The average

$$\left(\int_{G_{n,k}} f_{\pi_F \mu}(0) \, d\nu_{n,k}(F)\right)^{1/k}$$

has been shown in [26, Proposition 4.6] to be related to the negative moments of the Euclidean ball. The work of small ball probability estimates has established that for $k \leqslant \sqrt{n}$ this average is bounded by an absolute constant. Moreover, Eldan and Klartag in [7], continuing the work of [13] and [14], show that for $k \leqslant n^{c_1}$ this average is close to $1/\sqrt{2\pi}$, the average of the standard Gaussian density. In fact, something stronger than that is proved there: the density of the typical marginal $f_{\pi_E \mu}$ satisfies

$$|1 - f_{\pi_E \mu}(x)/g_k(x)| \leqslant n^{-c_3}$$

for all $x \in E$ with $\|x\|_2 \leqslant n^{c_2}$ with probability greater than $1 - e^{-n^{c_4}}$ in $G_{n,k}$, where $c_1, c_2, c_3, c_4 > 0$ are absolute constants. Here we consider higher moments. The next lemma shows that the quantity in (3.11) is also closely related to the affine quermassintegrals and in turn to the volume of the L_k-centroid body of μ.

Lemma 3.2. *Let μ be a log-concave isotropic measure in $\mathbb{R}^n$. For all $1 \leqslant k \leqslant n-1$ we have:*

$$\mathcal{A}_{[k]}(\mu) \simeq \left(\int_{G_{n,k}} L_{\pi_F \mu}^{kn} \, d\nu_{n,k}(F)\right)^{\frac{1}{kn}} \simeq \Phi_{[k]}(Z_k(\mu))^{-1}. \tag{3.12}$$

In particular, we have that:

$$\mathcal{A}_{[k]}(\mu) \leqslant c_1 \sqrt{\frac{k}{n}} \frac{1}{|Z_k(\mu)|^{1/n}}, \tag{3.13}$$

where $c_1 > 0$ is an absolute constant.

Proof. The first equivalence follows directly from (3.4) applied to $\pi_F \mu$ while the second equivalence from (3.3), (3.2) and (2.5) for $K = Z_k(\pi_F \mu)$. The estimate (3.13) follows from (3.12) and by Theorem 2.4 applied to $Z_k(\mu)$. □

So, if μ is isotropic then (3.13) and (3.9) imply that

$$\mathcal{A}_{[k]}(\mu) \leqslant C\beta \text{ if } 1 \leqslant k \leqslant q_v(\mu, \beta) \quad \text{and} \quad \mathcal{A}_{[k]}(\mu) \leqslant cL_\mu \text{ if } k \leqslant n-1. \tag{3.14}$$

An application of Markov's inequality together with (3.12) yields the following large deviation estimate:

Proposition 3.3. *Let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$, $\beta \geqslant 1$ and $1 \leqslant k \leqslant q_v(\mu,\beta)$. Then, we have:*

$$\nu_{n,k}(\{F \in G_{n,k} : L_{\pi_F\mu} \geqslant C\beta t\}) \leqslant t^{-kn}, \tag{3.15}$$

for all $t > 1$.

Now we are ready to prove the following:

Theorem 3.4. *Let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$ and let $\beta \geqslant 1$. For any $1 \leqslant k \leqslant q_v(\mu,\beta)$, any $E \in G_{n,k}$ and every $\varepsilon \in (0,1)$, there exists $F \in G_{n,k}$ such that $\rho(E,F) < \varepsilon$ and*

$$L_{\pi_F\mu} < \frac{C\beta}{\varepsilon^{1-\frac{k}{n}}},$$

where $C > 0$ is an absolute constant and ρ is a metric on $G_{n,k}$ as in Theorem 2.1.

Proof. Let $\varepsilon \in (0,1)$, $\beta \geqslant 1$, $1 \leqslant k \leqslant q_v(\mu,\beta)$ and let $E \in G_{n,k}$. For any $t > 1$ consider the set $A_t := \{F \in G_{n,k} : L_{\pi_F\mu} \geqslant C_1 t\beta\}$. Proposition 3.3 implies that $\nu_{n,k}(A_t) \leqslant t^{-kn}$. On the other hand from Corollary 2.2 we have $\nu_{n,k}(B_\rho(E,\varepsilon)) \geqslant (\varepsilon/c_1)^{k(n-k)}$. Choosing $t > 1$ such that $t^{-kn} = (\varepsilon/c_1)^{k(n-k)}$, that is $t \simeq 1/\varepsilon^{1-\frac{k}{n}}$, we conclude that $A_t^c \cap B_\rho(E,\varepsilon) \neq \emptyset$ and the result follows. □

Proof of Theorem 1.1. It follows from Theorem 3.4 applied for $\rho = d$ and Theorem 3.1. Moreover, if $L_\mu \leqslant C$ by (3.14) we can take $k \leqslant n-1$. □

Note that the proof shows that Theorem 1.1 holds true if we replace the distance d with any other distance ρ as in Theorem 2.1. Let $k = \lambda n$ for some $\lambda \in (0,1)$ and choose $\beta \simeq L_\mu$. We have also proved that for any $E \in G_{n,\lambda n}$ and every $\varepsilon \in (0,1)$ there exists $F \in G_{n,\lambda n}$ with $d(E,F) < \varepsilon$ and

$$L_{\pi_F\mu} < \frac{CL_\mu}{\varepsilon^{1-\lambda}}. \tag{3.16}$$

One should compare the above inequality with the following (optimal) pointwise estimate: For any isotropic log-concave measure μ on $\mathbb{R}^n$, every $\lambda \in (0,1)$ and every $F \in G_{n,\lambda n}$ one has

$$L_{\pi_F\mu} < (CL_\mu)^{1/\lambda}. \tag{3.17}$$

To see this recall the fact that for any isotropic log-concave probability measure μ on $\mathbb{R}^n$ there exists an isotropic convex body T in $\mathbb{R}^n$ with the properties $L_T \simeq L_\mu$ and

$$\frac{L_{\pi_F\mu}}{L_\mu} \simeq |T \cap F^\perp|^{1/k},$$

for all $F \in G_{n,k}$ (see [6, Proposition 2.1, Lemma 5.8] for details). Using the entropy estimate $N(L_T D_n, T) \leqslant (C_1 L_T)^n$ and the Rogers-Shephard inequality from [30]

$$|P_F T||T \cap F^\perp| \leqslant \binom{n}{k}|T| \tag{3.18}$$

for every $F \in G_{n,k}$ the estimate (3.17) easily follows. In order to see that (3.17) is optimal consider any isotropic probability measure ν on $\mathbb{R}^{\lambda n}$ and the measure μ on $\mathbb{R}^n$ with $\mu = \nu \otimes \gamma_{(1-\lambda)n}$, where γ_m is the standard Gaussian on $\mathbb{R}^m$. Then, one can check that $(e^{-1}L_\mu)^{1/\lambda} \leqslant L_\nu = L_{\pi_{\mathbb{R}^{\lambda n}}\mu}$. So, (**??**) is optimal in the sense that any improvement on the dependence on λ would settle the Hyperplane Conjecture.

We summarize the above discussion in the following:

Proposition 3.5. *Let μ be an isotropic* log*-concave probability measure in $\mathbb{R}^n$ and $\lambda \in (0,1)$ and let $E \in G_{n,\lambda n}$. Then*

$$L_{\pi_E \mu} < (c_1 L_\mu)^{\frac{1}{\lambda}} \tag{3.19}$$

and the inequality is sharp up to the constant c_1. Moreover, for every $\varepsilon > 0$, there exists $F \in G_{n,\lambda n}$ such that $d(E,F) \leqslant \varepsilon$ and

$$L_{\pi_F \mu} < \frac{c_2 L_\mu}{\varepsilon^{1-\lambda}}, \tag{3.20}$$

where $c_1, c_2 > 0$ are absolute constants.

4 On the dependence on ε in Theorem 1.1

In this section we discuss the dependence on ε in Theorem 3.4 and we conclude with some remarks on the quantity $q_v(\mu, \beta)$.

We show that an improvement on the dependence on these parameters would imply the Hyperplane Conjecture.

§1. As we mentioned on the Introduction, any improvement to $o(1/\varepsilon)$ on the dependence on ε in Theorem 1.1, would imply an affirmative answer to the Hyperplane Conjecture. More precisely we have the following:

> **Assumption.** *There exist $\alpha \in (0,1)$ and positive integers $k_n < n$, $k_n \to \infty$ with the following property: For all n, for all isotropic log-concave probability measures μ on $\mathbb{R}^n$, for all $\varepsilon \in (0,1)$ and for every $E \in G_{n,k_n}$ there exists $F \in G_{n,k_n}$ with $d(E,F) < \varepsilon$ and $L_{\pi_F \mu} < C/\varepsilon^\alpha$.*

The reader should notice the global character of the above assumption and that the technique utilized below is not applicable for a specific measure μ. In this setting we can prove that:

Proposition 4.1. *With the above assumption for any n, any isotropic measure μ on $\mathbb{R}^n$ satisfies $L_\mu < C$ for some absolute constant $C > 0$.*

In this paragraph, in view of (3.3), we define the isotropic constant as:

$$L_\nu := |Z_m(\nu)|^{-1/m},$$

for any isotropic log-concave probability measure ν on $\mathbb{R}^m$.

For the proof of Proposition 4.1 we shall need some lemmas. We start with the next stability result of the isotropic constant of marginals with respect to the distance d. Recall that the geometric distance $d_G(K,L)$ between any two centrally symmetric convex bodies is defined as:

$$d_G(K,L) = \inf\{ab \mid \exists\, a,b > 0 : a^{-1} L \subseteq K \subseteq bL\}.$$

With this notation we have the following:

Lemma 4.2. *Let K be a centrally symmetric convex body in $\mathbb{R}^n$ with $t = d_G(K, B_2^n)$. Let $E,F \in G_{n,k}$ with $d(E,F) = d$. Then, there exists $U \in O(n)$ such that $U(E) = F$ and*

$$(1+td)^{-1} P_F K \subseteq U(P_E K) \subseteq (1+td) P_F K. \tag{4.1}$$

In particular,

$$|P_E K|^{1/k} \leqslant (1+td) |P_F K|^{1/k}. \tag{4.2}$$

Proof. We consider $U \in O(n)$ such that $d = \|I - U\|$. Let $\theta \in S_F$. Then, $U^*\theta = \phi \in S_E$ therefore we have $\|\theta - \phi\|_2 \leqslant d$. We may write:

$$\frac{h_{P_F K}(\theta)}{h_{U(P_E K)}(\theta)} = \frac{h_K(\theta)}{h_K(\phi)} \leqslant 1 + \frac{h_K(\theta - \phi)}{h_K(\phi)} \leqslant 1 + \frac{dR(K)}{r(P_E K)},$$

where $R(\cdot), r(\cdot)$ are the circumradius and inradius respectively. Similarly, we have that:

$$\frac{h_{U(P_E K)}(\theta)}{h_{P_F K}(\theta)} = \frac{h_K(\phi)}{h_K(\theta)} \leqslant 1 + \frac{h_K(\theta - \phi)}{h_K(\theta)} \leqslant 1 + \frac{dR(K)}{r(P_F K)}.$$

Since $r(P_F K), r(P_E K) \geqslant r(K)$ and $d_G(K, B_2^n) = R(K)/r(K)$ the result follows. □

Applying Lemma 4.2 for $K = Z_k(\nu)$ and using the modified definition of the isotropic constant we arrive at the following:

Proposition 4.3. *Let ν be an isotropic log-concave probability measure in $\mathbb{R}^n$ and let $1 \leqslant k \leqslant n-1$. For any $E, F \in G_{n,k}$ we have:*

$$L_{\pi_E \nu}/L_{\pi_F \nu} \leqslant 1 + d_G(Z_k(\nu), B_2^n) d(E, F). \tag{4.3}$$

The above estimate also implies that the *length of the gradient* (see [16, Chapter 3] for a definition) of the function $h : (G_{n,k}, d) \to \mathbb{R}$ with $F \overset{h}{\longmapsto} \log L_{\pi_F \nu}$ at F is bounded by $d_G(Z_k(\nu), B_2^n)$, that is $|\nabla h|(F) \leqslant d_G(Z_k(\nu), B_2^n)$ or

$$|\nabla \log L_{\pi_F \nu}| \leqslant d_G(Z_k(\nu), B_2^n). \tag{4.4}$$

In particular, the function $F \mapsto \log L_{\pi_F \nu}$ is $d_G(Z_k(\nu), B_2^n)$-Lipschitz with respect to d.

The next step is for a given measure μ on $\mathbb{R}^k$ ($k = k_n$) to construct a measure ν on $\mathbb{R}^n$ such that the geometric distance of $Z_k(\nu)$ to B_2^n to be at most L_μ. We write $L_m = \sup_\nu L_\nu$ where the supremum is taken over all isotropic log-concave probability measures on $\mathbb{R}^m$.

Proposition 4.4. *For any $1 \leqslant k < n$ there exist an isotropic log-concave probability measure μ_1 on $\mathbb{R}^k$ with $L_{\mu_1} \simeq L_k$ and an isotropic log-concave measure μ_2 on $\mathbb{R}^n$ such that $\pi_{\mathbb{R}^k} \mu_2 = \mu_1$ and*

$$\frac{c_1}{L_k} \sqrt{k} B_2^n \subseteq Z_k(\mu_2) \subseteq c_2 \sqrt{k} B_2^n. \tag{4.5}$$

The next two lemmas will be needed for the proof of Proposition 4.4.

Lemma 4.5. *The L_q-centroid bodies enjoy the following properties:*

(i) *If ν_1, ν_2 are probability measures in $\mathbb{R}^k$ and at least one of them is symmetric, then for all $q \geqslant 1$ we have:*

$$Z_q(\nu_1 * \nu_2) \subseteq Z_q(\nu_1) + Z_q(\nu_2) \subseteq 2 Z_q(\nu_1 * \nu_2). \tag{4.6}$$

(ii) *If μ, ν are probability measures on $\mathbb{R}^k$ and $\mathbb{R}^m$ respectively and at least one of them is symmetric, then for all $q \geqslant 1$:*

$$Z_q(\mu \otimes \nu) \subseteq Z_q(\mu) \times Z_q(\nu) \subseteq 2 Z_q(\mu \otimes \nu). \tag{4.7}$$

Proof (Sketch). We prove the second statement (see also Proposition 6.2 in [15]), the first can be derived similarly - see [9, Lemma 3.3]. Since for any $(x,y) \in \mathbb{R}^k \times \mathbb{R}^m$ we may write:

$$h_{Z_q(\mu\otimes\nu)}(x,y) = \left(\int_{\mathbb{R}^k} \int_{\mathbb{R}^m} |\langle x, z_1 \rangle + \langle y, z_2 \rangle|^q \, d\mu(z_1)\, d\nu(z_2) \right)^{1/q},$$

the left-hand side inclusion follows from Minkowski's inequality applied on the corresponding product space $(\mathbb{R}^k \times \mathbb{R}^n, \mu \otimes \nu)$ for the functions $u(z_1,z_2) = \langle x, z_1 \rangle$ and $v(z_1,z_2) = \langle y, z_2 \rangle$ and the fact that for any two convex bodies K, L we have $h_{K\times L}(x,y) = h_K(x) + h_L(y)$. For the right-hand side inclusion we use the symmetry to write:

$$h_{Z_q(\mu\otimes\nu)}(x,y) = \left(\int_{\mathbb{R}^k} \int_{\mathbb{R}^m} \frac{|\langle x, z_1 \rangle + \langle y, z_2 \rangle|^q + |\langle x, z_1 \rangle - \langle y, z_2 \rangle|^q}{2} \, d\mu(z_1)\, d\nu(z_2) \right)^{1/q}.$$

Applying the elementary inequality:

$$|u+v|^q + |u-v|^q \geqslant |u|^q + |v|^q,$$

for all $u, v \in \mathbb{R}$, $q \geqslant 1$ we obtain:

$$h_{Z_q(\mu\otimes\nu)}(x,y) \geqslant \left(\frac{h^q_{Z_q(\mu)}(x) + h^q_{Z_q(\nu)}(y)}{2} \right)^{1/q}.$$

The concavity of $t \mapsto t^{1/q}$ completes the proof. □

Given an isotropic probability measure μ on $\mathbb{R}^k$ and any $\xi \in (0,1)$ we define the measure μ_ξ with density function:

$$f_{\mu_\xi}(x) := \int_{\mathbb{R}^k} f_\mu(\sqrt{1-\xi^2}x - \xi y) g_k(\xi x + \sqrt{1-\xi^2} y)\, dy, \quad x \in \mathbb{R}^k, \tag{4.8}$$

where $g_k(z) = (2\pi)^{-k/2} e^{-\|z\|_2^2/2}$ is the density of the standard k-dimensional Gaussian measure γ_k. We need to adapt the definition of M-position for convex bodies of V. Milman (see [20]) in the setting of probability measures. We say that an isotropic log-concave measure in $\mathbb{R}^k$ is in *M-position with constant* $A > 0$ if the body $K := L_\mu Z_k(\mu)$ satisfies: $|K + D_k|^{1/k} \leqslant A$.

Next lemma describes some properties of the measure μ_ξ.

Lemma 4.6. *Let μ be an isotropic* log-*concave probability measure on* $\mathbb{R}^k$.

1. *For any $\xi \in (0,1)$ the measure μ_ξ is log-concave and isotropic on $\mathbb{R}^k$ and*

$$L_{\mu_\xi} \lesssim \min\left\{ \frac{L_\mu}{\sqrt{1-\xi^2}}, \frac{1}{\xi} \right\}. \tag{4.9}$$

2. *If μ is in M-position with constant $A > 0$ then we have:*

$$L_{\mu_\xi} \gtrsim \frac{1}{A} \min\left\{ \frac{L_\mu}{\sqrt{1-\xi^2}}, \frac{1}{\xi} \right\}. \tag{4.10}$$

3. *If $Z_k(\mu) \subseteq D_k$ then for any $\xi \in (0,1)$ we have:*

$$c_2 \xi D_k \subseteq Z_k(\mu_\xi) \subseteq c_3 D_k. \tag{4.11}$$

Proof. The log-concavity follows from Prékopa-Leindler inequality [29]. The isotropicity is straightforward and follows from the fact that μ and γ_k are isotropic. We may write:

$$\begin{aligned} L_{\mu_\xi} \simeq f_{\mu_\xi}(0)^{1/k} &= \left(\int_{\mathbb{R}^k} f_\mu(-\xi y) g_k(\sqrt{1-\xi^2}y)\, dy \right)^{1/k} \\ &\leqslant \|f_\mu\|_\infty^{1/k} \left(\int_{\mathbb{R}^k} g_k(\sqrt{1-\xi^2}y)\, dy \right)^{1/k} = \frac{L_\mu}{\sqrt{1-\xi^2}}. \end{aligned}$$

Arguing similarly for γ_k, we can conclude that $L_{\mu_\xi} \leqslant C_1 \min\{L_\mu/\sqrt{1-\xi^2}, 1/\xi\}$. For the inverse estimate we employ the information that μ is in M-position. Considering the case $\xi \leqslant \sqrt{1-\xi^2}/L_\mu$ we may write:

$$|\sqrt{1-\xi^2} Z_k(\mu) + \xi D_k|^{1/k} \leqslant \frac{\sqrt{1-\xi^2}}{L_\mu} |K + D_k|^{1/k} \leqslant A \frac{\sqrt{1-\xi^2}}{L_\mu}.$$

In the case where $\xi \geqslant \sqrt{1-\xi^2}/L_\mu$ we also have that

$$|\sqrt{1-\xi^2} Z_k(\mu) + \xi D_k|^{1/k} \leqslant A \max\{\sqrt{1-\xi^2}/L_\mu, \xi\}.$$

Therefore, using the fact that $Z_k(\mu_\xi) \simeq \sqrt{1-\xi^2} Z_k(\mu) + \xi Z_k(\gamma_k)$ from Lemma 4.5 we obtain:

$$L_{\mu_\xi}^{-1} \simeq |Z_k(\mu_\xi)|^{1/k} \simeq |\sqrt{1-\xi^2} Z_k(\mu) + \xi Z_k(\gamma_k)|^{1/k} \lesssim A \max\left\{ \frac{\sqrt{1-\xi^2}}{L_\mu}, \xi \right\}$$

where we have also used the fact that $Z_k(\gamma_k) \simeq D_k$. The last assertion follows again from Lemma 4.5(i). □

Proof of Proposition 4.4. Let $1 \leqslant k < n$ and let μ be an isotropic probability measure on $\mathbb{R}^k$ with maximal isotropic constant. One can build (see [6] for a construction) a new isotropic measure μ_0 in $\mathbb{R}^k$ such that $L_{\mu_0} \simeq L_\mu$, $Z_k(\mu_0) \subseteq c_1 D_k$ and μ_0 is in M-position with absolute constant $A > 0$. Selecting $\xi \simeq L_\mu^{-1}$ and considering the measure μ_ξ induced by μ_0 and Gaussian we readily see that $L_{\mu_\xi} \simeq L_\mu \simeq L_k$ and

$$\frac{c_4}{L_\mu} \sqrt{k} B_2^k \subseteq Z_k(\mu_\xi) \subseteq c_5 \sqrt{k} B_2^k, \tag{4.12}$$

from Lemma 4.6. Finally, consider the probability measure $\mu_2 := \mu_\xi \otimes \gamma_{n-k}$. Clearly $\pi_{\mathbb{R}^k} \mu_2 = \mu_\xi$. From Lemma 4.5 we know that

$$Z_k(\mu_2) \simeq Z_k(\mu_\xi) \times Z_k(\gamma_{n-k}). \tag{4.13}$$

Since $Z_k(\gamma_{n-k}) \simeq \sqrt{k} B_2^{n-k}$ the inclusions in (4.5) follow directly if we combine (4.12) and (4.13) by setting $\mu_1 = \mu_\xi$. □

Proof of Proposition 4.1. Let $n \geqslant 1$ and $k = k_n < n$ the corresponding integer from the assumption. Consider the measure μ_1 on $\mathbb{R}^k$ and $\mu_2 \in \mathbb{R}^n$ as given by Proposition 4.4. Then for $E = \mathbb{R}^k \times \{\mathbf{0}\}$ and for $\varepsilon \simeq L_k^{-1}$ the assumption yields a subspace $F \in G_{n,k}$ with $L_{\pi_F \mu_2} \leqslant C \varepsilon^{-\alpha}$. By Proposition 4.3 we obtain:

$$L_k \simeq L_{\mu_1} = L_{\pi_E \mu_2} \leqslant (1 + c\varepsilon L_k) L_{\pi_F \mu_2} \leqslant C' L_k^\alpha \tag{4.14}$$

and the claim is proved. $\square$

§2. The auxiliary parameter q_v is one of the many parameters have been introduced so far for the study of isotropic log-concave measures. In [25] the parameter $q_*(\mu)$ was introduced for proving sharp large deviation estimates for the Euclidean norm with respect to a log-concave measure. In [26] the parameter $q_{-c}(\mu,\delta)$, $\delta>1$ was introduced for the study of small ball probability estimates. In [15] and [35] local (hereditary) version of these parameters was introduced for a unified approach to the Hyperplane Conjecture. We will not provide here all the definitions. The results of [15, Theorem 1.2] and [35, Theorem 1.1] show that the quantity q_v is larger than the hereditary parameters. Moreover it is not hard to construct examples (assuming that the Hyperplane Conjecture is false) that the q_v parameter is much larger than the hereditary ones. This is not the case when one compares with the q_{-c} parameter. For $\delta>1$, $q_{-c}(\mu,\delta)$ is defined as the largest p such that $I_{-p}(\mu)\geqslant I_2(\mu)/\delta$, where $I_q(\mu):=\left(\int_{\mathbb{R}^n}\|x\|_2^q d\mu(x)\right)^{\frac{1}{q}}$, $-(n-1)\leqslant q<\infty$. It is known (see [26] Proposition 4.6) that for $k\leqslant q_{-c}(\mu,\delta)$

$$\left(\int_{G_{n,k}} L^k_{\pi_F\mu}\, d\nu_{n,k}(F)\right)^{\frac{1}{k}}\leqslant c\delta.$$

So (3.14) implies that $q_{-c}(\mu,C\beta)\gtrsim q_v(\mu,\beta)$ for any isotropic log-concave probability measure μ. If one could prove that for all n and for all log-concave, isotropic measures μ in $\mathbb{R}^n$ we have $q_v(\mu,\beta)\gtrsim q_{-c}(\mu,\beta)$ for all $\beta>1$ the Hyperplane Conjecture would follow: if μ is an isotropic log-concave measure on $\mathbb{R}^n$ then we can build the isotropic log-concave measure $\nu=\mu\otimes\gamma_m$ in $\mathbb{R}^{n+m}$ where γ_m is the standard Gaussian and $m\simeq n/\log L_\mu$. Note that $L_\nu\simeq L_\mu^{\frac{n}{n+m}}L_{\gamma_m}^{\frac{m}{n+m}}\geqslant c_1L_\mu$. Moreover, we have $I_{-k}(\nu)\geqslant I_{-k}(\gamma_m)$ for all $1\leqslant k\leqslant m-1$ which shows that $q_{-c}(\nu,\sqrt{\log L_\mu})\geqslant c_2 n/\log L_\mu$. Then, by definition of q_v we may write:

$$L_\nu\leqslant c_3\sqrt{\log L_\mu}\sqrt{\frac{n+m}{q_v(\nu,\sqrt{\log L_\mu})}}\leqslant c_4\sqrt{\log L_\mu}\sqrt{\frac{n}{q_{-c}(\nu,\sqrt{\log L_\mu})}}\leqslant c_5\log L_\mu.$$

Moreover the quantities q_* and q_{-c} are equivalent (up to constants) (see Proposition 2.5 in [27]) for "truncated" isotropic measures (supported on ball of radius of order $\sqrt{n}$). So the quantity $q_v(\mu,\beta)$ is in general larger than the "hereditary" quantities, smaller than the quantity $q_{-c}(\mu,\beta)$ and if one could prove that q_{-c} and q_{-v} are comparable the Hyperplane Conjecture would follow.

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