Small-ball probabilities for the volume of random convex sets

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Abstract

We prove small-deviation estimates for the volume of random convex sets. The focus is on convex hulls and Minkowski sums of line segments generated by independent random points. The random models considered include (Lebesgue) absolutely continuous probability measures with bounded densities and the class of log-concave measures.

Contents

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1 Introduction

The focus of this paper is distributional inequalities for the volume of random convex sets. Typical models involve convex hulls or Minkowski sums of line segments generated by independent random points in $\mathbb{R}^n$. Specifically, let μ be a probability measure on $\mathbb{R}^n$. Sample $N \geqslant n$ independent points $X_1, \ldots, X_N$ according to μ. Let K_N be the absolute convex hull of the X_i's, i.e.,

$$K_N := \operatorname{conv}\{\pm X_1, \ldots, \pm X_N\} \tag{1.1}$$

and let Z_N be the zonotope, i.e., the Minkowski sum of the line segments $[-X_i, X_i]$,

$$Z_N := \sum_{i=1}^{N}[-X_i, X_i] = \Big\{\sum_{i=1}^{N}\lambda_i X_i : \lambda_i \in [-1,1],\ i = 1, \ldots, N\Big\}. \tag{1.2}$$

The literature contains a wealth of results aimed at quantifying the size of K_N and its non-symmetric analogue $\operatorname{conv}\{X_1, \ldots, X_N\}$ in terms of metric quantities such as volume, surface area and mean-width; especially in the asymptotic setting where the dimension n is fixed and $N \to \infty$. The measure μ strongly determines the corresponding properties of K_N and Z_N. Common models include the case when μ is the standard Gaussian measure, see e.g., [10], [38]; the uniform measure on a convex body, see e.g. the survey [7]; among many others, e.g., [68]. These are just a sample of recent articles and we refer the reader to the thorough list of references given therein.

A different asymptotic setting involves the case when the dimension n is large and one is interested in precise dependence on N and phenomena that hold uniformly for a large family of measures μ. In this setting, various geometric properties of K_N and Z_N such as Banach-Mazur distance, in-radius and other metric quantities have been analyzed. For zonotopes, see e.g. [14], [34]. Concerning K_N there have been a number of recent results with special attention paid to estimates that hold "with high probability." These include, for instance, the case when μ is the uniform measure on the vertices of the cube [28], measures with "Gaussian-like" features [49], [44] and the case when μ is the uniform measure on a convex body [29], [20]. We are interested in distributional inequalities for $\operatorname{vol}_n(K_N)$ and $\operatorname{vol}_n(Z_N)$, where $\operatorname{vol}_n(\cdot)$ denotes n-dimensional Lebesgue measure, with precise dependence on n and N for a broad class of measures.

Let $\mathcal{P}_n$ denote the set of all probability measures on $\mathbb{R}^n$ that are absolutely continuous with respect to Lebesgue measure. Our setting involves those μ in $\mathcal{P}_n$ whose densities $f_\mu = \frac{d\mu}{dx}$ are bounded. To fix the normalization, we set $\mathcal{P}_n^b := \big\{\mu \in \mathcal{P}_n : \|f_\mu\|_\infty = 1\big\}$, where $\|f\|_\infty$ is the essential supremum of f. In particular, our setting includes the Gaussian measure and the uniform measure on a convex body $K \subset \mathbb{R}^n$ but not the case of discrete measures. We assume that $\mu_1, \ldots, \mu_N \in \mathcal{P}_n^b$ and that $X_1, \ldots, X_N$ are independent random vectors with X_i distributed according μ_i. Since we will compare K_N and Z_N (which depend on the X_i's) for various underlying measures, we will write $\mathbb{P}_{\otimes_{i=1}^N \mu_i}$ (or simply $\mathbb{P}_{\otimes \mu_i}$) for the product measure associated with $\mu_1, \ldots, \mu_N$; the corresponding expectation by $\mathbb{E}_{\otimes_{i=1}^N \mu_i} = \mathbb{E}_{\otimes \mu_i}$.

Our main interest is in bounding the quantity

$$\mathbb{P}_{\otimes \mu_i} \left(\mathrm{vol}_n \left(K_N \right)^{1/n} \leqslant \varepsilon \right), \tag{1.3}$$

for small values of ε; in particular, the precise dependence on ε, n and N. Such estimates are often referred to as small-ball probabilities. Our aim is to find and quantify universal behavior of small-ball probabilities for $\mathrm{vol}_n(K_N)$, as well as $\mathrm{vol}_n(Z_N)$, for $\mu_i \in \mathcal{P}_n^b$. For the expectation $\mathbb{E}_{\otimes \mu_i} \mathrm{vol}_n(K_N)$, the behavior can be far from uniform. Indeed, even for the Euclidean norm $|X_1|$ of a single vector, the quantity $\mathbb{E}_\mu |X_1|$ need not be finite. Thus in such a general setting, searching for uniform concentration phenomena seems a lost cause. We will show, however, that small-ball-type estimates always hold and are surprisingly uniform.

To the best of our knowledge, apart from particular cases, general small-ball estimates are unknown. Surveying related results in the literature, it was unclear to us even the order of magnitude to expect. One reason for this is that the volume problem is often approached indirectly. Many cases involve stronger statements about, e.g., the in-radius of K_N or inclusion of other naturally associated sets. For instance, the main focus of [49] is singular values of certain random matrices; volume estimates for K_N arise as consequences. To put our problem in context, we state a sample result from the latter paper. Specifically, in [49], K_N is the absolute convex hull of the rows of a random matrix, the entries of which are symmetric, independent and identically distributed random variables with sub-Gaussian tail-decay. In this case, they prove that if $N \geqslant (1+\zeta)n$, where $\zeta > 1/\ln n$, and $\beta \in (0, 1/2)$, then

$$\mathbb{P} \left(\mathrm{vol}_n \left(K_N \right)^{1/n} \leqslant c(\zeta) \sqrt{\frac{\beta \ln(2N/n)}{n}} \right) \leqslant \exp(-c_1 N^{1-\beta} n^{\beta});$$

here $c(\zeta)$ is a constant that depends on ζ and the sub-Gaussian constant of the measure and c_1 is an absolute constant. The latter is proved by estimating the in-radius of K_N. The factor $N^{1-\beta} n^\beta$ in the exponent is the best possible for the analogous statement involving the in-radius of K_N in the class of measures they consider (see [49, Theorem 4.2 & subsequent remark]). In the class $\mathcal{P}_n^b$, however, the volume $\mathrm{vol}_n(K_N)$ behaves differently.

A similar result involves the case when μ_K is the uniform measure on a convex body $K \subset \mathbb{R}^n$ of volume one. In this case, it is known that if $N \geqslant n$, then

$$\mathbb{P}_{\otimes \mu_K}\left(\mathrm{vol}_n\left(K_N\right)^{1/n} \leqslant c\sqrt{\frac{\ln(2N/n)}{n}}\right) \leqslant e^{-n},$$

where c is an absolute constant. See the discussion in [20, §3.1] (and [65, Proposition 1] for the case $N = n$).

The quantity $\sqrt{\frac{\ln(2N/n)}{n}}$ that appears in both of the latter examples corresponds to the expectation of $\mathrm{vol}_n\left(K_N\right)^{1/n}$ for the uniform measure λ_{D_n} on the Euclidean ball of volume one. More precisely, for $n \leqslant N \leqslant e^n$, one has

$$\left(\mathbb{E}_{\otimes \lambda_{D_n}} \mathrm{vol}_n\left(K_N\right)\right)^{1/n} \simeq \sqrt{\frac{\ln(2N/n)}{n}};$$

see, e.g., [29] (see also the references in §4). Here $A \simeq B$ means that $c_1 B \leqslant A \leqslant c_2 B$ for some positive absolute constants c_1 and c_2. It is proved in [63] that among all measures $\mu \in \mathcal{P}_n^b$ the uniform measure λ_{D_n} on the Euclidean ball of volume one minimizes the expected volume of K_N, namely,

$$\mathbb{E}_{\otimes \mu_i} \mathrm{vol}_n\left(K_N\right) \geqslant \mathbb{E}_{\otimes \lambda_{D_n}} \mathrm{vol}_n\left(K_N\right). \tag{1.4}$$

Similarly, it is shown in [63] that

$$\mathbb{E}_{\otimes \mu_i} \mathrm{vol}_n\left(Z_N\right) \geqslant \mathbb{E}_{\otimes \lambda_{D_n}} \mathrm{vol}_n\left(Z_N\right). \tag{1.5}$$

It is easy to check that for $N \geqslant n$,

$$\left(\mathbb{E}_{\otimes \lambda_{D_n}} \mathrm{vol}_n\left(Z_N\right)\right)^{1/n} \simeq \frac{N}{\sqrt{n}};$$

(use, e.g., Lemma 4.6; see also (9.4) for a more general result). Thus it is always meaningful to ask for the dependence on ε, n and N in the following quantities

$$\mathbb{P}_{\otimes \mu_i}\left(\mathrm{vol}_n\left(K_N\right)^{1/n} \leqslant c\varepsilon\sqrt{\frac{\ln(2N/n)}{n}}\right) \tag{1.6}$$

and

$$\mathbb{P}_{\otimes \mu_i}\left(\mathrm{vol}_n\left(Z_N\right)^{1/n} \leqslant \frac{c\varepsilon N}{\sqrt{n}}\right) \tag{1.7}$$

for all measures in $\mathcal{P}_n^b$.

Our first main result is the following theorem.

Theorem 1.1. *Let $n \leqslant N \leqslant e^n$ and let $\mu_1, \ldots, \mu_N \in \mathcal{P}_n^b$. Let $\delta > 1$ and $\varepsilon \in (0,1)$. Then*

$$\mathbb{P}_{\otimes \mu_i}\left(\left\{\mathrm{vol}_n\left(K_N\right)^{1/n} \leqslant \frac{c_1\varepsilon}{\delta}\sqrt{\frac{\ln(2N/n)}{n}}\right\}\right) \leqslant \varepsilon^{c_2 N^{1-1/\delta^2} n^{1/\delta^2}}. \tag{1.8}$$

Moreover, if $N \leqslant ne^{\delta^2}$, then

$$\mathbb{P}_{\otimes \mu_i}\left(\left\{\mathrm{vol}_n\left(K_N\right)^{1/n} \leqslant \frac{c_3\varepsilon}{\delta}\sqrt{\frac{\ln(2N/n)}{n}}\right\}\right) \leqslant \varepsilon^{n(N-n+1-o(1))/4}, \tag{1.9}$$

where the c_i's are absolute constants.

Here and throughout the paper, we use the notation $o(1)$ to denote a quantity in $[0,1]$ that tends to 0 as $N, n \to \infty$. For zonotopes, we prove the following theorem.

Theorem 1.2. *Let $n \leqslant N \leqslant e^n$ and let $\mu_1, \ldots, \mu_N \in \mathcal{P}_n^b$. Then for each $\varepsilon \in (0,1)$,*

$$\mathbb{P}_{\otimes \mu_i}\left(\left\{\mathrm{vol}_n\left(Z_N\right)^{1/n} \leqslant \frac{c\varepsilon N}{\sqrt{n}}\right\}\right) \leqslant \varepsilon^{n(N-n+1-o(1))/4}, \tag{1.10}$$

where c is an absolute constant.

In §8, we also give lower bounds for the quantities in Theorems 1.1 and 1.2, which suggest that the estimates (1.9) and (1.10) are essentially optimal.

It has been observed in various other contexts that achieving the best bounds in small-ball estimates in high-dimensional geometry often requires different techniques than those used for proving large deviations e.g., [42], [32, Proposition 3], [44, Proposition 2.6]. To describe the techniques used in this paper, we outline our viewpoint.

As in [63], we adopt an operator-theoretic point of view from the Local Theory of Banach spaces, e.g., [52], [53], [54]. Namely, we view K_N and Z_N as the image of the cross-polytope B_1^N and the cube B_∞^N, respectively, under the random matrix $[X_1 \ldots X_N]$, i.e., $K_N = [X_1 \ldots X_N]B_1^N$ and $Z_N = [X_1 \ldots X_N]B_\infty^N$. In the same way, for any convex body $C \subset \mathbb{R}^N$, we generate a random n-dimensional convex body by applying $[X_1 \ldots X_N]$ to C:

$$[X_1 \cdots X_N]C = \left\{\sum_{i=1}^N c_i X_i : c = (c_i) \in C\right\}.$$

Our first step is to identify the extremal measures $\mu_i \in \mathcal{P}_n^b$ that maximize the small-ball probability

$$\mathbb{P}_{\otimes \mu_i}\left(\mathrm{vol}_n\left([X_1 \cdots X_N]C\right)^{\frac{1}{n}} \leqslant \varepsilon\right).$$

This is done by means of symmetrization as in [63]. We show that the probability in question is maximized for $\mu_i = \lambda_{D_n}$, the uniform measure on the Euclidean ball of volume one. While this simplifies the problem, computing the small-ball probability directly for λ_{D_n} is non-trivial. We turn instead to $\mu = \gamma_n$, the standard Gaussian measure. Working with γ_n allows us to recast the small-ball problem in more geometric terms by using the Gaussian representation of intrinsic volumes [69], [71] and a suitable extension. A key point in our approach is that purely geometric properties of C - its intrinsic volumes and natural generalizations - dictate the small-ball behavior for $\mathrm{vol}_n([X_1 \ldots X_N]C)$. In this way, we reduce Theorems 1.1 and 1.2 to questions from the realm of classical convexity about the cross-polytope and the cube. In particular, Theorem 1.2 depends on verification of an isomorphic version of a conjecture of E. Lutwak about affine quermassintegrals; a key tool here is a result due to E. Grinberg [36] (see §5). Wherever possible, we outline proofs for a general convex body $C \subset \mathbb{R}^N$. However, the focus of the paper is on B_1^N and B_∞^N.

A more common normalization than that which we use (although slightly more restrictive) is when the covariance matrix of μ is assumed to be the identity, i.e., μ is isotropic. We prove estimates analogous to those of Theorems 1.1 and 1.2 under this normalization in §9; we also treat the important subclass of log-concave measures (see §2 and §9 for definitions). In the last several years, there have been many important results concerning random matrices generated by log-concave measures, see e.g., [1], [2] and the references therein. In this important class we obtain more precise estimates, such as the following theorem.

Theorem 1.3. *Let $n \leqslant N \leqslant e^n$ and μ be an isotropic log-concave probability measure on $\mathbb{R}^n$ with bounded isotropic constant. Then for every $\varepsilon \in (0,1)$,*

$$\mathbb{P}_{\otimes\mu}\left(\mathrm{vol}_n\left(Z_N\right)^{\frac{1}{n}} \leqslant c\varepsilon\left(\mathbb{E}_{\otimes\mu}\,\mathrm{vol}_n\left(Z_N\right)\right)^{\frac{1}{n}}\right) \leqslant \varepsilon^{n(N-n+1-o(1))/4} \tag{1.11}$$

and

$$\mathbb{P}_{\otimes\mu}\left(\left\{\mathrm{vol}_n\left(Z_N\right)^{1/n} \leqslant c_1\varepsilon\left(\mathbb{E}_{\otimes\mu}\,\mathrm{vol}_n\left(Z_N\right)\right)^{\frac{1}{n}}\right\}\right) \geqslant \varepsilon^{nN}, \tag{1.12}$$

where c and c_1 are absolute constants.

See §9 for the corresponding result for K_N.

The paper is organized as follows. In §2 we give basic notation and definitions used in the paper. The reduction to the uniform measure on the Euclidean ball via symmetrization is described in §3; we simply sketch the main points from [63]. In §4 we discuss the Gaussian representation of intrinsic volumes and show how an extension thereof is connected to the small-ball problem. Generalizations of intrinsic volumes are discussed in §5, along with implications for small-ball estimates in the Gaussian case. §6 involves technical computations for the generalized intrinsic volumes of B_1^N and B_∞^N. In §7, we transfer the small-ball estimates obtained for γ_n to λ_{D_n}. In §8 we prove Theorems 1.1 and 1.2 and give complementary lower bounds. In §9 we deal with the isotropic normalization and the log-concave case. We conclude with a discussion in §10 about general random convex sets $[X_1 \dots X_N]C$ and show how results from the asymptotic theory of convex bodies ([57], [64]) can be applied to the general problem of small-ball estimates for random convex sets.

2 Preliminaries

In this section we record notation and definitions used throughout the paper. The setting is $\mathbb{R}^n$, where $n \geqslant 2$, with the usual inner-product $\langle \cdot, \cdot \rangle$, standard Euclidean norm $|\cdot|$ and standard unit vector basis $e_1, \dots, e_n$; n-dimensional Lebesgue measure $\mathrm{vol}_n(\cdot)$; Euclidean ball of radius one B_2^n with volume $\omega_n = \mathrm{vol}_n(B_2^n)$. We reserve D_n for the Euclidean ball of volume one, i.e., $D_n = \omega_n^{-1/n} B_2^n$; Lebesgue measure restricted to D_n is λ_{D_n}. The unit sphere is S^{n-1} and is equipped with the Haar measure σ. The Grassmannian manifold of all n-dimensional subspaces of $\mathbb{R}^N$ is denoted $G_{N,n}$, with Haar measure $\nu_{N,n}$. For a subspace $F \in G_{N,n}$, we write P_F for the orthogonal projection onto F. The standard Gaussian measure is γ_n, i.e., $d\gamma_n(x) = (2\pi)^{-n/2} e^{-|x|^2/2} dx$, while $\overline{\gamma}_n$ is the Gaussian measure with $d\overline{\gamma}_n(x) = e^{-\pi|x|^2} dx$.

Throughout the paper we reserve the symbols $c, c_1, c_2, \dots$ for positive absolute constants (not necessarily the same in each occurrence). We use the convention $A \simeq B$ to signify that $c_1 B \leqslant A \leqslant c_2 B$ for some positive absolute constants c_1 and c_2. Wherever necessary, we assume without loss of generality that n is larger than a fixed absolute constant. By adjusting the constants involved one can always force the results to hold for all $n \geq 2$.

A convex body $K \subset \mathbb{R}^n$ is a compact, convex set with non-empty interior. The support function of a convex body K is given by

$$h_K(y) = \sup\{\langle x, y \rangle : x \in K\} \quad (y \in \mathbb{R}^n)$$

and the mean-width of K is

$$W(K) = \int_{S^{n-1}} h_K(\theta) d\sigma(\theta) + \int_{S^{n-1}} h_K(-\theta) d\sigma(\theta) = 2 \int_{S^{n-1}} h_K(\theta) d\sigma(\theta).$$

We say that K is origin-symmetric if $K = -K$. If the origin is an interior point of K, the polar body K° of K is defined by $K^\circ = \{y \in \mathbb{R}^n : h_K(y) \leqslant 1\}$. A convex body is isotropic if its volume is one, its center of mass is the origin and

$$\int_K |\langle x, \theta\rangle|^2 dx = L_K^2; \tag{2.1}$$

the constant L_K is called the isotropic constant of K. We say that a convex body $K \subset \mathbb{R}^n$ is 1-symmetric (with respect to the standard basis $e_1, \ldots, e_n$), if

$$(\alpha_{\xi(1)} x_{\xi(1)}, \ldots, \alpha_{\xi(n)} x_{\xi(n)}) \in K \tag{2.2}$$

whenever $x = (x_1, \ldots, x_n) \in K$, $\alpha_i \in [-1, 1]$ for each $i = 1, \ldots, n$ and $\xi : \{1, \ldots, n\} \to \{1, \ldots, n\}$ is a permutation. We say that K is 1-unconditional if (2.2) holds whenever $x = (x_1, \ldots, x_n) \in K$, $\alpha_i \in [-1, 1]$ for each $i = 1, \ldots, N$ and ξ is the identity. We also let B_p^n denote the unit-ball in ℓ_p^n.

Let $\mathcal{P}_n$ denote the class of all probability measures on $\mathbb{R}^n$ that are absolutely continuous with respect to Lebesgue measure. The subclass $\mathcal{P}_n^b \subset \mathcal{P}_n$ consists of all those measures μ in $\mathcal{P}_n$ whose densities $f_\mu := \frac{d\mu}{dx}$ satisfy $\|f_\mu\|_\infty = 1$, where $\|\cdot\|_\infty$ is the essential supremum.

A Borel measure μ on $\mathbb{R}^n$ is said to be log-concave if for any compact sets $A, B \subset \mathbb{R}^n$ and $t \in [0, 1]$,

$$\mu(tA + (1-t)B) \geqslant \mu(A)^t \mu(B)^{1-t}.$$

Similarly, a function $f : \mathbb{R}^n \to \mathbb{R}^+$ is log-concave if $\log f$ is concave on its support. It is known that if μ is a log-concave measure on $\mathbb{R}^n$ that is not supported on any proper affine subspace, then $\mu \in \mathcal{P}_n$ and its density f_μ is log-concave [13].

If $A \subset \mathbb{R}^n$ is a Borel set with finite volume, the symmetric rearrangement A^* of A is the (open) Euclidean ball centered at the origin whose volume is equal to that of A. The symmetric decreasing rearrangement of χ_A is defined by $\chi_A^* := \chi_{A^*}$. If $f : \mathbb{R}^n \to \mathbb{R}^+$ is an integrable function, we define its symmetric decreasing rearrangement f^* by

$$f^*(x) = \int_0^\infty \chi^*_{\{f>t\}}(x) dt = \int_0^\infty \chi_{\{f>t\}^*}(x) dt.$$

The latter should be compared with the "layer-cake representation" of f:

$$f(x) = \int_0^\infty \chi_{\{f>t\}}(x) dt. \tag{2.3}$$

see [46, Theorem 1.13]. The function f^* is radially-symmetric, decreasing and equimeasurable with f, i.e., $\{f > \alpha\}$ and $\{f^* > \alpha\}$ have the same volume for each $\alpha \geqslant 0$. By equimeasurability and (2.3), one has $\|f\|_p = \|f^*\|_p$ for each $1 \leqslant p \leqslant \infty$, where $\|\cdot\|_p$ denotes the L_p-norm. If $\mu \in \mathcal{P}_n^b$ has density f_μ, we let μ^*

denote the measure in $\mathcal{P}_n^b$ with density f_μ^*. See [46] and [17] for further background material on rearrangements.

For the reader's convenience, we list a few basic linear algebra facts used in the paper.

Proposition 2.1. *Suppose that $N \geqslant n$ and that $T : \mathbb{R}^N \to \mathbb{R}^n$ is a linear operator. Denote the adjoint of T by T^*.*

(i) (Polar decomposition) There is an isometry $U : \mathbb{R}^n \to \mathbb{R}^N$ such that $T^* = U(TT^*)^{1/2}$.

(ii) If $v_1, \ldots, v_n \in \mathbb{R}^N$ denote the columns of T^* (as a matrix with respect to the standard unit vector basis), then

$$\begin{aligned} \mathrm{vol}_n\left(T^*[0,1]^n\right) &= \det\left(TT^*\right)^{1/2} &(2.4)\\ &= |v_1||P_{V_1^\perp}v_2||P_{V_2^\perp}v_3|\cdots|P_{V_{n-1}^\perp}v_n|, &(2.5)\end{aligned}$$

where

$$V_k := \mathrm{span}\{v_1, \ldots, v_k\} \quad V_0 = \{0\},$$

for $k = 1, \ldots, n-1$.

(iii) Let $E = \ker(T)^\perp$ and let $T|_E$ be the restriction of T to E. If $B \subset \mathbb{R}^N$ is a compact set then

$$\mathrm{vol}_n\left(TB\right) = |\det(T|_E)|\,\mathrm{vol}_n\left(P_E B\right), \tag{2.6}$$

where $|\det(T|_E)| = \det\left(TT^*\right)^{1/2}$.

For (i) see, e.g., [23, §3.2]; (2.4) follows from (i), while (2.5) is the well-known formula for the volume of the parallelpiped spanned by $v_1, \ldots, v_n$, which follows from Gram-Schmidt (see, e.g., [4, Theorem 7.5.1]). For (iii), note that $E = \mathrm{Range}(T^*)$ and

$$\det(TT^*) = \mathrm{vol}_n\left(TT^*[0,1]^n\right) = |\det(T|_E)|\,\mathrm{vol}_n\left(T^*[0,1]^n\right) = |\det(T|_E)|\det(TT^*)^{1/2},$$

hence $\det(T|_E) = \det(TT^*)^{1/2}$; (2.6) follows from the fact that $TB = T|_E P_E B$.

3 Distributional inequalities via symmetrization

The main goal of this section is to show that the small-ball probabilities in Theorems 1.1 and 1.2 are maximized for the uniform measure λ_{D_n} on the Euclidean ball of volume one. This is done by adapting the main result from [63], which concerns more general random sets. For $x_1, \ldots, x_N \in \mathbb{R}^n$, we denote the $n \times N$

matrix with columns $x_1, \dots, x_N$ by $[x_1 \cdots x_N]$ (with respect to the standard basis). For a convex body $C \subset \mathbb{R}^N$, we consider the set

$$[x_1 \cdots x_N]C = \left\{ \sum_{i=1}^{N} c_i x_i : c = (c_i) \in C \right\}.$$

It is proved in [63, Theorem 1.1] that if $\mu_1, \dots, \mu_N \in \mathcal{P}_n^b$, then

$$\mathbb{E}_{\otimes \mu_i} \operatorname{vol}_n \left([X_1 \dots X_N]C \right) \geqslant \mathbb{E}_{\otimes \lambda_{D_n}} \operatorname{vol}_n \left([X_1 \dots X_N]C \right). \tag{3.1}$$

In the notation of the introduction, $K_N = [X_1 \dots X_N]B_1^N$ and $Z_N = [X_1 \dots X_N]B_\infty^N$. The next theorem is a distributional form of (3.1) in the case when C is 1-unconditional (which suffices for our purposes).

Theorem 3.1. *Let $N \geqslant n$ and let $\mu_1, \dots, \mu_N \in \mathcal{P}_n^b$. Suppose that $C \subset \mathbb{R}^N$ is a 1-unconditional convex body. Then*

$$\mathbb{P}_{\otimes \mu_i} \left(\{ \operatorname{vol}_n \left([X_1 \dots X_N]C \right) \geqslant \alpha \} \right) \geqslant \mathbb{P}_{\otimes \lambda_{D_n}} \left(\{ \operatorname{vol}_n \left([X_1 \dots X_N]C \right) \geqslant \alpha \} \right).$$

Remark 3.2. The analogous result for the convex hull of random points sampled in a convex body of volume one was proved by A. Giannopoulos and A. Tsolomitis [31, Lemma 3.3].

Remark 3.3. In Theorem 3.1, one can replace $\operatorname{vol}_n(\cdot)$ by other intrinsic volumes (see [63, Remark 4.4]). In this paper we focus all of our efforts on $\operatorname{vol}_n(\cdot)$.

The proof of Theorem 3.1 is a straightforward modification of that of (3.1). To clarify the role of the extra unconditionality assumption in the present context, we sketch the main points. Recall that if $\mu \in \mathcal{P}_n^b$ has density f_μ, then μ^* denotes the measure in $\mathcal{P}_n^b$ whose density is the symmetric decreasing rearrangement f_μ^*.

Theorem 3.4. *Let N and n be positive integers. Let $\mu_1, \dots, \mu_N \in \mathcal{P}_n^b$ and let $\alpha > 0$. Suppose that $F : (\mathbb{R}^n)^N \to \mathbb{R}^+$ satisfies the following condition: for each $z \in S^{n-1}$, for all $y_1, \dots, y_N \in z^\perp$, the level set*

$$\{ t \in \mathbb{R}^N : F(y_1 + t_1 z, \dots, y_N + t_N z) \leqslant \alpha \}$$

is origin-symmetric and convex. Then

$$\mathbb{P}_{\otimes \mu_i} \left(\{ F > \alpha \} \right) \geqslant \mathbb{P}_{\otimes \mu_i^*} \left(\{ F > \alpha \} \right). \tag{3.2}$$

The latter theorem makes use of the Brascamp-Lieb-Luttinger rearrangement inequality [16] (see also [19]); the proof is given in detail in [63, Proposition 3.2] (use the fact that $\mathbb{P}_{\otimes \mu_i} \left(\{ F > \alpha \} \right) = \mathbb{E}_{\otimes \mu_i} \mathbb{1}_{\{F > \alpha\}}$).

If $K \subset \mathbb{R}^n$ is a compact set of volume one and all μ_i are equal to the uniform measure on K, then Theorem 3.4 gives immediately

$$\mathbb{P}_{\otimes \mu_i}(\{F > \alpha\}) \geqslant \mathbb{P}_{\otimes \lambda_{D_n}}(\{F > \alpha\}). \tag{3.3}$$

For general measures $\mu \in \mathcal{P}_n^b$, an additional step is required to pass to the uniform measure on the ball. We say that $F : (\mathbb{R}^n)^N \to \mathbb{R}^+$ is coordinate-wise increasing if for all $x_1, \ldots, x_N$ in $\mathbb{R}^n$,

$$F(s_1 x_1, \ldots, s_N x_N) \leqslant F(t_1 x_1, \ldots, t_N x_N) \tag{3.4}$$

whenever $0 \leqslant s_i \leqslant t_i$, $i = 1, \ldots, N$. For such functions, one can pass from rotationally-invariant measures $\mu \in \mathcal{P}_n^b$ to λ_{D_n}. Here and elsewhere, we use the term "increasing" in the non-strict sense.

Proposition 3.5. *Let $\mu_1, \ldots, \mu_N \in \mathcal{P}_n^b$ and suppose that $\mu_i = \mu_i^*$ for each $i = 1, \ldots, N$. Assume that F is coordinate-wise increasing as in (3.4). Then*

$$\mathbb{P}_{\otimes \mu_i}(\{F > \alpha\}) \geqslant \mathbb{P}_{\otimes \lambda_{D_n}}(\{F > \alpha\}).$$

Proof. Using spherical coordinates $x_i = r_i \theta_i$, where $r_i \in \mathbb{R}^+$ and $\theta_i \in S^{n-1}$ and writing $d\overline{r} = dr_1 \ldots dr_N$ and $d\overline{\theta} = d\sigma(\theta_1) \ldots d\sigma(\theta_N)$, we have

$$\begin{aligned} \mathbb{P}_{\otimes \mu_i}(\{F > \alpha\}) &= \int_{\mathbb{R}^n} \cdots \int_{\mathbb{R}^n} \mathbb{1}_{\{F > \alpha\}}(x_1, \ldots, x_N) \prod_{i=1}^N f_i(x_i) dx_1 \ldots dx_N \\ &= (n\omega_n)^N \int_{(\mathbb{R}^+)^N} \int_{(S^{n-1})^N} \mathbb{1}_{\{F > \alpha\}}(r_1\theta_1, \ldots, r_N\theta_N) \prod_{i=1}^N f_i(r_i\theta_i) d\overline{\theta} d\overline{r}. \end{aligned}$$

By our assumption on F,

$$\mathbb{R}^+ \ni r_j \mapsto \mathbb{1}_{\{F \geqslant \alpha\}}(r_1\theta_1, \ldots r_j\theta_j, \ldots, r_N\theta_N)$$

is increasing, hence

$$\begin{aligned} &\int_0^\infty \mathbb{1}_{\{F > \alpha\}}(r_1\theta_1, \ldots, r_j\theta_j, \ldots, r_N\theta_N) f_j(r_j\theta_j) dr_j \\ &\qquad \geqslant \int_0^{\omega_n^{1/n}} \mathbb{1}_{\{F > \alpha\}}(r_1\theta_1, \ldots, r_j\theta_j, \ldots, r_N\theta_N) dr_j; \end{aligned}$$

(see, e.g., [63, Lemma 3.5]). Applying the latter inequality for each j, together with Fubini's Theorem, yields the result. $\square$

Proof of Theorem 3.1. Let $F : (\mathbb{R}^n)^N \to \mathbb{R}^+$ be defined by

$$F(x_1, \dots, x_N) := \mathrm{vol}_n\left([x_1 \dots x_N]C\right).$$

Using an argument due to Groemer [37], it is shown in [63, Proposition 4.1] that F satisfies the assumption in Theorem 3.4, hence (3.2) holds. The unconditionality assumption on C guarantees that for each $x_1, \dots, x_N$ in $\mathbb{R}^n$,

$$[s_1 x_1 \dots s_N x_N]C \subset [t_1 x_1 \dots t_N x_N]C,$$

whenever $0 \leqslant s_i \leqslant t_i$, for $i = 1, \dots, N$, hence F is coordinate-wise increasing and Proposition 3.5 applies. □

While Theorem 3.1 reduces Theorems 1.1 and 1.2 to the case of $\mathbb{P}_{\otimes \lambda_{D_n}}$, our path will involve first calculating the small-ball probability for the Gaussian measure, to which we now turn our attention.

4 An extension of the Gaussian representation of intrinsic volumes

To calculate the small-ball probability in Theorems 1.1 and 1.2 for $\mu = \gamma_n$, the standard Gaussian measure, i.e.,

$$\mathbb{P}_{\otimes \gamma_n}\left(\mathrm{vol}_n\left(K_N\right)^{1/n} \leqslant \varepsilon \sqrt{\frac{\ln(2N/n)}{n}}\right),$$

we will prove a reverse-Hölder inequality for $\mathbb{E}_{\otimes \gamma_n} \mathrm{vol}_n\left(K_N\right)^{-p}$, for $p > 0$. As in the previous section, we work with random sets of the form $[X_1 \dots X_N]C$ for a general convex body $C \subset \mathbb{R}^N$. Our first ingredient is an extension of the Gaussian representation of intrinsic volumes.

Recall that the intrinsic volumes of a convex body $C \subset \mathbb{R}^N$ can be defined via the Steiner formula for the outer parallel volume of C:

$$\mathrm{vol}_N\left(C + \alpha B_2^N\right) = \sum_{n=0}^{N} \omega_n V_{N-n}(C) \alpha^n. \tag{4.1}$$

The quantities V_n, $n = 1, \dots, N$, are the n-th intrinsic volumes of C (we set $V_0 \equiv 1$). Of particular interest are V_1, V_{N-1} and V_N, which are multiples of the mean-width, surface area and volume, respectively. Intrinsic volumes are also referred to as quermassintegrals (under an alternate labelling and normalization). For further background on intrinsic volumes, we refer the reader to [67]. We will make use of the following fact, which is a special case of Kubota's integral recursion:

$$V_n(C) = \binom{N}{n} \frac{\omega_N}{\omega_n \omega_{N-n}} \int_{G_{N,n}} \mathrm{vol}_n\,(P_E C)\, d\nu_{N,n}(E). \tag{4.2}$$

The latter formula has a version using Gaussian random matrices rather than orthogonal projection and integration on the Grassmannian, known as the Gaussian representation of intrinsic volumes, as in [69], [71]. Namely, if $G = [\gamma_{ij}]$ is an $n \times N$ matrix with independent standard Gaussian entries, then the n-th intrinsic volume of $C \subset \mathbb{R}^N$ is given by

$$V_n(C) = \frac{(2\pi)^{n/2}}{\omega_n n!} \mathbb{E}\,\mathrm{vol}_n\,(GC)\,. \tag{4.3}$$

The next proposition is an extension of (4.3), which connects powers of $\mathrm{vol}_n\,(GC)$ and the following parameter $W_{[n,p]}(C)$, defined in [21],

$$W_{[n,p]}(C) := \left(\int_{G_{N,n}} \mathrm{vol}_n\,(P_F C)^p\, d\nu_{N,n}(F) \right)^{\frac{1}{np}}, \tag{4.4}$$

for $p \in [-\infty, \infty]$. The quantities $W_{[n,p]}(C)$ are discussed in greater detail in §5. The proof we give below is the same as that of [69, Theorem 6], although presented differently; see also [70, Theorem 1] for a probabilistic derivation of the Steiner formula (4.1), which led us to the connection.

Proposition 4.1. *Let $n \leqslant N$ and let G be an $n \times N$ random matrix with independent standard Gaussian entries. Let $C \subset \mathbb{R}^N$ be a compact set with non-empty interior and $p > -(N - n + 1)$. Then*

$$(\mathbb{E}\,\mathrm{vol}_n\,(GC)^p)^{\frac{1}{p}} = (\mathbb{E}\det(GG^*)^{\frac{p}{2}})^{\frac{1}{p}} W^n_{[n,p]}(C). \tag{4.5}$$

If C is a convex body and $p = 1$, then (4.5) reduces to

$$\mathbb{E}\,\mathrm{vol}_n\,(GC) = \frac{1}{(2\pi)^{n/2}} \frac{N!}{(N-n)!} \frac{\omega_N}{\omega_{N-n}} \int_{G_{N,n}} \mathrm{vol}_n\,(P_E C)\, d\nu_{N,n}(E),$$

which is the Gaussian representation of intrinsic volumes. The random matrix GG^* in Proposition 4.1 is distributed according to the Wishart density and explicit formulas for $\mathbb{E}\det(GG^*)^{\frac{p}{2}}$ are well-known, e.g., [4, Chapter 7]; a direct argument giving the order of magnitude of $\mathbb{E}\det(GG^*)^{\frac{p}{2}}$ is given below in Lemma 4.2. For a strong stochastic equivalence involving projections of regular simplices on $G_{N,n}$ and Gaussian vectors, see [11, Theorem 1].

In a different context, passage between Gaussian random operators and random projections on the Grassmannian manifold has been used to great effect in studying volumetric invariants that arise in Banach-Mazur distance investigations; see [52] and [54].

Proof of Proposition 4.1. Let $h_1, \ldots, h_n \in \mathbb{R}^N$ be the columns of G^*. Then $G^*[0,1]^n$ is the parallelpiped generated by $h_1, \ldots, h_n$ and $\mathrm{vol}_n\left(G^*[0,1]^n\right) = \det(GG^*)^{1/2}$, by Proposition 2.1(ii). Let H be the subspace spanned by $h_1, \ldots, h_n$ so that

$$H = \mathrm{Range}(G^*) = \ker(G)^{\perp}.$$

Let U be a random matrix distributed uniformly on the orthogonal group $\mathcal{O}(N)$, independent of G. Note that $(GU)^*[0,1]^n$ is the parallelpiped spanned by the vectors $U^*h_1, \ldots, U^*h_n$, hence

$$\mathrm{vol}_n\left((GU)^*[0,1]^n\right) = \det((GU)(GU)^*)^{1/2} = \det(GG^*)^{1/2}.$$

Combining the latter equality with Proposition 2.1(iii), we have

$$\mathrm{vol}_n\left(GUC\right) = \det(GG^*)^{\frac{1}{2}}\,\mathrm{vol}_n\left(P_{U^*H}C\right).$$

Let $\mathbb{E}_{\otimes_{i=1}^N \gamma_n} = \mathbb{E}_{\otimes_{i=1}^n \gamma_N}$ denote expectation with respect to G; similarly let $\mathbb{E}_U$ denote expectation with respect to U. By rotational invariance of γ_N, G and GU have the same distribution, hence

$$\begin{aligned}
\mathbb{E}_{\otimes_{i=1}^n \gamma_N}\mathrm{vol}_n\left(GC\right)^p &= \mathbb{E}_{\otimes_{i=1}^n \gamma_N}\mathbb{E}_U\,\mathrm{vol}_n\left(GUC\right)^p \\
&= \mathbb{E}_{\otimes_{i=1}^n \gamma_N}\left(\det(GG^*)^{\frac{p}{2}}\mathbb{E}_U\,\mathrm{vol}_n\left(P_{U^*H}C\right)\right) \\
&= \mathbb{E}_{\otimes_{i=1}^n \gamma_N}\det(GG^*)^{\frac{p}{2}}\int_{G_{N,n}}\mathrm{vol}_n\left(P_E C\right)^p\,d\nu_{N,n}(E).
\end{aligned}$$

□

As mentioned above, we give the order of magnitude of $\mathbb{E}\det\left(GG^*\right)^{\frac{p}{2}}$. Since the resulting estimate is closely connected to the small-ball estimate in the Gaussian case, we include a detailed proof.

Lemma 4.2. *Let $N \geqslant n$ and let G be an $n \times N$ random matrix with independent standard Gaussian entries. Then for all $p \in [-(N-n+1-e^{-n(N-n+1)}), N]$,*

$$\left(\mathbb{E}\det\left(GG^*\right)^{\frac{p}{2}}\right)^{\frac{1}{pn}} \simeq \sqrt{N}.$$

Proof. Let $X = (x_1, \ldots, x_N)$ be an N-dimensional standard Gaussian vector. Let $m \in \{1, \ldots, N\}$ and $F \in G_{N,m}$. For each $\eta > 0$ and for all $p \in [-(m - e^{-\eta m}), m]$, we have

$$ce^{-\eta}\sqrt{m} \leqslant (\mathbb{E}|P_F X|^p)^{\frac{1}{p}} \leqslant c_1\sqrt{m}. \tag{4.6}$$

Indeed, note that for $a \in (0,1)$, $\mathbb{E}_{\gamma_1}|x_1|^{-a} \simeq \frac{1}{1-a}$. Then, for $p_0 = m - e^{-\eta m}$, we have

$$\begin{aligned}
\left(\mathbb{E}|P_F X|^{-p_0}\right)^{-\frac{1}{p_0}} &= \left(\mathbb{E}_{\gamma_m}|(x_1, \ldots, x_m)|^{-p_0}\right)^{-\frac{1}{p_0}} \\
&= \left(\frac{m\omega_m}{(2\pi)^{\frac{m}{2}}}\int_0^\infty r^{m-(m-e^{-\eta m})-1}e^{-\frac{r^2}{2}}\,dr\right)^{-\frac{1}{p_0}} \\
&= \frac{1}{(m\omega_m)^{\frac{1}{p_0}}}(2\pi)^{\frac{m-1}{2p_0}}\left(\frac{1}{2}\mathbb{E}_{\gamma_1}|x_1|^{-(1-e^{-\eta m})}\right)^{-\frac{1}{p_0}} \\
&\geqslant ce^{-\eta}\sqrt{m}.
\end{aligned}$$

For the positive range,

$$\begin{aligned}
(\mathbb{E}|P_F X|^m)^{\frac{1}{m}} &= (\mathbb{E}_{\gamma_m}|(x_1, \ldots, x_m)|^m)^{\frac{1}{m}} \\
&= \left(\frac{m\omega_m}{(2\pi)^{\frac{m}{2}}}\int_0^\infty r^{2m-1}e^{-\frac{r^2}{2}}\,dr\right)^{-\frac{1}{m}} \\
&\simeq \sqrt{m}.
\end{aligned}$$

As in the proof of Proposition 4.1, let $h_1, \ldots, h_n \in \mathbb{R}^N$ be the columns of G^*. Let $H_0 = \{0\}$. For $k = 1, \ldots, n-1$, set

$$H_k := \operatorname{span}\{h_1, \ldots, h_k\}.$$

By Proposition 2.1 (ii), we have

$$\det(GG^*)^{\frac{p}{2}} = \prod_{k=1}^{n}|P_{H_{k-1}^{\perp}}h_k|^p. \tag{4.7}$$

Let $p_1 = -\left(N - n + 1 - e^{-n(N-n+1)}\right)$. Integrating first with respect to h_n, then h_{n-1} and so forth, at each stage applying (4.6) with $m = N - k + 1$ and

$\eta_k = 2^{-k}$ for $k \geqslant 2$ and $\eta_1 = n$, we obtain

$$\begin{aligned}\left(\mathbb{E}\det(GG^*)^{\frac{p_1}{2}}\right)^{\frac{1}{p_1 n}} &= \left(\mathbb{E}\prod_{k=1}^{n}|P_{H_{k-1}^{\perp}}h_k|^{p_1}\right)^{\frac{1}{p_1 n}}\\ &\geqslant \left(\prod_{k=1}^{n}(N-k+1)\right)^{\frac{1}{2n}} e^{-\frac{1}{n}\sum_{k=1}^{n}\eta_k}\\ &\geqslant \left(\binom{N}{n}n!\right)^{\frac{1}{2n}} e^{-n-1/2}\\ &\geqslant c\sqrt{N}.\end{aligned}$$

Similarly, for the positive range, we have

$$\left(\mathbb{E}\det(GG^*)^{\frac{N}{2}}\right)^{\frac{1}{N}} \simeq \sqrt{N}.$$

The result follows by Hölder's inequality. $\square$

Proposition 4.3. *Let $N \geqslant n$ and let G be an $n \times N$ random matrix with independent standard Gaussian entries. Then for any $\varepsilon \in (0,1)$,*

$$\mathbb{P}\left(\det(GG^*)^{1/(2n)} \leqslant c\varepsilon\sqrt{N}\right) \geqslant \varepsilon^{n(N-n+1)},$$

where c is an absolute constant.

Proof. Let X be an N-dimensional standard Gaussian vector. Let $m \in \{1,\dots,N\}$ and $F \in G_{N,m}$. By Chebyshev's inequality,

$$\mathbb{P}\left(|P_F X| \leqslant \sqrt{2m}\right) \geqslant \frac{1}{2}. \tag{4.8}$$

Moreover, for any $\varepsilon \in (0,1)$,

$$\mathbb{P}\left(|P_F X| \leqslant c_1\varepsilon\sqrt{m}\right) \geqslant \varepsilon^m, \tag{4.9}$$

where c_1 is an absolute constant.

As in the previous proof, let $h_1,\dots,h_n$ denote the columns of G^*; set $H_0 = \{0\}$ and $H_k = \mathrm{span}\{h_1,\dots,h_k\}$. For each $k = 1,\dots,n-1$, let $a_k = \sqrt{2(N-k+1)}$ and let $a_n = \varepsilon^n\sqrt{N-n+1}$. Using (4.7), we have

$$\mathbb{P}\left(\det(GG^*)^{\frac{1}{2n}} \leqslant c\varepsilon\sqrt{N}\right) \;\geqslant\; \mathbb{P}\left(|P_{H_{k-1}^{\perp}}h_k| \leqslant c^n a_k \text{ for each } k=1,\dots,n\right),$$

where c is an absolute constant. Applying Fubini's theorem iteratively (integrating first with respect to h_n, then h_{n-1} and so on), using (4.9) with $m = N-n+1$ and (4.8) for $m = N-k+1$ (for $k = n-1,\dots,1$) gives the desired result. $\square$

We finish this section with known bounds for the intrinsic volumes of B_1^N and B_∞^N, stated here in their Gaussian form (cf. (4.3)) as this is more convenient for our purpose. It is also a well-known result from the perspective of Gaussian random polytopes.

Proposition 4.4. *Let $N \geqslant n$ and let G be an $n \times N$ matrix with independent standard Gaussian entries. Then, for $N \leqslant e^n$, we have*

$$\left(\mathbb{E}\,\mathrm{vol}_n\left(GB_1^N\right)\right)^{\frac{1}{n}} \simeq \sqrt{\frac{\ln(2N/n)}{n}}. \tag{4.10}$$

For any $N \geqslant n$, we have

$$\left(\mathbb{E}\,\mathrm{vol}_n\left(GB_\infty^N\right)\right)^{\frac{1}{n}} \simeq \frac{N}{\sqrt{n}}. \tag{4.11}$$

The intrinsic volumes of B_1^N are computed explicitly in [12]. For B_∞^N, one has $V_n(B_\infty^N) = 2^n\binom{N}{n}$. Alternatively, taking the view of random sets generated by the Gaussian measure, the estimates in Proposition 4.4 have been proved by numerous methods. One approach for the upper bounds involves volume estimates for the convex hull and Minkowski sum of arbitrary points in $\mathbb{R}^n$. As these will be needed again in §8, we record them here.

Theorem 4.5. *Let $N \geqslant n$ and let $x_1, \ldots, x_N \in \mathbb{R}^n$ with $|x_i| \leqslant M$ for $i = 1, \ldots, N$. Then*

$$\left(\mathrm{vol}_n\left([x_1 \ldots x_N]B_1^N\right)\right)^{1/n} \leqslant \frac{cM\sqrt{\ln(2N/n)}}{n},$$

where c is an absolute constant.

The latter theorem can be proved in a number of ways, see [18], [33], [8], [9], [6]. For zonotopes, we use the following elementary lemma. Here we use $|I|$ to denote the cardinality of the set I.

Lemma 4.6. *Let $N \geqslant n$ and let $x_1, \ldots, x_N \in \mathbb{R}^n$. Then*

$$\mathrm{vol}_n\left(\sum_{i=1}^N [-x_i, x_i]\right) = 2^n \sum_{\substack{I \subset \{1,\ldots,N\} \\ |I|=n}} |\det[x_i]_{i\in I}|. \tag{4.12}$$

Moreover, if $|x_i| \leqslant M$ for each $i = 1, \ldots, N$, then

$$\mathrm{vol}_n\left([x_1 \ldots x_N]B_\infty^N\right)^{1/n} \leqslant \frac{cNM}{n},$$

where c is an absolute constant.

Remark 4.7. Analogous volume estimates for $\mathrm{vol}_n\left([x_1 \dots x_N]B_p^N\right)$, where $1 \leqslant p \leqslant \infty$, are proved in [35].

Proof. (Sketch) The first assertion (4.12) is the well-known zonotope volume formula (see, e.g., [56, page 73]). The second assertion follows from the first since

$$\mathrm{vol}_n\left([x_1 \dots x_N]B_\infty^N\right) = 2^n \sum_{|I|=n} d_I \leqslant 2^n \binom{N}{n} \max_{i \in I} d_I$$

where $d_I = |\det([x_i]_{i\in I})|$. We conclude by using the estimate $\binom{N}{n} \leqslant (eN/n)^n$ together with Hadamard's determinant inequality: $d_I \leqslant \prod_{i\in I}|x_i|$. $\square$

Thus if $g_1, \dots, g_N$ denote the columns of G in Proposition 4.4, then the upper bound for $\mathbb{E}\,\mathrm{vol}_n\left(GB_1^N\right)$ follows from Theorem 4.5 and the fact that with high probability, $|g_i| \simeq \sqrt{n}$ (cf. (4.6)). The lower bound, for $N \geqslant 2n$, follows from Gluskin's lemma [33] (see also [58], [47]) or by computing the inradius of GB_1^N as in [29] (which treats the case of vectors distributed according to λ_{D_n}); for $N = n$, one can simply estimate the determinant: $(\mathbb{E}\det([g_1 \dots g_n]))^{\frac{1}{n}} \simeq \sqrt{n}$ (e.g., take $N = n$ in Lemma 4.2). For asymptotic values as $N \to \infty$ (in the non-symmetric case), see [3]. Similarly, for $GB_\infty^N = \sum_{i=1}^N [-g_i, g_i]$ one applies (4.12) and the fact that $(\mathbb{E}\det([g_1 \dots g_n]))^{\frac{1}{n}} \simeq \sqrt{n}$.

5 Generalized intrinsic volumes

As suggested by Proposition 4.1, a reverse-Hölder inequality for the quantity $\mathbb{E}\,\mathrm{vol}_n(GC)^{-p}$, for $p > 0$, can be obtained by capturing the asymptotics of the generalized intrinsic volumes $W_{[n,-p]}(C)$ defined in §4. In this section we delve further into properties of the $W_{[n,-p]}$.

Let $C \subset \mathbb{R}^N$ be a convex body and let $1 \leqslant n \leqslant N-1$. As in §4, for every $p \in [-\infty, \infty]$ we set

$$W_{[n,p]}(C) := \left(\int_{G_{N,n}} \mathrm{vol}_n\left(P_F C\right)^p \, d\nu_{N,n}(F)\right)^{\frac{1}{np}}. \tag{5.1}$$

Note that $W_{[n]}(C) := W_{[n,1]}(C)$ is a constant multiple (depending on N and n) of the n-th intrinsic volume of C. We also set $W_{[N]}(C) := \mathrm{vol}_N(C)^{\frac{1}{N}}$. The Aleksandrov-Fenchel inequality (e.g. [67, Chapter 6]) implies that for $1 \leqslant n_1 \leqslant n_2 \leqslant N$,

$$\frac{W_{[n_2]}(C)}{W_{[n_2]}(B_2^N)} \leqslant \frac{W_{[n_1]}(C)}{W_{[n_1]}(B_2^N)}.$$

The latter inequality, together with the fact that $\mathrm{vol}_N\left(B_2^N\right)^{\frac{1}{N}} \simeq \frac{1}{\sqrt{N}}$, implies that

$$c_1\sqrt{\frac{N}{n}}\,\mathrm{vol}_N\,(C)^{\frac{1}{N}} \leqslant W_{[n]}(C) \leqslant \frac{c_2}{\sqrt{n}}W(C). \tag{5.2}$$

We now define variants of the normalized affine quermassintegrals, introduced by E. Lutwak [50]. For a convex body $C \subset \mathbb{R}^N$ of volume one, set

$$\Phi_{[n]}(C) := W_{[n,-N]}(C) := \left(\int_{G_{N,n}} \mathrm{vol}_n\,(P_F C)^{-N}\,d\nu_{N,n}(F)\right)^{-\frac{1}{nN}}. \tag{5.3}$$

The fact that $\Phi_{[n]}(C)$ is invariant under volume-preserving affine transformations was proved by E. Grinberg [36, Theorem 2] (see also [25]). It was conjectured by E. Lutwak in [51] that if $C \subset \mathbb{R}^N$ is a convex body of $\mathrm{vol}_N\,(C) = 1$, then for $1 < n < N-1$,

$$\Phi_{[n]}(C) \geqslant \Phi_{[n]}(D_N), \tag{5.4}$$

where $D_N \subset \mathbb{R}^N$ is the Euclidean ball of volume one, with equality if and only if C is an ellipsoid. Here we follow the normalization used in [21]. When $n = N-1$, inequality (5.4) is true and known as the Petty projection inequality; when $n = 1$ and the centroid of C is the origin, (5.4) is the Blaschke-Santalo inequality; see [26, Chapter 9] and the references and notes therein. In [21], it is conjectured that the quantities $\Phi_{[n]}(C)$ are asymptotically of the same order as $\Phi_{[n]}(D_N)$, i.e., if $C \subset \mathbb{R}^N$ is a convex body of $\mathrm{vol}_N\,(C) = 1$, then for $1 < n < N-1$,

$$\Phi_{[n]}(C) \simeq \sqrt{\frac{N}{n}}.$$

In [21], the upper bound is shown to be correct up to a logarithmic factor. In this section, we verify that the lower bound holds as well.

Theorem 5.1. *Let $C \subset \mathbb{R}^N$ be a convex body of volume one. Then for $1 \leqslant n \leqslant N-1$,*

$$\Phi_{[n]}(C) \geqslant c\sqrt{\frac{N}{n}},$$

where c is an absolute constant.

The proof uses a duality argument. The first ingredient is the following theorem due to Grinberg [36]; see also [27].

Theorem 5.2. *Let $K \subset \mathbb{R}^N$ be a compact set of volume 1. Then*

$$\left(\int_{G_{N,n}} \mathrm{vol}_n\,(K \cap F)^N\,d\nu_{N,n}(F)\right)^{\frac{1}{nN}} \leqslant \left(\int_{G_{N,n}} \mathrm{vol}_n\,(D_N \cap F)^N\,d\nu_{N,N}(F)\right)^{\frac{1}{nN}}.$$

We will also use the Blaschke-Santaló inequality [66].

Theorem 5.3. *Let $C \subset \mathbb{R}^N$ be a convex body with center of mass at the origin. Then*

$$(\mathrm{vol}_N(C)\,\mathrm{vol}_N(C^\circ))^{\frac{1}{N}} \leqslant \omega_N^{\frac{2}{N}}, \tag{5.5}$$

with equality if and only if C is an ellipsoid

The proof in the origin-symmetric case can be found in, e.g., [26], together with additional notes and references; we also refer to the introduction of [30] for a discussion relating the role of the center of mass and the Santalo point of C.

The reverse inequality, proved by Bourgain and Milman [15], will also be used.

Theorem 5.4. *Let $C \subset \mathbb{R}^N$ be a convex body with the origin in its interior. Then*

$$c\omega_N^{\frac{2}{N}} \leqslant (\mathrm{vol}_N(C)\,\mathrm{vol}_N(C^\circ))^{\frac{1}{N}}, \tag{5.6}$$

where c is an absolute constant.

See [43] for the best-known constant c in the latter theorem in the origin-symmetric case; for recent developments and further references, see [30].

Proof of Theorem 5.1. Without loss of generality we can assume that the center of mass of C is the origin. Let $F \in G_{N,n}$. Applying Theorem 5.4, we have

$$\mathrm{vol}_n(P_F C)^{-\frac{1}{n}} \leqslant cn\,\mathrm{vol}_n((P_F C)^\circ)^{\frac{1}{n}} = cn\,\mathrm{vol}_n(C^\circ \cap F)^{\frac{1}{n}},$$

where c is an absolute constant. Set $K = C^\circ$ and write $\widetilde{K} := K/\mathrm{vol}_N(K)^{1/N}$. Since $\mathrm{vol}_N(C) = 1$, Theorem 5.3 gives the upper bound $\mathrm{vol}_N(K)^{\frac{1}{N}} \leqslant c/N$, where c is an absolute constant, hence

$$\mathrm{vol}_n(K \cap F)^{\frac{1}{n}} = \mathrm{vol}_N(K)^{\frac{1}{N}}\,\mathrm{vol}_n\left(\widetilde{K} \cap F\right)^{\frac{1}{n}} \leqslant \frac{c}{N}\,\mathrm{vol}_n\left(\widetilde{K} \cap F\right)^{\frac{1}{n}}.$$

The latter two inequalities imply that

$$\left(\int_{G_{N,n}} \mathrm{vol}_n(P_F C)^{-N}\, d\nu_{N,n}(F)\right)^{\frac{1}{Nn}} \leqslant \frac{c_1 n}{N}\left(\int_{G_{N,n}} \mathrm{vol}_n\left(\widetilde{K} \cap F\right)^N d\nu_{N,n}(F)\right)^{\frac{1}{nN}},$$

where c_1 is an absolute constant. Now we apply Theorem 5.2 to obtain

$$\begin{aligned}\left(\int_{G_{N,n}} \mathrm{vol}_n(P_F C)^{-N}\, d\nu_{N,n}(F)\right)^{\frac{1}{Nn}} &\leqslant \frac{c_1 n}{N}\left(\int_{G_{N,n}} \mathrm{vol}_n(D_N \cap F)^N\, d\nu_{N,n}(F)\right)^{\frac{1}{nN}} \\ &\leqslant c_2\sqrt{\frac{n}{N}},\end{aligned}$$

where c_2 is an absolute constant, from which the result follows. □

Lastly, we will make use of a result from [21] (Theorem 3.2 and the subsequent remark (3.22)). For completeness, we give the proof. If $C \subset \mathbb{R}^N$ is a convex body with the origin in its interior and $p \in [-\infty, \infty]$, define its generalized mean-width by

$$W_p(C) := \left(\int_{S^{n-1}} h_C(\theta)^p d\sigma(\theta) \right)^{\frac{1}{p}}. \tag{5.7}$$

Proposition 5.5. *Let $C \subset \mathbb{R}^N$ be a convex body with the origin in its interior. Then for each $p \geqslant 1$,*

$$W_{[n,-p]}(C) \geqslant \frac{c}{\sqrt{n}} W_{-np}(C), \tag{5.8}$$

where c is an absolute constant.

Proof. Let $F \in G_{N,n}$ and write $S_F = S^{N-1} \cap F$; let σ_F denote the Haar measure on S_F. By Theorem 5.4,

$$\mathrm{vol}_n\left(P_F C\right)^{-p} \leqslant \frac{\mathrm{vol}_n\left((P_F C)^\circ\right)^p}{c^{np} \omega_n^{2p}}.$$

Using the fact that $h_{P_F C}(\theta) = h_C(\theta)$ for $\theta \in S_F$, together with Hölder's inequality, we have

$$\mathrm{vol}_n\left((P_F C)^\circ\right)^p = \left(\omega_n \int_{S_F} h_C^{-n}(\theta) d\sigma_F(\theta) \right)^p \leqslant \omega_n^p \int_{S_F} h_C^{-np}(\theta) d\sigma_F(\theta).$$

The latter two inequalities imply that

$$\begin{aligned} \left(\int_{G_{N,n}} \mathrm{vol}_n\left(P_F C\right)^{-p} d\nu_{N,n}(F) \right)^{\frac{1}{np}} &\leqslant c_1 \sqrt{n} \left(\int_{G_{N,n}} \int_{S_F} h_C^{-np}(\theta) d\sigma_F(\theta) d\nu_{N,n}(F) \right)^{\frac{1}{np}} \\ &= c_1 \sqrt{n} \left(\int_{S^{N-1}} h_C^{-np}(\theta) d\sigma(\theta) \right)^{\frac{1}{np}} \\ &= c_1 \sqrt{n} W_{-np}^{-1}(C), \end{aligned}$$

where c_1 is an absolute constant.

□

We refer the reader to [21] for further information on the quantities $W_{[n,p]}$.

5.1 Connection to small-ball estimates for the Gaussian case

For a convex body $C \subset \mathbb{R}^N$, a positive integer $n \leqslant N$ and $p \in [-1, \infty]$, we define

$$A_{n,p}(C) := \frac{W_{[n,1]}(C)}{W_{[n,-p]}(C)} = \frac{\left(\int_{G_{N,n}} \mathrm{vol}_n\left(P_F C\right) d\nu_{N,n}(F)\right)^{\frac{1}{n}}}{\left(\int_{G_{N,n}} \mathrm{vol}_n\left(P_F C\right)^{-p} d\nu_{N,n}(F)\right)^{-\frac{1}{pn}}}. \tag{5.9}$$

By Hölder's inequality, $A_{n,p}(C) \geqslant 1$ and

$$[-1, \infty) \ni p \mapsto A_{n,p}(C)$$

is an increasing function. The significance of $A_{n,p}(C)$ for estimating small-ball probabilities for the Gaussian measure is captured in the next proposition.

Proposition 5.6. *Let $N \geqslant n$ and let G be an $n \times N$ random matrix with independent standard Gaussian entries. Let $C \subset \mathbb{R}^N$ be a convex body and $p \in [0, N-n+1-e^{-n(N-n+1)}]$. Then*

$$\left(\mathbb{E}_{\otimes\gamma_n} \mathrm{vol}_n\left(GC\right)^{-p}\right)^{-\frac{1}{pn}} \geqslant \frac{\left(\mathbb{E}_{\otimes\gamma_n} \mathrm{vol}_n\left(GC\right)\right)^{\frac{1}{n}}}{c_0 A_{n,p}(C)}, \tag{5.10}$$

where c_0 is an absolute constant. Consequently, for each $\varepsilon \in (0,1)$,

$$\mathbb{P}_{\otimes\gamma_n}\left(\mathrm{vol}_n\left(GC\right)^{1/n} \leqslant \frac{\varepsilon}{c A_{n,p}(C)}\left(\mathbb{E}\, \mathrm{vol}_n\left(GC\right)\right)^{1/n}\right) \leqslant \varepsilon^{pn}, \tag{5.11}$$

where c is an absolute constant.

Proof. By Proposition 4.1 and Lemma 4.2, we get that for $p \in [-1, N-n+1)$,

$$A_{n,p}(C) \simeq \frac{\left(\mathbb{E}\, \mathrm{vol}_n\left(GC\right)\right)^{\frac{1}{n}}}{\left(\mathbb{E}\, \mathrm{vol}_n\left(GC\right)^{-p}\right)^{-\frac{1}{pn}}}, \tag{5.12}$$

which implies (5.10). Using the latter equivalence and Markov's inequality, for any $\eta > 0$, we have

$$\begin{aligned} &\mathbb{P}\left(\mathrm{vol}_n\left(GC\right)^{1/n} \leqslant \frac{\eta}{A_{n,p}(C)}\left(\mathbb{E}\, \mathrm{vol}_n\left(GC\right)\right)^{\frac{1}{n}}\right) \\ &\qquad \leqslant \mathbb{P}\left(\mathrm{vol}_n\left(GC\right)^{1/n} \leqslant c\eta\left(\mathbb{E}\, \mathrm{vol}_n\left(GC\right)^{-p}\right)^{-\frac{1}{pn}}\right) \\ &\qquad \leqslant (c\eta)^{pn}, \end{aligned}$$

where c is an absolute constant. The small-ball estimate (5.11) follows on substituting $\varepsilon = c\eta$. ∎

6 Bounds for generalized intrinsic volumes of B_1^N and B_∞^N

By Proposition 5.6 in the previous section, we can obtain small-ball estimates in the Gaussian case by bounding the quantities $A_{n,p}(B_1^N)$ and $A_{n,p}(B_\infty^N)$. We will invoke Proposition 5.5, which relates $W_{[n,-p]}(C)$ and the generalized mean-width $W_{-p}(C)$ (defined in (5.7)) and thus we start by estimating $W_{-p}(B_1^N)$.

Proposition 6.1. *Let $1 \leqslant p \leqslant N$. Then*

$$W_{-p}(B_1^N) \simeq \frac{\sqrt{\ln \frac{2N}{p}}}{\sqrt{N}}. \tag{6.1}$$

Proof. Using integration in spherical coordinates, one may verify that

$$W_{-p}(C) \simeq \frac{1}{\sqrt{N}} \left(\int_{\mathbb{R}^N} h_C^{-p}(x) d\gamma_N(x) \right)^{-\frac{1}{p}}$$

for all $0 < p \leqslant \frac{N}{2}$. Note that for all $r > 0$,

$$\gamma_N \left(\{x : h_{B_1^N}(x) \leqslant r\} \right) = \gamma_N \left(r[-1,1]^N \right) = (1 - 2\Phi(r))^N,$$

where

$$\Phi(r) := \frac{1}{\sqrt{2\pi}} \int_r^\infty e^{-x^2/2} dx.$$

Assume first that $p \leqslant c_1 N$ for some absolute constant $c_1 \in (0,1)$ to be specified later. Write

$$\begin{aligned} \int_{\mathbb{R}^N} h_{B_1^N}^{-p}(x) d\gamma_N(x) &= p \int_0^\infty \frac{(1-2\Phi(s))^N}{s^{p+1}} ds \\ &= p \int_0^1 \frac{(1-2\Phi(s))^N}{s^{p+1}} ds + p \int_1^\infty \frac{(1-2\Phi(s))^N}{s^{p+1}} ds. \end{aligned}$$

Using the inequality $1 - 2\Phi(r) \leqslant \sqrt{\frac{2}{\pi}} r$ for $r \in [0,1]$, we choose $c_1 \in (0,1)$ to ensure that

$$p \int_0^1 \frac{(1-2\Phi(s))^N}{s^{p+1}} ds \leqslant p \left(\frac{2}{\pi} \right)^{\frac{N}{2}} \int_0^1 s^{N-p-1} ds \leqslant \left(\frac{2}{\pi} \right)^{\frac{N}{2}}.$$

For the remainder of the integral, we use the rough estimate

$$p \int_1^\infty \frac{(1-2\Phi(s))^N}{s^{p+1}} ds \leqslant p \int_1^\infty \frac{(1-2e^{-8s^2})^N}{s^{p+1}} ds.$$

A routine calculation shows that the integrand $g(s) := \frac{(1-2e^{-8s^2})^N}{s^{p+1}}$ is increasing on $(1, s_0)$ where $s_0 := \frac{1}{3}\sqrt{\ln(2N/p)}$. Thus

$$p\int_1^{s_0} g(s)dsds \leqslant p(s_0-1)g(s_0) \leqslant \frac{p}{s_0^p}$$

and

$$p\int_{s_0}^{\infty} \frac{1}{s^{p+1}}ds = \frac{1}{s_0^p}.$$

Combining each of the estimates yields

$$p\int_0^{\infty} \frac{(1-2\Phi(s))^N}{s^{p+1}}ds \leqslant \frac{p+2}{s_0^p}. \tag{6.2}$$

The reverse inequality is proved similarly.

Lastly, we treat the case $c_1 N \leqslant p \leqslant N$. Note that

$$W_{-N}(B_1^N) = \left(\frac{\mathrm{vol}_N\left(B_2^N\right)}{\mathrm{vol}_N\left(B_\infty^N\right)}\right)^{\frac{1}{N}} \simeq \frac{1}{\sqrt{N}},$$

hence Hölder's inequality yields

$$\frac{1}{\sqrt{N}} \simeq W_{-c_1 N}(B_1^N) \geqslant W_{-p}(B_1^N) \geqslant W_{-N}(B_1^N) \simeq \frac{1}{\sqrt{N}}.$$

□

Proposition 6.2. *Let $N \geqslant n$ and let $\delta \geqslant 1$. Then for $1 \leqslant p \leqslant \left(\frac{N}{n}\right)^{1-\frac{1}{\delta^2}}$, we have*

$$A_{n,p}(B_1^N) \leqslant c'\delta.$$

Moreover, for $N \leqslant ne^{\delta^2}$,

$$A_{n,N}(B_1^N) \leqslant c''\delta,$$

where c' and c'' are absolute constants.

Proof. Set $p_0 = \left(\frac{N}{n}\right)^{1-\frac{1}{\delta^2}}$. By Proposition 5.5 and Hölder's inequality, for $p \leqslant p_0$, we have

$$W_{[n,p]}(B_1^N) \geqslant \frac{c}{\sqrt{n}}W_{-np}(B_1^N) \geqslant \frac{c}{\sqrt{n}}W_{-np_0}(B_1^N).$$

By Proposition 6.1, the latter quantity is at least as large as

$$\frac{c'\sqrt{\ln\left(2N/(np_0)\right)}}{\sqrt{nN}} = \frac{c'}{\delta}\frac{\sqrt{\ln\left(2N/n\right)}}{\sqrt{nN}}.$$

Moreover, by Proposition 4.1, Lemma 4.2 and Proposition 4.4, we have

$$W_{[n,1]}(B_1^N) \simeq \frac{\sqrt{\ln{(2N/n)}}}{\sqrt{nN}}. \tag{6.3}$$

Combining the latter two estimates, we have

$$A_{n,p}(B_1^N) = \frac{W_{[n,1]}(B_1^N)}{W_{[n,-p]}(B_1^N)} \leqslant c\delta.$$

Finally, for any $p \leqslant N$, Hölder's inequality and Theorem 5.1 imply that

$$W_{[n,-p]}(B_1^N) \geqslant W_{[n,-N]}(B_1^N) = W_{[n,-N]}(\widetilde{B_1^N})\,\mathrm{vol}_n\left(B_1^N\right)^{\frac{1}{N}} \geqslant \frac{c}{\sqrt{nN}}, \tag{6.4}$$

where $\widetilde{B_1^N}$ is the volume-one homothet of B_1^N. Thus by (6.3), (6.4) and the definition of $A_{n,p}$ we get that

$$A_{n,p}(B_1^N) \leqslant c\sqrt{\ln{(2N/n)}} \leqslant c''\delta,$$

provided that $N \leqslant ne^{\delta^2}$. □

Proposition 6.3. *Let* $n \leqslant N$ *and let* $0 < p \leqslant N$*. Then*

$$A_{n,p}(B_\infty^N) \leqslant c_0,$$

where c_0 *is an absolute constant.*

Proof. Since $W(B_\infty^N) \leqslant \mathrm{diam}(B_\infty^N) = 2\sqrt{N}$, (5.2) yields

$$W_{[n,1]}(B_\infty^N) = W_{[n]}(B_\infty^N) \leqslant \frac{c_2}{\sqrt{n}} W(B_\infty^N) \leqslant 2c_2\sqrt{\frac{N}{n}}.$$

By Theorem 5.1, we have

$$W_{[n,-N]}(B_\infty^N) = 2\Phi_{[n]}((1/2)B_\infty^N) \geqslant c_1\sqrt{\frac{N}{n}},$$

where c_1 is an absolute constant. Since $W_{[n,-p]}(B_\infty^N) \geqslant W_{[n,-N]}(B_\infty^N)$ whenever $0 < p \leqslant N$, we obtain

$$A_{n,p}(B_\infty^N) = \frac{W_{[n,1]}(B_\infty^N)}{W_{[n,-p]}(B_\infty^N)} \leqslant \frac{2c_2}{c_1}.$$

□

Remark 6.4. The proof of Proposition 6.3 shows that if $C \subset \mathbb{R}^N$ is a convex body with $\mathrm{vol}_N(C) = 1$ and $W(C) \leqslant c\sqrt{N}$, then for any $0 < p \leqslant N$ we have $A_{n,p}(C) \leqslant c'$, where c' is a constant that depends only on c. Any zonoid in Lowner's position satisfies this property (see [60]). In particular, there is an absolute constant c_1 such that $A_{n,p}(B_q^N) \leqslant c_1$ whenever $0 < p \leqslant N$ and $2 \leqslant q \leqslant \infty$. Note that by Urysohn's inequality (see, e.g., [64, Corollary 1.4]), the inequality $W(C) \geqslant c\sqrt{N}$ holds for any convex body C satisfying $\mathrm{vol}_N(C) = 1$.

6.1 Small-ball estimates in the Gaussian case

The results of the previous subsection lead to the following small-ball estimates.

Proposition 6.5. *Let $n \leqslant N \leqslant e^n$ and let $\varepsilon \in (0,1)$ and $\delta > 1$. Then*

$$\mathbb{P}_{\otimes\gamma_n}\left(\left\{\mathrm{vol}_n(K_N)^{\frac{1}{n}} \leqslant \frac{\varepsilon}{c_1\delta}\mathbb{E}_{\otimes\gamma_n}\mathrm{vol}_n(K_N)^{\frac{1}{n}}\right\}\right) \leqslant \varepsilon^{N^{1-1/\delta^2}n^{1/\delta^2}},$$

where c_1 is an absolute constant. Moreover, if $N \leqslant ne^{\delta^2}$, then

$$\mathbb{P}_{\otimes\gamma_n}\left(\left\{\mathrm{vol}_n(K_N)^{\frac{1}{n}} \leqslant \frac{\varepsilon}{c_2\delta}\mathbb{E}_{\otimes\gamma_n}\mathrm{vol}_n(K_N)^{\frac{1}{n}}\right\}\right) \leqslant \varepsilon^{n(N-n+1-o(1))},$$

where c_2 is an absolute constant.

Proof. Let $p_0 = \left(\frac{N}{n}\right)^{1-1/\delta^2}$. Then $p_0 \leqslant N - n + 1$, hence Propositions 5.6 and 6.2 imply that

$$\begin{aligned}\mathbb{P}_{\otimes\gamma_n}&\left(\left\{\mathrm{vol}_n(K_N)^{\frac{1}{n}} \leqslant \frac{\varepsilon}{c_1\delta}\mathbb{E}_{\otimes\gamma_n}\mathrm{vol}_n(K_N)^{\frac{1}{n}}\right\}\right)\\ &\leqslant \mathbb{P}_{\otimes\gamma_n}\left(\left\{\mathrm{vol}_n(K_N)^{\frac{1}{n}} \leqslant \frac{\varepsilon}{cA_{n,p_0}(B_1^N)}\left(\mathbb{E}_{\otimes\gamma_n}\mathrm{vol}_n(K_N)\right)^{\frac{1}{n}}\right\}\right)\\ &\leqslant \varepsilon^{np_0},\end{aligned}$$

where c_1 is an absolute constant. If $N \leqslant ne^{\delta^2}$, we take $p_1 = N-n+1-e^{-n(N-n+1)}$ and argue as above. □

For zonotopes generated by the Gaussian measure we have the following.

Proposition 6.6. *Let $N \geqslant n$ and let $\varepsilon \in (0,1)$. Then*

$$\mathbb{P}_{\otimes\gamma_n}\left(\left\{\mathrm{vol}_n(Z_N)^{\frac{1}{n}} \leqslant \frac{\varepsilon}{c_1}\mathbb{E}_{\otimes\gamma_n}\mathrm{vol}_n(Z_N)^{\frac{1}{n}}\right\}\right) \leqslant \varepsilon^{n(N-n+1-o(1))},$$

where c_1 is an absolute constant.

Proof. Use Propositions 5.6 and 6.3 and argue as in the proof of the previous proposition. □

We conclude this section with a complementary lower bound that shows Proposition 6.6 is essentially optimal.

Proposition 6.7. *Let $N \geqslant n$ and let $\varepsilon \in (0,1)$. Then*

$$\mathbb{P}_{\otimes\gamma_n}\left(\left\{\mathrm{vol}_n\left(Z_N\right)^{\frac{1}{n}} \leqslant \frac{\varepsilon}{c_2}\mathbb{E}_{\otimes\gamma_n}\mathrm{vol}_n\left(Z_N\right)^{\frac{1}{n}}\right\}\right) \geqslant \varepsilon^{n(N-n+1)},$$

where the c_2 is an absolute constant.

Proof. Let G be an $n \times N$ matrix with independent standard Gaussian entries. Then $Z_N = GB_\infty^N \subset \sqrt{N}GB_2^N$, hence

$$\mathrm{vol}_n\left(Z_N\right) \leqslant N^{n/2}\det(GG^*)^{1/2}\omega_n.$$

Using the latter inequality and Proposition 4.4, we have

$$\mathbb{P}_{\otimes\gamma_n}\left(\left\{\mathrm{vol}_n\left(Z_N\right)^{\frac{1}{n}} \leqslant \frac{\varepsilon}{c_2}\mathbb{E}_{\otimes\gamma_n}\mathrm{vol}_n\left(Z_N\right)^{\frac{1}{n}}\right\}\right) \geqslant \mathbb{P}_{\otimes\gamma_n}\left(\left\{\det(GG^*)^{\frac{1}{2n}} \leqslant \frac{\varepsilon}{c_2}\sqrt{N}\right\}\right),$$

where c_2 is an absolute constant. The result follows from Proposition 4.3. □

7 From the Gaussian measure to the ball

With estimates for Gaussian-measure in hand, we proceed to transfer them to the uniform measure on the Euclidean ball. Let $\overline{\gamma}_n$ be the Gaussian measure on $\mathbb{R}^n$ with density $d\overline{\gamma}_n(x) = e^{-\pi|x|^2}dx$; in particular, $\overline{\gamma}_n$ belongs to the class $\mathcal{P}_n^b$.

The main goal of this section is to establish the following proposition.

Proposition 7.1. *Let $n < N \leqslant e^n$ and set $m = N/2 + (n-1)/2$. Then for any $p \in (0, (N-n+1)/4)$, we have*

$$\left(\mathbb{E}_{\otimes_{i=1}^m\overline{\gamma}_n}\mathrm{vol}_n\left(K_m\right)^{-p}\right)^{-\frac{1}{pn}} \leqslant c\left(\mathbb{E}_{\otimes_{i=1}^N\lambda_{D_n}}\mathrm{vol}_n\left(K_N\right)^{-p}\right)^{-\frac{1}{pn}} \tag{7.1}$$

and

$$\left(\mathbb{E}_{\otimes_{i=1}^m\overline{\gamma}_n}\mathrm{vol}_n\left(Z_m\right)^{-p}\right)^{-\frac{1}{pn}} \leqslant c\left(\mathbb{E}_{\otimes_{i=1}^N\lambda_{D_n}}\mathrm{vol}_n\left(Z_N\right)^{-p}\right)^{-\frac{1}{pn}}, \tag{7.2}$$

where c is an absolute constant.

For the case when $N = n$, see Remark 7.3. For simplicity, we assume throughout that $m = N/2 + (n-1)/2$ is an integer; simple modifications will yield the result for all m.

As in the previous sections, we will prove a more general statement. Let $C \subset \mathbb{R}^N$ be a 1-symmetric convex body. For convenience of notation, we write $\overline{x} = (x_1, \dots, x_N) \in (\mathbb{R}^n)^N$ and set

$$F(\overline{x}) := F(x_1, \cdots, x_N) := \mathrm{vol}_n\left([x_1 \cdots x_N]C\right). \tag{7.3}$$

The main properties of F used here are the following:

i. F is coordinate-wise increasing: for fixed $x_1, \dots, x_N \in \mathbb{R}^n$ and for $0 < s_i \leqslant t_i$, $i \leqslant N$, we have

$$F(s_1 x_1, \cdots, s_N x_N) \leqslant F(t_1 x_1, \cdots, t_N x_N); \tag{7.4}$$

see the proof of Theorem 3.1.

ii. F is n-homogeneous, i.e., $F(a\overline{x}) = a^n F(\overline{x})$ for $a > 0$;

iii. F is invariant under permutation of its coordinates, i.e., $F(x_1, \dots, x_N) = F(x_{\xi(1)}, \dots, x_{\xi(N)})$ for any permutation $\xi : \{1, \dots, N\} \to \{1, \dots, N\}$.

Proposition 7.2. *Let $F : (\mathbb{R}^n)^N \to \mathbb{R}^+$ be defined by (7.3). Let $n < N \leqslant e^n$ and set $m = N/2 + (n-1)/2$. If $p \in (0, (N-n+1)/4)$, then*

$$\left(\mathbb{E}_{\otimes_{i=1}^m \overline{\gamma}_n} F(X_1, \dots, X_m, 0, \dots, 0)^{-p}\right)^{-\frac{1}{pn}} \leqslant c \left(\mathbb{E}_{\otimes_{i=1}^N \lambda_{D_n}} F(X_1, \dots, X_N)^{-p}\right)^{-\frac{1}{pn}}, \tag{7.5}$$

where c is an absolute constant.

The complementary inequality

$$\left(\mathbb{E}_{\otimes_{i=1}^N \overline{\gamma}_n} F(X_1, \dots, X_N)^{-p}\right)^{-\frac{1}{pn}} \geqslant \left(\mathbb{E}_{\otimes_{i=1}^N \lambda_{D_n}} F(X_1, \dots, X_N)^{-p}\right)^{-\frac{1}{pn}}$$

follows from Theorem 3.1.

To prove the theorem, we will express the expectations in (7.5) in spherical coordinates and compare them with the corresponding expectations on the N-fold product of spheres $S_n^N := S^{n-1} \times \ldots \times S^{n-1}$, equipped with the product of the Haar measures σ, denoted here by $\mathbb{P}_{\otimes_{i=1}^N \sigma}$. Before doing so, we discuss the case $N = n$.

Remark 7.3. If $N = n$, then $F(x_1, \dots, x_n) = |\det([x_1 \dots x_N])| \,\mathrm{vol}_n(C)$. In this case, if $X_1, \dots, X_N$ are independent and distributed according $\overline{\gamma}_n$, then one can write $X_i = |X_i|\theta_i$, where $\theta_i = X_i/|X_i|$ is uniformly distributed on the sphere and is independent of $|X_i|$. Thus for any $p \in (0,1)$,

$$\mathbb{E}_{\otimes_{i=1}^n \overline{\gamma}_n} F(X_1, \dots, X_n)^{-p} = \mathbb{E}_{\otimes_{i=1}^n \overline{\gamma}_n} |X_1|^{-p} \cdots |X_n|^{-p} \mathbb{E}_{\otimes_{i=1}^n \sigma} F(\theta_1, \dots, \theta_n)^{-p}.$$

Similarly, if the X_i's are sampled according to λ_{D_n}, we have

$$\mathbb{E}_{\otimes_{i=1}^n \lambda_{D_n}} F(X_1, \dots, X_n)^{-p} = \mathbb{E}_{\otimes_{i=1}^n \lambda_{D_n}} |X_1|^{-p} \cdots |X_n|^{-p} \mathbb{E}_{\otimes_{i=1}^n \sigma} F(\theta_1, \dots, \theta_n)^{-p}.$$

Proof of Proposition 7.2: Assume first that $X_1, \dots, X_N$ are independent random vectors distributed according to $\overline{\gamma}_n$ and write $\overline{X} = (X_1, \dots, X_N)$. Then for each $t_0 > 0$, we have

$$\begin{aligned}
&\mathbb{E}_{\otimes_{i=1}^N \overline{\gamma}_n} F(\overline{X})^{-p} \\
&\quad \geqslant \int_{t_0 B_2^n} \cdots \int_{t_0 B_2^n} F^{-p}(x_1, \cdots, x_N) d\overline{\gamma}_n(x_N) \cdots d\overline{\gamma}_n(x_1) \\
&\quad = (n\omega_n)^N \int_{[0,t_0]^N} \int_{S_n^N} F^{-p}(r_1\theta_1, \cdots, r_N\theta_N) \prod_{i=1}^N r_i^{n-1} e^{-\pi r_i^2} d\sigma_n^N(\overline{\theta}) d\overline{r} \\
&\quad \geqslant (n\omega_n)^N \int_{[0,t_0]^N} \int_{S_n^N} F^{-p}(t_0\theta_1, \cdots, t_0\theta_N) \prod_{i=1}^N r_i^{n-1} e^{-\pi r_i^2} d\sigma_n^N(\overline{\theta}) d\overline{r} \\
&\quad = t_0^{-pn} (n\omega_n)^N \int_{[0,t_0]^N} \prod_{i=1}^N r_i^{n-1} e^{-\pi r_i^2} d\overline{r} \int_{S_n^N} F^{-p}(\theta_1, \cdots, \theta_N) d\sigma_n^N(\overline{\theta}) \\
&\quad = t_0^{-pn} \overline{\gamma}_n(t_0 B_2^n)^N \mathbb{E}_{\otimes_{i=1}^N \sigma} F(\overline{\theta})^{-p}, \qquad (7.6)
\end{aligned}$$

where $\overline{\theta} = (\theta_1, \dots, \theta_N)$ is distributed according to $\mathbb{P}_{\otimes_{i=1}^N \sigma}$.

At this point, we choose t_0 such that $\overline{\gamma}_n(t_0 B_2^n) = 1 - e^{-n}$; one can check that $t_0 \simeq \sqrt{n}$. Then, for $N \leqslant e^n$, we have

$$1 \geqslant (\overline{\gamma}_n(t_0 B_2^n))^N = (1 - e^{-n})^N \geqslant \frac{1}{e}. \qquad (7.7)$$

Combining (7.6) and (7.7) yields

$$\left(\mathbb{E}_{\otimes_{i=1}^N \overline{\gamma}_n} F^{-p}(\overline{X})\right)^{-\frac{1}{pn}} \leqslant c\sqrt{n} \left(\mathbb{E}_{\otimes_{i=1}^N \sigma} F^{-p}(\overline{\theta})\right)^{-\frac{1}{pn}}, \qquad (7.8)$$

where $c > 0$ is an absolute constant.

Assume now that $X_1, X_2, \dots, X_N$ are independent random vectors distributed uniformly in D_n and write $\overline{X} = (X_1, \dots, X_N)$. Note that for each $i = 1, \dots, N$, we can write $X_i = |X_i|\theta_i$, where $|X_i|$ is the Euclidean norm of X_i, and $\theta_i = X_i/|X_i|$ is distributed uniformly on the sphere S^{n-1} and is independent of $|X_i|$.

Let s_0 be such that $\mathbb{P}_{\lambda_{D_n}}(|X_1| \geq s_0) = 1 - e^{-n}$ and note that $s_0 \simeq \sqrt{n}$. Since $N \leqslant e^n$,

$$\mathbb{P}_{\otimes_{i=1}^N \lambda_{D_n}}(|X_i| \geqslant s_0 \text{ for each } i = 1, \dots, N) = (1 - e^{-n})^N \geqslant \frac{1}{e}.$$

Denote the decreasing rearrangement of the sequence $(|X_i|)$ by $(|X_i|^*)$. Then

$$\mathbb{E}_{\otimes_{i=1}^N \lambda_{D_n}} |X_N|^* = \mathbb{E}_{\otimes_{i=1}^N \lambda_{D_n}} \min_{i \leqslant N} |X_i| \geqslant s_0/e. \tag{7.9}$$

Since F is invariant under permutations, we have

$$F(X_1 \dots X_N) = F(\theta_1 |X_1|^*, \dots, \theta_N |X_N|^*).$$

We partition the sequence $(|X_i|^*)$ into three blocks as follows:

$$\underbrace{|X_1|^*, \dots,}_{n-1} \; \underbrace{|X_n|^* \dots,}_{(N-n+1)/2} \; \underbrace{\dots, |X_N|^*}_{(N-n+1)/2}.$$

Taking $m = n - 1 + (N - n + 1)/2 = N/2 + (n - 1)/2$ and using monotonicity and homogeneity of F, we have

$$F(X_1 \dots X_N) \geqslant (|X_m|^*)^n F(\theta_1, \dots, \theta_m, 0, \dots, 0). \tag{7.10}$$

Since $N - m = (N - n + 1)/2$, we have

$$\begin{aligned} \mathbb{P}_{\otimes_{i=1}^N \lambda_{D_n}}\left(\{|X_m|^* \leqslant c\varepsilon\sqrt{n}\}\right) &\leqslant \sum_{|I|=(N-n+1)/2} \mathbb{P}_{\otimes_{i=1}^N \lambda_{D_n}}\left(\bigcap_{i \in I} \{|X_i| \leqslant c\varepsilon\sqrt{n}\}\right) \\ &\leqslant \binom{N}{(N-n+1)/2} \mathbb{P}_{\lambda_{D_n}}\left(|X_1| \leqslant c\varepsilon\sqrt{n}\right)^{(N-n+1)/2} \\ &\leqslant \left(\frac{2eN}{N-n+1}\right)^{(N-n+1)/2} \varepsilon^{n(N-n+1)/2}. \end{aligned}$$

By the distribution formula for non-negative random variables, we obtain

$$\left(\mathbb{E}_{\otimes_{i=1}^N \lambda_{D_n}} |X_m|^*)^{-pn}\right)^{-\frac{1}{pn}} \geqslant c_0 \mathbb{E}_{\otimes_{i=1}^N \lambda_{D_n}} |X_m|^* \tag{7.11}$$

for all $0 \leqslant p \leqslant (N - n + 1)/4$, where c_0 is an absolute constant. By (7.9) we have

$$\mathbb{E}_{\otimes_{i=1}^N \lambda_{D_n}} |X_m|^* \geqslant \mathbb{E}_{\otimes_{i=1}^N \lambda_{D_n}} |X_N|^* \geqslant c_1 \sqrt{n},$$

where c_1 is an absolute constant. Taking powers and then expectations in (7.10) and applying (7.11), we get that for $0 < p < (N - n + 1)/4$,

$$\left(\mathbb{E}_{\otimes_{i=1}^N \lambda_{D_n}} F(\overline{X})^{-p}\right)^{-\frac{1}{pn}} \geqslant c_1 \sqrt{n} \left(\mathbb{E}_{\otimes_{i=1}^m \sigma} F(\overline{\theta})^{-p}\right)^{-\frac{1}{pn}},$$

where $\overline{\theta} = (\theta_1, \ldots, \theta_m, 0, \ldots, 0)$ and $\theta_1, \ldots, \theta_m$ are independent and uniformly distributed on the sphere S^{n-1}. The proposition now follows by applying (7.8) (with N replaced by m). □

Remark 7.4. 1.) The assumption $N \leqslant e^n$ in Proposition 7.1 is essential for K_N since after this point, $\mathbb{E}_{\otimes\overline{\gamma}_n} \mathrm{vol}_n(K_N)$ is much larger than $\mathbb{E}_{\otimes\lambda_{D_n}} \mathrm{vol}_n(K_N)$.
2.) We do not believe the constant 4 in Proposition 7.1 is necessary; perhaps the optimal constant is $1 + o(1)$. Any improvement here will lead to better constants in the exponents of the small-ball estimates in Theorems 1.1 - 1.3.

8 Proof of the main theorems and further remarks

We are now ready to prove the two main results of this paper.

Theorem 8.1. *Let $n \leqslant N \leqslant e^n$ and let $\mu_1, \ldots, \mu_N \in \mathcal{P}_n^b$. Let $\delta > 1$ and let $\varepsilon \in (0,1)$. Then*

$$\mathbb{P}_{\otimes\mu_i}\left(\left\{\mathrm{vol}_n(K_N)^{1/n} \leqslant \frac{c\varepsilon}{\delta}\sqrt{\frac{\ln(2N/n)}{n}}\right\}\right) \leqslant \varepsilon^{c_1 N^{1-1/\delta^2} n^{1/\delta^2}} \tag{8.1}$$

and, if $N \leqslant ne^{\delta^2}$, then

$$\mathbb{P}_{\otimes\mu_i}\left(\left\{\mathrm{vol}_n(K_N)^{1/n} \leqslant \frac{c\varepsilon}{\delta}\sqrt{\frac{\ln(2N/n)}{n}}\right\}\right) \leqslant \varepsilon^{n(N-n+1-o(1))/4}. \tag{8.2}$$

Proof. Let $m = N/2 + (n-1)/2$ and let $p_0 = \left(\frac{m}{n}\right)^{1-1/\delta^2}$. By (5.12) and Proposition 6.2,

$$\frac{\left(\mathbb{E}_{\otimes_{i=1}^m \gamma_n} \mathrm{vol}_n(K_m)\right)^{\frac{1}{n}}}{\left(\mathbb{E}_{\otimes_{i=1}^m \gamma_n} \mathrm{vol}_n(K_m)^{-p_0}\right)^{-\frac{1}{p_0 n}}} \simeq A_{n,p_0}(B_1^m) \leqslant c'\delta, \tag{8.3}$$

where c' is an absolute constant. Since $p_0 \leqslant (N-n+1)/4$, by Proposition 7.2 and (8.3), we have

$$\begin{aligned}
\left(\mathbb{E}_{\otimes_{i=1}^N \lambda_{D_n}} \mathrm{vol}_n(K_N)^{-p_0}\right)^{-\frac{1}{p_0 n}} &\geqslant c_0\left(\mathbb{E}_{\otimes_{i=1}^m \overline{\gamma}_n} \mathrm{vol}_n(K_m)^{-p_0}\right)^{-\frac{1}{p_0 n}} \\
&\geqslant c_1\left(\mathbb{E}_{\otimes_{i=1}^m \gamma_n} \mathrm{vol}_n(K_m)^{-p_0}\right)^{-\frac{1}{p_0 n}} \\
&\geqslant \frac{c_2}{\delta}\left(\mathbb{E}_{\otimes_{i=1}^m \gamma_n} \mathrm{vol}_n(K_m)\right)^{1/n} \\
&\geqslant \frac{c_3}{\delta}\sqrt{\frac{\ln(2N/n)}{n}}.
\end{aligned}$$

By Markov's inequality, we obtain

$$\mathbb{P}_{\otimes_{i=1}^{N}\lambda_{D_n}}\left(\left\{\mathrm{vol}_n\left(K_N\right)^{1/n}\leqslant\frac{c\varepsilon}{\delta}\sqrt{\frac{\ln(2N/n)}{n}}\right\}\right)\leqslant\varepsilon^{c_1N^{1-1/\delta^2}n^{1/\delta^2}}.$$

Lastly, apply Theorem 3.1. The proof of (8.2) follows the same argument. □

Theorem 8.2. *Let $n\leqslant N\leqslant e^n$ and let $\mu_1,\ldots,\mu_N\in\mathcal{P}_n^b$. Then for each $\varepsilon\in(0,1)$,*

$$\mathbb{P}_{\otimes\mu_i}\left(\left\{\mathrm{vol}_n\left(Z_N\right)^{1/n}\leqslant\frac{c\varepsilon N}{\sqrt{n}}\right\}\right)\leqslant\varepsilon^{n(N-n+1-o(1))/4}.\tag{8.4}$$

Proof. Argue as in the proof of the previous theorem and apply Proposition 6.3 instead of Proposition 6.2. □

Remark 8.3. Note that when $N=2n$ the estimate in (8.2) is much stronger than the estimate in (8.1), which suggests that a better exponent can be achieved in general. As we will see in the next subsection, the estimates in (8.2) and (8.4) are sharp up to the absolute constants in the theorem.

8.1 Complementary small-ball estimates

In this section we give lower bounds for the probabilities in Theorems 8.1 and 8.2. We make use of known bounds for the volume of the convex hull and zonotope generated by arbitrary points in $\mathbb{R}^n$ (which we stated in §4).

Let $\mu\in\mathcal{P}_n^b$ and assume that $f_\mu(0)=\|f_\mu\|_\infty=1$. Suppose there exists $\varepsilon_0=\varepsilon_0(\mu)$ such that

$$\mathbb{P}_\mu\left(|X|\leq c\varepsilon\sqrt{n}\right)\geqslant\varepsilon^n\ \text{ whenever }\ \varepsilon\leqslant\varepsilon_0,\tag{8.5}$$

where c is an absolute constant. For instance, if f_μ is continuous at 0 then there exists $\varepsilon_0=\varepsilon_0(\mu)$ such that $|f_\mu(x)|\geqslant 1/2$ whenever $|x|\leq\varepsilon_0 c\sqrt{n}$, hence (8.5) holds (with c replaced by $2^{1/n}c$). Given $\varepsilon_0(\mu)$, we can apply Theorem 4.5 to obtain, for each $\varepsilon\leqslant\varepsilon_0(\mu)$,

$$\begin{aligned}\mathbb{P}_{\otimes\mu}\left(\mathrm{vol}_n\left(K_N\right)^{1/n}\leqslant c\varepsilon\sqrt{\frac{\ln(2N/n)}{n}}\right)&\geqslant\mathbb{P}_{\otimes\mu}\left(|X_i|\leqslant c\varepsilon\sqrt{n}\text{ for }i=1,\ldots,N\right)\\&=\mathbb{P}_\mu\left(|X_1|\leqslant c\varepsilon\sqrt{n}\right)^N\\&\geqslant\varepsilon^{nN}.\end{aligned}$$

Similarly, for Z_N we apply Lemma 4.6: for any $\varepsilon \leqslant \varepsilon_0(\mu)$,

$$
\begin{aligned}
\mathbb{P}_{\otimes\mu}\left(\mathrm{vol}_n\left(Z_N\right)^{1/n} \leqslant \frac{c\varepsilon N}{\sqrt{n}}\right) &\geqslant \mathbb{P}_{\otimes\mu}\left(|X_i| \leqslant c\varepsilon\sqrt{n} \text{ for } i=1,\dots,N\right)\\
&= \mathbb{P}_{\mu}\left(|X_1| \leqslant c\varepsilon\sqrt{n}\right)^N\\
&\geqslant \varepsilon^{nN}.
\end{aligned}
$$

Thus even though $\varepsilon_0(\mu)$ depends on μ and $\inf\{\varepsilon_0(\mu) : \mu \in \mathcal{P}_n^b\} = 0$, the asymptotic behavior of the small-ball estimates for K_N and Z_N as $\varepsilon \to 0$ is at least ε^{nN}. In some classes of measures, one can control the value of $\varepsilon_0(\mu)$; in particular, for the class of isotropic log-concave probability measures (see §9.1).

9 Isotropicity and log-concavity

In most cases in the literature about similar results, the measures under consideration are assumed to be isotropic rather than the normalization used in this paper. However, one can easily deduce results for isotropic measures from our main theorems.

Let $\mathcal{P}_n^{\mathrm{cov}}$ denote the set of measures $\mu \in \mathcal{P}_n$ with bounded densities such that the covariance matrix of μ is well-defined. We say that a probability measure $\mu \in \mathcal{P}_n^{\mathrm{cov}}$ is isotropic if its covariance matrix is the identity. When μ is isotropic, we define its isotropic constant L_μ by

$$L_\mu := \|f_\mu\|_\infty^{1/n},$$

where f_μ is the density of μ. Given any measure $\mu \in \mathcal{P}_n^{\mathrm{cov}}$ with barycenter at the origin, one can find a linear map $T : \mathbb{R}^n \to \mathbb{R}^n$ (unique modulo orthogonal transformations) of determinant one such that $\mu \circ T^{-1}$ is an isotropic probability measure; in this way, the isotropic constant is uniquely defined for all $\mu \in \mathcal{P}_n^{\mathrm{cov}}$.

Let $a > 0$ and $\mu \in \mathcal{P}_n^{\mathrm{cov}}$ with density f_μ. We define a new probability measure μ_a on $\mathbb{R}^n$ as the measure that has density $f_{\mu_a}(x) = a^n f_\mu(ax)$. One can check that

$$\|f_{\mu_a}\|_\infty = a^n \|f_\mu\|_\infty .$$

Moreover, if $F : (\mathbb{R}^n)^N \to \mathbb{R}^+$, is p-homogeneous, then

$$\mathbb{E}_{\otimes\mu} F(X_1,\dots,X_N) = a^p \mathbb{E}_{\otimes\mu_a} F(X_1,\dots,X_N).$$

Thus if $\mu \in \mathcal{P}_n^{\mathrm{cov}}$ is isotropic then $\mu' := \mu_{\frac{1}{L_\mu}}$ satisfies $\|f_{\mu'}\|_\infty = 1$,

$$\left(\mathbb{E}_{\otimes\mu}\,\mathrm{vol}_n\left([X_1\dots X_N]C\right)\right)^{\frac{1}{n}} = \frac{1}{L_\mu}\left(\mathbb{E}_{\otimes\mu'}\,\mathrm{vol}_n\left([X_1\dots X_N]C\right)\right)^{\frac{1}{n}}.$$

and

$$\frac{\left(\mathbb{E}_{\otimes\mu}\operatorname{vol}_n\left([X_1\dots X_N]C\right)\right)^{\frac{1}{n}}}{\left(\mathbb{E}_{\otimes\mu}\operatorname{vol}_n\left([X_1\dots X_N]C\right)^p\right)^{\frac{1}{pn}}}=\frac{\left(\mathbb{E}_{\otimes\mu'}\operatorname{vol}_n\left([X_1\dots X_N]C\right)\right)^{\frac{1}{n}}}{\left(\mathbb{E}_{\otimes\mu'}\operatorname{vol}_n\left([X_1\dots X_N]C\right)^p\right)^{\frac{1}{pn}}}.$$

By a change of variables, note that for any $S\in SL(n)$, we have

$$\mathbb{E}_{\otimes\mu\circ S^{-1}}\operatorname{vol}_n\left([X_1\dots X_N]C\right)^p=\mathbb{E}_{\otimes\mu}\operatorname{vol}_n\left([X_1\dots X_N]C\right)^p$$

for any p for which the expressions are defined. Thus there is no loss in generality in assuming that μ is isotropic.

Following the proof of our main theorem we obtain a corresponding result for isotropic probability measures.

Theorem 9.1. *Let $n\leqslant N\leqslant e^n$. Let $\mu\in\mathcal{P}_n^{\mathrm{cov}}$ and assume that μ is isotropic. Then for every $\varepsilon\in(0,1)$,*

$$\mathbb{P}_{\otimes\mu}\left(\left\{\operatorname{vol}_n\left(K_N\right)^{1/n}\leqslant\frac{c\varepsilon}{\delta L_\mu}\sqrt{\frac{\ln(2N/n)}{n}}\right\}\right)\leqslant\varepsilon^{c_1N^{1-1/\delta^2}n^{1/\delta^2}}$$

and, if $N\leqslant ne^{\delta^2}$, then

$$\mathbb{P}_{\otimes\mu}\left(\left\{\operatorname{vol}_n\left(K_N\right)^{1/n}\leqslant\frac{c\varepsilon}{\delta L_\mu}\sqrt{\frac{\ln(2N/n)}{n}}\right\}\right)\leqslant\varepsilon^{n(N-n+1-o(1))/4},$$

where c and c_1 are absolute constants. Similarly, for each $\varepsilon\in(0,1)$,

$$\mathbb{P}_{\otimes\mu}\left(\left\{\operatorname{vol}_n\left(Z_N\right)^{\frac{1}{n}}\leqslant\frac{c_2\varepsilon}{L_\mu}\frac{N}{\sqrt{n}}\right\}\right)\leqslant\varepsilon^{n(N-n+1-o(1))/4},$$

where c_2 is an absolute constant.

One can check that $\inf\{L_\mu:\mu\in\mathcal{P}_n^{\mathrm{cov}}\}\geqslant L_{\lambda_{D_n}}\simeq 1$. On the other hand, L_μ does not admit a uniform upper bound as μ varies in $\mathcal{P}_n^{\mathrm{cov}}$. However, in the important class of log-concave probability measures $\mathcal{LP}_n$ it has been conjectured that

$$\sup\{L_\mu:n\in\mathbb{N},\mu\in\mathcal{LP}_n\}\leqslant c, \tag{9.1}$$

where $c>0$ is an absolute constant. This is known to be equivalent to a famous open problem in convex geometry, namely, the Hyperplane Conjecture. We refer to [56] for an introductory survey and to [39], [22], [41] for the best known results. In many large subclasses of $\mathcal{LP}_n$, it has been verified that L_μ admits a uniform upper bound, independent of the dimension; see, e.g., the references given in [59]. Henceforth, we say that $\mu\in\mathcal{LP}_n$ has bounded isotropic constant if $L_\mu\leqslant c$, where c is an absolute constant (independent of μ and n).

It is known that if μ is an isotropic log-concave probability measure on $\mathbb{R}^n$ with bounded isotropic constant and $n \leqslant N \leqslant e^n$, then

$$(\mathbb{E}_{\otimes\mu} \operatorname{vol}_n (K_N))^{\frac{1}{n}} \simeq \frac{\sqrt{\ln(2N/n)}}{\sqrt{n}} \simeq \left(\mathbb{E}_{\otimes\lambda_{D_n}} \operatorname{vol}_n (K_N)\right)^{\frac{1}{n}};$$

see [20]. In this case, we obtain the following result.

Theorem 9.2. *Let $n \leqslant N \leqslant e^n$ and let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$ with bounded isotropic constant. Then for every $\varepsilon \in (0,1)$,*

$$\mathbb{P}_{\otimes\mu}\left(\operatorname{vol}_n (K_N)^{\frac{1}{n}} \leqslant \frac{c\varepsilon}{\delta} (\mathbb{E}_{\otimes\mu} \operatorname{vol}_n (K_N))^{\frac{1}{n}}\right) \leqslant \varepsilon^{c_1 N^{1-1/\delta^2} n^{1/\delta^2}} \tag{9.2}$$

and, if $N \leqslant ne^{\delta^2}$, then

$$\mathbb{P}_{\otimes\mu}\left(\operatorname{vol}_n (K_N)^{\frac{1}{n}} \leqslant \frac{c\varepsilon}{\delta} (\mathbb{E}_{\otimes\mu} \operatorname{vol}_n (K_N))^{\frac{1}{n}}\right) \leqslant \varepsilon^{n(N-n+1-o(1))/4}, \tag{9.3}$$

where c and c_1 are absolute constants.

A similar theorem is true for random zonotopes. If μ is an isotropic log-concave probability measure on $\mathbb{R}^n$, then

$$\mathbb{E}_{\otimes\mu} \operatorname{vol}_n (Z_N)^{1/n} \simeq \frac{N}{\sqrt{n}}; \tag{9.4}$$

the latter equivalence is proved in [65, Proposition 6 & Remark 3]. For the reader's convenience we sketch the proof. Note that for any subspace $E \subset \mathbb{R}^n$, the isotropicity of μ implies that

$$\mathbb{E}_\mu |P_E X_1|^2 = \dim(E).$$

Thus

$$\mathbb{E}_{\otimes\mu} |\det[X_1 \dots X_n]|^2 = n!$$

(apply (2.5) and use Fubini's theorem, integrating with respect to X_n, then X_{n-1}, and so on). For $I \subset \{1, \dots, N\}$, write $d_I := |\det([X_i]_{i\in I})|$ and apply the zonotope volume formula (4.12) and Jensen's inequality:

$$\mathbb{E}_{\otimes\mu}\left(\sum_{|I|=n} d_I\right)^{1/n} \leqslant \binom{N}{n}^{1/n} (\mathbb{E}_{\otimes\mu} d_{I_0})^{1/n} \leqslant \frac{eN}{n}(n!)^{1/(2n)} \leqslant \frac{cN}{\sqrt{n}},$$

where c is a positive absolute constant and $I_0 = \{1, \dots, n\}$. For the lower bound, we use concavity of $x \mapsto x^{1/n}$ in (4.12):

$$\mathbb{E}_{\otimes\mu}\left(\sum_{|I|=n} d_I\right)^{1/n} \geqslant \binom{N}{n}^{1/n-1} \sum_{|I|=n} \mathbb{E}_{\otimes\mu} d_I^{1/n} \geqslant \frac{N}{n} \mathbb{E}_{\otimes\mu} d_I^{1/n}.$$

One completes the proof of (9.4) by using the fact that $\mathbb{E}_{\otimes\mu}|\det([X_1\dots X_n])|^{1/n} \simeq \sqrt{n}$ (see [65, Corollary 1]).

Theorem 9.1 and the equivalence in (9.4) leads to the following.

Theorem 9.3. *Let $n \leqslant N \leqslant e^n$ and let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$ with bounded isotropic constant. Then, for every $\varepsilon \in (0,1)$,*

$$\mathbb{P}_{\otimes\mu}\left(\mathrm{vol}_n\left(Z_N\right)^{\frac{1}{n}} \leqslant c\varepsilon\left(\mathbb{E}_{\otimes\mu}\,\mathrm{vol}_n\left(Z_N\right)\right)^{\frac{1}{n}}\right) \leqslant \varepsilon^{n(N-n+1-o(1))/4}. \tag{9.5}$$

where c is an absolute constant.

9.1 Complementary lower bounds

In this section we prove that the small-ball probabilities in (9.3) (for $N = 2n$) and (9.5) (for $N \geqslant 2n$) are essentially sharp (up to the absolute constants involved).

Proposition 9.4. *Let $n \leqslant N \leqslant e^n$ and let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$. Then for every $\varepsilon \in (0,1)$,*

$$\mathbb{P}_{\otimes\mu}\left(\mathrm{vol}_n\left(K_N\right)^{\frac{1}{n}} \leqslant c_1\varepsilon\sqrt{\frac{\ln(2N/n)}{n}}\right) \geqslant \varepsilon^{Nn},$$

and

$$\mathbb{P}_{\otimes\mu}\left(\mathrm{vol}_n\left(Z_N\right)^{\frac{1}{n}} \leqslant \frac{c_2\varepsilon N}{\sqrt{n}}\right) \geqslant \varepsilon^{Nn},$$

where c_1, c_2 are absolute constants.

Note that a sharper bound for the Gaussian measure γ_n was given in Proposition 6.7. The proof is analogous to the general case, which we gave in §8.1. All that remains is to show the following proposition.

Proposition 9.5. *Let μ be an isotropic log-concave probability measure on $\mathbb{R}^n$. Let X be a random vector distributed according to μ. Then for every $\varepsilon \in (0,1)$,*

$$\mathbb{P}_{\mu}\left(|X| \leqslant c_1\varepsilon\sqrt{n}\right) \geqslant \varepsilon^n,$$

where $c_1 > 0$ is an absolute constant.

Proposition 9.5 shows that the quantity $\varepsilon_0(\mu)$, defined in (8.5), satisfies $\varepsilon_0(\mu) = 1$ for any isotropic log-concave probability measure μ on $\mathbb{R}^n$. By the argument given in §8.1, the small-ball estimates in (9.3) and (9.5) are essentially sharp.

The first step in the proof of Proposition 9.5 involves covering numbers. Recall that if C and D are convex bodies in $\mathbb{R}^N$, the covering number of C with respect to D is the minimum number $N(C,D)$ of translates of D whose union covers C, i.e.,

$$N(C,D) := \inf\left\{M : \exists x_1,\dots,x_M \in \mathbb{R}^N\ C \subset \bigcup_{i=1}^{M}(D+x_i)\right\}. \tag{9.6}$$

For further information on covering numbers see, e.g., [64].

The second ingredient is the following technical lemma about log-concave functions, which is essentially shown in [40, Lemmas 4.4, 5.2].

Lemma 9.6. *Let $f:\mathbb{R}^+\to\mathbb{R}^+$ be a C^2 log-concave function with $\int_0^\infty f(t)dt<\infty$. Suppose that $\|f\|_\infty \leqslant e^n f(0)$. Then for $n\geqslant 2$ and any $b>0$,*

$$\int_0^b t^{n-1}f(t)dt \geqslant c^n \min\left\{\int_0^\infty t^{n-1}f(t)dt, f(0)b^n\right\} \tag{9.7}$$

Proof. For convenience, let $g(t)=t^{n-1}f(t)$ and $h:=\int_0^\infty g(t)dt$. Let t_n be the (unique) positive real such that $g'(t_n)=0$. Let $\varepsilon\in(0,1)$ and $a\geqslant 5$. Set $t_0:=\sup\{s>0: f(s)\geqslant e^{-an}f(0)\}$. It is shown in [40, Lemmas 4.4, 5.2] that

$$\int_{t_n(1-\varepsilon)}^{t_n(1+\varepsilon)} t^{n-1}f(t)dt \geqslant \left(1-c_1e^{-c\varepsilon^2 n}\right)\int_0^\infty t^{n-1}f(t)dt \tag{9.8}$$

where $c_1>1$ and $0<c<1$ are absolute constants, and

$$\int_0^{t_0} t^{n-1}f(t)dt \geqslant \left(1-e^{-an/8}\right)\int_0^\infty t^{n-1}f(t)dt. \tag{9.9}$$

Taking $\varepsilon=1/2$ in (9.8) and using the definition of t_n, we have

$$h\leqslant c_2\int_{t_n/2}^{\frac{3t_n}{2}} t^{n-1}f(t)dt \leqslant c_2t_n^n f(t_n)\leqslant c_2t_n^n\|f\|_\infty \leqslant c_3^nt_n^nf(0), \tag{9.10}$$

where c_2 and c_3 are absolute constants. Taking $a=5$ in (9.9) and using (9.8), we have $t_0\geqslant t_n/2$, which means that for $s\leqslant t_n/2$,

$$f(s)\geqslant e^{-5n}f(0). \tag{9.11}$$

Applying (9.8) once more, together with (9.11), we have

$$hc_1e^{-\frac{c}{4}n}\geqslant \int_0^{t_n/2} t^{n-1}f(t)dt \geqslant e^{-5n}f(0)\int_0^{t_n/2}t^{n-1}dt=\frac{e^{-5n}}{n2^n}f(0)t_n^n,$$

which implies $h\geqslant c_1^nf(0)t_n^n$. Finally, if $0<b\leqslant t_n/2$, then (9.11) yields

$$\int_0^b t^{n-1}f(t)dt\geqslant \frac{e^{-5n}}{n}f(0)b^n.$$

On the other hand, if $b \geqslant t_n/2$, we apply (9.10) to get

$$\int_0^b t^{n-1} f(t)dt \geqslant \int_0^{t_n/2} t^{n-1} f(t)dt \geqslant \frac{e^{-5n}}{n} f(0)t_n^n \geqslant c^n h, \tag{9.12}$$

from which the result follows. □

Proof of Proposition 9.5. Let K be an isotropic convex body with isotropic constant L_K (cf. (2.1)). We will first show that for every $\varepsilon \in (0,1)$,

$$\mathrm{vol}_n\,(K \cap \varepsilon L_K D_n) \geqslant (c\varepsilon)^n, \tag{9.13}$$

where $c > 0$ is an absolute constant. By [48, Lemma 4], the covering number $N(K, L_K D_n)$ satisfies

$$N(K, L_K D_n) \leqslant e^{c_0 n},$$

where $c_0 > 0$ is an absolute constant. Standard estimates give

$$\begin{aligned} N(K, \varepsilon L_K D_n) &\leqslant \frac{4^n \,\mathrm{vol}_n\,(K + \varepsilon L_K D_n)}{\varepsilon^n \,\mathrm{vol}_n\,(L_K D_n)} \leqslant \frac{4^n \,\mathrm{vol}_n\,(K + L_K D_n)}{\varepsilon^n \,\mathrm{vol}_n\,(L_K D_n)} \\ &\leqslant \left(\frac{c_1}{\varepsilon}\right)^n N(K, L_K D_n) \leqslant \left(\frac{c_2}{\varepsilon}\right)^n. \end{aligned}$$

By the Brunn-Minkowski inequality and [24, Theorem 4], if $C_1, C_2 \subset \mathbb{R}^N$ are convex bodies such that the center of mass of C_1 is the origin, $\mathrm{vol}_n\,(C_1) = 1$, and C_2 is origin-symmetric, then

$$1 \leqslant \max_{x \in \mathbb{R}^n} \mathrm{vol}_n\,(C_1 \cap (x + C_2))\, N(C_1, C_2) \leqslant e^n \,\mathrm{vol}_n\,(C_1 \cap C_2)\, N(C_1, C_2).$$

Thus

$$\mathrm{vol}_n\,(K \cap \varepsilon L_K D_n) \geqslant \frac{e^{-n}}{N(K, \varepsilon L_K D_n)} \geqslant (c\varepsilon)^n,$$

which establishes (9.13).

Without loss of generality we may assume that the density f of μ is C^2. Let $b_n := \omega_n^{-\frac{1}{n}}$ and set

$$\rho_K(\theta) := \left(\frac{n}{f(0)} \int_0^\infty t^{n-1} f(t\theta)dt\right)^{\frac{1}{n}}$$

By [5], ρ_K is the radial function of a convex body K. It is known that $\mathrm{vol}_n\,(K)^{\frac{1}{n}} = f(0)^{-\frac{1}{n}}$, $L_K \simeq f(0)^{1/n}$ and there exists $T \in SL(n)$ satisfying $|Tx| \simeq |x|$ for all $x \in S^{n-1}$ such that TK is an isotropic convex body (see, e.g., [61, Propositions 3.3, 3.5]). Thus if $\widetilde{K}$ is the volume-one homothet of K, we have

$$\rho_{\widetilde{K}}(\theta) = n^{\frac{1}{n}} \left(\int_0^\infty t^{n-1} f(t)dt\right)^{\frac{1}{n}}.$$

Note that

$$\rho_{\widetilde{K}\cap\varepsilon f(0)^{\frac{1}{n}}D_n}(\theta) = \min\left\{ n^{\frac{1}{n}}\left(\int_0^\infty t^{n-1}f(t)dt\right)^{\frac{1}{n}}, \varepsilon f(0)^{\frac{1}{n}}b_n \right\}. \tag{9.14}$$

Since μ is isotropic, [24, Theorem 4] gives $\|f\|_\infty \leqslant e^n f(0)$. Using Lemma 9.6 and (9.14) we have

$$\begin{aligned}
\mu(\varepsilon D_n) &= n\omega_n \int_{S^{n-1}}\int_0^{\varepsilon b_n} t^{n-1}f(t\theta)dtd\sigma(\theta) \\
&\geqslant c^n\omega_n \int_{S^{n-1}} \min\{\rho^n_{\widetilde{K}}(\theta), \varepsilon^n f(0)b_n^n\}d\sigma(\theta) \\
&= c^n\omega_n \int_{S^{n-1}} \rho^n_{\widetilde{K}\cap\varepsilon f(0)^{\frac{1}{n}}D_n}(\theta)d\sigma(\theta) \\
&= c^n \operatorname{vol}_n\left(\widetilde{K}\cap\varepsilon f(0)^{\frac{1}{n}}D_n\right) \\
&\geqslant c^n \operatorname{vol}_n\left(\widetilde{K}\cap\varepsilon c' L_K D_n\right).
\end{aligned}$$

By adjusting the constants and applying Lemma 9.6 and (9.13) for $\widetilde{K}$, we conclude the proof. □

10 Bounds for a general convex body C

A large part of this paper has involved general random convex sets $[X_1 \dots X_N]C$ and we have emphasized the small-ball probabilities for $C = B_1^N$ and $C = B_\infty^N$ only. The approach of applying a random linear operator $[X_1 \dots X_N]$ to a general convex body C has led to several applications [63, §4,5], [62] and we feel it is of interest to outline how to obtain small-ball probabilities for $\operatorname{vol}_n([X_1 \dots X_N]C)$ in the general case.

If $C \subset \mathbb{R}^N$ is nearly degenerate, one cannot expect to control the small-ball probability

$$\mathbb{P}_{\otimes\mu_i}\left(\operatorname{vol}_n([X_1 \dots X_N]C)^{1/n} \leqslant \varepsilon\right).$$

To ensure that C is not degenerate, we make assumptions about its "position." By a position of a convex body, we mean a linear image, chosen to satisfy certain conditions. As Proposition 5.6 indicates, a key part of the proof is to bound the quantity $A_{n,p}(C)$. As we did for B_1^N and B_∞^N, we will give nearly optimal estimates when N is proportional to n, assuming that C is in a suitable position. We will also provide non-trivial estimates in the general case.

10.1 M-position and the proportional case

Our first method for bounding $A_{n,p}(C)$ is applicable when N is proportional to n and depends on a deep result due to V. Milman [55]; see also [64, Chapter 7]. V. Milman proved that given any convex body C, one can find a suitable position such that the covering number of C by a ball of the same volume is of minimal possible order. As in §9.1, we use $N(C, D)$ to denote the covering number of C with respect to D (cf. (9.6)). Using the above notation, V. Milman's theorem reads as follows.

Theorem 10.1. *There exists a constant $\beta > 0$ such that for any convex body $C \subset \mathbb{R}^N$ there exists a linear operator $T : \mathbb{R}^N \to \mathbb{R}^N$ such that $\mathrm{vol}_N(TC) = 1$ and*

$$N(TC, D_N) \leqslant e^{\beta N}. \tag{10.1}$$

We say that C is in M-position if T is the identity operator. Note that any 1-symmetric convex body of volume one is in M-position. We refer to [64] for further information about M-position.

The following proposition is a well-known property of bodies in M-position; the proof is included for completeness.

Proposition 10.2. *Let $C \subset \mathbb{R}^N$ be an origin-symmetric convex body in M-position with constant $\beta > 0$. Let $\lambda \in (0, 1)$ and set $n = \lambda N$. Then*

$$\frac{\max_{F \in G_{N,n}} \mathrm{vol}_n (P_F C)^{\frac{1}{n}}}{\min_{F \in G_{N,n}} \mathrm{vol}_n (P_F C)^{\frac{1}{n}}} \leqslant c(\lambda, \beta), \tag{10.2}$$

where $c(\lambda, \beta) > 0$ depends only on λ, β.

Proof. Let $F \in G_{N,n}$. Then

$$\frac{\mathrm{vol}_n (P_F C)}{\mathrm{vol}_n (P_F D_N)} \leqslant N(P_F C, P_F D_N) \leqslant N(C, D_N) \leqslant e^{\beta N},$$

hence

$$\mathrm{vol}_n (P_F C)^{\frac{1}{n}} \leqslant e^{\beta \frac{N}{k}} \mathrm{vol}_n (P_F D_N)^{\frac{1}{n}}. \tag{10.3}$$

Since $\mathrm{vol}_{N-n}\left(C \cap F^{\perp}\right) \mathrm{vol}_n (P_F C) \geqslant 1$, we have

$$\mathrm{vol}_{N-n}\left(C \cap F^{\perp}\right) \geqslant \frac{1}{\mathrm{vol}_n (P_F C)} \geqslant \frac{1}{e^{\beta N} \mathrm{vol}_n (P_F D_N)}.$$

Thus for every $1 \leqslant \ell < N$ and $E \in G_{N,N-\ell}$ we obtain

$$\mathrm{vol}_{N-\ell}(P_E C) \geqslant \mathrm{vol}_{N-\ell}(C \cap E) \geqslant e^{-\beta N} \frac{1}{\mathrm{vol}_\ell (P_{E^\perp} D_N)}.$$

Applying the latter inequality for $\ell := N - n$ and $E \in G_{N,n}$ yields

$$\begin{aligned} \mathrm{vol}_n\left(P_E C\right)^{\frac{1}{n}} &\geqslant e^{-\beta\frac{N}{n}} \frac{1}{\mathrm{vol}_{N-n}\left(P_{E^\perp} D_N\right)^{\frac{1}{n}}} \\ &= e^{-\beta\frac{N}{n}} \frac{1}{\mathrm{vol}_N\left(D_N \cap E^\perp\right)^{\frac{1}{n}}} \\ &\geqslant ce^{-\frac{\beta N}{n}}. \end{aligned} \tag{10.4}$$

By (10.3), (10.4) and the fact that $\mathrm{vol}_n\left(P_F D_N\right)^{\frac{1}{n}} \simeq \sqrt{N/n}$, we conclude that

$$\frac{\max_{F\in G_{N,n}} \mathrm{vol}_n\left(P_F C\right)^{\frac{1}{n}}}{\min_{F\in G_{N,n}} \mathrm{vol}_n\left(P_F C\right)^{\frac{1}{n}}} \leqslant ce^{2\frac{\beta N}{n}} \sqrt{\frac{N}{n}}.$$

This yields (10.2) with $c(\lambda, \beta) := \frac{c}{\sqrt{\lambda}} e^{\frac{2\beta}{\lambda}}$. □

Proposition 10.3. *Let $C \subset \mathbb{R}^N$ be an origin-symmetric convex body in M-position with constant β. Let $\lambda \in (0,1)$ and let $n = \lambda N$. Then for all $p \in [1, \infty]$,*

$$A_{n,p}(C) \leqslant c_1 e^{3\beta/\lambda}. \tag{10.5}$$

Proof. Recalling the definition of $A_{n,p}(C)$ (cf. (5.9)), we have

$$A_{n,p}(C) \leqslant \frac{\max_{F\in G_{N,n}} \mathrm{vol}_n\left(P_F C\right)^{\frac{1}{n}}}{\min_{F\in G_{N,n}} \mathrm{vol}_n\left(P_F C\right)^{\frac{1}{n}}}.$$

Applying Proposition 10.2 gives the result. □

By applying Proposition 5.6, one obtains small-ball estimates for $\mathrm{vol}_n(GC)$ when N is proportional to n and C is in M-position. Proceeding to the case of arbitrary measures $\mu_i \in \mathcal{P}_n^b$ then depends the comparison in Proposition 7.2 (where we have assumed C is 1-symmetric) and the proof follows that of Theorem 8.1. It is not difficult to show that any 1-symmetric convex body of volume one is in M-position.

10.2 Small-ball estimates for norms: implications for generalized intrinsic volumes

Our second method for bounding $A_{n,p}(C)$ involves Proposition 5.5 and therefore depends on lower bounds for generalized mean-widths $W_{-p}(C)$; this, in turn, depends on small-ball estimates for norms. The study of small-ball probabilities for norms was initiated in [42] and [45] and shown to have close connections to V. Milman's proof of Dvoretzky's theorem on nearly-Euclidean sections of convex bodies. We will give bounds for $A_{n,p}(C)$ in terms of the Dvoretzky dimension of C (defined below). Actually, one can replace the Dvoretzky dimension by a potentially larger quantity. For this we will make use of a theorem from [42], which we state below in terms of support functions (dual to the setting there).

If $C \subset \mathbb{R}^N$ is a convex body, the Dvoretzky dimension $k_*(C)$ is defined by

$$k_*(C) = N\left(\frac{W(C)}{\mathrm{diam}(C)}\right)^2,$$

where $\mathrm{diam}(C)$ is the diameter of C and $W(C)$ is the mean-width of C. The parameter $k_*(C)$ is the dimension up to which "most" projections of C are nearly Euclidean; more precisely, for $n \leqslant k_*(C)$ the $\nu_{N,n}$-measure of subspaces $E \in G_{N,n}$ satisfying

$$c_1 W(C) P_E B_2^N \subset P_E C \subset c_2 W(C) P_E B_2^N \tag{10.6}$$

for some absolute constants c_1 and c_2 is at least $1 - e^{-n}$; see [57] or [64] for further background information.

It has been observed that if one requires only the left-hand inclusion of (10.6), then the dimension at which this holds can increase dramatically. The critical dimension depends on the following quantity, introduced in [42],

$$d_*(C) := \min\{-\ln \sigma\{\theta \in S^{N-1} : h_C(\theta) \leqslant W(C)/2\}, N\}.$$

One has $d_*(C) \geqslant c_1 k_*(C)$ for some absolute constant $c_1 > 0$, see [42].

Theorem 10.4 ([42])**.** *Let C be an origin-symmetric convex body in $\mathbb{R}^N$. Assume that $0 < p \leqslant d_*(C)$. Then*

$$c_1 W(C) \leqslant W_{-p}(C) \leqslant c_2 W(C)$$

where c, c_1, c_2 are positive absolute constants.

When C is in a suitable position, for instance when C° is in John's position (see, e.g., [57, Chapter 3]), we have $k_*(C) \geqslant c \ln N$, where c is an absolute constant.

Proposition 10.5. *Let $C \subset \mathbb{R}^N$ be an origin-symmetric convex body. If $np \leqslant k_*(C) \leqslant d_*(C)$, then*

$$A_{n,p}(C) \leqslant c, \tag{10.7}$$

where c is an absolute constant. In particular, if C is a convex body such that C° is in John's position and $0 \leqslant p \leqslant \frac{\ln N}{n}$, then (10.7) holds.

Proof. By (5.2), we have

$$W_{[n,1]}(C) \leqslant \frac{c_2}{\sqrt{n}} W(C).$$

On the other hand, Proposition 5.5 gives

$$W_{[n,-p]}(C) \geqslant \frac{c_1}{\sqrt{n}} W_{-np}(C).$$

Thus

$$A_{n,p}(C) = \frac{W_{[n,1]}(C)}{W_{[n,-p]}(C)} \leqslant \frac{c_2 W(C)}{c_1 W_{-np}(C)}$$

Applying Theorem 10.4 yields $A_{n,p}(C) \leqslant c$. □

Remark 10.6. It is shown in [42] that $d_*(B_1^N)$ is much larger than $k_*(B_1^N)$. In fact, the calculation in [42, Remark 2 on page 204] led us to consider Proposition 6.1 and our proof is based on similar estimates.

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