# From intersection bodies to dual centroid bodies: a stochastic approach to isoperimetry 

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#### Abstract

We establish a family of isoperimetric inequalities for sets that interpolate between intersection bodies and dual $L_{p}$ centroid bodies. This provides a bridge between the Busemann intersection inequality and the Lutwak-Zhang inequality. The approach depends on new empirical versions of these inequalities.


## 1 Introduction

The focus of this paper is on connections between fundamental inequalities in Brunn-Minkowski theory and dual Brunn-Minkowski theory. The former details the behavior of the volume of Minkowski sums of convex bodies. The standard isoperimetric inequality is emblematic of deep principles within Alexandrov's theory of mixed volumes [83. A central line of research is on affine-invariant strengthenings of kindred isoperimetric principles, especially around projections of convex sets; as a sample, see Lutwak's survey [58], Schneider's monograph [83], the fundamental papers [60, 64, 65], and [71] for a recent breakthrough. In dual Brunn-Minkowski theory, the emphasis is on star-shaped sets and radial addition. Dual mixed volumes, put forth by Lutwak [53, parallel many aspects of mixed volumes. They provide a rich framework for studying intersections of star bodies with subspaces; for example, see [54, [55] for foundational results; the monographs by Koldobsky 47 and Gardner [24] for the resolution of the Busemann-Petty problem and interplay with geometric tomography; the papers [38, 39, 7, 8, for striking new developments. Establishing an important family of isoperimetric inequalities linking the two theories has remained a principle challenge.

A common root for the inequalities we treat is the Busemann intersection inequality [14 for the volume of central slices of a compact set $K \subseteq \mathbb{R}^{n}$ :

$$
\begin{equation*}
\int_{S^{n-1}}\left|K \cap u^{\perp}\right|^{n} d u \leq \frac{\omega_{n-1}^{n}}{\omega_{n}^{n-1}}|K|^{n-1}, \tag{1.1}
\end{equation*}
$$

where $d u$ denotes integration with respect to the normalized Haar probability measure on the sphere $S^{n-1},|\cdot|$ is volume and $\omega_{n}$ is the volume of the Euclidean unit ball $B_{2}^{n}$. The result

[^0]itself (with hindsight) is an invariant inequality for the volume of the intersection body $I(K)$ of $K$, which is defined by its radial function via $\rho(I(K), u)=\left|K \cap u^{\perp}\right|$ (see $\S 3$ for notation and definitions). Intersection bodies were introduced by Lutwak [55] in connection with the Busemann-Petty problem and play a crucial role in dual Brunn-Minkowski theory [24, 47]. The proof of (1.1) used an essential ingredient known as the Busemann random simplex inequality, which says that the expected volume of certain random simplices in a convex body are minimal for ellipsoids. Petty used the latter to establish a conjecture of Blaschke on the volume of centroid bodies [79, which is now known as the Busemann-Petty centroid inequality. Geometrically, given an origin-symmetric convex body $K$ in $\mathbb{R}^{n}$, the centroids of halves of $K$ cut by hyperplanes through the origin form the surface of its centroid body. Centroid bodies are zonoids i.e., Hausdorff limits of Minkowski sums of segments, and thus naturally belong to Brunn-Minkowski theory. Zonoids play an important role in functional analysis and related fields, e.g., [4, 84, 9, 72.

Lutwak raised the question of connecting the Busemann intersection inequality and the Busemann-Petty centroid inequality in 58. The latter is one of several fundamental results that lead to strengthenings of the standard isoperimetric inequality; in particular, it is equivalent to an inequality of Petty [80] on polar projection bodies, as shown in [58]. Projection bodies are also zonoids and play a central role in Brunn-Minkowski theory [24].

A functional analytic perspective has shaped the development of both intersection bodies and polar projection bodies. Early work in the isometric theory of Banach spaces, going back to Lévy, introduced stable laws in connection with embeddings in $L_{p}$ for $p \in(0,2]$. Positive definite distributions, stable laws and associated change of density arguments play a central role [3, 85, 74, 68. Koldobsky developed a parallel theory, based on a Fourier-analytic approach, for embedding in $L_{p}$, for $p<0$. This led to fundamental characterizations of intersection bodies and their higher-dimensional analogues 44, 46, 47]. With this view, intersection bodies are unit "balls" of finite-dimensional subspaces of $L_{-1}$. At the other end, polar projection bodies arise naturally as unit balls of subspaces of $L_{1}[4]$. In between $L_{-1}$ and $L_{1}$ is a continuum of spaces that are no longer Banach spaces. A result of Koldobsky shows that the classes in between decrease as $p$ varies from -1 to 1 ; in particular, every polar projection body is an intersection body [45, 47]. A longstanding question of Kwapień from 1970 [50], in geometric form, asks if every intersection body is isomorphic to a polar projection body, which remains unsolved; see work of Kalton and Koldobsky 40 for progress on this question.

A rich theory of isoperimetric inequalities has flourished around centroid bodies and polar projection bodies. Two fundamental papers in this development are those of Lutwak-Zhang [66] and Lutwak-Yang-Zhang [60]. For a star-shaped body $K$ and $1 \leq p \leq \infty$, the $L_{p}$ centroid body $Z_{p}(K)$ is defined by its support function (see $\$ 3$ ) via

$$
\begin{equation*}
h^{p}\left(Z_{p}(K), u\right)=\frac{1}{|K|} \int_{K}|\langle x, u\rangle|^{p} d x \tag{1.2}
\end{equation*}
$$

Lutwak and Zhang proved that for $1 \leq p \leq \infty$,

$$
\begin{equation*}
\left|Z_{p}^{\circ}(K)\right| \leq\left|Z_{p}^{\circ}\left(K^{*}\right)\right| \tag{1.3}
\end{equation*}
$$

here $K^{*}$ is the dilate of the unit ball centered at the origin of the same volume as $K$. When $p=\infty$, (1.3) is the Blaschke-Santaló inequality, which is equivalent to the affine isoperimetric inequality [58. When $p=1$, (1.3) follows from the Busemann-Petty centroid inequality. Lutwak, Yang, and Zhang [60] later proved a stronger inequality for $Z_{p}(K)$ itself. These are central results within the framework of $L_{p}$-Brunn-Minkowski theory, which is governed by a different elemental notion of summation, called $L_{p}$-addition [23, 57, 59. This theory provides a basis for wideranging inequalities in geometry, analysis, and probability, e.g., 61, 62, 63, 35, 34, 37. Campi and Gronchi developed an alternate approach to isoperimetric inequalities for $L_{p}$-centroid bodies in [15, 16. In particular, they further developed the notion and applications of shadow systems, as introduced by Rogers and Shephard [82]. These systems generalize Steiner symmetrization
and have far-reaching extensions and applications; see, e.g., [17, 18]. There is significant interest in $L_{p}$-Brunn-Minkowski theory for the challenging setting of $p<1$ [6]; see the survey [8], and recent advances in 49, 70, and the references therein.

A common framework for polar projection bodies and intersection bodies has been pursued from several perspectives. Drawing on [66], the notion of the dual $L_{p}$-centroid body was extended by Gardner and Giannopoulos in $[25]$ to $p \in(-1,1)$ via

$$
\begin{equation*}
\rho^{-p}\left(Z_{p}^{\diamond}(K), u\right)=\frac{1}{|K|} \int_{K}|\langle x, u\rangle|^{p} d x \tag{1.4}
\end{equation*}
$$

The bodies $Z_{p}^{\diamond}(K)$ interpolate between intersection bodies and polar $L_{p}$-centroid bodies using

$$
\rho(I(K), u)=\left|K \cap u^{\perp}\right|=\lim _{p \rightarrow-1^{+}} \frac{p+1}{2} \int_{K}|\langle x, u\rangle|^{p} d x
$$

see [25, 29, 47, 31. For $p<1, Z_{p}^{\diamond}(K)$ need not be convex, which we emphasize here by the use of the.$\diamond$ notation. Busemann-Petty type volume comparison problems for $Z_{p}^{\diamond}(K)$, motivated by earlier work of Grinberg and Zhang [29] and Lutwak [56] were treated by Yaskin and Yaskina in 88. For $p<0$, these bodies have also been termed $L_{p}$-intersection bodies and characterizations of such operators as radial valuations were established by Haberl and Ludwig [33]; see also [32] for $p>-1$. Properties of $L_{p}$-intersection bodies were further developed by Haberl in 31]. When $K$ is an origin-symmetric convex body, a result of Berck [2] shows that $Z_{p}^{\diamond}(K)$ is actually convex for $-1<p<1$, which extends Busemann's seminal result for intersection bodies 13 .

We develop methods to bridge the gap between the Busemann intersection inequality (1.1) and the Lutwak-Zhang theorem (1.3). Each of these can be proved using Steiner symmetrization, but in very different ways. The former applies to the star bodies $I(K)$ and uses integral geometric identities (of Blaschke-Petkantschin type) that are particular to slices of $K$. The latter relies on convexity of the polar centroid bodies $Z_{p}^{\circ}(K)$ for $p \geq 1$. We develop a new approach that applies to star bodies in between these two classes, that sees (1.1) and (1.3) from the same viewpoint. We will show that (1.1) is one of a large family of inequalities for unit balls of finitedimensional subspaces of $L_{p}$. We merge several techniques that have been used for $p= \pm 1$. These include symmetrization, embedding via random linear operators, and a classical change of density technique used in Koldobsky's Fourier analytic treatment of intersection bodies.

We follow a probabilistic approach in which $L_{p}$-centroid bodies are attached to probability densities rather than sets. This view was put forth by the second-named author [75] in the study of high-dimensional measures and their concentration properties; see also 42, 51. Fundamental inequalities of Lutwak, Yang and Zhang, in 60, were extended to probability measures by the second and third-named authors in [76, 77]. An empirical approach to dual $L_{p}$-centroid bodies, for $p \geq 1$ was developed in further joint work with Cordero-Erausquin and Fradelizi [20], motivated by [17]. To fix the notation, we set

$$
\mathcal{P}_{n}=\left\{f: \mathbb{R}^{n} \rightarrow[0, \infty) \mid \int_{\mathbb{R}^{n}} f(x) d x=1,\|f\|_{\infty}<\infty\right\}
$$

where $\|f\|_{\infty}$ denotes the essential supremum. For $f \in \mathcal{P}_{n}$, the empirical $L_{p}$ centroid body $\mathcal{Z}_{p, N}(f)$ is defined by its support function via

$$
\begin{equation*}
h^{p}\left(\mathcal{Z}_{p, N}(f), u\right)=\frac{1}{N} \sum_{i=1}^{N}\left|\left\langle X_{i}, u\right\rangle\right|^{p} \tag{1.5}
\end{equation*}
$$

where $X_{1}, \ldots, X_{N}$ are independent random vectors with density $f$. In [20, a stronger stochastic version of (1.3) was established for radial measures $\nu$ with decreasing densities,

$$
\begin{equation*}
\mathbb{E} \nu\left(\mathcal{Z}_{p, N}^{\circ}(f)\right) \leq \mathbb{E} \nu\left(\mathcal{Z}_{p, N}^{\circ}\left(f^{*}\right)\right) \tag{1.6}
\end{equation*}
$$

where $f^{*}$ is the symmetric decreasing rearrangement of $f$ (see 33). By the law of large numbers, (1.6) implies the Lutwak-Zhang inequalities (1.3) when $N \rightarrow \infty$ and $f=\frac{1}{|K|} \chi_{K}$.

The empirical inequality (1.6) follows from a general theorem about random operators acting in normed spaces [20]. The random operator viewpoint is from the asymptotic theory of normed spaces. In seminal work, Gluskin used random operators to construct counter-examples to a longstanding question on the maximal Banach-Mazur distance between finite-dimensional spaces [26]. The expository article of Mankiewicz and Tomczak-Jaegermann 67] details its far-reaching extensions in Banach space theory. This viewpoint was also fruitful in developing stochastic versions of a number of isoperimetric inequalities [77, 78]. However, inherent in the method was a restriction to convex sets. The main new feature we develop here is its applicability to star-shaped sets. We will show how this change provides a bridge between the aforementioned inequalities in Brunn-Minkowski theory and dual Brunn-Minkowski theory.

## 2 Main results

Our first result establishes a sharp isoperimetric inequality that extends the Lutwak-Zhang inequality (1.3) to the case $p \in(0,1)$. For $f \in \mathcal{P}_{n}$ and $p \in(0,1)$, define the dual $L_{p}$-centroid body $Z_{p}^{\diamond}(f)$ via its radial function:

$$
\begin{equation*}
\rho^{-p}\left(Z_{p}^{\diamond}(f), u\right)=\int_{\mathbb{R}^{n}}|\langle x, u\rangle|^{p} f(x) d x \tag{2.1}
\end{equation*}
$$

To define the empirical version $\mathcal{Z}_{p, N}^{\diamond}(f)$, we let $N>n$ and consider independent random vectors $X_{1}, \ldots, X_{N}$ according to $f$ as above, and set

$$
\begin{equation*}
\rho^{-p}\left(\mathcal{Z}_{p, N}^{\diamond}(f), u\right)=\frac{1}{N} \sum_{i=1}^{N}\left|\left\langle X_{i}, u\right\rangle\right|^{p} \tag{2.2}
\end{equation*}
$$

Theorem 2.1. Let $f \in \mathcal{P}_{n}$ and let $0<p<1$. Then

$$
\begin{equation*}
\left|Z_{p}^{\diamond}(f)\right| \leq\left|Z_{p}^{\diamond}\left(f^{*}\right)\right| \tag{2.3}
\end{equation*}
$$

## Moreover,

$$
\begin{equation*}
\mathbb{E}\left|\mathcal{Z}_{p, N}^{\diamond}(f)\right| \leq \mathbb{E}\left|\mathcal{Z}_{p, N}^{\diamond}\left(f^{*}\right)\right| \tag{2.4}
\end{equation*}
$$

Theorem 2.1] relies on first establishing the empirical version (2.4), while (2.3) is derived as a consequence. This is a key difference from the empirical approach in [77, 20, 78, in which (non-random) inequalities of Lutwak, Yang and Zhang 66, 60, 64 inspired the development of their empirical versions (e.g., (1.3) motivated its stochastic form (1.6)). Recently, Yaskin proved (2.3) and extensions raised in [48] in the case when $f=\chi_{K}$, where $K$ is an origin-symmetric star body sufficiently close to the Euclidean ball [87].

Our original inspiration is a recent volume formula for sections of finite-dimensional $L_{p}$ balls by Nayar and Tkocz [73] that builds on ideas involving Gaussian mixtures of random variables from [22]. Kindred probabilistic representations have been indispensible in the study of sections of convex bodies, e.g., 69, 43, 1]. In our case, it allows for a reduction from star-shaped sets to convex sets that interfaces well with the empirical approach from [77, 20, 78].

The methods we develop here go beyond centroid bodies, to families of subspaces of $L_{p}$. For $f \in \mathcal{P}_{n}$, an origin-symmetric convex body $C$ in $\mathbb{R}^{m}, m \geq 1$, and $p \neq 0$, we define $Z_{p, C}^{\diamond}(f) \subseteq \mathbb{R}^{n}$ by its radial function: for $p \neq 0$,

$$
\begin{equation*}
\rho^{-p}\left(Z_{p, C}^{\diamond}(f), u\right)=\int_{\left(\mathbb{R}^{n}\right)^{m}} h^{p}\left(C,\left(\left\langle x_{i}, u\right\rangle\right)_{i=1}^{m}\right) \prod_{i=1}^{m} f\left(x_{i}\right) d \bar{x} \tag{2.5}
\end{equation*}
$$

where $d \bar{x}=d x_{1} \ldots d x_{m}$, and for $p=0$,

$$
\begin{equation*}
\log \rho\left(Z_{0, C}^{\diamond}(f), u\right)=-\int_{\left(\mathbb{R}^{n}\right)^{m}} \log h\left(C,\left(\left\langle x_{i}, u\right\rangle\right)_{i=1}^{m}\right) \prod_{i=1}^{m} f\left(x_{i}\right) d \bar{x} \tag{2.6}
\end{equation*}
$$

We also define empirical versions involving multiple bodies $C$ and densities $f$. Specifically, let $C_{1}, \ldots, C_{N}$ be origin-symmetric convex bodies with $m_{i}=\operatorname{dim}\left(C_{i}\right) \geq 1$ for $i \in[N]=\{1, \ldots, N\}$, where $N>n$. Let $\left(X_{i j}\right), i \in[N], j \in\left[m_{i}\right]$ be independent random vectors with $X_{i j}$ distributed according to $f_{i j} \in \mathcal{P}_{n}$. Write $\mathcal{C}=\left(C_{1}, \ldots, C_{N}\right)$ and $\mathcal{F}=\left(\left(f_{i j}\right)_{j}\right)_{i}$. For $p \neq 0$, we define a star-shaped set $Z_{p, \mathcal{C}}^{\diamond}(\mathcal{F}) \subseteq \mathbb{R}^{n}$ by

$$
\begin{equation*}
\rho^{-p}\left(\mathcal{Z}_{p, \mathcal{C}}^{\diamond}(\mathcal{F}), u\right)=\frac{1}{N} \sum_{i=1}^{N} h^{p}\left(C_{i},\left(\left\langle X_{i j}, u\right\rangle\right)_{j=1}^{m_{i}}\right) ; \tag{2.7}
\end{equation*}
$$

for $p=0$, we define $\mathcal{Z}_{0, \mathcal{C}}^{\diamond}(\mathcal{F}) \subseteq \mathbb{R}^{n}$ by its radial function

$$
\begin{equation*}
\rho^{-N}\left(\mathcal{Z}_{0, \mathcal{C}}^{\diamond}(\mathcal{F}), u\right)=\prod_{i=1}^{N} h\left(C_{i},\left(\left\langle X_{i j}, u\right\rangle\right)_{j=1}^{m_{i}}\right) \tag{2.8}
\end{equation*}
$$

For $p \geq 1$, the.$\diamond$ notation agrees with usual polarity. When $p>0$ and $C=[-1,1]$, then $Z_{p,[-1,1]}^{\diamond}(f)=Z_{p}^{\diamond}(f)$; similarly, if $p>0, \mathcal{F}=(f)_{i=1}^{N}$ and $\mathcal{C}=([-1,1])_{i=1}^{N}$, then $\mathcal{Z}_{p, \mathcal{C}}^{\diamond}(\mathcal{F})=$ $\mathcal{Z}_{p, N}^{\diamond}(f)$. For $p \geq 0$, we have the following generalization of Theorem 2.1] going from $\mathcal{F}=\left(f_{i j}\right)$ to the family of rearranged densities $\mathcal{F}^{\#}=\left(f_{i j}^{*}\right)$.

Theorem 2.2. Let $f \in \mathcal{P}_{n}$ and let $p \geq 0$. If $C$ is an origin-symmetric convex body of dimension $m \geq 1$, then

$$
\begin{equation*}
\left|Z_{p, C}^{\diamond}(f)\right| \leq\left|Z_{p, C}^{\diamond}\left(f^{*}\right)\right| \tag{2.9}
\end{equation*}
$$

Moreover, if $\mathcal{F}=\left(f_{i j}\right) \subseteq \mathcal{P}_{n}$ and $\mathcal{C}=\left(C_{1}, \ldots, C_{N}\right)$, where each $C_{i}$ is an origin-symmetric convex body of dimension $m_{i} \geq 1$, then

$$
\begin{equation*}
\mathbb{E}\left|\mathcal{Z}_{p, \mathcal{C}}^{\diamond}(\mathcal{F})\right| \leq \mathbb{E}\left|\mathcal{Z}_{p, \mathcal{C}}^{\diamond}\left(\mathcal{F}^{\#}\right)\right| \tag{2.10}
\end{equation*}
$$

The theorem is new for all values of $p$. For $p \geq 1$, the proof uses tools that have already been developed in 20]. The main novelty here is in techniques to deal with the star-shaped sets $\mathcal{Z}_{p, \mathcal{C}}^{\diamond}(\mathcal{F})$ in the range $p \in[0,1)$. In particular, we provide a separate treatment for $p=0$ including a new volume formula for $\mathcal{Z}_{0, \mathcal{C}}^{\diamond}(\mathcal{F})$. For $p<0$, the expected volume of the empirical bodies $\mathcal{Z}_{p, \mathcal{C}}^{\diamond}(\mathcal{F})$ need not be finite when $\operatorname{dim}\left(C_{i}\right)<n$ (see Remark 5.8); here the use of higherdimensional convex bodies $C_{1}, \ldots, C_{N}$ is essential. For certain values of $p$, namely when $p \in$ $[-1,0)$ and $n / p$ is an integer, we establish the following theorem.
Theorem 2.3. Let $f \in \mathcal{P}_{n}$ and let $p \in[-1,0)$. Let $C$ be an origin-symmetric convex body with $\operatorname{dim}(C) \geq 1$. If $p>-1$ and $n /|p| \in \mathbb{N}$, then

$$
\begin{equation*}
\left|Z_{p, C}^{\diamond}(f)\right| \leq\left|Z_{p, C}^{\diamond}\left(f^{*}\right)\right| \tag{2.11}
\end{equation*}
$$

Furthermore, let $\mathcal{F}=\left(f_{i j}\right) \subseteq \mathcal{P}_{n}$ and $\mathcal{C}=\left(C_{1}, \ldots, C_{N}\right)$, where each $C_{i}$ is an origin-symmetric convex body of dimension $m_{i} \geq n+1$. If $p \geq-1$ and $n /|p| \in \mathbb{N}$, then

$$
\begin{equation*}
\mathbb{E}\left|\mathcal{Z}_{p, \mathcal{C}}^{\diamond}(\mathcal{F})\right| \leq \mathbb{E}\left|\mathcal{Z}_{p, \mathcal{C}}^{\diamond}\left(\mathcal{F}^{\#}\right)\right| \tag{2.12}
\end{equation*}
$$

Empirical versions of isoperimetric inequalities from [77, 20, 78] have involved operations in Brunn-Minkowski theory; e.g., for $p \geq 1$, the sets $\mathcal{Z}_{p, \mathcal{C}}(f)$ in (1.5) are $L_{p}$ sums of random line segments (see §40). Theorems 2.1-2.3 are the first to treat empirical forms of inequalities for star-shaped sets in dual Brunn-Minkowski theory. In particular, we develop randomized analogues of approximation results of Goodey and Weil [27, and Kalton, Koldobsky, Yaskin and Yaskina [41, in which intersection bodies and their $L_{p}$ analogues are limits of radial sums of ellipsoids. The use of higher-dimensional bodies $C_{i}$ in Theorem 2.3 is needed for this purpose and such bodies are crucial for establishing the corresponding isoperimetric inequalities. In particular, we define a variant of the $L_{p}$-intersection body as follows: for $f \in \mathcal{P}_{n}, \alpha>0$ and $p \in[-1,0)$, we set

$$
\rho^{|p|}\left(I_{|p|}^{\alpha}(f), u\right)=\int_{\mathbb{R}^{n}}\left(|\langle x, u\rangle|^{2}+\alpha^{2}\|u\|_{2}^{2}\right)^{-|p| / 2} f(x) d x .
$$

For the empirical version, we consider $N>n$ independent random vectors $X_{1}, \ldots, X_{N}$ from $f \in \mathcal{P}_{n}$ and define $\mathcal{I}_{|p|, N}^{\alpha}(f)$ via

$$
\rho^{|p|}\left(\mathcal{I}_{|p|, N}^{\alpha}(f), u\right)=\frac{1}{N} \sum_{i=1}^{N}\left(\left|\left\langle X_{i}, u\right\rangle\right|^{2}+\alpha^{2}\|u\|_{2}^{2}\right)^{-|p| / 2}
$$

The star-shaped bodies $\mathcal{I}_{|p|, N}^{\alpha}(f)$ are $L_{p}$-radial sums of ellipsoids (see $\$ 3$ for definitions). In fact, the bodies $\mathcal{I}_{|p|, N}^{\alpha}(f)$ are special (limiting) cases of $\mathcal{Z}_{p, \mathcal{C}}^{\diamond}(\mathcal{F})$ for a suitable choice of $\mathcal{C}$ and $\mathcal{F}$, involving ellipsoids and uniform measures on balls.
Corollary 2.4. Let $f \in \mathcal{P}_{n}, \alpha>0, p \in[-1,0)$ and $n /|p| \in \mathbb{N}$. Then

$$
\begin{equation*}
\left|I_{|p|}^{\alpha}(f)\right| \leq\left|I_{|p|}^{\alpha}\left(f^{*}\right)\right| \tag{2.13}
\end{equation*}
$$

Moreover,

$$
\begin{equation*}
\mathbb{E}\left|\mathcal{I}_{|p|, N}^{\alpha}(f)\right| \leq \mathbb{E}\left|\mathcal{I}_{|p|, N}^{\alpha}\left(f^{*}\right)\right| \tag{2.14}
\end{equation*}
$$

When $p=-1,(2.14)$ is a stochastic form of the Busemann intersection inequality (1.1), as it implies the latter when $N \rightarrow \infty$ and $\alpha \rightarrow 0$. Indeed, if $f \in \mathcal{P}_{n}$, we write $I(f)$ for the intersection body of $f$, defined by its radial function via

$$
\rho(I(f), u)=\int_{u^{\perp}} f(x) d x
$$

and (2.14) implies the following functional version of (1.1).
Corollary 2.5. Let $f$ be a continuous and compactly supported function in $\mathcal{P}_{n}$. Then

$$
\begin{equation*}
|I(f)| \leq\left|I\left(f^{*}\right)\right| \tag{2.15}
\end{equation*}
$$

Thus the Busemann intersection inequality (1.1) is one limiting case of a family of extremal inequalities about $L_{p}$-radial sums in Theorem 2.3. For (non-random) functional versions of the Busemann intersection inequality, see [21, and [36] for recent developments.

Lastly, we can further reduce inequalities to uniform measures on balls in each of the above theorems whenever the convex bodies $C$ and $C_{i}$ are unconditional, i.e., invariant under reflections in the coordinate hyperplanes.
Theorem 2.6. Let $f \in \mathcal{P}_{n}$. Suppose that $p \in[0,1]$, or $p \in[-1,0)$ and $n /|p| \in \mathbb{N}$. Let $C$ be an unconditional convex body in $\mathbb{R}^{m}, m \geq 1$. Set $g=\|f\|_{\infty} \chi_{r B_{2}^{n}}$, where $r>0$ satisfies $\int g=1$. Then for $p>-1$,

$$
\left|Z_{p, C}^{\diamond}(f)\right| \leq\left|Z_{p, C}^{\diamond}(g)\right|
$$

while for $p \geq-1$,

$$
\mathbb{E}\left|\mathcal{I}_{|p|, N}^{\alpha}(f)\right| \leq \mathbb{E}\left|\mathcal{I}_{|p|, N}^{\alpha}(g)\right| .
$$

Furthermore, assume that $\mathcal{F}=\left(f_{i j}\right) \subseteq \mathcal{P}_{n}$ and $\mathcal{G}=\left(g_{i j}\right)$, where $g_{i j}=\left\|f_{i j}\right\|_{\infty} \chi_{r_{i j} B_{2}^{n}}$ with $r_{i j}>0$ satisfying $\int g_{i j}=1$. If $\mathcal{C}=\left(C_{1}, \ldots, C_{N}\right)$, where each $C_{i}$ is an unconditional convex body of dimension $m_{i}$ as above, then

$$
\mathbb{E}\left|\mathcal{Z}_{p, \mathcal{C}}^{\diamond}(\mathcal{F})\right| \leq \mathbb{E}\left|\mathcal{Z}_{p, \mathcal{C}}^{\diamond}(\mathcal{G})\right|
$$

The paper is organized as follows: $\S 3$ introduces notation and basic tools; $\S 4$ is devoted to the non-random bodies $Z_{p, C}^{\diamond}(f)$ and variants of $L_{p}$-intersection bodies; $\$ 5$ develops the randomized versions of these objects. New volume formulas and representations for radial functions are developed in §6. The theorems are proved in $\$ 7$.

## 3 Preliminaries

### 3.1 Notation and definitions

For a compact set $K \subseteq \mathbb{R}^{n}$, we denote its convex hull by $\operatorname{conv}(K)$. The set of all compact, convex sets in $\mathbb{R}^{n}$ will be denoted by $\mathcal{K}^{n}$. For $K \in \mathcal{K}^{n}$, its support function is defined by $h(K, u)=\sup _{x \in K}\langle x, u\rangle, u \in \mathbb{R}^{n}$. The Hausdorff metric on $\mathcal{K}^{n}$ is defined by

$$
\delta^{H}(K, L)=\sup _{\theta \in S^{n-1}}|h(K, \theta)-h(L, \theta)|
$$

where $S^{n-1}$ is the unit sphere. We call $K \in \mathcal{K}^{n}$ a convex body if it has interior points. We say that $K \in \mathcal{K}^{n}$ is origin-symmetric if $-x \in K$ whenever $x \in K$. The set of all origin-symmetric convex bodies in $\mathbb{R}^{n}$ will be denoted by $\mathcal{K}_{s}^{n}$. Each $K \in \mathcal{K}_{s}^{n}$ gives rise to a norm on $\mathbb{R}^{n}$ given by

$$
\|u\|_{K}=\inf \{\lambda>0: u \in \lambda K\}
$$

The polar body of $K \in \mathcal{K}_{s}^{n}$ is defined by $K^{\circ}=\left\{u \in K: h_{K}(u) \leq 1\right\}$.
For measurable sets $A \subseteq \mathbb{R}^{n}$, we use $|A|$ for the Lebesgue measure of $A$. By $\omega_{n}$, we mean the volume of the Euclidean ball in $\mathbb{R}^{n}$ with radius 1 , i.e., $\omega_{n}=\pi^{n / 2} / \Gamma(n / 2+1)$.

We will call a set $K$ in $\mathbb{R}^{n}$ star-shaped if $0 \in K$ and $\alpha x \in K$ whenever $x \in K$ and $\alpha \in[0,1]$. The radial function of a star-shaped set $K$ is defined as $\rho(K, u)=\sup \{\alpha \geq 0: \alpha u \in K\}$ for $u \in S^{n-1}$. Here we allow $K$ to be unbounded and $\rho(K, u)$ may take the value $+\infty$. As our focus is volumetric inequalities, we are particularly interested in radial functions of star-shaped sets $K$ with $\rho(K, \cdot) \in L_{n}\left(S^{n-1}, \sigma\right)$ in which case we write

$$
\|\rho(K, \cdot)\|_{n}=\left(\int_{S^{n-1}} \rho^{n}(K, u) d u\right)^{1 / n}=\omega_{n}^{-1 / n}|K|^{1 / n}
$$

Throughout, $d u$ denotes $d \sigma(u)$, where $\sigma$ is the normalized Haar probability measure on $S^{n-1}$.
We will call $K$ a star-body if it is a compact, star-shaped set with the origin in its interior and its radial function is continuous. When $K \in \mathcal{K}_{s}^{n}$, we have for $u \in \mathbb{R}^{n} \backslash\{0\}$,

$$
\rho(K, u)=\|u\|_{K}^{-1} \quad \text { and } \quad \rho\left(K^{\circ}, u\right)=h_{K}^{-1}(u)
$$

We recall a core notion of addition of convex bodies from $L_{p}$ Brunn-Minkowski theory, e.g. [23, 57, 59]. For $K, L \in \mathcal{K}^{n}$ containing the origin and $p \geq 1$, we will write $K+{ }_{p} L$ for their $L_{p}$ sum, i.e.,

$$
h^{p}\left(K+{ }_{p} L, u\right)=h^{p}(K, u)+h^{p}(L, u) \quad\left(u \in \mathbb{R}^{n}\right)
$$

In dual Brunn-Minkowski theory, (e.g., [66, 83]), for star-bodies $K, L$, and $p \neq 0$, their $L_{p}$-radial $\operatorname{sum} K \tilde{+}_{p} L$ is defined by

$$
\rho^{p}\left(K \tilde{+}_{p} L, u\right)=\rho^{p}(K, u)+\rho^{p}(L, u) \quad\left(u \in S^{n-1}\right)
$$

For a measurable set $A$ in $\mathbb{R}^{n}$ with finite volume, we define its rearrangement $A^{*}$ to be the (open) Euclidean ball centered at the origin satisfying $\left|A^{*}\right|=|A|$. We will use the following bracket notation for indicator functions:

$$
\begin{equation*}
[u \in A]=\chi_{A}(u) \tag{3.1}
\end{equation*}
$$

For a non-negative integrable function $f$ on $\mathbb{R}^{n}$, its layer-cake representation is given by

$$
\begin{equation*}
f(x)=\int_{0}^{\infty} \chi_{\{f>t\}}(x) d t=\int_{0}^{\infty}[x \in\{f>t\}] d t \tag{3.2}
\end{equation*}
$$

The symmetric decreasing rearrangement of a non-negative integrable function $f$ on $\mathbb{R}^{n}$ is defined using rearrangement of its level sets $\left\{x \in \mathbb{R}^{n}: f(x)>t\right\}=\{f>t\}(t>0)$ via

$$
\begin{equation*}
f^{*}(x)=\int_{0}^{\infty} \chi_{\{f>t\}^{*}}(x) d t=\int_{0}^{\infty}\left[x \in\{f>t\}^{*}\right] d t \tag{3.3}
\end{equation*}
$$

For a general reference on rearrangements, we refer the reader to [52. We will use the fact that $f$ and $f^{*}$ are equimeasurable; in particular, $f^{*}$ preserves all $L_{p}$ norms of $f$. Note also that if $f \leq g$, then $f^{*} \leq g^{*}$. Moreover, rearrangements satisfy the following contractive property: for $1 \leq p \leq \infty$ and for $f, g \in L_{p}$,

$$
\begin{equation*}
\left\|f^{*}-g^{*}\right\|_{p} \leq\|f-g\|_{p} \tag{3.4}
\end{equation*}
$$

For $f \in \mathcal{P}_{n}$, the marginal density of $f$ on a subspace $E$ of dimension $k$, is defined as

$$
\begin{equation*}
\pi_{E}(f)(x)=\int_{E^{\perp}+x} f(y) d y \tag{3.5}
\end{equation*}
$$

where $E^{\perp}$ denotes the orthogonal complement of $E$. Note that when $f \in \mathcal{P}_{n}$ and has compact support, then $\pi_{E}(f)$ is also bounded and has compact support.

### 3.2 Probabilistic tools

We will make repeated use of the following fact about uniformly integrable collections of random variables (e.g., [86, pg. 189]).
Proposition 3.1. Let $\eta, \eta_{1}, \eta_{2}, \ldots$ be non-negative random variables on a probability space $(\Omega, \mathcal{M}, \mathbb{P})$ such that $\eta_{k} \rightarrow \eta$ as $k \rightarrow \infty$ almost surely. If $\left\{\eta_{k}\right\}$ is uniformly integrable, then

$$
\lim _{k \rightarrow \infty} \mathbb{E} \eta_{k}=\mathbb{E} \eta<\infty
$$

Remark 3.2. A sufficient condition for uniform integrability of a family of random variables $\left\{\eta_{k}\right\}$ is boundedness in $L_{1+\delta}(\Omega, \mathcal{M}, \mathbb{P})$, for some $\delta>0$ ([86, pg. 190]).

We will also use Kolmogorov's strong law of large numbers ([86, pg. 391]).
Proposition 3.3. Let $\eta_{1}, \eta_{2}, \ldots$ be independent identically distributed random variables on a probability space $(\Omega, \mathcal{M}, \mathbb{P})$ such that $\mathbb{E}\left|\eta_{1}\right|<\infty$. Then, almost surely, as $N \rightarrow \infty$,

$$
\frac{1}{N} \sum_{k=1}^{N} \eta_{k} \rightarrow \mathbb{E} \eta_{1}
$$

We will frequently use a.s. as an abbreviation for almost sure convergence; similarly, we use i.i.d. for a sequence of independent identically distributed random variables.

### 3.3 Volume in terms of Gaussian integrals

We will use the following elementary lemma which relates the volume of star-shaped sets to certain Gaussian integrals.
Lemma 3.4. Let $K$ be a star-shaped set with $0 \in \operatorname{int}(K)$ and $\rho(K, \cdot) \in L_{n}\left(S^{n-1}, \sigma\right)$. If $\xi$ is a standard Gaussian vector in $\mathbb{R}^{n}$, and $s \in(0, n)$, then

$$
\begin{equation*}
\mathbb{E}_{\xi} \rho^{s}(K, \xi)=b_{n, s} \int_{S^{n-1}} \rho^{s}(K, u) d u \tag{3.6}
\end{equation*}
$$

where

$$
\begin{equation*}
b_{n, s}=\mathbb{E}_{\xi}\|\xi\|_{2}^{-s}=\frac{n \Gamma\left(\frac{n-s}{2}\right)}{2^{s / 2+1} \Gamma\left(\frac{n}{2}+1\right)} . \tag{3.7}
\end{equation*}
$$

Furthermore, if $\rho(K, \cdot)$ is additionally the pointwise limit of an increasing sequence of radial functions $\left\{\rho\left(K_{\ell}, \cdot\right)\right\}$ of star shaped sets $\left\{K_{\ell}\right\}$, then

$$
\begin{equation*}
|K|=\lim _{\ell \rightarrow \infty} \frac{\mathbb{E}_{\xi} \rho^{n-1 / \ell}\left(K_{\ell}, \xi\right)}{b_{n, n-1 / \ell}} \tag{3.8}
\end{equation*}
$$

Proof. Using polar coordinates, we have for $0<s<n$,

$$
\begin{aligned}
\mathbb{E}_{\xi} \rho^{s}(K, \xi) & =\frac{n \omega_{n}}{(2 \pi)^{n / 2}} \int_{0}^{\infty} r^{n-s-1} e^{-r^{2} / 2} d r \int_{S^{n-1}} \rho^{s}(K, u) d u \\
& =\frac{n \Gamma\left(\frac{n-s}{2}\right)}{2^{s / 2+1} \Gamma\left(\frac{n}{2}+1\right)} \int_{S^{n-1}} \rho^{s}(K, u) d u
\end{aligned}
$$

The conditions $0 \in \operatorname{int}(K)$ and $\rho(K, \cdot) \in L_{n}\left(S^{n-1}, \sigma\right)$ ensure that $\rho(K, u)$ is positive and finite for all $u$ outside of a null set on $S^{n-1}$. For such $u$, since $\rho\left(K_{\ell}, u\right) \rightarrow \rho(K, u)$, we have

$$
\rho^{n-1 / \ell}\left(K_{\ell}, u\right)=\rho^{n}\left(K_{\ell}, u\right) \exp \left(-\frac{\log \rho\left(K_{\ell}, u\right)}{\ell}\right) \rightarrow \rho^{n}(K, u)
$$

Next, since $\left\{\rho\left(K_{\ell}, u\right)\right\}$ is increasing,

$$
\begin{equation*}
\rho^{n-1 / \ell}\left(K_{\ell}, u\right) \leq \max \left(1, \rho^{n}\left(K_{\ell}, u\right)\right) \leq \max \left(1, \rho^{n}(K, u)\right) \leq 1+\rho^{n}(K, u) \tag{3.9}
\end{equation*}
$$

By dominated convergence and (3.6), we get

$$
\omega_{n}^{-1}|K|=\int_{S^{n-1}} \rho^{n}(K, u) d u=\lim _{\ell \rightarrow \infty} \int_{S^{n-1}} \rho^{n-1 / \ell}\left(K_{\ell}, u\right) d u=\lim _{\ell \rightarrow \infty} \frac{\mathbb{E}_{\xi} \rho^{n-1 / \ell}\left(K_{\ell}, \xi\right)}{b_{n, n-1 / \ell}}
$$

## 4 Dual $L_{p, C}$-centroid bodies

Let $f \in \mathcal{P}_{n}, p>-1$ and let $C$ be an origin-symmetric convex body in $\mathbb{R}^{m}, m \geq 1$. For ease of reference, we recall that for $p \neq 0$,

$$
\begin{equation*}
\rho^{-p}\left(Z_{p, C}^{\diamond}(f), u\right)=\int_{\left(\mathbb{R}^{n}\right)^{m}} h^{p}\left(C,\left(\left\langle x_{i}, u\right\rangle\right)_{i=1}^{m}\right) \prod_{i=1}^{m} f\left(x_{i}\right) d \bar{x} \tag{4.1}
\end{equation*}
$$

and for $p=0$,

$$
\begin{equation*}
\log \rho\left(Z_{0, C}^{\diamond}(f), u\right)=-\int_{\left(\mathbb{R}^{n}\right)^{m}} \log h_{C}\left(\left(\left\langle x_{i}, u\right\rangle\right)_{i=1}^{m}\right) \prod_{i=1}^{m} f\left(x_{i}\right) d \bar{x} \tag{4.2}
\end{equation*}
$$

As noted in the introduction, the latter bodies are not convex in general. We will use the term dual $L_{p, C}$ centroid body as these bodies fit within dual Brunn-Minkowski theory. This agrees with the convex case when $p \geq 1$, however, the term here is meant in a broader sense than duality for convex bodies. When $p \geq 1, Z_{p, N}^{\diamond}(f)=Z_{p, N}^{\circ}(f)$.

We start by noting a few elementary properties of the bodies $Z_{p, C}^{\diamond}(f)$.
Lemma 4.1. Let $f \in \mathcal{P}_{n}, p, p_{1}, p_{2}>-1$ and $C \in \mathcal{K}_{s}^{m}, m \geq 1$.
(a) If $p_{1} \leq p_{2}$, then

$$
Z_{p_{2}, C}^{\diamond}(f) \subseteq Z_{p_{1}, C}^{\diamond}(f)
$$

(b) If $D \in \mathcal{K}_{s}^{m_{1}}$, and $C \subseteq D$, then

$$
Z_{p, C}^{\diamond}(f) \supseteq Z_{p, D}^{\diamond}(f)
$$

(c) $\rho^{|p|}\left(Z_{p, C}^{\diamond}(f), \cdot\right) \in L_{1}\left(S^{n-1}, \sigma\right)$.
(d) For $k \in \mathbb{N}$ such that $\int_{k B_{2}^{n}} f(x) d x>0$, let $\varphi^{(k)}=\left.f\right|_{k B_{2}^{n}}$ and $\phi^{(k)}=\varphi^{(k)} / \int \varphi^{(k)}$. Then

$$
\rho\left(Z_{p, C}^{\diamond}(f), u\right)=\lim _{k \rightarrow \infty} \rho\left(Z_{p, C}^{\diamond}\left(\phi^{(k)}\right), u\right) \quad\left(u \in S^{n-1}\right)
$$

Proof. Part (a) is a consequence of Hölder's inequality. For (b), the condition $C \subseteq D$ is equivalent to $h(C, \cdot) \leq h(D, \cdot)$, hence $\rho\left(Z_{p, D}^{\diamond}(f), u\right) \leq \rho\left(Z_{p, C}^{\diamond}(f), u\right)$ for each $u \in S^{n-1}$.

By using (a), it is sufficient to treat (c) for $p \in(-1,0)$. Since $C \in \mathcal{K}_{s}^{m}$, we can assume $C \subseteq \operatorname{span}\left\{e_{1}, \ldots, e_{m}\right\}$, and there exists $r_{0}>0$ such that $r_{0}\left[-e_{1}, e_{1}\right] \subseteq C$, hence $\rho\left(Z_{p, C}^{\diamond}(f), u\right) \leq$ $r_{0}^{-1} \rho\left(Z_{p}^{\diamond}(f), u\right)$ for $u \in S^{n-1}$. For $p \in(-1,0)$, we have for each $u \in S^{n-1}$,

$$
\|x\|_{2}^{-|p|}=\beta_{n, p} \int_{S^{n-1}}|\langle x, u\rangle|^{-|p|} d u
$$

where $\beta_{n, p}=b_{n,|p|} / b_{1,|p|}$ (cf. (3.7)). Since $x \mapsto\|x\|_{2}^{-|p|}$ is locally integrable and $f \in \mathcal{P}_{n}$, we have

$$
\int_{\mathbb{R}^{n}}\|x\|_{2}^{-|p|} f(x) d x \leq\|f\|_{\infty} \int_{B_{2}^{n}}\|x\|_{2}^{-|p|} d x+\int_{\mathbb{R}^{n} \backslash B_{2}^{n}} f(x) d x<\infty
$$

Thus part (c) follows from

$$
\begin{aligned}
\int_{S^{n-1}} \rho^{|p|}\left(Z_{p}^{\diamond}(f), u\right) d u & =\int_{S^{n-1}} \int_{\mathbb{R}^{n}}|\langle x, u\rangle|^{-|p|} f(x) d x d u \\
& =\beta_{n, p}^{-1} \int_{\mathbb{R}^{n}}\|x\|_{2}^{-|p|} f(x) d x
\end{aligned}
$$

To prove (d), we note that part (c) implies $\rho\left(Z_{p, C}^{\diamond}(f), u\right)<\infty$ for a.e. $u \in S^{n-1}$. Since $\varphi^{(k)} \rightarrow f$, and $f \in \mathcal{P}_{n}$, we have $\int \varphi^{(k)} \rightarrow \int f=1$. For $p \neq 0$, (d) follows by monotone convergence:

$$
\begin{equation*}
\int_{\left(\mathbb{R}^{n}\right)^{m}} h^{p}\left(C,\left(\left\langle x_{i}, u\right\rangle\right)_{i=1}^{m}\right) \prod_{i=1}^{m} \varphi^{(k)}\left(x_{i}\right) d \bar{x} \rightarrow \int_{\left(\mathbb{R}^{n}\right)^{m}} h^{p}\left(C,\left(\left\langle x_{i}, u\right\rangle\right)_{i=1}^{m}\right) \prod_{i=1}^{m} f\left(x_{i}\right) d \bar{x} \tag{4.3}
\end{equation*}
$$

the latter holds even when the righthand side of (4.3) is infinite, which may occur for $p>0$, in which case $\rho\left(Z_{p, C}^{\diamond}(f), u\right)=0$. To treat $p=0$, we set

$$
P_{1}(u)=\left\{\left(x_{i}\right)_{i=1}^{m} \in\left(\mathbb{R}^{n}\right)^{m}: h\left(C,\left(\left\langle x_{i}, u\right\rangle\right)_{i=1}^{m}\right)>1\right\}
$$

and $P_{2}(u)=\left(\mathbb{R}^{n}\right)^{m} \backslash P_{1}(u)$. Applying the same argument to each factor in

$$
\rho\left(Z_{0, C}^{\diamond}\left(\phi^{(k)}\right), u\right)=\prod_{i=1}^{2} \exp \left(-\int_{P_{i}(u)} \log h\left(C,\left(\left\langle x_{i}, u\right\rangle\right)_{i=1}^{m} \prod_{i=1}^{m} \phi^{(k)}\left(x_{i}\right) d \bar{x}\right)\right.
$$

shows that (d) holds for $p=0$ as well.

## 4.1 $L_{p}^{\alpha}$-intersection bodies

For $f \in \mathcal{P}_{n}$, we write $I(f)$ for its intersection body, defined by its radial function via

$$
\begin{equation*}
\rho(I(f), u)=\int_{u^{\perp}} f(x) d x \tag{4.4}
\end{equation*}
$$

where the integration is with respect to Lebesgue measure on $u^{\perp}$; for background on intersection bodies, see [55, 47, 24]. Motivated by approximation results for intersection bodies involving radial sums of ellipsoids [27, 41, we define a variant of (4.4): for $\alpha>0$ and $p \in[-1,0$ ), the $L_{p}^{\alpha}$-intersection body of $f$ is given by

$$
\rho^{|p|}\left(I_{|p|}^{\alpha}(f), u\right)=\int_{\mathbb{R}^{n}}\left(|\langle x, u\rangle|^{2}+\alpha^{2}\|u\|_{2}^{2}\right)^{-|p| / 2} f(x) d x
$$

As mentioned, when $f$ is the indicator of a star-body and $\alpha=0$, the latter bodies were studied in [88, 33, 31. When $p=-1$ and $\alpha>0$, we write $I^{\alpha}(f)=I_{1}^{\alpha}(f)$.
Proposition 4.2. Let $f$ be a continuous, compactly supported function in $\mathcal{P}_{n}$. For $\alpha>0$, let $s_{\alpha}=\sinh ^{-1}(1 / \alpha)$. Then

$$
|I(f)|=\lim _{\alpha \rightarrow 0}\left(2 s_{\alpha}\right)^{-n}\left|I^{\alpha}(f)\right|
$$

We will prove this using an approximate identity, i.e., a family of non-negative functions $\left(k_{\alpha}\right)_{\alpha \in(0,1)}$ on $\mathbb{R}$ satisfying the following conditions, for each $\alpha \in(0,1)$,
(i) $\int_{\mathbb{R}} k_{\alpha}(t) d t=1$;
(ii) for any $\delta>0, \lim _{\alpha \rightarrow 0} \int_{|t|>\delta} k_{\alpha}(t) d t=0$.

In this case, if $g$ is continuous and supported on a compact set $K$, then $\left\|\left(k_{\alpha} * g\right)-g\right\|_{L_{\infty}(K)} \rightarrow 0$ (see, e.g., [28, pg. 27]).

Proof. For $\alpha>0$, let

$$
k_{\alpha}(t)=\left(2 s_{\alpha}\right)^{-1}\left(t^{2}+\alpha^{2}\right)^{-1 / 2} \chi_{[-1,1]}(t)
$$

Standard computations show that $\left(k_{\alpha}\right)_{\alpha}$ is an approximate identity. Fix $u \in S^{n-1}$ and recall the notation for the marginal of $f$ on $[u]=\operatorname{span}\{u\}(\operatorname{cf}(3.5))$, and set $f_{u}(t)=\pi_{[u]}(f)(t)$. Then

$$
\begin{align*}
\left(2 s_{\alpha}\right)^{-1} \rho\left(I^{\alpha}(f), u\right) & =\left(2 s_{\alpha}\right)^{-1} \int_{\mathbb{R}}\left(t^{2}+\alpha^{2}\right)^{-1 / 2} f_{u}(t) d t \\
& =\int_{|t| \leq 1} k_{\alpha}(t) f_{u}(t) d t+\left(2 s_{\alpha}\right)^{-1} \int_{|t|>1}\left(t^{2}+\alpha^{2}\right)^{-1 / 2} f_{u}(t) d t  \tag{4.5}\\
& =\left(k_{\alpha} * f_{u}\right)(0)+\left(2 s_{\alpha}\right)^{-1} \int_{|t|>1}\left(t^{2}+\alpha^{2}\right)^{-1 / 2} f_{u}(t) d t
\end{align*}
$$

We have $k_{\alpha} * f_{u}(0) \rightarrow f_{u}(0)=\rho(I(f), u)$. Since $\int_{\mathbb{R}} f_{u}(t) d t=1$ and $s_{\alpha} \rightarrow \infty$ as $\alpha \rightarrow 0$, we have

$$
\lim _{\alpha \rightarrow 0}\left(2 s_{\alpha}\right)^{-1} \int_{|t|>1}\left(t^{2}+\alpha^{2}\right)^{-1 / 2} f_{u}(t) d t=0
$$

It follows that

$$
\left(2 s_{\alpha}\right)^{-n} \rho^{n}\left(I^{\alpha}(f), u\right) \rightarrow \rho^{n}(I(f), u)
$$

Using (4.5), the latter convergence is dominated on $\left(S^{n-1}, \sigma\right)$ by $\left(\sup _{u}\left\|f_{u}\right\|_{\infty}+\left(2 s_{1}\right)^{-1}\right)^{n}$, hence

$$
|I(f)|=\omega_{n} \int_{S^{n-1}} \lim _{\alpha \rightarrow 0} \rho^{n}\left(\left(2 s_{\alpha}\right)^{-1} I^{\alpha}(f), u\right) d u=\lim _{\alpha \rightarrow 0}\left(2 s_{\alpha}\right)^{-n}\left|I^{\alpha}(f)\right|
$$

## 5 Empirical dual $L_{p, C}$-centroid bodies

An empirical approach to $L_{p}$-centroid bodies was initiated in [77] and developed further in [20, 78. It relies on random linear operators acting on various sets in finite-dimensional normed spaces. In this section, we recall the main theorem from [20]. We lay the groundwork to reinterpret the random star-shaped bodies $\mathcal{Z}_{p, \mathcal{C}}^{\diamond}(\mathcal{F})$ of our main theorems as random sections of $\ell_{p}$-balls. We also develop new notions of randomly generated intersection bodies.

### 5.1 Tools from the empirical approach

It will be useful to fix some notation for matrices acting as linear operators. For an $n \times N$ matrix $\mathbf{X}=\left[x_{1} \ldots x_{N}\right]$, we write $\mathbf{X}^{T}$ for the transpose of $\mathbf{X}$ and we view $\mathbf{X}: \mathbb{R}^{N} \rightarrow \mathbb{R}^{n}$ and $\mathbf{X}^{T}: \mathbb{R}^{n} \rightarrow \mathbb{R}^{N}$ as linear operators. In particular, for an origin-symmetric convex body $C \subseteq \mathbb{R}^{N}$,

$$
\mathbf{X} C=\{\mathbf{X} c: c \in C\}=\left\{\sum_{i=1}^{N} c_{i} x_{i}: c=\left(c_{i}\right) \in C\right\}
$$

Principal examples include (i) $C=B_{1}^{N}=\operatorname{conv}\left\{ \pm e_{1}, \ldots, \pm e_{N}\right\}$ and (ii) $C=B_{\infty}^{N}=[-1,1]^{N}$ in which case

$$
\text { (i) } \mathbf{X} B_{1}^{N}=\operatorname{conv}\left\{ \pm X_{1}, \ldots, \pm X_{N}\right\} \quad \text { (ii) } \mathbf{X} B_{\infty}^{N}=\sum_{i=1}^{N}\left[-X_{i}, X_{i}\right]
$$

Volumetric inequalities for convex hulls of random points and random zonotopes (30, 10 motivated work in [77] to interpolate between these two extremes and led to an empirical study of $L_{p}$ centroid bodies; see the survey [78] and the references therein.

All of the theorems in the introduction will be derived from the following result about polars of convex bodies from [20]. It concerns radial measures with decreasing densities ("decreasing" is meant in a non-strict sense).
Theorem 5.1. Let $\mathbf{X}$ and $\mathbf{X}^{\#}$ be $n \times N$ random matrices with independent columns drawn from $\mathcal{F}=\left(f_{i}\right)_{i=1}^{N} \subseteq \mathcal{P}_{n}$ and $\mathcal{F}^{\#}=\left(f_{i}^{*}\right)_{i=1}^{N}$, respectively. Let $\nu$ be a radial measure with a decreasing density, i.e., $d \nu(x)=h\left(\|x\|_{2}\right) d x$ with $h:[0, \infty) \rightarrow[0, \infty)$ decreasing. Then for any origin-symmetric convex body $C$ in $\mathbb{R}^{N}$,

$$
\begin{equation*}
\mathbb{E} \nu\left((\mathbf{X} C)^{\circ}\right) \leq \mathbb{E} \nu\left(\left(\mathbf{X}^{\#} C\right)^{\circ}\right) \tag{5.1}
\end{equation*}
$$

Assume additionally that each $f_{i}$ is bounded and $\mathbf{Z}$ is an $n \times N$ random matrix with independent columns drawn from $g_{i}=\left\|f_{i}\right\|_{\infty} \chi_{r_{i} B_{2}^{n}}$, where $r_{i}>0$ satisfies $\int g_{i}=1$. Then for any unconditional convex body $C$ in $\mathbb{R}^{N}$,

$$
\begin{equation*}
\mathbb{E} \nu\left((\mathbf{X} C)^{\circ}\right) \leq \mathbb{E} \nu\left((\mathbf{Z} C)^{\circ}\right) \tag{5.2}
\end{equation*}
$$

The latter theorem relies on rearrangement inequalities of Rogers [81], Brascamp-LiebLuttinger [12] and Christ [19]. It also relies on the Borell-Brascamp-Lieb inequalities [5, 11]. It was motivated by work of Campi and Gronchi on symmetrization of polar convex bodies [17.

The following lemma is a useful re-interpretation of the bodies $(\mathbf{X} C)^{\circ}$, stated in terms of the transpose $\mathbf{X}^{T}$ and its pre-image $\mathbf{X}^{-T}$.
Lemma 5.2. Let $\mathbf{X}$ be an $n \times N$ matrix of full rank, viewed as a linear operator $\mathbf{X}: \mathbb{R}^{N} \rightarrow \mathbb{R}^{n}$. Then for $C \in \mathcal{K}_{s}^{N}$,

$$
\begin{equation*}
(\mathbf{X} C)^{\circ}=\mathbf{X}^{-T}\left[C^{\circ}\right] . \tag{5.3}
\end{equation*}
$$

Proof. When $N=n$, then $\mathbf{X}$ and $\mathbf{X}^{T}$ are invertible and (5.3) is a standard identity:

$$
\begin{aligned}
(\mathbf{X} C)^{\circ} & =\left\{x \in \mathbb{R}^{n}:\langle x, \mathbf{X} c\rangle \leq 1 \text { for all } c \in C\right\} \\
& =\left\{x \in \mathbb{R}^{n}:\left\langle\mathbf{X}^{T} x, c\right\rangle \leq 1 \text { for all } c \in C\right\} \\
& =\mathbf{X}^{-T}\left[C^{\circ}\right]
\end{aligned}
$$

When $N<n$, the same computation is valid by viewing $\mathbf{X}^{-T}\left[C^{\circ}\right]$ as the pre-image of $C^{\circ}$ under $\mathbf{X}^{T}$. When $N>n, \mathbf{X}^{T}$ is injective and $\mathbf{X}^{-T}$ is also the inverse of $\mathbf{X}^{T}$ on $\operatorname{Im}\left(\mathbf{X}^{T}\right)=\operatorname{ker}(\mathbf{X})^{\perp}$, in which case

$$
\begin{equation*}
\mathbf{X}^{-T}\left[C^{\circ}\right]=\mathbf{X}^{-T}\left[C^{\circ} \cap \operatorname{Im}\left(\mathbf{X}^{T}\right)\right] \tag{5.4}
\end{equation*}
$$

Remark 5.3. When $N<n$, we note that $(\mathbf{X} C)^{\circ}$ denotes polarity in $\mathbb{R}^{n}$ and $(\mathbf{X} C)^{\circ}$ may be unbounded. When $N \geq n$, (5.4) implies that

$$
\begin{equation*}
\left|(\mathbf{X} C)^{\circ}\right|=\operatorname{det}\left(\mathbf{X} \mathbf{X}^{T}\right)^{-1 / 2}\left|C^{\circ} \cap \operatorname{Im}\left(\mathbf{X}^{T}\right)\right| \tag{5.5}
\end{equation*}
$$

### 5.2 Random slices of finite-dimensional $\ell_{p}$-balls

For $p \geq 1$, the centroid body $Z_{p}(f)$ can be viewed in terms of limits of images of finitedimensional $\ell_{q}$ balls, where $1 / p+1 / q=1$. To fix the notation, for $p \neq 0$, we denote by $B_{p}^{N}$, the $\ell_{p}$ ball in $\mathbb{R}^{N}$, i.e.,

$$
\begin{equation*}
B_{p}^{N}=\left\{x \in \mathbb{R}^{N}:\left(\sum_{i=1}^{N}\left|\left\langle x, e_{i}\right\rangle\right|^{p}\right)^{1 / p} \leq 1\right\} \tag{5.6}
\end{equation*}
$$

where $e_{1}, \ldots, e_{N}$ is the standard unit vector basis for $\mathbb{R}^{N}$. For $p=0$, we set

$$
\begin{equation*}
B_{0}^{N}=\left\{x \in \mathbb{R}^{N}:\left(\prod_{i=1}^{N}\left|\left\langle x, e_{i}\right\rangle\right|\right)^{1 / N} \leq 1\right\} \tag{5.7}
\end{equation*}
$$

Note that $B_{p}^{N}$ is a convex body when $p \in[1, \infty)$ and a star-body when $p>0$. When $p \leq 0, B_{p}^{N}$ is unbounded but remains star-shaped.

Let $\mathbf{X}$ be an $n \times N$ random matrix with independent column vectors $X_{1}, \ldots, X_{N}$ drawn from $f \in \mathcal{P}_{n}$. For $1 \leq p<\infty$, the empirical $L_{p}$-centroid body $\mathcal{Z}_{p, N}(f)$ defined above in (1.5) has the equivalent description

$$
\mathcal{Z}_{p, N}(f)=N^{-1 / p} \mathbf{X} B_{q}^{N}
$$

where $1 / p+1 / q=1$. Indeed,

$$
h\left(\mathbf{X} B_{q}^{N}, u\right)=h\left(B_{q}^{N}, \mathbf{X}^{T} u\right)=\left(\sum_{i=1}^{N}\left|\left\langle X_{i}, u\right\rangle\right|^{p}\right)^{1 / p}
$$

Using Lemma 5.2 and $1 / p+1 / q=1$, we have

$$
\begin{equation*}
\mathcal{Z}_{p, N}^{\circ}(f)=N^{1 / p} \mathbf{X}^{-T}\left[B_{p}^{N}\right] \tag{5.8}
\end{equation*}
$$

where, as above, $\mathbf{X}^{-T}[A]$ denotes the pre-image of $A$ under $\mathbf{X}^{T}$. We will mimic identity (5.8) to realize the bodies $\mathcal{Z}_{p, N}^{\diamond}(f)$ defined in (2.2) as sections of $B_{p}^{N}$ for $p \in(0,1)$.
Lemma 5.4. Let $\mathbf{X}$ be an $n \times N$ random matrix with independent columns distributed according to $f \in \mathcal{P}_{n}$. Then for $p \in(0,1)$,

$$
\mathcal{Z}_{p, N}^{\diamond}(f)=N^{1 / p} \mathbf{X}^{-T}\left[B_{p}^{N}\right]
$$

Proof. For $u \in S^{n-1}$, we have

$$
\rho\left(\mathcal{Z}_{p, N}^{\diamond}(f), u\right)=\rho\left(N^{1 / p} B_{p}^{N}, \mathbf{X}^{T} u\right)=\rho\left(N^{1 / p} \mathbf{X}^{-T}\left[B_{p}^{N}\right], u\right)
$$

We can similarly view the bodies $\mathcal{Z}_{p, \mathcal{C}}^{\diamond}(\mathcal{F})$ (cf. (2.7)) using random linear operators. For $\mathcal{C}=\left(C_{1}, \ldots, C_{N}\right)$ with $C_{i} \in \mathcal{K}_{s}^{m_{i}}$, we place them in orthogonal subspaces $\mathbb{R}^{m_{i}}=\operatorname{span}\left\{e_{i j}\right\}_{j=1}^{m_{i}}$, $i=1, \ldots, N$. Then for $p \neq 0$, we define

$$
\begin{equation*}
\mathcal{B}_{p}^{N}(\mathcal{C})=\left\{\left(x_{1}, \ldots, x_{N}\right) \in \bigoplus_{i=1}^{N} \mathbb{R}^{m_{i}}:\left(\sum_{i=1}^{N} h^{p}\left(C_{i}, x_{i}\right)\right)^{1 / p} \leq 1\right\} \tag{5.9}
\end{equation*}
$$

when $p=\infty$, we replace the sum by $\max _{i} h\left(C_{i}, x_{i}\right)$. For $p=0$, we set

$$
\begin{equation*}
\mathcal{B}_{0}^{N}(\mathcal{C})=\left\{\left(x_{1}, \ldots, x_{N}\right) \in \bigoplus_{i=1}^{N} \mathbb{R}^{m_{i}}:\left(\prod_{i=1}^{N} h\left(C_{i}, x_{i}\right)\right)^{1 / N} \leq 1\right\} \tag{5.10}
\end{equation*}
$$

When the $C_{i}^{\prime} s$ are identical copies of $[-1,1]$, we have $B_{p}^{N}=\mathcal{B}_{p}^{N}\left(\left(\left[-e_{1}, e_{1}\right], \ldots,\left[-e_{N}, e_{N}\right]\right)\right)$. As for $B_{p}^{N}$, the set $\mathcal{B}_{p}^{N}(\mathcal{C})$ is a convex body, star-body or unbounded star-shaped set, according to whether $p \geq 1, p \in(0,1)$ or $p \leq 0$, respectively. Note that we have defined $\mathcal{B}_{p}^{N}(\mathcal{C})$ using support functions $h\left(C_{i}, \cdot\right)$ rather than norms associated to the $C_{i}$ 's, as some computations are more convenient with this convention. By standard duality arguments, for $1 \leq p, q \leq \infty$ with $1 / p+1 / q=1$, we have for $\mathcal{C}=\left(C_{1}, \ldots, C_{N}\right)$,

$$
\begin{equation*}
\left(\mathcal{B}_{p}^{N}(\mathcal{C})\right)^{\circ}=\mathcal{B}_{q}^{N}\left(\mathcal{C}^{\circ}\right) \tag{5.11}
\end{equation*}
$$

where we have set $\mathcal{C}^{\circ}=\left(C_{1}^{\circ}, \ldots, C_{N}^{\circ}\right)$. We will use the particular case of $p=1$ and $q=\infty$, combined with Lemma 5.2 in the following form.
Lemma 5.5. Let $\mathcal{C}=\left(C_{1}, \ldots, C_{N}\right)$, where $C_{i} \in \mathcal{K}_{s}^{m_{i}}, m_{i} \geq 1$, and $\mathcal{C}^{\circ}=\left(C_{1}^{\circ}, \ldots, C_{N}^{\circ}\right)$. Set $M=m_{1}+\ldots+m_{N}$. Let $\mathbf{X}=\left[\mathbf{X}_{1} \ldots \mathbf{X}_{N}\right]$ be an $n \times M$ matrix with $n \times m_{i}$ blocks $\mathbf{X}_{i}$ of full rank. Then

$$
\bigcap_{i=1}^{N}\left(\mathbf{X}_{i} C_{i}\right)^{\circ}=\left(\mathbf{X} \mathcal{B}_{1}^{N}\left(\mathcal{C}^{\circ}\right)\right)^{\circ}
$$

Proof. By Lemma 5.3.

$$
\bigcap_{i=1}^{N}\left(\mathbf{X}_{i} C_{i}\right)^{\circ}=\bigcap_{i=1}^{N} \mathbf{X}_{i}^{-T}\left[C_{i}^{\circ}\right]=\bigcap_{i=1}^{N}\left\{u \in \mathbb{R}^{n}: \mathbf{X}_{i}^{T} u \in C_{i}^{\circ}\right\}
$$

while

$$
\left(\mathbf{X} \mathcal{B}_{1}^{N}\left(\mathcal{C}^{\circ}\right)\right)^{\circ}=\mathbf{X}^{-T}\left[\mathcal{B}_{\infty}^{N}(\mathcal{C})\right]=\left\{u \in \mathbb{R}^{n}: \max _{i \leq N} h\left(C_{i}, \mathbf{X}_{i}^{T} u\right) \leq 1\right\}
$$

Using the above notation, the empirical bodies $\mathcal{Z}_{p, \mathcal{C}}^{\diamond}(\mathcal{F})$ defined in (2.7) and (2.8) can be realized as sections of $\mathcal{B}_{p}^{N}(\mathcal{C})$ as follows.
Lemma 5.6. For $i \in[N]$, let $C_{i} \in \mathcal{K}_{s}^{m_{i}}$, $m_{i} \geq 1$ and let $M=m_{1}+\ldots+m_{N}$. Let $\mathbf{X}=\left[\mathbf{X}_{1} \cdots \mathbf{X}_{N}\right]$ be an $n \times M$ random matrix with $n \times m_{i}$ blocks $\mathbf{X}_{i}=\left[X_{i 1} \ldots X_{i m_{i}}\right]$ having independent columns $X_{i j}$ distributed according to $f_{i j} \in \mathcal{P}_{n}$. Then for $p \neq 0$,

$$
\begin{equation*}
\mathcal{Z}_{p, \mathcal{C}}^{\diamond}(\mathcal{F})=N^{1 / p} \mathbf{X}^{-T}\left[\mathcal{B}_{p}^{N}(\mathcal{C})\right] \tag{5.12}
\end{equation*}
$$

and, for $p=0$,

$$
\begin{equation*}
\mathcal{Z}_{0, \mathcal{C}}^{\diamond}(\mathcal{F})=\mathbf{X}^{-T}\left[\mathcal{B}_{0}^{N}(\mathcal{C})\right] \tag{5.13}
\end{equation*}
$$

Proof. For $u \in S^{n-1}$,

$$
\mathbf{X}^{T} u=\left(\mathbf{X}_{1}^{T} u, \ldots, \mathbf{X}_{N}^{T} u\right)=\left(\left(\left\langle X_{1 j}, u\right\rangle\right)_{j=1}^{m_{1}}, \ldots,\left(\left\langle X_{N j}, u\right\rangle\right)_{j=1}^{m_{N}}\right)
$$

For any set $\mathcal{S}$ in $\mathbb{R}^{n M}$, we have

$$
\mathbf{X}^{-T}[\mathcal{S}]=\left\{u \in \mathbb{R}^{n}: \mathbf{X}^{T} u \in \mathcal{S}\right\}
$$

For $p \neq 0$, we have

$$
\begin{aligned}
\rho\left(N^{1 / p} \mathbf{X}^{-T}\left[\mathcal{B}_{p}^{N}(\mathcal{C})\right], u\right) & =\rho\left(N^{1 / p} \mathcal{B}_{p}^{N}(\mathcal{C}), \mathbf{X}^{T} u\right) \\
& =\left(\frac{1}{N} \sum_{i=1}^{N} h^{p}\left(C_{i}, \mathbf{X}_{i}^{T} u\right)\right)^{-1 / p} \\
& =\rho\left(\mathcal{Z}_{p, \mathcal{C}}^{\diamond}(\mathcal{F}), u\right)
\end{aligned}
$$

For $p=0$, we have

$$
\rho\left(\mathbf{X}^{-T}\left[\mathcal{B}_{0}^{N}(\mathcal{C})\right], u\right)=\prod_{i=1}^{N} h\left(C_{i}, \mathbf{X}_{i}^{T} u\right)^{-1 / N}=\rho\left(\mathcal{Z}_{0, \mathcal{C}}^{\diamond}(\mathcal{F}), u\right)
$$

Remark 5.7. For $p \geq 1$, we have by Lemma 5.2 and (5.11),

$$
\begin{equation*}
\mathcal{Z}_{p, \mathcal{C}}^{\diamond}(\mathcal{F})=\mathcal{Z}_{p, \mathcal{C}}^{\circ}(\mathcal{F})=N^{1 / p} \mathbf{X}^{-T}\left[\mathcal{B}_{p}^{N}(\mathcal{C})\right]=N^{1 / p}\left(\mathbf{X} \mathcal{B}_{q}^{N}\left(\mathcal{C}^{\circ}\right)\right)^{\circ} \tag{5.14}
\end{equation*}
$$

Remark 5.8. For $p \leq 0$, the bodies $\mathcal{Z}_{p, \mathcal{C}}^{\diamond}(\mathcal{F})$ are pre-images of slices of unbounded sets and hence need not be bounded. This is reflected in our notation as their radial functions take the value $+\infty$. When $m_{j}=\operatorname{dim}\left(C_{j}\right)<n$, the matrix $\mathbf{X}_{j}^{T}$ has a non-trivial kernel and, for $p \neq 0$,

$$
\rho\left(\mathbf{X}^{-T}\left[\mathcal{B}_{p}^{N}(\mathcal{C})\right], u\right)=\left(\sum_{i=1}^{N} h^{-|p|}\left(C_{i}, \mathbf{X}_{i}^{T} u\right)\right)^{1 /|p|} \geq h^{-1}\left(C_{j}, \mathbf{X}_{j}^{T} u\right)
$$

which is infinite for $u \in \operatorname{ker}\left(\mathbf{X}_{j}^{T}\right)$ and arbitrarily large in any neighboorhood of such $u$. When each $C_{i}$ has dimension $m_{i} \geq n$, the absolute continuity of $f_{i j}$ ensures that the $n \times m_{i}$ matrix $\mathbf{X}_{\mathbf{i}}$ has rank $n$ a.s.. This implies that $h\left(C, \mathbf{X}_{i}^{T} \cdot\right)>0$ a.s., hence each summand in the radial function $\rho\left(\mathcal{Z}_{p, \mathcal{C}}^{\diamond}(\mathcal{F}), \cdot\right)$ is necessarily finite a.s..

For $N$-tuples of origin-symmetric convex sets $\mathcal{C}=\left(C_{1}, \ldots, C_{N}\right)$ and $\mathcal{D}=\left(D_{1}, \ldots, D_{N}\right)$, with $\operatorname{dim}\left(C_{i}\right) \leq \operatorname{dim}\left(D_{i}\right)$, we will write

$$
\mathcal{C} \subseteq \mathcal{D} \Longleftrightarrow C_{i} \subseteq D_{i} \text { for all } i=1, \ldots, N
$$

Lemma 5.9. Let $\mathcal{F}=\left(f_{i j}\right) \subseteq \mathcal{P}_{n}$ and $-1 \leq p, p_{1}, p_{2} \leq \infty$. Let $\mathcal{C}=\left(C_{1}, \ldots, C_{N}\right)$ with $C_{i} \in \mathcal{K}_{s}^{m_{i}}, m_{i} \geq 1$, for $i \in[N]$.
(a) If $p_{1} \leq p_{2}$, then

$$
\begin{equation*}
\mathcal{Z}_{p_{2}, \mathcal{C}}^{\diamond}(\mathcal{F}) \subseteq \mathcal{Z}_{p_{1}, \mathcal{C}}^{\diamond}(\mathcal{F}) \tag{5.15}
\end{equation*}
$$

(b) If $\mathcal{D}=\left(D_{1}, \ldots, D_{N}\right)$ with $D_{i} \in \mathcal{K}_{s}^{m_{i}^{\prime}}, m_{i}^{\prime} \geq m_{i}$, for $i \in[N]$, and $\mathcal{C} \subseteq \mathcal{D}$, then

$$
\begin{equation*}
\mathcal{Z}_{p, \mathcal{C}}^{\diamond}(\mathcal{F}) \supseteq \mathcal{Z}_{p, \mathcal{D}}^{\diamond}(\mathcal{F}) \tag{5.16}
\end{equation*}
$$

Proof. Part (a) is a consequence of Hölder's inequality, which gives monotonicity of the normalized means in the definition of $\rho\left(\mathcal{Z}_{p, \mathcal{C}}^{\diamond}(\mathcal{F}), u\right)$ (cf. (2.7) and (2.8)).

For part $(b), C_{i} \subseteq D_{i}$ is equivalent to $h\left(C_{i}, \cdot\right) \leq h\left(D_{i}, \cdot\right)$ for each $i$, which implies (5.16).

### 5.3 Convergence of volumes

The next proposition provides sufficient conditions to obtain the volume of $Z_{p, C}^{\diamond}(f)$ as a limit of the expected volumes of the random bodies $\mathcal{Z}_{p, \mathcal{C}}^{\diamond}(\mathcal{F})$.
Proposition 5.10. For $i \in \mathbb{N}$, let $C_{i} \in \mathcal{K}_{s}^{m_{i}}, m_{i} \geq 1$ and $\left(f_{i j}\right) \subseteq \mathcal{P}_{n}, j \in\left[m_{i}\right]$. For $N \in \mathbb{N}$, let $\mathcal{C}_{N}=\left(C_{1}, \ldots, C_{N}\right)$ and $\mathcal{F}_{N}=\left(\left(f_{i j}\right)_{j=1}^{m_{i}}\right)_{i=1}^{N}$. Assume that
a. there is an $r_{0}>0$ such that $r_{0} B_{2}^{m_{i}} \subseteq C_{i}$ for each $i$.
b. $f_{i j}$ are supported on a common compact set and $\sup _{i, j}\left\|f_{i j}\right\|_{\infty}<\infty$.

If $p \in[0,1]$, or $p \in[-1,0)$ and $m_{i} \geq n+1$ for each $i$, then for any $\varepsilon \in(0,1)$,

$$
\begin{equation*}
\sup _{N \geq n+1} \sup _{u \in S^{n-1}} \mathbb{E} \rho^{n+\varepsilon}\left(\mathcal{Z}_{p, \mathcal{C}_{N}}^{\diamond}\left(\mathcal{F}_{N}\right), u\right)<\infty \tag{5.17}
\end{equation*}
$$

and hence

$$
\begin{equation*}
\sup _{N \geq n+1} \mathbb{E}\left|\mathcal{Z}_{p, \mathcal{C}_{N}}^{\diamond}\left(\mathcal{F}_{N}\right)\right|<\infty \tag{5.18}
\end{equation*}
$$

Furthermore, if $C_{1}, C_{2}, \ldots$ are copies of a given convex body $C$ of dimension $m$ and $f_{i j}$ are identical and satisfy (5.17), then

$$
\begin{equation*}
\left|Z_{p, C}^{\diamond}(f)\right|=\lim _{N \rightarrow \infty} \mathbb{E}\left|\mathcal{Z}_{p, \mathcal{C}_{N}}^{\diamond}\left(\mathcal{F}_{N}\right)\right| \tag{5.19}
\end{equation*}
$$

Proof. Without loss of generality, we may assume that $r_{0}=1$. By assumption (b), we can fix a Gaussian density $\phi_{\alpha}$ and a constant $A>0$ such that for each $i, j$,

$$
\begin{equation*}
\frac{1}{A} f_{i j}(x) \leq \phi_{\alpha}(x)=\frac{1}{\left(2 \pi \alpha^{2}\right)^{n / 2}} e^{-\|x\|_{2}^{2} / 2 \alpha^{2}} \quad\left(x \in \mathbb{R}^{n}\right) \tag{5.20}
\end{equation*}
$$

Fix $\varepsilon>0$ and $u \in S^{n-1}$. Assume first that $p \in[0,1]$. By Lemma 5.9, we need only treat the case $p=0, m_{i}=1$ for $i=1, \ldots, N$, and $\mathcal{C}_{N}=\left(\left[-e_{i}, e_{i}\right]\right)_{i=1}^{N}$. In the notation of Lemma [5.6, this means that $\mathcal{F}_{N}=\left(f_{i 1}\right)_{i=1}^{N}$ and $\mathbf{X}_{i}=\left[X_{i 1}\right]$ are $n \times 1$ matrices. By Fubini's theorem,

$$
\mathbb{E} \rho^{n+\varepsilon}\left(\mathcal{Z}_{0, \mathcal{C}_{N}}^{\diamond}\left(\mathcal{F}_{N}\right), u\right)=\prod_{i=1}^{N} \mathbb{E}\left|\left\langle X_{i 1}, u\right\rangle\right|^{-(n+\varepsilon) / N}
$$

Set $\tau=(n+\varepsilon) / N$. Let $g_{1}, \ldots, g_{N}$ be i.i.d. standard Gaussian vectors in $\mathbb{R}^{n}$. Fix $i \in\{1, \ldots, N\}$. Then $\left\langle g_{i}, u\right\rangle$ is a standard Gaussian random variable. Assume first that $N \geq 2(n+\varepsilon)$ so that $\tau \leq 1 / 2$. By Hölder's inequality,

$$
\mathbb{E}_{X_{i 1}}\left|\left\langle X_{i 1}, u\right\rangle\right|^{-\tau} \leq\left(\mathbb{E}\left|\left\langle X_{i 1}, u\right\rangle\right|^{-1 / 2}\right)^{2 \tau}
$$

Using (5.20) and the notation for $b_{n, s}$ from (3.7), we have

$$
A^{-1} \alpha^{1 / 2} \mathbb{E}_{X_{i 1}}\left|\left\langle X_{i 1}, u\right\rangle\right|^{-1 / 2} \leq \mathbb{E}_{g_{i}}\left|\left\langle g_{i}, u\right\rangle\right|^{-1 / 2}=b_{1,1 / 2}
$$

hence for $N \geq 2(n+\varepsilon)$,

$$
\begin{equation*}
\mathbb{E} \rho^{n+\varepsilon}\left(\mathcal{Z}_{0, \mathcal{C}_{N}}^{\diamond}\left(\mathcal{F}_{N}\right), u\right) \leq\left(A \alpha^{-1 / 2} b_{1,1 / 2}\right)^{2 \tau N}=\left(A \alpha^{-1 / 2} b_{1,1 / 2}\right)^{2(n+\varepsilon)} \tag{5.21}
\end{equation*}
$$

Assume now that $n+1 \leq N<2(n+\varepsilon)$. Then $\tau<1$ and $b_{1, \tau}<\infty$. By (5.20),

$$
A^{-1} \alpha^{\tau} \mathbb{E}_{X_{i 1}}\left|\left\langle X_{i 1}, u\right\rangle\right|^{-\tau} \leq \mathbb{E}_{g_{i}}\left|\left\langle g_{i}, u\right\rangle\right|^{-\tau}=b_{1, \tau}
$$

It follows that for $n+1 \leq N<2(n+\varepsilon)$,

$$
\begin{equation*}
\mathbb{E} \rho^{n+\varepsilon}\left(\mathcal{Z}_{0, \mathcal{C}_{N}}^{\diamond}\left(\mathcal{F}_{N}\right), u\right) \leq\left(A \alpha^{-\tau} b_{1, \tau}\right)^{N} \leq \max \left(1,\left(A \alpha^{-\tau} b_{1, \tau}\right)^{2(n+\varepsilon)}\right) \tag{5.22}
\end{equation*}
$$

The bounds in (5.21) and (5.22) are independent of $u$ and $N$. This establishes (5.17) for $p \in[0,1]$.
Assume now that $p \in[-1,0)$. By Lemma 5.9, we can assume that $p=-1, m_{i}=n+1$, for $i=1, \ldots, N$ and $\mathcal{C}=\left(B_{2}^{n+1}\right)_{i=1}^{N}$. Set $s=n+\varepsilon$ and let $s^{\prime}$ be defined by $1 / s+1 / s^{\prime}=1$. By Hölder's inequality,

$$
\rho^{n+\varepsilon}\left(\mathcal{Z}_{-1, \mathcal{C}_{N}}^{\diamond}\left(\mathcal{F}_{N}\right), u\right)=\left(\frac{1}{N} \sum_{i=1}^{N}\left\|\mathbf{X}_{i}^{T} u\right\|_{2}^{-1}\right)^{n+\varepsilon} \leq N^{s / s^{\prime}-s} \sum_{i=1}^{N}\left\|\mathbf{X}_{i}^{T} u\right\|_{2}^{-(n+\varepsilon)}
$$

For $i \in[N]$, let $\mathbf{G}_{i}$ be i.i.d. $n \times(n+1)$ random matrices with i.i.d. $N(0,1)$ entries. For $i \in[N]$,

$$
A^{-(n+1)} \alpha^{n+\varepsilon} \mathbb{E}_{\mathbf{X}_{\mathbf{i}}}\left\|\mathbf{X}_{i}^{T} u\right\|_{2}^{-(n+\varepsilon)} \leq \mathbb{E}_{\mathbf{G}_{\mathbf{i}}}\left\|\mathbf{G}_{i}^{T} u\right\|_{2}^{-(n+\varepsilon)}=b_{n+1, n+\varepsilon}
$$

Using $N^{1-s} N^{s / s^{\prime}}=1$, (5.17) now follows from

$$
\mathbb{E}_{\mathbf{X}} \rho^{n+\varepsilon}\left(\mathcal{Z}_{-1, \mathcal{C}_{N}}^{\diamond}\left(\mathcal{F}_{N}\right), u\right) \leq A^{n+1} \alpha^{-(n+\varepsilon)} b_{n+1, n+\varepsilon}
$$

here we have used that $m_{i}=\operatorname{dim}\left(C_{i}\right) \geq n+1$, which ensures finiteness of $b_{n+1, n+\varepsilon}$. To justify (5.18), for general $\mathcal{C}_{N}$ and $\mathcal{F}_{N}$, set $\delta=\varepsilon / n$ so that $n(1+\delta)=n+\varepsilon$. By Hölder's inequality,

$$
\begin{aligned}
\left(\int_{S^{n-1}} \mathbb{E} \rho^{n}\left(\mathcal{Z}_{p, \mathcal{C}_{N}}^{\diamond}\left(\mathcal{F}_{N}\right), u\right) d u\right)^{1+\delta} & \leq \int_{S^{n-1}}\left(\mathbb{E} \rho^{n}\left(\mathcal{Z}_{p, \mathcal{C}_{N}}^{\diamond}\left(\mathcal{F}_{N}\right), u\right)\right)^{1+\delta} d u \\
& \leq \int_{S^{n-1}} \mathbb{E} \rho^{n+\varepsilon}\left(\mathcal{Z}_{p, \mathcal{C}_{N}}^{\diamond}\left(\mathcal{F}_{N}\right), u\right) d u
\end{aligned}
$$

Therefore, (5.18) follows from

$$
\begin{equation*}
\left(\mathbb{E}\left|\mathcal{Z}_{p, \mathcal{C}_{N}}^{\diamond}\left(\mathcal{F}_{N}\right)\right|\right)^{1+\delta} \leq \omega_{n}^{1+\delta} \sup _{u \in S^{n-1}} \mathbb{E} \rho^{n+\varepsilon}\left(\mathcal{Z}_{p, \mathcal{C}_{N}}^{\diamond}\left(\mathcal{F}_{N}\right), u\right) \tag{5.23}
\end{equation*}
$$

Towards proving (5.19), we fix $u \in S^{n-1}$, identical bodies $C_{i}=C$ of dimension $m$ and $f_{i j}=f$. For $p \neq 0$, the family of i.i.d. random variables $\left\{h^{p}\left(C, \mathbf{X}_{i}^{T} u\right)\right\}_{i \in \mathbb{N}}$ has finite first moment, i.e.,

$$
\begin{equation*}
\mathbb{E} h^{p}\left(C, \mathbf{X}_{i}^{T} u\right)=\int_{\left(\mathbb{R}^{n}\right)^{m}} h^{p}\left(C,\left(\left\langle x_{i}, u\right\rangle\right)_{i=1}^{m}\right) \prod_{i=1}^{m} f\left(x_{i}\right) d \bar{x}<\infty \tag{5.24}
\end{equation*}
$$

Indeed, for $p>0$, this is a direct consequence of $f$ being bounded and compactly supported. For $p<0$, the function $\mathbb{E} h^{p}\left(C, \mathbf{X}_{i}^{T} \cdot\right)=\rho^{-p}\left(Z_{p, C}^{\diamond}(f), \cdot\right)$ is integrable by part (c) of Lemma 4.1,
in particular, (5.24) holds for all $u$ outside of a null set on $S^{n-1}$ (henceforth disregarded). Thus by Proposition 3.3 for our fixed $u \in S^{n-1}$,

$$
\left.\frac{1}{N} \sum_{i=1}^{N} h^{p}\left(C, \mathbf{X}_{i}^{T} u\right) \rightarrow \mathbb{E} h^{p}\left(C, \mathbf{X}_{i}^{T} u\right)=\rho^{-p}\left(Z_{p, C}^{\diamond}(f), u\right) \quad \text { (a.s. }\right)
$$

similarly, for $p=0$, the i.i.d. collection $\left\{\log h\left(C, \mathbf{X}_{i}^{T} u\right)\right\}_{i \in \mathbb{N}}$ satisfies

$$
\mathbb{E}\left|\log h\left(C, \mathbf{X}_{i}^{T} u\right)\right|=\int_{\left(\mathbb{R}^{n}\right)^{m}}\left|\log h\left(C,\left(\left\langle x_{i}, u\right\rangle\right)\right)\right| \prod_{i=1}^{m} f\left(x_{i}\right) d \bar{x}<\infty
$$

hence

$$
\frac{1}{N} \sum_{i=1}^{N} \log h\left(C, \mathbf{X}_{i}^{T} u\right) \rightarrow \mathbb{E} \log h\left(C, \mathbf{X}_{i}^{T} u\right) \quad \text { (a.s.). }
$$

In all cases, we have

$$
\left.\rho^{n}\left(\mathcal{Z}_{p, \mathcal{C}_{N}}^{\diamond}\left(\mathcal{F}_{N}\right), u\right) \rightarrow \rho^{n}\left(Z_{p, C}^{\diamond}(f), u\right) \quad \text { (a.s. }\right)
$$

Using (5.17), the collection $\left\{\rho^{n}\left(\mathcal{Z}_{p, \mathcal{C}_{N}}^{\diamond}\left(\mathcal{F}_{N}\right), u\right): N \geq n+1\right\}$ (for our fixed $u$ ) is bounded in $L_{1+\delta}$, where, as above, $\delta=\varepsilon / n$. By Proposition 3.1 and Remark 3.2, as $N \rightarrow \infty$,

$$
\begin{equation*}
\mathbb{E} \rho^{n}\left(\mathcal{Z}_{p, \mathcal{C}_{N}}^{\diamond}\left(\mathcal{F}_{N}\right), u\right) \rightarrow \mathbb{E} \rho^{n}\left(Z_{p, C}^{\diamond}(f), u\right)=\rho^{n}\left(Z_{p, C}^{\diamond}(f), u\right) \tag{5.25}
\end{equation*}
$$

Lastly, the collection $\left\{\mathbb{E} \rho^{n}\left(\mathcal{Z}_{p, \mathcal{C}_{N}}\left(\mathcal{F}_{N}\right), \cdot\right): N \geq n+1\right\}$ is uniformly integrable on $\left(S^{n-1}, \sigma\right)$ (by the inequality preceding (5.23)). Using (5.25), Proposition 3.1 and Fubini's theorem, we get

$$
\begin{aligned}
\left|Z_{p, C}^{\diamond}(f)\right| & =\omega_{n} \int_{S^{n-1}} \rho^{n}\left(Z_{p, C}^{\diamond}(f), u\right) d u \\
& =\omega_{n} \lim _{N \rightarrow \infty} \int_{S^{n-1}} \mathbb{E} \rho^{n}\left(\mathcal{Z}_{p, \mathcal{C}_{N}}^{\diamond}\left(\mathcal{F}_{N}\right), u\right) d u \\
& =\lim _{N \rightarrow \infty} \mathbb{E}\left|\mathcal{Z}_{p, \mathcal{C}_{N}}^{\diamond}\left(\mathcal{F}_{N}\right)\right|
\end{aligned}
$$

which establishes (5.19) and completes the proof of the proposition.

### 5.4 Empirical $L_{p}$-intersection bodies

In this section, we show how particular choices of $\mathcal{C}$ and $\mathcal{F}$ in the bodies $\mathcal{Z}_{p, \mathcal{C}}^{\diamond}(\mathcal{F})$ lead naturally to empirical versions of $L_{p}$-intersection bodies. As mentioned, unit balls of normed spaces that embed in $L_{p}, p \in[-1,1]$ can be obtained as limits of $p$-radial sums of ellipsoids [27, 41]. The next proposition can be seen as a complementary volumetric random approximation. Since our main interest is when $p=-1$, we have stated this only for $p \in[-1,0)$; similar considerations lead to an analogous result for $p>0$.

For $p \in[-1,0)$ and $\alpha>0$, we define the empirical $L_{p}^{\alpha}$-intersection body $\mathcal{I}_{|p|, N}^{\alpha}(f)$ via

$$
\rho\left(\mathcal{I}_{|p|, N}^{\alpha}(f), u\right)=\frac{1}{N} \sum_{i=1}^{N} \rho^{|p|}\left(\mathcal{E}^{\alpha}\left(X_{i}\right), u\right)
$$

where $X_{1}, \ldots, X_{N}$ are i.i.d. with density $f \in \mathcal{P}_{n}$ and $\mathcal{E}^{\alpha}\left(X_{i}\right)=\left(\left[-X_{i}, X_{i}\right]+{ }_{2} \alpha B_{2}^{n}\right)^{\circ}$.

Proposition 5.11. Let $f$ be a compactly supported function in $\mathcal{P}_{n}$. Let $p \in[-1,0)$ and $\alpha>0$. Then for $N \geq n+1$,

$$
\mathbb{E}\left|\mathcal{I}_{|p|, N}^{\alpha}(f)\right|=\lim _{m \rightarrow \infty} \mathbb{E}\left|\mathcal{Z}_{p, \mathcal{C}_{m}^{\alpha}}^{\diamond}\left(\mathcal{F}_{m}\right)\right|,
$$

where $\mathcal{C}_{m}^{\alpha}=\left(C_{m}^{\alpha}\right)_{i=1}^{N}$ and $\mathcal{F}_{m}=\left(\left(f_{i j}\right)_{j=1}^{m+1}\right)_{i=1}^{N}$ are given by

$$
C_{m}^{\alpha}=\left[-e_{1}, e_{1}\right]+{ }_{2} \alpha \operatorname{conv}\left\{ \pm e_{j}\right\}_{j=2}^{m+1}, \quad f_{i j}= \begin{cases}f & \text { if } i \in[N], j=1 \\ \omega_{n}^{-1} \chi_{B_{2}^{n}} & \text { if } i \in[N], j>1\end{cases}
$$

Proof. Let $\mathbf{X}_{1}, \ldots \mathbf{X}_{N}$ be i.i.d. $n \times(m+1)$ random matrices with $\mathbf{X}_{i}=\left[X_{i 1} Z_{i 1} \cdots Z_{i m}\right]$, where $X_{i 1}$ has density $f_{i 1}=f$, and $Z_{i j}$ has density $f_{i(j+1)}=\omega_{n}^{-1} \chi_{B_{2}^{n}}$, and the columns are independent. Then for $i=1, \ldots, N$,

$$
\mathbf{X}_{i} C_{m}^{\alpha}=\left[-X_{i 1}, X_{i 1}\right]+{ }_{2} \alpha \operatorname{conv}\left\{ \pm Z_{i j}\right\}_{j=1}^{m}
$$

which for $m \rightarrow \infty$, converges a.s. in the Hausdorff metric to $\left[-X_{i 1}, X_{i 1}\right]+{ }_{2} \alpha B_{2}^{n}$. For $u \in S^{n-1}$, we have as $m \rightarrow \infty$,

$$
\left.\frac{1}{N} \sum_{i=1}^{N} h^{-|p|}\left(\mathbf{X}_{i} C_{m}^{\alpha}, u\right) \rightarrow \frac{1}{N} \sum_{i=1}^{N} h^{-|p|}\left(\left[-X_{i 1}, X_{i 1}\right]+{ }_{2} \alpha B_{2}^{n}, u\right) \quad \text { (a.s. }\right)
$$

and hence

$$
\rho^{n}\left(\underset{\mathcal{Z}_{p, \mathcal{C}_{m}^{\alpha}}^{\diamond}}{ }\left(\mathcal{F}_{m}\right), u\right) \rightarrow \rho^{n}\left(\mathcal{I}_{|p|, N}^{\alpha}(f), u\right) \quad \text { (a.s.). }
$$

For $m \geq n$, the latter convergence is dominated by $\rho^{n}\left(\mathcal{Z}_{-1, \mathcal{C}_{n}^{\alpha}}^{\diamond}\left(\mathcal{F}_{n}\right), u\right)$ (cf. (5.15)). The inradius of $C_{n}^{\alpha}$ is $\min (1, \alpha / \sqrt{n})$. Using Proposition 5.10 with fixed $N \geq n+1$,

$$
\int_{S^{n-1}} \mathbb{E} \rho^{n}\left(\mathcal{Z}_{-1, \mathcal{C}_{n}^{\alpha}}^{\diamond}\left(\mathcal{F}_{n}\right), u\right) d u<\infty
$$

By dominated convergence, we get

$$
\mathbb{E} \int_{S^{n-1}} \rho^{n}\left(\mathcal{I}_{|p|, N}^{\alpha}(f), u\right) d u=\lim _{m \rightarrow \infty} \mathbb{E} \int_{S^{n-1}} \rho^{n}\left(\mathcal{Z}_{p, \mathcal{C}_{m}^{\alpha}}^{\diamond}\left(\mathcal{F}_{m}\right), u\right) d u
$$

Proposition 5.12. Let $f \in \mathcal{P}_{n}, p \in[-1,0)$, and $\alpha>0$. Then

$$
\left|I_{|p|}^{\alpha}(f)\right|=\lim _{N \rightarrow \infty} \mathbb{E}\left|\mathcal{I}_{|p|, N}^{\alpha}(f)\right| .
$$

Proof. Fix $u \in S^{n-1}$. Since $f \in \mathcal{P}_{n}$, the random variables $\left(\left|\left\langle X_{i}, u\right\rangle\right|^{2}+\alpha^{2}\|u\|_{2}^{2}\right)^{-|p| / 2}$ have finite first moment. By the law of large numbers, as $N \rightarrow \infty$, we have

$$
\frac{1}{N} \sum_{i=1}^{N}\left(\left|\left\langle X_{i}, u\right\rangle\right|^{2}+\alpha^{2}\|u\|_{2}^{2}\right)^{-1 / 2} \rightarrow \int_{\mathbb{R}^{n}}\left(|\langle x, u\rangle|^{2}+\alpha^{2}\|u\|_{2}^{2}\right)^{-1 / 2} f(x) d x \quad \text { (a.s.) }
$$

hence

$$
\left.\rho^{n}\left(\mathcal{I}_{|p|, N}^{\alpha}(f), u\right) \rightarrow \rho^{n}\left(I_{|p|}^{\alpha}(f), u\right) \quad \text { a.s. }\right) .
$$

Since $\rho\left(I_{|p|, N}^{\alpha}(f), u\right) \leq 1 / \alpha$ for each $u$, we can use dominated convergence to get

$$
\mathbb{E} \int_{S^{n-1}} \rho^{n}\left(\mathcal{I}_{|p|, N}^{\alpha}(f), u\right) d u \rightarrow \mathbb{E} \int_{S^{n-1}} \rho^{n}\left(I_{|p|}^{\alpha}(f), u\right) d u=\left|I_{|p|}^{\alpha}(f)\right|
$$

## 6 Volume formulas

As mentioned, our work is inspired by a formula for the volume of sections of $B_{p}^{N}, p \in(0,2)$, due to Nayar and Tkocz [73. We will recall the basic ingredients and then derive a formula for the volume of the random sets $\underset{p, \mathcal{C}}{\diamond}(\mathcal{F})$. For $p \leq 0$, we will present an alternative path and complementary volume formulas.

### 6.1 Volume via Gaussian mixtures for $p>0$

Recall that for $0<\alpha<1$, a positive random variable $w$ is called normalized positive $\alpha$-stable if

$$
\begin{equation*}
\mathbb{E} e^{-t w}=e^{-t^{\alpha}} \quad(t>0) \tag{6.1}
\end{equation*}
$$

We will denote the density of such a random variable by $g_{\alpha}$; for background on stable random variables, see [89. The following Nayar-Tkocz volume formula was proved in [73], where it is stated explicitly for $p=1$ and explained how the same method applies to $p \in(0,2)$.
Proposition 6.1. Let $0<p<2$ and let $X$ be a $n \times N$ matrix with columns $x_{1}, \cdots, x_{N}$ spanning $\mathbb{R}^{n}$. Let $W=\left(w_{1}, \cdots, w_{N}\right)$ be a random vector with i.i.d. entries $w_{i}$ having common density proportional to $s \mapsto s^{-1 / 2} g_{p / 2}(s)$. Then

$$
\begin{equation*}
\frac{\left|B_{p}^{N} \cap \operatorname{Im}\left(X^{T}\right)\right|}{\operatorname{det}\left(X X^{T}\right)^{\frac{1}{2}}}=a_{N, n, p} \pi^{n / 2} \mathbb{E}_{W} \sqrt{w_{1} \cdots w_{N}}\left(\operatorname{det}\left(\sum_{i=1}^{N} w_{i} x_{i} x_{i}^{T}\right)\right)^{-\frac{1}{2}} \tag{6.2}
\end{equation*}
$$

where $a_{N, n, p}=\pi^{-N / 2} \Gamma(1+1 / p)^{N} \Gamma(1+n / p)^{-1}$.
The proof of the formula relies on two ingredients. The first is that the volume of a star-body $K$ in $\mathbb{R}^{n}$ with radial function $\rho(K, \cdot)$ is given by

$$
\begin{equation*}
|K|=c_{n, p} \int_{\mathbb{R}^{n}} \exp \left(-\rho^{-p}(K, x)\right) d x \tag{6.3}
\end{equation*}
$$

where $c_{n, p}=\Gamma(1+n / p)^{-1}$. The second ingredient is the following fact from [22, Lemma 23]: if $\xi$ is a standard Gaussian random variable, independent of a positive random variable $w$ with density proportional to $t \mapsto t^{-1 / 2} g_{p / 2}(t)$, then $\frac{1}{\sqrt{2 w}} \xi$ has density $[2 \Gamma(1+1 / p)]^{-1} e^{-|t|^{p}}$ and

$$
\begin{equation*}
e^{-|x|^{p}}=d_{p} \mathbb{E}_{w} \sqrt{w} e^{-w x^{2}} \quad(x \in \mathbb{R}) \tag{6.4}
\end{equation*}
$$

where $d_{p}=\Gamma(1+1 / p) / \sqrt{\pi}$ (as can be seen by integrating (6.4) on $\mathbb{R}$ ).
We will adapt the Nayar-Tkocz argument to derive a volume formula for $\mathcal{Z}_{p, \mathcal{C}}^{\diamond}(\mathcal{F})$ for $p \in$ $(0,2)$, using the pre-image interpretation in (5.12).
Proposition 6.2. Let $\mathcal{C}=\left(C_{1}, \ldots, C_{N}\right), \mathcal{F}$ and $\mathbf{X}$ be as in Lemma 5.6. Let $0<p<2$ and let $W=\left(w_{1}, \cdots, w_{N}\right)$ be a random vector with i.i.d. entries $w_{i}$ having a common density proportional to $s \mapsto s^{-1 / 2} g_{p / 2}(s)$. Set $\mathcal{C}_{W}^{\circ}=\left(\left(\sqrt{w_{1}} C_{1}\right)^{\circ}, \ldots,\left(\sqrt{w_{N}} C_{N}\right)^{\circ}\right)$. Then

$$
\left|\mathcal{Z}_{p, \mathcal{C}}^{\diamond}(\mathcal{F})\right|=a_{N, n, p} c_{n, 2}^{-1} N^{n / p} \mathbb{E}_{W} \sqrt{w_{1} \cdots w_{N}}\left|\left(\mathbf{X} \mathcal{B}_{2}^{N}\left(\mathcal{C}_{W}^{\circ}\right)\right)^{\circ}\right|
$$

Proof. Note that $\left(\mathcal{B}_{2}^{N}\left(\mathcal{C}_{W}\right)\right)^{\circ}=\mathcal{B}_{2}^{N}\left(\mathcal{C}_{W}^{\circ}\right)($ cf. (5.9) $)$, hence

$$
\begin{equation*}
\sum_{i=1}^{N} h^{2}\left(\sqrt{w_{i}} C_{i}, \mathbf{X}_{i}^{T} u\right)=h^{2}\left(\mathbf{X} \mathcal{B}_{2}^{N}\left(\mathcal{C}_{W}^{\circ}\right), u\right) \tag{6.5}
\end{equation*}
$$

Using (6.3)-(6.5), we have

$$
\begin{aligned}
c_{n, p}^{-1}\left|\mathbf{X}^{-T}\left[\mathcal{B}_{p}^{N}(\mathcal{C})\right]\right| & =\int_{\mathbb{R}^{n}} \exp \left(-\rho^{-p}\left(\mathcal{B}_{p}^{N}(\mathcal{C}), \mathbf{X}^{T} u\right)\right) d u \\
& =\int_{\mathbb{R}^{n}} \prod_{i=1}^{N} \exp \left(-h^{p}\left(C_{i}, \mathbf{X}_{i}^{T} u\right)\right) d u \\
& =d_{p}^{N} \int_{\mathbb{R}^{n}} \mathbb{E}_{W} \prod_{i=1}^{N} \sqrt{w_{i}} \exp \left(-w_{i} h^{2}\left(C_{i}, \mathbf{X}_{i}^{T} u\right)\right) d u \\
& =d_{p}^{N} \int_{\mathbb{R}^{n}} \mathbb{E}_{W} \sqrt{w_{1} \cdots w_{N}} \exp \left(-\sum_{i=1}^{N} h^{2}\left(\sqrt{w_{i}} C_{i}, \mathbf{X}_{i}^{T} u\right)\right) d u \\
& =c_{n, 2}^{-1} d_{p}^{N} \mathbb{E}_{W} \sqrt{w_{1} \cdots w_{N}}\left|\mathbf{X}^{-T}\left[\mathcal{B}_{2}^{N}\left(\mathcal{C}_{W}\right)\right]\right|
\end{aligned}
$$

where $\mathcal{C}_{W}=\left(\sqrt{w_{1}} C_{1}, \ldots, \sqrt{w_{N}} C_{N}\right)$ and we used (6.3) again in the last equality. The result now follows from Lemmas 5.2 and 5.6 and the identity $a_{N, n, p}=c_{n, p} d_{p}^{N}$.

Remark 6.3. To see that the latter proposition implies (6.2), we take $C_{i}=\left[-e_{i}, e_{i}\right]$ and write $\mathbf{X}_{W}=\left[\sqrt{w_{1}} X_{1}, \ldots, \sqrt{w}_{N} X_{N}\right]$ so that $\mathbf{X}_{W} B_{2}^{N}=\mathbf{X} \mathcal{B}_{2}^{N}\left(\mathcal{C}_{W}^{\circ}\right)$. Ву (5.5),

$$
\left|\left(\mathbf{X}_{W} B_{2}^{N}\right)^{\circ}\right|=\omega_{n}\left(\operatorname{det}\left(\sum_{i=1}^{N} w_{i} X_{i} X_{i}^{T}\right)\right)^{-\frac{1}{2}}
$$

When $p=1$ in (6.2), $w_{i}$ is the reciprocal of an exponential random variable [73] and we have maintained this convention here, though the exact normalization is immaterial in what follows.

### 6.2 Volume via Gaussian measure for $p=0$

The set $\mathcal{Z}_{0, \mathcal{C}}^{\diamond}(\mathcal{F})$ can be treated as a limiting case of $\mathcal{Z}_{p, \mathcal{C}}^{\diamond}(\mathcal{F})$ when $p \rightarrow 0$ but it will be handy to derive a different volume formula using the pre-image representation (5.13) directly. This approach will also be helpful for $p<0$. The formula involves standard Gaussian measure $\gamma_{n}$ and negative moments of the Gaussian random vectors $b_{n, s}$ defined in (3.7).
Proposition 6.4. Let $\mathcal{C}=\left(C_{1}, \ldots, C_{N}\right), \mathcal{F}$ and $\mathbf{X}$ be as in Lemma 5.6. For $t=\left(t_{1}, \ldots, t_{N}\right)$ in $\mathbb{R}_{+}^{N}$ and $s>0$, set $\mathcal{C}_{s, t}^{\circ}=\left(\left(t_{1}^{N / s} C_{1}\right)^{\circ}, \ldots,\left(t_{N}^{N / s} C_{N}\right)^{\circ}\right)$. Then

$$
\begin{equation*}
\mathbb{E}\left|\mathcal{Z}_{0, \mathcal{C}}^{\diamond}(\mathcal{F})\right|=\lim _{s \rightarrow n^{-}} b_{n, s}^{-1} \int_{\mathbb{R}_{+}^{N}} \mathbb{E}_{\mathbf{X}} \gamma_{n}\left(\left(\mathbf{X} \mathcal{B}_{1}^{N}\left(\mathcal{C}_{s, t}^{\circ}\right)\right)^{\circ}\right) d t \tag{6.6}
\end{equation*}
$$

Proof. We will first show that for $u \in \mathbb{R}^{n} \backslash\{0\}$,

$$
\begin{equation*}
\rho^{s}\left(\mathcal{Z}_{0, \mathcal{C}}^{\diamond}(\mathcal{F}), u\right)=\int_{\mathbb{R}_{+}^{N}}\left[u \in\left(\mathbf{X} \mathcal{B}_{1}^{N}\left(\mathcal{C}_{s, t}^{\circ}\right)\right)^{\circ}\right] d t \tag{6.7}
\end{equation*}
$$

Note that

$$
\begin{aligned}
\rho^{s}\left(\mathcal{Z}_{0, \mathcal{C}}^{\diamond}(\mathcal{F}), u\right) & =\prod_{i=1}^{N} h^{-s / N}\left(C_{i}, \mathbf{X}_{i}^{T} u\right) \\
& =\int_{\mathbb{R}_{+}^{N}} \prod_{i=1}^{N}\left[u \in\left\{h^{-s / N}\left(C_{i}, \mathbf{X}_{i}^{T} \cdot\right)>t_{i}\right\}\right] d t
\end{aligned}
$$

For each $i=1, \ldots, N$, we have

$$
\left\{h^{-s / N}\left(C_{i}, \mathbf{X}_{i}^{T} \cdot\right)>t_{i}\right\}=\left\{t_{i}^{N / s} h\left(\mathbf{X}_{i} C_{i}, \cdot\right)<1\right\}=\left(t_{i}^{N / s} \mathbf{X}_{i} C_{i}\right)^{\circ}
$$

By Lemma 5.5.

$$
\bigcap_{i=1}^{N}\left(t_{i}^{N / s} \mathbf{X}_{i} C_{i}\right)^{\circ}=\left(\mathbf{X} \mathcal{B}_{1}^{N}\left(\mathcal{C}_{s, t}^{\circ}\right)\right)^{\circ}
$$

Therefore,

$$
\begin{aligned}
\rho^{s}\left(\mathcal{Z}_{0, \mathcal{C}}^{\diamond}(\mathcal{F}), u\right) & =\int_{\mathbb{R}_{+}^{N}}\left[u \in \bigcap_{i=1}^{N}\left\{h^{-s / N}\left(C_{i}, \mathbf{X}_{i}^{T} y\right)>t_{i}\right\}\right] d t \\
& =\int_{\mathbb{R}_{+}^{N}}\left[u \in\left(\mathbf{X} \mathcal{B}_{1}^{N}\left(\mathcal{C}_{s, t}^{\circ}\right)\right)^{\circ}\right] d t
\end{aligned}
$$

Let $\xi$ be a standard Gaussian random vector in $\mathbb{R}^{n}$ and $s \in(0, n)$. Using Lemma 3.4 and (6.7),

$$
\begin{equation*}
\mathbb{E}_{\xi} \rho^{s}\left(\mathcal{Z}_{0, \mathcal{C}}^{\diamond}(\mathcal{F}), \xi\right)=\int_{\mathbb{R}_{+}^{N}} \gamma_{n}\left(\left(\mathbf{X} \mathcal{B}_{1}^{N}\left(\mathcal{C}_{s, t}^{\circ}\right)\right)^{\circ}\right) d t \tag{6.8}
\end{equation*}
$$

Assume first that

$$
\begin{equation*}
\mathbb{E}\left|\mathcal{Z}_{0, \mathcal{C}}^{\diamond}(\mathcal{F})\right|=\omega_{n} \mathbb{E} \int_{S^{n-1}} \rho^{n}\left(\mathcal{Z}_{0, \mathcal{C}}^{\diamond}(\mathcal{F}), u\right) d u<\infty \tag{6.9}
\end{equation*}
$$

Then $\rho\left(\mathcal{Z}_{0, \mathcal{C}}^{\diamond}(\mathcal{F}), \cdot\right) \in L_{n}\left(S^{n-1}, \sigma\right)$ a.s.. Arguing as in the proof of Lemma 3.4,

$$
\begin{equation*}
\left.\int_{S^{n-1}} \rho^{s}\left(\mathcal{Z}_{0, \mathcal{C}}^{\diamond}(\mathcal{F}), u\right) d u \rightarrow \int_{S^{n-1}} \rho^{n}\left(\mathcal{Z}_{0, \mathcal{C}}^{\diamond}(\mathcal{F}), u\right) d u \quad \text { a.s. }\right) \tag{6.10}
\end{equation*}
$$

and the convergence is dominated by $1+\omega_{n}^{-1}\left|\mathcal{Z}_{0, \mathcal{C}}^{\diamond}(\mathcal{F})\right|$ (cf. (3.9). Thus, using (3.6), we get

$$
\begin{aligned}
\mathbb{E}\left|\mathcal{Z}_{0, \mathcal{C}}^{\diamond}(\mathcal{F})\right| & =\omega_{n} \mathbb{E}_{\mathbf{X}} \lim _{s \rightarrow n^{-}} \int_{S^{n-1}} \rho^{s}\left(\mathcal{Z}_{0, \mathcal{C}}^{\diamond}(\mathcal{F}), u\right) d u \\
& =\omega_{n} \lim _{s \rightarrow n^{-}} \mathbb{E}_{\mathbf{X}} \int_{S^{n-1}} \rho^{s}\left(\mathcal{Z}_{0, \mathcal{C}}^{\diamond}(\mathcal{F}), u\right) d u \\
& =\omega_{n} \lim _{s \rightarrow n^{-}} b_{n, s}^{-1} \mathbb{E}_{\mathbf{X}} \mathbb{E}_{\xi} \rho^{s}\left(\mathcal{Z}_{0, \mathcal{C}}^{\diamond}(\mathcal{F}), \xi\right)
\end{aligned}
$$

Applying (6.8) gives the proposition when $\mathbb{E}\left|\mathcal{Z}_{0, \mathcal{C}}^{\diamond}(\mathcal{F})\right|$ is finite. The proposition also remains valid when $\mathbb{E}\left|\mathcal{Z}_{0, \mathcal{C}}^{\diamond}(\mathcal{F})\right|$ is infinite. Indeed, we can replace $S^{n-1}$ in (6.10) by $\left\{\rho\left(\mathcal{Z}_{0, \mathcal{C}}^{\diamond}(\mathcal{F}), \cdot\right) \geq 1\right\}$, in which case the convergence is monotone and both sides of (6.6) are divergent.

### 6.3 Volume via Gaussian measure for $p<0$

We will start with a volume formula for the non-random bodies $Z_{p, C}^{\diamond}(f)$.
Proposition 6.5. Let $f \in \mathcal{P}_{n}$ and $C \in \mathcal{K}_{s}^{m}$, where $m \geq 1$. Let $p \in(-1,0)$ and set $n(p)=$ $n /|p| \in \mathbb{N}$. Let $\mathbf{X}$ be an $n \times n(p) m$ random matrix with independent columns distributed according to $f$. For $\ell \in \mathbb{N}$, let $p_{\ell}=p(1-1 /(\ell n))$. For $t_{1}, \ldots, t_{n(p)}>0$ and $\ell \in \mathbb{N}$, let $\mathcal{C}_{t, p_{\ell}}^{\circ}=\left(\left(t_{1}^{1 /\left|p_{\ell}\right|} C\right)^{\circ}, \ldots,\left(t_{n(p)}^{1 /\left|p_{\ell}\right|} C\right)^{\circ}\right)$. Then

$$
\begin{equation*}
\left|Z_{p, C}^{\diamond}(f)\right|=\lim _{\ell \rightarrow \infty} b_{n, n-1 / \ell}^{-1} \int_{\mathbb{R}_{+}^{n(p)}} \mathbb{E}_{\mathbf{X}} \gamma_{n}\left(\left(\mathbf{X} \mathcal{B}_{1}^{n(p)}\left(\mathcal{C}_{t, p_{\ell}}^{\circ}\right)\right)^{\circ}\right) d t \tag{6.11}
\end{equation*}
$$

Proof. Fix $k \in \mathbb{N}$. Let $\mathbf{X}_{1}, \ldots, \mathbf{X}_{k}$ be independent $n \times m$ random matrices with independent columns drawn from $f$. We will first show that for $u \in \mathbb{R}^{n} \backslash\{0\}$,

$$
\begin{equation*}
\rho^{k|p|}\left(Z_{p, C}^{\diamond}(f), u\right)=\int_{\mathbb{R}_{+}^{k}} \mathbb{E}\left[u \in\left(\mathbf{X} \mathcal{B}_{1}^{k}\left(\mathcal{C}_{t, p}^{\circ}\right)\right)^{\circ}\right] d t \tag{6.12}
\end{equation*}
$$

Note that

$$
\rho^{k|p|}\left(Z_{p, C}^{\diamond}(f), u\right)=\left(\mathbb{E}_{\mathbf{X}_{1}} h^{-|p|}\left(C, \mathbf{X}_{1}^{T} u\right)\right)^{k}=\mathbb{E}_{\mathbf{X}_{1}} \cdots \mathbb{E}_{\mathbf{X}_{k}} \prod_{i=1}^{k} h^{-|p|}\left(C, \mathbf{X}_{i}^{T} u\right)
$$

and

$$
\prod_{i=1}^{k} h^{-|p|}\left(C, \mathbf{X}_{i}^{T} u\right)=\int_{\mathbb{R}_{+}^{k}} \prod_{i=1}^{k}\left[u \in\left\{h^{-|p|}\left(C, \mathbf{X}_{i}^{T} \cdot\right)>t_{i}\right\}\right] d t
$$

For each $i=1, \ldots, k$,

$$
\left\{h^{-|p|}\left(C, \mathbf{X}_{i}^{T} \cdot\right)>t_{i}\right\}=\left\{h\left(\mathbf{X}_{i} C, \cdot\right)<t_{i}^{-1 /|p|}\right\}=\left(t_{i}^{1 /|p|} \mathbf{X}_{i} C\right)^{\circ}
$$

By Lemma 5.5.

$$
\bigcap_{i=1}^{k}\left(t_{i}^{1 /|p|} \mathbf{X}_{i} C\right)^{\circ}=\left(\mathbf{X} \mathcal{B}_{1}^{k}\left(\mathcal{C}_{t, p}^{\circ}\right)\right)^{\circ}
$$

Therefore,

$$
\mathbb{E}_{\mathbf{X}_{1}} \cdots \mathbb{E}_{\mathbf{X}_{k}} \prod_{i=1}^{k} h^{-|p|}\left(C, \mathbf{X}_{i}^{T} u\right)=\int_{\mathbb{R}_{+}^{k}} \mathbb{E}_{\mathbf{X}}\left[u \in\left(\mathbf{X} \mathcal{B}_{1}^{k}\left(\mathcal{C}_{t, p}^{\circ}\right)\right)^{\circ}\right] d t
$$

which implies (6.12). If $\xi$ is a standard Gaussian vector in $\mathbb{R}^{n}$, then

$$
\begin{equation*}
\mathbb{E}_{\xi} \rho^{k|p|}\left(Z_{p, C}^{\diamond}(f), \xi\right)=\int_{\mathbb{R}_{+}^{k}} \mathbb{E}_{\mathbf{X}} \gamma_{n}\left(\left(\mathbf{X} \mathcal{B}_{1}^{k}\left(\mathcal{C}_{t, p}^{\circ}\right)\right)^{\circ}\right) d t \tag{6.13}
\end{equation*}
$$

Note that $n(p)=\frac{n}{|p|}=\frac{n-1 / \ell}{\left|p_{\ell}\right|}$. It remains to apply Lemma 3.4 with $K=Z_{p, C}^{\diamond}(f)$ and the increasing sequence $K_{\ell}=Z_{p_{\ell}, C}^{\diamond}(f)$ (cf. Lemma 4.1(a)). With an eye on (6.13) with $p_{\ell}$ and $n(p)$ in place of $p$ and $k$, respectively, we conclude by

$$
\left|Z_{p, C}^{\diamond}(f)\right|=\omega_{n} \lim _{\ell \rightarrow \infty} b_{n, n-1 / \ell}^{-1} \mathbb{E}_{\xi} \rho^{n(p)\left|p_{\ell}\right|}\left(Z_{p_{\ell}, C}^{\diamond}(f), \xi\right)
$$

### 6.4 Radial function representation for $p<0$

The volume formulas for $\mathcal{Z}_{0, \mathcal{C}}^{\diamond}(\mathcal{F})$ and $Z_{p, C}^{\diamond}(f)$ each rely on a representation of the radial function as a mixture of indicator functions of origin-symmetric convex bodies. In this subsection, we develop an analogous representation for the radial function of the empirical bodies $\mathcal{Z}_{p, \mathcal{C}}^{\diamond}(\mathcal{F})$ for $p<0$ and $n /|p| \in \mathbb{N}$. A similar volume formula for $\mathcal{Z}_{p, \mathcal{C}}^{\diamond}(\mathcal{F})$ holds but the notation becomes lengthy, so we will derive only the radial function for later use.

To fix the notation, for $k \in \mathbb{N}$, we let $\underline{\mathbf{k}}=\left(k_{1}, \ldots, k_{N}\right) \in[k]^{N}$ and define $S(\underline{\mathbf{k}})=k_{1}+\ldots+k_{N}$ and $m(\underline{\mathbf{k}})=\left\{i \in[N]: k_{i} \neq 0\right\}$; we write $|m(\underline{\mathbf{k}})|$ for the cardinality of $m(\underline{\mathbf{k}})$.

Proposition 6.6. Let $\mathcal{C}=\left(C_{1}, \ldots, C_{N}\right), \mathcal{F}$ and $\mathbf{X}$ be as in Lemma 5.6. Let $p \in(-1,0)$ and $k \in \mathbb{N}$. Then for $u \in \mathbb{R}^{n} \backslash\{0\}$,

$$
\rho^{k|p|}\left(\mathcal{Z}_{p, \mathcal{C}}^{\diamond}(\mathcal{F}), u\right)=N^{k} \sum_{\substack{\mathbf{k} \in[k)^{N} \\ S \\(\underline{\mathbf{k}})=k}}\binom{k}{\underline{\mathbf{k}}} \int_{\mathbb{R}_{+}^{|m(\underline{k})|}}\left[u \in\left(\mathbf{X}_{\underline{\mathbf{k}}} \mathcal{B}_{1}^{|m(\underline{\mathbf{k}})|}\left(\mathcal{C}_{\underline{\mathbf{k}}, t, p}^{\circ}\right)\right)^{\circ}\right] d t
$$

where $\binom{k}{\underline{\mathbf{k}}}=\frac{k!}{k_{1}!\cdots k_{N}!}, \mathbf{X}_{\underline{\mathbf{k}}}=\left[\mathbf{X}_{k_{i}}\right]_{i \in m(\mathbf{k})}$ and $\mathcal{C}_{\underline{\mathbf{k}}, t, p}^{\circ}=\left(\left(t_{i}^{\frac{1}{k_{i}|p|}} C_{i}\right)^{\circ}\right)_{i \in m(\underline{\mathbf{k}})}$.
Proof. Using the fact that $k \in \mathbb{N}$, we have for any $u \in \mathbb{R}^{n}$,

$$
\rho^{k|p|}\left(\mathbf{X}^{-T}\left[\mathcal{B}_{p}^{N}(\mathcal{C})\right], u\right)=\sum_{\substack{\underline{\mathbf{k}} \in[k]^{N} \\ S(\underline{\mathbf{k}})=k}}\binom{k}{\underline{\mathbf{k}}} \prod_{i \in m(\underline{\mathbf{k}})} h^{-k_{i}|p|}\left(C_{i}, \mathbf{X}_{i}^{T} u\right) .
$$

Fix $\underline{\mathbf{k}}=\left(k_{1}, \ldots, k_{N}\right)$ with $S(\underline{\mathbf{k}})=k$. Then

$$
\prod_{i \in m(\underline{\mathbf{k}})} h^{-k_{i}|p|}\left(C_{i}, \mathbf{X}_{i}^{T} u\right)=\int_{\mathbb{R}_{+}^{|m(\underline{\mathbf{k}})|}} \prod_{i \in m(\underline{\mathbf{k}})}\left[u \in\left\{h^{-k_{i}|p|}\left(C_{i}, \mathbf{X}_{i}^{T} \cdot\right)>t_{i}\right\}\right] d t
$$

For each $i \in m(\underline{\mathbf{k}})$,

$$
\left\{h^{-k_{i}|p|}\left(C_{i}, \mathbf{X}_{i}^{T} \cdot\right)>t_{i}\right\}=\left\{h\left(\mathbf{X}_{i} C_{i}, \cdot\right)<t_{i}^{-\frac{1}{k_{i}|p|}}\right\}=\left(t_{i}^{\frac{1}{k_{i} \mid p}} \mathbf{X}_{i} C_{i}\right)^{\circ}
$$

By Lemma 5.5.

$$
\bigcap_{i \in m(\underline{\mathbf{k}})}\left(t_{i}^{\frac{1}{k_{i}|p|}} \mathbf{X}_{i} C_{i}\right)^{\circ}=\left(\mathbf{X}_{\underline{\mathbf{k}}} \mathcal{B}_{1}^{|m(\underline{\mathbf{k}})|}\left(\mathcal{C}_{\underline{\mathbf{k}}, t, p}^{\circ}\right)\right)^{\circ}
$$

Thus the proposition follows from

$$
\prod_{i \in m(\underline{\mathbf{k}})} h^{-k_{i}|p|}\left(C_{i}, \mathbf{X}_{i}^{T} u\right)=\int_{\mathbb{R}_{+}^{|m(\underline{\mathbf{k}})|}}\left[u \in\left(\mathbf{X}_{\underline{\mathbf{k}}} \mathcal{B}_{1}^{|m(\underline{\mathbf{k}})|}\left(\mathcal{C}_{\underline{\mathbf{k}}, t, p}^{\circ}\right)\right)^{\circ}\right] d t
$$

## 7 Main proofs

Proof of Theorem 2.2. Suppose that $\mathbf{X}$ and $\mathbf{X}^{\#}$ are $n \times M$ random matrices with independent columns drawn from $\mathcal{F}=\left(f_{i j}\right) \subseteq \mathcal{P}_{n}$ and $\mathcal{F}^{\#}=\left(f_{i j}^{*}\right)$ respectively, where $M=m_{1}+\ldots+m_{N}$. Suppose that each $f_{i j}$ is supported on a Euclidean ball $R B_{2}^{n}$. Denote the expectation in $\mathbf{X}$ and $\mathbf{X}^{\#}$ by $\mathbb{E}_{\mathbf{X}}$ and $\mathbb{E}_{\mathbf{X} \#}$, respectively.

We will use Theorem [5.1] combined with the volume formulas for $\mathcal{Z}_{p, \mathcal{C}}^{\diamond}(\mathcal{F})$ as indicated. For $p \geq 1$, we have by Remark 5.7.

$$
\begin{aligned}
\mathbb{E}\left|\mathcal{Z}_{p, \mathcal{C}}^{\diamond}(\mathcal{F})\right| & =\mathbb{E}_{\mathbf{X}}\left|N^{1 / p}\left(\mathbf{X} \mathcal{B}_{q}^{N}\left(\mathcal{C}^{\circ}\right)\right)^{\circ}\right| \\
& \leq \mathbb{E}_{\mathbf{X} \#}\left|N^{1 / p}\left(\mathbf{X}^{\#} \mathcal{B}_{q}^{N}\left(\mathcal{C}^{\circ}\right)\right)^{\circ}\right| \\
& =\mathbb{E}\left|\mathcal{Z}_{p, \mathcal{C}}^{\diamond}\left(\mathcal{F}^{\#}\right)\right|
\end{aligned}
$$

For $p \in(0,1)$, using Proposition 6.2 and Fubini's theorem,

$$
\mathbb{E}\left|\mathcal{Z}_{p, \mathcal{C}}^{\diamond}(\mathcal{F})\right|=a_{N, n, p} \mathbb{E}_{W} \mathbb{E}_{\mathbf{X}} \sqrt{w_{1} \cdots w_{N}}\left|\left(\mathbf{X} \mathcal{B}_{2}^{N}\left(\mathcal{C}_{W}^{\circ}\right)\right)^{\circ}\right|
$$

$$
\begin{aligned}
& \leq a_{N, n, p} \mathbb{E}_{W} \mathbb{E}_{\mathbf{X} \#} \sqrt{w_{1} \cdots w_{N}}\left|\left(\mathbf{X}^{\#} \mathcal{B}_{2}^{N}\left(\mathcal{C}_{W}^{\circ}\right)\right)^{\circ}\right| \\
& =\mathbb{E}\left|\mathcal{Z}_{p, \mathcal{C}}^{\diamond}\left(\mathcal{F}^{\#}\right)\right|
\end{aligned}
$$

For $p=0$, we apply Proposition 6.4 to get

$$
\begin{aligned}
\mathbb{E}\left|\mathcal{Z}_{0, \mathcal{C}}^{\diamond}(\mathcal{F})\right| & =\lim _{s \rightarrow n^{-}} \int_{\mathbb{R}_{+}^{N}} \mathbb{E}_{\mathbf{X}} \gamma_{n}\left(\left(\mathbf{X} \mathcal{B}_{1}^{N}\left(\mathcal{C}_{s, t}^{\circ}\right)\right)^{\circ}\right) d t \\
& \leq \lim _{s \rightarrow n^{-}} \int_{\mathbb{R}_{+}^{N}} \mathbb{E}_{\mathbf{X} \#} \gamma_{n}\left(\left(\mathbf{X}^{\#} \mathcal{B}_{1}^{N}\left(\mathcal{C}_{s, t}^{\circ}\right)\right)^{\circ}\right) d t \\
& =\mathbb{E}\left|\mathcal{Z}_{0, \mathcal{C}}^{\diamond}\left(\mathcal{F}^{\#}\right)\right|
\end{aligned}
$$

When the $C_{i}$ 's are identical, we have by Proposition 5.10,

$$
\begin{equation*}
\left|Z_{p, C}^{\diamond}(f)\right|=\lim _{N \rightarrow \infty} \mathbb{E}\left|\mathcal{Z}_{p, \mathcal{C}_{N}}^{\diamond}\left(\mathcal{F}_{N}\right)\right| \tag{7.1}
\end{equation*}
$$

which proves (2.9) for $f$ compactly supported. For a general $f \in \mathcal{P}_{n}$, we define $\left\{\phi^{(k)}\right\}$ as in Lemma 4.1 By Fatou's lemma and the compactly supported case,

$$
\left|Z_{p, C}^{\diamond}(f)\right| \leq \liminf _{k \rightarrow \infty}\left|Z_{p, C}^{\diamond}\left(\phi^{(k)}\right)\right| \leq \liminf _{k \rightarrow \infty}\left|Z_{p, C}^{\diamond}\left(\left(\phi^{(k)}\right)^{*}\right)\right|=\left|Z_{p, C}^{\diamond}\left(f^{*}\right)\right|
$$

where the last equality holds as each set $Z_{p, C}^{\diamond}\left(\left(\phi^{(k)}\right)^{*}\right)$ is a Euclidean ball and the convergence is ensured by (3.4).

Lastly, we turn to the case when $\mathcal{F}=\left(f_{i j}\right)$ consists of functions that are not supported on a common compact set. In the notation of Lemma 4.1(d), we set $\varphi_{i j}^{(k)}=\left.f_{i j}\right|_{k B_{2}^{n}}$ and $\phi_{i j}^{(k)}=\varphi_{i j}^{(k)} / \int \varphi_{i j}^{(k)}$ and set $\mathcal{F}_{k}=\left(\phi_{i j}^{(k)}\right)$. Then

$$
\mathbb{E}\left|\mathcal{Z}_{p, C}^{\diamond}\left(\mathcal{F}_{k}\right)\right|=\int_{\left(\left(\mathbb{R}^{n}\right)^{m}\right)^{N}} \int_{S^{n-1}}\left(\frac{1}{N} \sum_{i=1}^{N} h^{p}\left(C,\left(\left\langle x_{i j}, u\right\rangle\right)_{j=1}^{m_{i}}\right)\right)^{n / p} \prod_{i, j} \phi_{i j}^{(k)}\left(x_{i j}\right) d u d \bar{x}
$$

Using $\int \varphi_{i j}^{(k)} \rightarrow \int f_{i j}=1$ and monotone convergence for $\varphi_{i j}^{(k)}$,

$$
\mathbb{E}\left|\mathcal{Z}_{p, \mathcal{C}}^{\diamond}\left(\mathcal{F}_{k}\right)\right|=\lim _{k \rightarrow \infty} \mathbb{E}\left|\mathcal{Z}_{p, \mathcal{C}}^{\diamond}\left(\mathcal{F}_{k}\right)\right| \leq \lim _{k \rightarrow \infty} \mathbb{E}\left|\mathcal{Z}_{p, \mathcal{C}}^{\diamond}\left(\mathcal{F}_{k}^{\#}\right)\right|=\mathbb{E}\left|\mathcal{Z}_{p, \mathcal{C}}^{\diamond}\left(\mathcal{F}^{\#}\right)\right|
$$

Proof of Theorem 2.1. Taking $C=C_{i}=[-1,1]$ and $\mathcal{F}=(f)$ gives $\mathcal{Z}_{p, N}^{\diamond}(f)=\mathcal{Z}_{p, \mathcal{C}}^{\diamond}(\mathcal{F})$ and $Z_{p, C}^{\diamond}(f)=Z_{p}^{\diamond}(f)$, hence Theorem 2.1 follows from Theorem 2.2,

Proof of Theorem 2.3. Let $\mathbf{X}$ and $\mathbf{X}^{\#}$ be $n \times n(p) m$ random matrices with i.i.d. columns drawn from $f$ and $f^{*}$, respectively. By Proposition 6.5 and Theorem 5.1.

$$
\begin{aligned}
\left|Z_{p, C}^{\diamond}(f)\right| & =\lim _{\ell \rightarrow \infty} b_{n, n-1 / \ell}^{-1} \int_{\mathbb{R}_{+}^{n(p)}} \mathbb{E}_{\mathbf{X}} \gamma_{n}\left(\left(\mathbf{X} \mathcal{B}_{1}^{n(p)}\left(\mathcal{C}_{t, p_{\ell}}^{\circ}\right)\right)^{\circ}\right) d t \\
& \leq \lim _{\ell \rightarrow \infty} b_{n, n-1 / \ell}^{-1} \int_{\mathbb{R}_{+}^{n(p)}} \mathbb{E}_{\mathbf{X} \#} \gamma_{n}\left(\left(\mathbf{X}^{\#} \mathcal{B}_{1}^{n(p)}\left(\mathcal{C}_{t, p_{\ell}}^{\circ}\right)\right)^{\circ}\right) d t \\
& =\left|Z_{p, C}^{\diamond}\left(f^{*}\right)\right|
\end{aligned}
$$

Next, we prove (2.12). Fix origin-symmetric convex bodies $C_{1}, \ldots, C_{N}$ with $\operatorname{dim}\left(C_{i}\right)=m_{i} \geq$ $n+1$. Set $M=m_{1}+\ldots+m_{N}$. Suppose that $\mathbf{X}$ and $\mathbf{X}^{\#}$ are $n \times M$ random matrices with
independent columns drawn from $\mathcal{F}=\left(f_{i j}\right)$ and $\mathcal{F}^{\#}=\left(f_{i j}^{*}\right)$, respectively. Fix $k \in \mathbb{N}$ and $p \in[-1,0)$ with $k|p|<n$.

Assume first that $f$ is supported on a Euclidean ball $R B_{2}^{n}$. By Proposition 5.10

$$
\mathbb{E}\left|\mathcal{Z}_{p, \mathcal{C}}^{\diamond}(\mathcal{F})\right|=\omega_{n} \mathbb{E}_{\mathbf{X}} \int_{S^{n-1}} \rho^{n}\left(\mathbf{X}^{-T}\left[\mathcal{B}_{p}^{N}(\mathcal{C})\right], u\right) d u<\infty
$$

Applying Proposition 6.6 for a standard Gaussian random vector $\xi$ in $\mathbb{R}^{n}$, we have

$$
\mathbb{E}_{\xi} \rho^{k|p|}\left(\mathcal{Z}_{p, \mathcal{C}}^{\diamond}(\mathcal{F}), \xi\right)=\sum_{\substack{\mathbf{k} \in[k]^{N} \\ S(\underline{\mathbf{k}})=k}}\binom{k}{\underline{\mathbf{k}}} \int_{\mathbb{R}_{+}^{|m(\underline{\mathbf{k}})|}} \gamma_{n}\left(\left(\mathbf{X}_{\underline{\mathbf{k}}} \mathcal{B}_{1}^{|m(\underline{\mathbf{k}})|}\left(\mathcal{C}_{\underline{\mathbf{k}}, t}^{\circ}\right)\right)^{\circ}\right) d t
$$

Fix $\underline{\mathbf{k}}=\left(k_{1}, \ldots, k_{N}\right) \in[k]^{N}$ with $S(\underline{\mathbf{k}})=k$ and $t_{i} \in(0, \infty)$ for $i \in m(\underline{\mathbf{k}})$. By Theorem 5.1,

$$
\left.\mathbb{E}_{\mathbf{X}_{\underline{\mathbf{k}}}} \gamma_{n}\left(\left(\mathbf{X}_{\underline{\mathbf{k}}} \mathcal{B}_{1}^{|m(\underline{\mathbf{k}})|}\left(\mathcal{C}_{\underline{\mathbf{k}}, t}^{\circ}\right)\right)^{\circ}\right) \leq \mathbb{E}_{\mathbf{X}_{\underline{\mathbf{k}}}^{\#} \gamma_{n}}\left(\left(\mathbf{X}_{\underline{\mathbf{k}}}^{\#} \mathcal{B}_{1}^{\mid m(\underline{\mathbf{k}} \mid} \mid \mathcal{C}_{\underline{\mathbf{k}}, t}^{\circ}\right)\right)^{\circ}\right) .
$$

Consequently,

$$
\begin{equation*}
\mathbb{E}_{\mathbf{X}} \mathbb{E}_{\xi} \rho^{k|p|}\left(\mathcal{Z}_{p, \mathcal{C}}^{\diamond}(\mathcal{F}), \xi\right) \leq \mathbb{E}_{\mathbf{X} \#} \mathbb{E}_{\xi} \rho^{k|p|}\left(\mathcal{Z}_{p, \mathcal{C}}^{\diamond}\left(\mathcal{F}^{\#}\right), \xi\right) \tag{7.2}
\end{equation*}
$$

As in the proof of Proposition 6.5, when $n /|p| \in \mathbb{N}$, we choose $p_{\ell} \in \mathbb{Q} \cap(p, 0)$ such that $\frac{n}{|p|}=\frac{n-1 / \ell}{\left|p_{\ell}\right|}$ for $j \in \mathbb{N}$. For $u \in S^{n-1}$, we have

$$
\rho\left(\mathcal{Z}_{p_{\ell}, \mathcal{C}}^{\diamond}(\mathcal{F}), u\right) \rightarrow \rho\left(\mathcal{Z}_{p, \mathcal{C}}^{\diamond}(\mathcal{F}), u\right) \quad \text { (a.s.) }
$$

As in the proof of Lemma 3.4 using (3.9) with $K_{\ell}=\mathcal{Z}_{p_{\ell}, \mathcal{C}}^{\diamond}(\mathcal{F})$, we have

$$
\int_{S^{n-1}} \rho^{\ell\left|p_{\ell}\right|}\left(\mathcal{Z}_{p_{\ell}, \mathcal{C}}^{\diamond}(\mathcal{F}), u\right) d u \rightarrow \int_{S^{n-1}} \rho^{n}\left(\mathcal{Z}_{p, \mathcal{C}}^{\diamond}(\mathcal{F}), u\right) d u \quad \text { (a.s.) }
$$

and the convergence is dominated by $1+\omega_{n}^{-1}\left|\mathcal{Z}_{p, \mathcal{C}}^{\diamond}(\mathcal{F})\right|$. Thus,

$$
\begin{aligned}
\mathbb{E}_{\mathbf{X}}\left|\mathcal{Z}_{p, \mathcal{C}}^{\diamond}(\mathcal{F})\right| & =\omega_{n} \mathbb{E}_{\mathbf{X}} \lim _{\ell \rightarrow \infty} \int_{S^{n-1}} \rho^{\ell\left|p_{\ell}\right|}\left(\mathcal{Z}_{p_{\ell}, \mathcal{C}}^{\diamond}(\mathcal{F}), u\right) d u \\
& =\omega_{n} \lim _{\ell \rightarrow \infty} \mathbb{E}_{\mathbf{X}} \int_{S^{n-1}} \rho^{n-1 / \ell}\left(\mathcal{Z}_{p_{\ell}, \mathcal{C}}^{\diamond}(\mathcal{F}), u\right) d u \\
& =\omega_{n} \lim _{\ell \rightarrow \infty} b_{n, n-1 / \ell}^{-1} \mathbb{E}_{\mathbf{X}} \mathbb{E}_{\xi} \rho^{n-1 / \ell}\left(\mathcal{Z}_{p_{\ell}, \mathcal{C}}^{\diamond}(\mathcal{F}), \xi\right),
\end{aligned}
$$

where $b_{n, n-1 / \ell}$ is the constant in (3.7). The same identities apply for $\mathbf{X}^{\#}$ and $\mathcal{F}^{\#}$. Thus applying (7.2), we get

$$
\mathbb{E}\left|\mathcal{Z}_{p, \mathcal{C}}^{\diamond}(\mathcal{F})\right| \leq \mathbb{E}\left|\mathcal{Z}_{p, \mathcal{C}}^{\diamond}\left(\mathcal{F}^{\#}\right)\right|
$$

Lastly, we can remove the assumption that the functions are compactly supported by arguing as in the proof of Theorem 2.2.

Proof of Corollary 2.4. By Proposition 5.11 and Theorem [2.3,

$$
\begin{equation*}
\mathbb{E}\left|\mathcal{I}_{|p|, N}^{\alpha}(f)\right|=\lim _{m \rightarrow \infty} \mathbb{E}\left|\mathcal{Z}_{p, \mathcal{C}_{m}^{\alpha}}^{\diamond}\left(\mathcal{F}_{m}\right)\right| \leq \lim _{m \rightarrow \infty} \mathbb{E}\left|\mathcal{Z}_{p, \mathcal{C}_{m}^{\alpha}}^{\diamond}\left(\mathcal{F}_{m}^{\#}\right)\right|=\mathbb{E}\left|\mathcal{I}_{|p|, N}^{\alpha}\left(f^{*}\right)\right| \tag{7.3}
\end{equation*}
$$

Using (7.3) with Proposition 5.12 we get

$$
\begin{equation*}
\left|I_{|p|}^{\alpha}(f)\right|=\lim _{N \rightarrow \infty} \mathbb{E}\left|\mathcal{I}_{|p|, N}^{\alpha}(f)\right| \leq \lim _{N \rightarrow \infty} \mathbb{E}\left|\mathcal{I}_{|p|, N}^{\alpha}\left(f^{*}\right)\right|=\left|I_{|p|}^{\alpha}\left(f^{*}\right)\right| \tag{7.4}
\end{equation*}
$$

Finally, we apply Proposition 4.2 and (7.4) for $p=-1$, to obtain

$$
|I(f)|=\lim _{\alpha \rightarrow 0^{+}}\left|\left(2 s_{\alpha}\right)^{-1} I_{\alpha}(f)\right| \leq \lim _{\alpha \rightarrow 0^{+}}\left|\left(2 s_{\alpha}\right)^{-1} I_{\alpha}\left(f^{*}\right)\right|=\left|I\left(f^{*}\right)\right|
$$

Proof of Theorem 2.6. We have reduced Theorems 2.1]to 2.3] and Corollaries [2.4, 2.5 to a suitable application of (5.1) in Theorem 5.1. When the convex bodies $C_{1}, \ldots, C_{N}$ are unconditional, we can instead apply (5.2) in Theorem 5.1.

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