

COMPLEX INTERSECTION BODIES

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ABSTRACT. We introduce complex intersection bodies and show that their properties and applications are similar to those of their real counterparts. In particular, we generalize Busemann's theorem to the complex case by proving that complex intersection bodies of symmetric complex convex bodies are also convex. Other results include stability in the complex Busemann-Petty problem for arbitrary measures and the corresponding hyperplane inequality for measures of complex intersection bodies.

1. INTRODUCTION

The concept of an intersection body was introduced by Lutwak [37], as part of his dual Brunn-Minkowski theory. In particular, these bodies played an important role in the solution of the Busemann-Petty problem. Many results on intersection bodies have appeared in recent years (see [10, 22, 34] and references there), but almost all of them apply to the real case. The goal of this paper is to extend the concept of an intersection body to the complex case.

Let K and L be origin symmetric star bodies in \mathbb{R}^n . Following [37], we say that K is the *intersection body of L* if the radius of K in every direction is equal to the volume of the central hyperplane section of L perpendicular to this direction, i.e. for every $\xi \in S^{n-1}$,

$$\|\xi\|_K^{-1} = |L \cap \xi^\perp|, \tag{1}$$

where $\|x\|_K = \min\{a \geq 0 : x \in aK\}$, $\xi^\perp = \{x \in \mathbb{R}^n : (x, \xi) = 0\}$, and $|\cdot|$ stands for volume. By a theorem of Busemann [8] the intersection body of an origin symmetric convex body is also convex. However, intersection bodies of convex bodies form just a small part of the class of intersection bodies. In particular, by results of Hensley [20] and Borell [4], intersection bodies of symmetric convex bodies are isomorphic to an ellipsoid, i.e. $d_{BM}(I(K), B_2^n) \leq c$ where d_{BM} is the Banach-Mazur distance and $c > 0$ is a universal constant.

The right-hand side of (1) can be written using the polar formula for volume:

$$\|\xi\|_K^{-1} = \frac{1}{n-1} \int_{S^{n-1} \cap \xi^\perp} \|\theta\|_L^{-n+1} d\theta = \frac{1}{n-1} \mathcal{R}(\|\cdot\|_L^{-n+1})(\xi),$$

where the operator $\mathcal{R} : C(S^{n-1}) \rightarrow C(S^{n-1})$ is the spherical Radon transform defined by

$$\mathcal{R}f(\xi) = \int_{S^{n-1} \cap \xi^\perp} f(x) dx.$$

This means that a star body K is the intersection body of a star body if and only if the function $\|\cdot\|_K^{-1}$ is the spherical Radon transform of a continuous positive function on S^{n-1} .

A more general class of bodies was introduced in [18]. A star body K in \mathbb{R}^n is called an *intersection body* if there exists a finite Borel measure μ on the sphere S^{n-1} so that $\|\cdot\|_K^{-1} = \mathcal{R}\mu$ as functionals on $C(S^{n-1})$, i.e. for every continuous function f on S^{n-1} ,

$$\int_{S^{n-1}} \|x\|_K^{-1} f(x) dx = (\mathcal{R}\mu, f) = \int_{S^{n-1}} \mathcal{R}f(x) d\mu(x). \quad (2)$$

We introduce complex intersection bodies along the same lines. In Section 2 we define complex intersection bodies of complex star bodies, and in Section 7 we study complex intersection bodies of convex bodies. While the complex version of Busemann's theorem requires a serious effort, the extension of the Hensley-Borell theorem to the complex case follows from a result of Ball [2]. In Section 3 we prove that the complex spherical Radon transform and the Fourier transform of distributions coincide (up to a constant) on a class of $(-2n+2)$ -homogeneous functions on \mathbb{R}^{2n} with symmetries determined by the complex structure. A similar result in the real case was crucial for the study of real intersection bodies. We use this result in Section 4, where we define complex intersection bodies and prove a Fourier characterization of intersection bodies: an origin symmetric complex star body K in \mathbb{R}^{2n} is a complex intersection body if and only if the function $\|\cdot\|_K^{-2}$ represents a positive definite distribution. We use this characterization in Section 5 to show that the class of complex intersection bodies coincides with the class of real 2-intersection bodies in \mathbb{R}^{2n} and, at the same time, with the class of generalized 2-intersection bodies, provided that bodies from the real classes possess symmetries determined by the complex structure of \mathbb{R}^{2n} . The latter allows to extend to the complex case a result of Goodey and Weil [19] by showing that all symmetric complex intersection bodies can be obtained as limits in the radial metric of complex radial sums of ellipsoids. Finally, Section 6 deals with stability in the

complex Busemann-Petty problem for arbitrary measures and related hyperplane inequalities.

2. COMPLEX INTERSECTION BODIES OF STAR BODIES

The theory of real convex bodies goes back to ancient times and continues to be a very active field now. The situation with complex convex bodies is different, as no systematic studies of these bodies have been carried out, and results appear only occasionally; see for example [31, 35, 1, 42, 49, 50].

origin symmetric convex bodies in \mathbb{C}^n are the unit balls of norms on \mathbb{C}^n . We denote by $\|\cdot\|_K$ the norm corresponding to the body K :

$$K = \{z \in \mathbb{C}^n : \|z\|_K \leq 1\}.$$

In order to define volume, we identify \mathbb{C}^n with \mathbb{R}^{2n} using the standard mapping

$$\xi = (\xi_1, \dots, \xi_n) = (\xi_{11} + i\xi_{12}, \dots, \xi_{n1} + i\xi_{n2}) \mapsto (\xi_{11}, \xi_{12}, \dots, \xi_{n1}, \xi_{n2}).$$

Since norms on \mathbb{C}^n satisfy the equality

$$\|\lambda z\| = |\lambda| \|z\|, \quad \forall z \in \mathbb{C}^n, \forall \lambda \in \mathbb{C},$$

origin symmetric complex convex bodies correspond to those origin symmetric convex bodies K in \mathbb{R}^{2n} that are invariant with respect to any coordinate-wise two-dimensional rotation, namely for each $\theta \in [0, 2\pi]$ and each $\xi = (\xi_{11}, \xi_{12}, \dots, \xi_{n1}, \xi_{n2}) \in \mathbb{R}^{2n}$

$$\|\xi\|_K = \|R_\theta(\xi_{11}, \xi_{12}), \dots, R_\theta(\xi_{n1}, \xi_{n2})\|_K, \quad (3)$$

where R_θ stands for the counterclockwise rotation of \mathbb{R}^2 by the angle θ with respect to the origin. We shall say that K is a *complex convex body in \mathbb{R}^{2n}* if K is a convex body and satisfies equations (3).

A compact set K in \mathbb{R}^n is called a star body if the origin is an interior point of K , every straight line passing through the origin crosses the boundary of K at exactly two points and the Minkowski functional of K defined by

$$\|x\|_K = \min\{a \geq 0 : x \in aK\}, \quad \forall x \in \mathbb{R}^n$$

is a continuous function on \mathbb{R}^n . The radial function of K is given by

$$\rho_K(x) = \max\{a > 0 : ax \in K\}.$$

If $x \in S^{n-1}$, then $\rho_K(x)$ is the radius of K in the direction of x . Note that for any unit vector ξ , $\rho_K(\xi) = \|\xi\|_K^{-1}$. The radial metric in the class of star bodies is defined by

$$\rho(K, L) = \max_{\xi \in S^{n-1}} |\rho_K(\xi) - \rho_L(\xi)|.$$

If the Minkowski functional of a star body K in \mathbb{R}^{2n} is R_θ -invariant (i.e. satisfies equations (3)), we say that K is a *complex star body* in \mathbb{R}^{2n} .

For $\xi \in \mathbb{C}^n$, $|\xi| = 1$, denote by

$$H_\xi = \{z \in \mathbb{C}^n : (z, \xi) = \sum_{k=1}^n z_k \bar{\xi}_k = 0\}$$

the complex hyperplane through the origin, perpendicular to ξ . Under the standard mapping from \mathbb{C}^n to \mathbb{R}^{2n} the hyperplane H_ξ turns into a $(2n - 2)$ -dimensional subspace of \mathbb{R}^{2n} orthogonal to the vectors

$$\xi = (\xi_{11}, \xi_{12}, \dots, \xi_{n1}, \xi_{n2}) \quad \text{and} \quad \xi^\perp = (-\xi_{12}, \xi_{11}, \dots, -\xi_{n2}, \xi_{n1}).$$

The orthogonal two-dimensional subspace H_ξ^\perp has orthonormal basis $\{\xi, \xi^\perp\}$. A star (convex) body K in \mathbb{R}^{2n} is a complex star (convex) body if and only if, for every $\xi \in S^{2n-1}$, the section $K \cap H_\xi^\perp$ is a two-dimensional Euclidean circle with radius $\rho_K(\xi) = \|\xi\|_K^{-1}$.

We introduce complex intersection bodies of complex star bodies using a definition under which these bodies play the same role in complex convexity, as their real counterparts in the real case. We use the notation $|K|$ for the volume of K ; the dimension where we consider volume is clear in every particular case.

Definition 1. *Let K, L be origin symmetric complex star bodies in \mathbb{R}^{2n} . We say that K is the complex intersection body of L and write $K = I_c(L)$ if for every $\xi \in \mathbb{R}^{2n}$*

$$|K \cap H_\xi^\perp| = |L \cap H_\xi|. \quad (4)$$

Since $K \cap H_\xi^\perp$ is the two-dimensional Euclidean circle with radius $\|\xi\|_K^{-1}$, (4) can be written as

$$\pi \|\xi\|_{I_c(L)}^{-2} = |L \cap H_\xi|. \quad (5)$$

All the bodies K that appear as complex intersection bodies of different complex star bodies form *the class of complex intersection bodies of star bodies*. In Section 4, we will introduce a more general class of *complex intersection bodies*.

3. THE RADON AND FOURIER TRANSFORMS OF R_θ -INVARIANT FUNCTIONS

Denote by $C_c(S^{2n-1})$ the space of R_θ -invariant continuous functions, i.e. continuous real-valued functions f on the unit sphere S^{2n-1} in \mathbb{R}^{2n} satisfying $f(\xi) = f(R_\theta(\xi))$ for all $\xi \in S^{2n-1}$ and all $\theta \in [0, 2\pi]$. The

complex spherical Radon transform is an operator $\mathcal{R}_c : C_c(S^{2n-1}) \rightarrow C_c(S^{2n-1})$ defined by

$$\mathcal{R}_c f(\xi) = \int_{S^{2n-1} \cap H_\xi} f(x) dx.$$

Writing volume in polar coordinates, we get that for every complex star body L in \mathbb{R}^{2n} and every $\xi \in S^{2n-1}$,

$$|L \cap H_\xi| = \frac{1}{2n-2} \int_{S^{2n-1} \cap H_\xi} \|x\|_L^{-2n+2} dx = \frac{1}{2n-2} \mathcal{R}_c (\|\cdot\|_L^{-2n+2})(\xi), \quad (6)$$

so the condition (5) reads as

$$\|\xi\|_{I_c(L)}^{-2} = \frac{1}{2\pi(n-1)} \mathcal{R}_c (\|\cdot\|_L^{-2n+2})(\xi). \quad (7)$$

This means that a complex star body K is a complex intersection body of a star body if and only if the function $\|\cdot\|_K^{-2}$ is the complex spherical Radon transform of a continuous positive R_θ -invariant function on S^{2n-1} . We use this observation in Section 4, where we introduce a more general class of complex intersection bodies (not depending on the underlying star body), like it was done in the real case in [18]. But before that we need several facts connecting the Radon transform to the Fourier transform in the complex setting.

We use the techniques of the Fourier transform of distributions; see [14] for details. As usual, we denote by $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space of rapidly decreasing infinitely differentiable functions (test functions) in \mathbb{R}^n , and $\mathcal{S}'(\mathbb{R}^n)$ is the space of distributions over $\mathcal{S}(\mathbb{R}^n)$.

Suppose that f is a locally integrable complex-valued function on \mathbb{R}^n with *power growth at infinity*, i.e. there exists a number $\beta > 0$ so that

$$\lim_{|x|_2 \rightarrow \infty} \frac{f(x)}{|x|_2^\beta} = 0,$$

where $|\cdot|_2$ stands for the Euclidean norm on \mathbb{R}^n . Then f represents a distribution acting by integration: for every $\phi \in \mathcal{S}$,

$$\langle f, \phi \rangle = \int_{\mathbb{R}^n} f(x) \phi(x) dx.$$

The Fourier transform of a distribution f is defined by $\langle \hat{f}, \phi \rangle = \langle f, \hat{\phi} \rangle$ for every test function ϕ . If ϕ is an even test function, then $(\hat{\phi})^\wedge = (2\pi)^n \phi$, so the Fourier transform is self-invertible (up to a constant) for even distributions.

A distribution f is called even homogeneous of degree $p \in \mathbb{R}$ if

$$\langle f(x), \phi(x/\alpha) \rangle = |\alpha|^{n+p} \langle f, \phi \rangle$$

for every test function ϕ and every $\alpha \in \mathbb{R}$, $\alpha \neq 0$. The Fourier transform of an even homogeneous distribution of degree p is an even homogeneous distribution of degree $-n - p$.

We say that a distribution is *positive definite* if its Fourier transform is a positive distribution in the sense that $\langle \hat{f}, \phi \rangle \geq 0$ for every non-negative test function ϕ . Schwartz's generalization of Bochner's theorem (see, for example, [15, p.152]) states that a distribution is positive definite if and only if it is the Fourier transform of a tempered measure on \mathbb{R}^n . Recall that a (non-negative, not necessarily finite) measure μ is called tempered if

$$\int_{\mathbb{R}^n} (1 + |x|_2)^{-\beta} d\mu(x) < \infty$$

for some $\beta > 0$.

Our definition of a star body K assumes that the origin is an interior point of K . If $0 < p < n$, then $\|\cdot\|_K^{-p}$ is a locally integrable function on \mathbb{R}^n and represents an even homogeneous of degree $-p$ distribution. If $\|\cdot\|_K^{-p}$ represents a positive definite distribution for some $p \in (0, n)$, then its Fourier transform is a tempered measure which is at the same time a homogeneous distribution of degree $-n + p$. One can express such a measure in polar coordinates, as follows.

Proposition 1. ([22, Corollary 2.26]) *Let K be an origin symmetric convex body in \mathbb{R}^n and $p \in (0, n)$. The function $\|\cdot\|_K^{-p}$ represents a positive definite distribution on \mathbb{R}^n if and only if there exists a finite Borel measure μ on S^{n-1} so that for every even test function ϕ ,*

$$\int_{\mathbb{R}^n} \|x\|_K^{-p} \phi(x) dx = \int_{S^{n-1}} \left(\int_0^\infty t^{p-1} \hat{\phi}(t\xi) dt \right) d\mu(\xi).$$

For any even continuous function f on the sphere S^{n-1} and any non-zero number $p \in \mathbb{R}$, we denote by $f \cdot r^p$ the extension of f to an even homogeneous function of degree p on \mathbb{R}^n defined as follows. If $x \in \mathbb{R}^n$, then $x = r\theta$, where $r = |x|_2$ and $\theta = x/|x|_2$. We put

$$f \cdot r^p(x) = f(\theta) r^p.$$

It was proved in [22, Lemma 3.16] that, for any $p \in (-n, 0)$ and infinitely smooth function f on S^{n-1} , the Fourier transform of $f \cdot r^{-p}$ is equal to another infinitely smooth function h on S^{n-1} extended to an even homogeneous of degree $-n + p$ function $h \cdot r^{-n+p}$ on the whole of \mathbb{R}^n . The following Parseval formula on the sphere was proved in [22, Corollary 3.22].

Proposition 2. *Let f, g be even infinitely smooth functions on S^{n-1} , and $p \in (0, n)$. Then*

$$\int_{S^{n-1}} (f \cdot r^{-p})^\wedge(\theta) (g \cdot r^{-n+p})^\wedge(\theta) = (2\pi)^n \int_{S^{n-1}} f(\theta)g(\theta) d\theta. \quad (8)$$

We need a simple observation that will, however, provide the basis for applications of the Fourier transform to complex bodies.

Lemma 1. *Suppose that $f \in C_c(S^{2n-1})$ is an even infinitely smooth function. Then for every $0 < p < 2n$ and $\xi \in S^{2n-1}$ the Fourier transform of the distribution $f \cdot r^{-p}$ is a constant function on $S^{2n-1} \cap H_\xi^\perp$.*

Proof : By [22, Lemma 3.16], the Fourier transform of $f \cdot r^{-p}$ is a continuous function outside of the origin in \mathbb{R}^{2n} . The function f is invariant with respect to all R_θ , so by the connection between the Fourier transform of distributions and linear transformations, the Fourier transform of $f \cdot r^{-p}$ is also invariant with respect to all R_θ . Recall that the two-dimensional space H_ξ^\perp is spanned by vectors ξ and ξ^\perp (see the Introduction). Every vector in $S^{2n-1} \cap H_\xi^\perp$ is the image of ξ under one of the coordinate-wise rotations R_θ , so the Fourier transform of $f \cdot r^{-p}$ is a constant function on $S^{2n-1} \cap H_\xi^\perp$. □

The following connection between the Fourier and Radon transforms is well-known; see for example [22, Lemma 3.24].

Proposition 3. *Let $1 \leq k < n$, and let $\phi \in \mathcal{S}(\mathbb{R}^n)$ be an even test function. Then for any $(n - k)$ -dimensional subspace H of \mathbb{R}^n*

$$\int_H \phi(x)dx = \frac{1}{(2\pi)^k} \int_{H^\perp} \hat{\phi}(x)dx.$$

We also use the spherical version of Proposition 3; see [22, Lemma 3.25].

Proposition 4. *Let ϕ be an even infinitely smooth function on S^{n-1} , let $0 < k < n$, and let H be an arbitrary $(n - k)$ -dimensional subspace of \mathbb{R}^n . Then*

$$\int_{S^{n-1} \cap H} \phi(\theta)d\theta = \frac{1}{(2\pi)^k} \int_{S^{n-1} \cap H^\perp} (\phi \cdot r^{-n+k})^\wedge(\theta)d\theta.$$

Let us translate the latter fact to the complex situation.

Lemma 2. *Let $\phi \in C_c(S^{2n-1})$ be an even infinitely smooth function. Then for every $\xi \in S^{2n-1}$*

$$\mathcal{R}_c\phi(\xi) = \frac{1}{2\pi} (\phi \cdot r^{-2n+2})^\wedge(\xi).$$

Proof : By Proposition 4,

$$\mathcal{R}_c\phi(\xi) = \int_{S^{2n-1} \cap H_\xi} \phi(\theta) d\theta = \frac{1}{(2\pi)^2} \int_{S^{2n-1} \cap H_\xi^\perp} (\phi \cdot r^{-2n+2})^\wedge(\theta) d\theta,$$

and, by Lemma 1, the function under the integral in the right-hand side is constant on $S^{2n-1} \cap H_\xi^\perp$. The value of this constant is the function value at $\xi \in S^{2n-1} \cap H_\xi^\perp$. Also, recall that $S^{2n-1} \cap H_\xi^\perp$ is the two-dimensional Euclidean unit circle, so

$$\int_{S^{2n-1} \cap H_\xi^\perp} (\phi \cdot r^{-2n+2})^\wedge(\theta) d\theta = 2\pi (\phi \cdot r^{-2n+2})^\wedge(\xi).$$

□

Lemma 3. *The complex spherical Radon transform is self-dual, i.e. for any even functions $f, g \in C_c(S^{2n-1})$,*

$$\int_{S^{2n-1}} \mathcal{R}_c f(\xi) g(\xi) d\xi = \int_{S^{2n-1}} f(\theta) \mathcal{R}_c g(\theta) d\theta.$$

Proof : By approximation, it is enough to consider the case where f, g are infinitely smooth. For some infinitely smooth even function $h \in C_c(S^{2n-1})$, we have $g \cdot r^{-2n+2} = (h \cdot r^{-2})^\wedge$, then $(g \cdot r^{-2n+2})^\wedge = (2\pi)^{2n} h \cdot r^{-2}$. By Lemma 2 and the spherical Parseval formula (8),

$$\begin{aligned} \int_{S^{2n-1}} \mathcal{R}_c f(\xi) g(\xi) d\xi &= \frac{1}{2\pi} \int_{S^{2n-1}} (f \cdot r^{-2n+2})^\wedge(\xi) (g \cdot r^{-2n+2})(\xi) d\xi \\ &= \frac{(2\pi)^{2n}}{2\pi} \int_{S^{2n-1}} (f \cdot r^{-2n+2})^\wedge(\xi) (h \cdot r^{-2})^\wedge(\xi) d\xi \\ &= \frac{1}{2\pi} \int_{S^{2n-1}} f(\theta) (g \cdot r^{-2n+2})^\wedge(\theta) d\theta = \int_{S^{2n-1}} f(\theta) \mathcal{R}_c g(\theta) d\theta. \end{aligned}$$

□

We now prove Lemma 2 without smoothness assumption. This result is a complex version of [22, Lemma 3.7]. We say that a distribution f on \mathbb{R}^{2n} is R_θ -invariant if $\langle f, \phi(R_\theta(\cdot)) \rangle = \langle f, \phi \rangle$ for every test function $\phi \in \mathcal{S}(\mathbb{R}^{2n})$ and every $\theta \in [0, 2\pi]$. If f and g are R_θ -invariant distributions, and $\langle f, \phi \rangle = \langle g, \phi \rangle$ for any test function ϕ that is invariant with respect to all R_θ , then $f = g$. This follows from the observation that the value of an R_θ -invariant distribution on a test function ϕ does not change if ϕ is replaced by the function $\frac{1}{2\pi} \int_0^{2\pi} \phi(R_\theta(\cdot)) d\theta$.

Lemma 4. *Let $f \in C_c(S^{2n-1})$ be an even function. Then the Fourier transform of $f \cdot r^{-2n+2}$ is a continuous function on the sphere extended to a homogeneous function of degree -2 on the whole \mathbb{R}^{2n} . Moreover, on the sphere this function is equal (up to a constant) to the complex spherical Radon transform of f : for any $\xi \in S^{2n-1}$,*

$$\mathcal{R}_c f(\xi) = \frac{1}{2\pi} (f \cdot r^{-2n+2})^\wedge(\xi).$$

Proof : Let $\phi \in \mathcal{S}(\mathbb{R}^{2n})$ be any even R_θ -invariant test function. Then $\hat{\phi}$ is also an even R_θ -invariant test function. By Lemma 1, for any $\xi \in S^{2n-1}$,

$$\int_{H_\xi^\perp} \hat{\phi}(x) dx = \int_{S^{2n-1} \cap H_\xi^\perp} \left(\int_0^\infty r \hat{\phi}(r\theta) dr \right) d\theta = 2\pi \int_0^\infty r \hat{\phi}(r\xi) dr. \quad (9)$$

By Proposition 3 and Lemma 3,

$$\begin{aligned} \langle (f \cdot r^{-2n+2})^\wedge, \phi \rangle &= \int_{\mathbb{R}^{2n}} |x|_2^{-2n+2} f(x/|x|_2) \hat{\phi}(x) dx \\ &= \int_{S^{2n-1}} f(\xi) \left(\int_0^\infty r \hat{\phi}(r\xi) dr \right) d\xi = \frac{1}{2\pi} \int_{S^{2n-1}} f(\xi) \left(\int_{H_\xi^\perp} \hat{\phi}(x) dx \right) d\xi \\ &= 2\pi \int_{S^{2n-1}} f(\xi) \left(\int_{H_\xi} \phi(x) dx \right) d\xi \\ &= 2\pi \int_{S^{2n-1}} f(\xi) \left(\int_{S^{2n-1} \cap H_\xi} \left(\int_0^\infty r^{2n-3} \phi(r\theta) dr \right) d\theta \right) d\xi \\ &= 2\pi \int_{S^{2n-1}} f(\xi) \mathcal{R}_c \left(\int_0^\infty r^{2n-3} \phi(r \cdot) dr \right) (\xi) d\xi \\ &= 2\pi \int_{S^{2n-1}} \left(\int_0^\infty r^{2n-3} \phi(r\theta) dr \right) \mathcal{R}_c f(\theta) d\theta \\ &= 2\pi \int_{\mathbb{R}^{2n}} |x|_2^{-2} \mathcal{R}_c f(x/|x|_2) \phi(x) dx. \end{aligned}$$

We get that for every even R_θ -invariant test function ϕ ,

$$\langle (f \cdot r^{-2n+2})^\wedge, \phi \rangle = 2\pi \langle |x|_2^{-2} \mathcal{R}_c f(x/|x|_2), \phi \rangle,$$

so even R_θ -invariant distributions $(f \cdot r^{-2n+2})^\wedge$ and $2\pi |x|_2^{-2} \mathcal{R}_c f(x/|x|_2)$ are equal.

□

Lemma 4 implies the following Fourier transform formula for the volume of sections of star bodies. Note that the real version of this formula was proved in [23], and that the complex formula below was proved in [31] for infinitely smooth bodies by a different method; here we remove the smoothness condition.

Theorem 1. *Let K be an origin symmetric complex star body in \mathbb{R}^{2n} , $n \geq 2$. For every $\xi \in S^{2n-1}$, we have*

$$|K \cap H_\xi| = \frac{1}{4\pi(n-1)} (\|x\|_K^{-2n+2})^\wedge(\xi).$$

Proof : By (6) and Lemma 4 applied to the function $f(\theta) = \|\theta\|_K^{-2n+2}$,

$$|K \cap H_\xi| = \frac{1}{2n-2} \mathcal{R}_c(\|\cdot\|_K^{-2n+2})(\xi) = \frac{1}{4\pi(n-1)} (\|x\|_K^{-2n+2})^\wedge(\xi).$$

□

We use Theorem 1 to prove the complex version of the Minkowski-Funk theorem saying that an origin symmetric star body is uniquely determined by volume of its central hyperplane sections; see [22, Corollary 3.9].

Corollary 1. *If K, L are origin symmetric complex star bodies in \mathbb{R}^{2n} , and their intersection bodies $I_c(K)$ and $I_c(L)$ coincide, then $K = L$.*

Proof : The equality of intersection bodies means that, for every $\xi \in S^{2n-1}$, $|K \cap H_\xi| = |L \cap H_\xi|$. By Theorem 1, homogeneous of degree -2 continuous on $\mathbb{R}^{2n} \setminus \{0\}$ functions $(\|\cdot\|_K^{-2n+2})^\wedge$ and $(\|\cdot\|_L^{-2n+2})^\wedge$ coincide on the sphere S^{2n-1} , so they are also equal as distributions on the whole of \mathbb{R}^{2n} . The result follows from the uniqueness theorem for the Fourier transform of distributions.

□

4. COMPLEX INTERSECTION BODIES

We are going to define the class of complex intersection bodies by extending the equality (7) to measures, as it was done in the real case in [18]. We say that a finite Borel measure μ on S^{2n-1} is R_θ -invariant if for any continuous function f on S^{2n-1} and any $\theta \in [0, 2\pi]$,

$$\int_{S^{2n-1}} f(x) d\mu(x) = \int_{S^{2n-1}} f(R_\theta x) d\mu(x).$$

The complex spherical Radon transform of an R_θ -invariant measure μ is defined as a functional $\mathcal{R}_c\mu$ on the space $C_c(S^{2n-1})$ acting by

$$(\mathcal{R}_c\mu, f) = \int_{S^{2n-1}} \mathcal{R}_c f(x) d\mu(x).$$

Definition 2. *An origin symmetric complex star body K in \mathbb{R}^{2n} is called a complex intersection body if there exists a finite Borel R_θ -invariant measure μ on S^{2n-1} so that $\|\cdot\|_K^{-2}$ and $\mathcal{R}_c\mu$ are equal as functionals on $C_c(S^{2n-1})$, i.e. for any $f \in C_c(S^{2n-1})$,*

$$\int_{S^{2n-1}} \|x\|_K^{-2} f(x) dx = \int_{S^{2n-1}} \mathcal{R}_c f(\theta) d\mu(\theta).$$

Clearly, $\mathcal{R}_c\mu$ is a finite Borel R_θ -invariant measure on S^{2n-1} . Also, an easy consequence of self-duality of the complex spherical Radon transform (see Lemma 3) is that if μ has density f on S^{2n-1} , then the measure $\mathcal{R}_c\mu$ has density $\mathcal{R}_c f$. The latter, in conjunction with (7), immediately implies that every complex intersection body of a star body is a complex intersection body in the sense of Definition 2.

Many results on real intersection bodies depend on the following Fourier characterization (see [26, Theorem 1]): an origin symmetric star body K in \mathbb{R}^n is an intersection body if and only if the function $\|\cdot\|_K^{-1}$ represents a positive definite distribution. Complex intersection bodies admit a similar characterization. To see the connection with the Fourier transform, combine the definition of the intersection body of a star body (4) with the result of Theorem 1: for every $\xi \in S^{2n-1}$,

$$\|\xi\|_{I_c(L)}^{-2} = \frac{1}{\pi} |L \cap H_\xi| = \frac{1}{4\pi^2(n-1)} (\|x\|_L^{-2n+2})^\wedge(\xi).$$

Both sides of the latter equality are even homogeneous functions of degree -2, so these functions are equal as distributions on the whole of \mathbb{R}^{2n} . Since the Fourier transform of even distributions is self-invertible (up to a constant), we get

$$\left(\|\cdot\|_{I_c(L)}^{-2}\right)^\wedge = \frac{(2\pi)^{2n}}{4\pi^2(n-1)} \|\cdot\|_L^{-2n+2} > 0,$$

so the distribution $\|\cdot\|_{I_c(L)}$ is positive definite. Moreover, if the Fourier transform of $\|\cdot\|_K^{-2}$ is an even strictly positive R_θ -invariant function on the sphere, then one can use the latter equality to construct a complex star body L such that $K = I_c(L)$. This connection holds for arbitrary complex intersection bodies, as shown in the following theorem.

Theorem 2. *An origin symmetric complex star body K in \mathbb{R}^{2n} is a complex intersection body if and only if the function $\|x\|_K^{-2}$ represents a positive definite distribution on \mathbb{R}^{2n} .*

Proof : Suppose that K is a complex intersection body with the corresponding measure μ . To prove that $\|\cdot\|_K^{-2}$ is a positive definite distribution, it is enough to show that $\langle (\|\cdot\|_K^{-2})^\wedge, \phi \rangle \geq 0$ for every

even R_θ -invariant non-negative test function ϕ . By Definition 2 and Proposition 3,

$$\begin{aligned}
\langle (\|\cdot\|_K^{-2})^\wedge, \phi \rangle &= \int_{\mathbb{R}^{2n}} \|x\|_K^{-2} \hat{\phi}(x) dx \\
&= \int_{S^{2n-1}} \|\theta\|_K^{-2} \left(\int_0^\infty r^{2n-3} \hat{\phi}(r\theta) dr \right) d\theta \\
&= \int_{S^{2n-1}} \mathcal{R}_c \left(\int_0^\infty r^{2n-3} \hat{\phi}(r\cdot) dr \right) (\xi) d\mu(\xi) \\
&= \int_{S^{2n-1}} \left(\int_{H_\xi} \hat{\phi}(x) dx \right) d\mu(\xi) \\
&= (2\pi)^{2n-2} \int_{S^{2n-1}} \left(\int_{H_\xi^\perp} \phi(x) dx \right) d\mu(\xi) \geq 0.
\end{aligned}$$

Now suppose that $\|\cdot\|_K^{-2}$ is a positive definite distribution. By Proposition 2, there exists a finite Borel measure μ on S^{2n-1} such that for any even test function ϕ

$$\int_{\mathbb{R}^{2n}} \|x\|_K^{-2} \phi(x) dx = \int_{S^{2n-1}} \left(\int_0^\infty t \hat{\phi}(t\xi) dt \right) d\mu(\xi). \quad (10)$$

Recall that K is R_θ -invariant, so we can assume that μ is R_θ -invariant and the latter equality holds only for even R_θ -invariant test functions ϕ . For each such test function, we have by (9) that

$$\int_0^\infty t \hat{\phi}(t\xi) dt = \frac{1}{2\pi} \int_{H_\xi^\perp} \hat{\phi}(x) dx.$$

Using this and Proposition 3 and then writing the interior integral in polar coordinates, we get that the right-hand side of (10) is equal to

$$\begin{aligned}
\frac{1}{2\pi} \int_{S^{2n-1}} \left(\int_{H_\xi^\perp} \hat{\phi}(x) dx \right) d\mu(\xi) &= 2\pi \int_{S^{2n-1}} \left(\int_{H_\xi} \phi(x) dx \right) d\mu(\xi) \\
&= 2\pi \int_{S^{2n-1}} \mathcal{R}_c \left(\int_0^\infty r^{2n-3} \phi(r\cdot) dr \right) (\xi) d\mu(\xi).
\end{aligned}$$

Writing the left-hand side of (10) in polar coordinates we get

$$\begin{aligned}
&\int_{S^{2n-1}} \|\theta\|_K^{-2} \left(\int_0^\infty r^{2n-3} \phi(r\theta) dr \right) d\theta \\
&= 2\pi \int_{S^{2n-1}} \mathcal{R}_c \left(\int_0^\infty r^{2n-3} \phi(r\cdot) dr \right) (\xi) d\mu(\xi) \quad (11)
\end{aligned}$$

for any even R_θ -invariant test function ϕ . Now let $\phi(x) = u(r)v(\theta)$ for every $x \in \mathbb{R}^{2n}$, where $x = r\theta$, $r \in [0, \infty)$, $\theta \in S^{2n-1}$, $u \in \mathcal{S}(\mathbb{R})$ is a non-negative test function on \mathbb{R} , and v is an arbitrary infinitely differentiable R_θ -invariant function on S^{2n-1} . Then

$$\int_0^\infty r^{2n-3} \phi(r\theta) dr = v(\theta) \int_0^\infty r^{2n-3} u(r) dr,$$

so the equality (11) turns into

$$\int_{S^{2n-1}} \|\theta\|_K^{-2} v(\theta) d\theta = 2\pi \int_{S^{2n-1}} \mathcal{R}_c v(\xi) d\mu(\xi).$$

Since infinitely smooth functions are dense in $C_c(S^{2n-1})$, the latter equality also holds for any function $v \in C_c(S^{2n-1})$, which means that K is a complex intersection body.

□

5. CHARACTERIZATIONS OF COMPLEX INTERSECTION BODIES

Complex intersection bodies are related to two generalizations of the concept of a real intersection body. These relations allow to apply to the complex case many results established originally in the real case.

The concept of a k -intersection body was introduced in [24, 25]. For an integer k , $1 \leq k < n$ and star bodies D, L in \mathbb{R}^n , we say that D is the k -intersection body of L if for every $(n - k)$ -dimensional subspace H of \mathbb{R}^n ,

$$|D \cap H^\perp| = |L \cap H|.$$

The class of k -intersection bodies was defined in [25] (see also [22, Section 4.2]) as follows.

Definition 3. *Let $1 \leq k < n$. We say that an origin symmetric star body D in \mathbb{R}^n is a k -intersection body if there exists a finite Borel measure μ on S^{n-1} so that for every even test function ϕ in \mathbb{R}^n ,*

$$\int_{\mathbb{R}^n} \|x\|_D^{-k} \phi(x) dx = \int_{S^{n-1}} \left(\int_0^\infty t^{k-1} \hat{\phi}(t\xi) dt \right) d\mu(\xi). \quad (12)$$

The class of k -intersection bodies is related to a certain generalization of the Busemann-Petty problem in the same way as intersection bodies are related to the original problem (see [22] for details; this generalization offers a condition that allows to compare volumes of two bodies in arbitrary dimensions). An equivalent and probably more geometric way to define k -intersection bodies would be to say that these bodies are limits in the radial metric of k -intersection bodies of star bodies (see [38] or [41] for a proof of equivalence of this property to the original definition from [25]).

It was shown in [25] that an origin symmetric star body K in \mathbb{R}^n is a k -intersection body if and only if the function $\|\cdot\|_K^{-k}$ represents a positive definite distribution. By Theorem 2,

Corollary 2. *An origin symmetric complex star body K in \mathbb{R}^{2n} is a complex intersection body if and only if it is a 2-intersection body in \mathbb{R}^{2n} satisfying (3).*

It was proved in [31, Theorem 3] that every origin symmetric complex convex body K in \mathbb{R}^{2n} is a $(2n-4)$ -, $(2n-3)$ - and $(2n-2)$ -intersection body in \mathbb{R}^{2n} . It follows that

Corollary 3. *Every origin symmetric complex convex body in \mathbb{R}^6 and \mathbb{R}^4 is a complex intersection body.*

This is no longer true in \mathbb{R}^{2n} , $n \geq 4$ as shown in [31, Theorem 4]. The unit balls of complex ℓ_q -balls with $q > 2$ are not k -intersection bodies for any $1 \leq k < 2n-4$.

Zhang in [46] introduced another generalization of intersection bodies. For $1 \leq k \leq n-1$, the $(n-k)$ -dimensional spherical Radon transform is an operator $\mathcal{R}_{n-k} : C(S^{n-1}) \mapsto C(G(n, n-k))$ defined by

$$\mathcal{R}_{n-k}(f)(H) = \int_{S^{n-1} \cap H} f(x) dx, \quad H \in G(n, n-k).$$

Here $G(n, n-k)$ is the Grassmannian of $(n-k)$ -dimensional subspaces of \mathbb{R}^n . Denote the image of the operator \mathcal{R}_{n-k} by X :

$$\mathcal{R}_{n-k}(C(S^{n-1})) = X \subset C(G(n, n-k)).$$

Let $M^+(X)$ be the space of linear positive continuous functionals on X , i.e. for every $\nu \in M^+(X)$ and non-negative function $f \in X$, we have $\nu(f) \geq 0$.

Definition 4. (Zhang [46]) *An origin symmetric star body K in \mathbb{R}^n is called a generalized k -intersection body if there exists a functional $\nu \in M^+(X)$ so that for every $f \in C(S^{n-1})$,*

$$\int_{S^{n-1}} \|x\|_K^{-k} f(x) dx = \nu(\mathcal{R}_{n-k} f).$$

It is easy to see that every complex intersection body in \mathbb{R}^{2n} is a generalized 2-intersection body in \mathbb{R}^{2n} . If K is a complex intersection body, then there exists an even R_θ -invariant measure μ on S^{2n-1} such that for every $f \in C(S^{2n-1})$,

$$\int_{S^{2n-1}} \|x\|_K^{-2} f(x) dx = \int_{S^{2n-1}} \mathcal{R}_c f(\xi) d\mu(\xi)$$

$$= \int_{G(2n, 2n-2)} \mathcal{R}_{2n-2} f(H) d\nu(H),$$

where ν is a measure on $G(2n, 2n - 2)$ which is the image of μ under the mapping $\xi \mapsto H_\xi$ from S^{2n-1} to $G(2n, 2n - 2)$. The measure ν can be considered as a positive continuous functional on X acting by

$$\nu(\mathcal{R}_{n-k} f) = \int_{G(2n, 2n-2)} \mathcal{R}_{n-k} f(H) d\nu(H),$$

which implies that K is a generalized 2-intersection body in \mathbb{R}^{2n} .

On the other hand, it was shown in [25] (see also [22, Theorem 4.23]) that every generalized k -intersection body in \mathbb{R}^n is a k -intersection body. So we have shown the following

Proposition 5. *An origin symmetric complex star body K in \mathbb{R}^{2n} is a complex intersection body if and only if it is a generalized 2-intersection body in \mathbb{R}^{2n} satisfying (3).*

Let us point out that the latter result is surprising. Combined with Corollary 2 implies that under the invariance assumption (3) the class of 2-intersection bodies is exactly the class of generalized 2-intersection bodies. Without the invariance assumption (3) this is no longer true as it follows from an example of E. Milman [39].

Goodey and Weil in [19] proved that any intersection body is the limit (in the radial metric topology) of finite radial sums of ellipsoids. This result has been extended by Grinberg and Zhang [17] (see another proof in [38]) to the case of generalized k -intersection bodies where the radial sum is replaced by the k -radial sum. Now we are going to prove a complex version of the result of Goodey and Weil. We do it by adjusting to the complex case the proofs from [38] and [34, Theorem 3.10].

We define the complex radial sum $K_1 +^c K_2$ of two complex star bodies K_1, K_2 as the complex star body that has radial function

$$\rho_{K_1 +^c K_2}^2 = \rho_{K_1}^2 + \rho_{K_2}^2.$$

The latter can be written as

$$\|\cdot\|_{K_1 +^c K_2}^{-2} = \|\cdot\|_{K_1}^{-2} + \|\cdot\|_{K_2}^{-2}.$$

Theorem 3. *Let K be an origin symmetric complex star body in \mathbb{R}^{2n} . Then K is a complex intersection body if and only if $\|\cdot\|_K^{-2}$ is the limit (in the metric of the space $C_c(S^{2n-1})$) of finite sums $\|\cdot\|_{E_1}^{-2} + \dots + \|\cdot\|_{E_m}^{-2}$, where E_1, \dots, E_m are complex ellipsoids in \mathbb{R}^{2n} (i.e. those ellipsoids in \mathbb{R}^{2n} that are complex convex bodies).*

In other words, an origin symmetric complex star body is an intersection body if and only if it is the limit (in the radial metric) of complex radial sums of complex ellipsoids.

To prove this result we need a few lemmas. For fixed $\xi \in S^{2n-1}$, $a > 0$, $b > 0$, let $E_{a,b}(\xi)$ be an ellipsoid in \mathbb{R}^{2n} with the norm

$$\|x\|_{E_{a,b}(\xi)} = \left(\frac{(x, \xi)^2 + (x, \xi^\perp)^2}{a^2} + \frac{|x|_2^2 - (x, \xi)^2 - (x, \xi^\perp)^2}{b^2} \right)^{\frac{1}{2}}, \quad x \in \mathbb{R}^{2n}.$$

Clearly, $E_{a,b}(\xi)$ is a complex ellipsoid. In fact, $(x, \xi)^2 + (x, \xi^\perp)^2 = |(x, \xi)_c|^2$ is the modulus squared of the complex scalar product of x and ξ considered as vectors from \mathbb{C}^n . The latter does not change when x is multiplied by any complex number of modulus 1, which means that the norm of $E_{a,b}(\xi)$ is invariant with respect to all rotations R_θ .

Using the formula for the Fourier transform of powers of the Euclidean norm in \mathbb{R}^{2n} (see [14, p.192]), we get

$$(|x|_2^{-2})^\wedge(\theta) = C(n)|\theta|_2^{-2n+2},$$

where $C(n) = 2^{2n-3}\pi^n\Gamma(n-1)$. By the connection between linear transformations and the Fourier transform,

$$(\|Tx\|^{-1})^\wedge(y) = |\det T|^{-1} (\|x\|^{-1})^\wedge((T^*)^{-1}y) \quad (13)$$

one can easily compute the following:

Lemma 5. *For all $\theta \in S^{2n-1}$,*

$$\left(\|x\|_{E_{a,b}(\xi)}^{-2} \right)_x^\wedge(\theta) = \frac{C(n)}{a^{2n-4}} \|\theta\|_{E_{b,a}(\xi)}^{-2n+2}.$$

Proof : By (13) with T being the composition of a rotation and a diagonal operator,

$$\left(\|x\|_{E_{a,b}(\xi)}^{-2} \right)_x^\wedge(\theta) = C(n)a^2b^{2n-2}\|\theta\|_{E_{1/a,1/b}(\xi)}^{-2n+2} = \frac{C(n)}{a^{2n-4}}\|\theta\|_{E_{b,a}(\xi)}^{-2n+2}.$$

□

Lemma 6. *Let K be an origin symmetric complex star body, then the function $\|\xi\|_K^{-2}$ can be approximated in the space $C_c(S^{2n-1})$ by functions of the form*

$$f_{a,b}(\xi) = \frac{C(n)}{a^{2n-4}} \int_{S^{n-1}} \|\theta\|_K^{-2} \|\theta\|_{E_{b,a}(\xi)}^{-2n+2} d\theta, \quad (14)$$

as $a \rightarrow 0$ and b is chosen appropriately.

Proof : Using Parseval's formula (8) and the previous Lemma we get

$$\begin{aligned}
 & \frac{C(n)}{a^{2n-4}} \int_{S^{2n-1}} \|\theta\|_{E_{b,a}(\xi)}^{-2n+2} d\theta \\
 &= \frac{1}{a^{2n-4}} \int_{S^{2n-1}} \|\theta\|_{E_{b,a}(\xi)}^{-2n+2} (|x|_2^{-2})^\wedge(\theta) d\theta \\
 &= \frac{1}{a^{2n-4}} \int_{S^{2n-1}} \left(\|x\|_{E_{b,a}(\xi)}^{-2n+2} \right)^\wedge(\theta) |\theta|_2^{-2} d\theta = \frac{1}{C(n)} \int_{S^{n-1}} \|x\|_{E_{a,b}(\xi)}^{-2} dx \\
 &= \frac{1}{C(n)} \int_{S^{2n-1}} \left(\frac{(x, \xi)^2 + (x, \xi^\perp)^2}{a^2} + \frac{1 - (x, \xi)^2 - (x, \xi^\perp)^2}{b^2} \right)^{-1} dx.
 \end{aligned}$$

For every fixed a , the latter integral goes to infinity when $b \rightarrow \infty$, and it goes to zero when $b \rightarrow 0$, so for every $a > 0$ there exists $b = b(a)$ such that

$$\frac{C(n)}{a^{2n-4}} \int_{S^{2n-1}} \|\theta\|_{E_{b(a),a}(\xi)}^{-2n+2} d\theta = 1.$$

Note that, by rotation invariance, the value of $b(a)$ does not depend on the choice of ξ .

Now for every $\xi \in S^{2n-1}$ we have

$$\begin{aligned}
 & \left| \|\xi\|_K^{-2} - \frac{C(n)}{a^{2n-4}} \int_{S^{2n-1}} \|\theta\|_K^{-2} \|\theta\|_{E_{b(a),a}(\xi)}^{-2n+2} d\theta \right| \\
 & \leq \frac{C(n)}{a^{2n-4}} \int_{S^{2n-1}} \left| \|\xi\|_K^{-2} - \|\theta\|_K^{-2} \right| \|\theta\|_{E_{b(a),a}(\xi)}^{-2n+2} d\theta \\
 & = \frac{C(n)}{a^{2n-4}} \int_{(\theta, \xi)^2 + (\theta, \xi^\perp)^2 \geq \delta} \left| \|\xi\|_K^{-2} - \|\theta\|_K^{-2} \right| \|\theta\|_{E_{b(a),a}(\xi)}^{-2n+2} d\theta \\
 & \quad + \frac{C(n)}{a^{2n-4}} \int_{(\theta, \xi)^2 + (\theta, \xi^\perp)^2 < \delta} \left| \|\xi\|_K^{-2} - \|\theta\|_K^{-2} \right| \|\theta\|_{E_{b(a),a}(\xi)}^{-2n+2} d\theta = I_1 + I_2.
 \end{aligned}$$

for every $\delta \in (0, 1)$.

Since K is a complex star body, the norm of K is constant on vectors of the form $u\xi + v\xi^\perp$ with $u^2 + v^2 = 1$. Vectors of this form are the only solutions on the sphere of the equation $(\theta, \xi)^2 + (\theta, \xi^\perp)^2 = 1$. Using this and the uniform continuity of $\|\cdot\|_K^{-2}$ on the sphere S^{2n-1} , for any given $\epsilon > 0$ we can find $\delta \in (0, 1)$, close to 1, so that $(\theta, \xi)^2 + (\theta, \xi^\perp)^2 \geq \delta$ implies $\left| \|\xi\|_K^{-2} - \|\theta\|_K^{-2} \right| < \epsilon/2$. Therefore

$$\begin{aligned}
 I_1 &= \frac{C(n)}{a^{2n-4}} \int_{(\theta, \xi)^2 + (\theta, \xi^\perp)^2 \geq \delta} \left| \|\xi\|_K^{-2} - \|\theta\|_K^{-2} \right| \|\theta\|_{E_{b(a),a}(\xi)}^{-2n+2} d\theta \\
 &\leq \frac{\epsilon}{2} \left[\frac{C(n)}{a^{2n-4}} \int_{(\theta, \xi)^2 + (\theta, \xi^\perp)^2 \geq \delta} \|\theta\|_{E_{b(a),a}(\xi)}^{-2n+2} d\theta \right] \leq \frac{\epsilon}{2}.
 \end{aligned}$$

Now fix δ chosen above and estimate the integral I_2 as follows

$$\begin{aligned} I_2 &= \frac{C(n)}{a^{2n-4}} \int_{(\theta, \xi)^2 + (\theta, \xi^\perp)^2 < \delta} \left| \|\xi\|_K^{-2} - \|\theta\|_K^{-2} \right| \|\theta\|_{E_{b(a), a}(\xi)}^{-2n+2} d\theta \\ &\leq \frac{C_1(n, K)}{a^{2n-4}} \int_{(\theta, \xi)^2 + (\theta, \xi^\perp)^2 < \delta} \|\theta\|_{E_{b(a), a}(x)}^{-2n+2} d\theta = \frac{C_1(n, K)}{a^{2n-4}} \times \\ &\int_{(\theta, \xi)^2 + (\theta, \xi^\perp)^2 < \delta} \left(\frac{(\theta, \xi)^2 + (\theta, \xi^\perp)^2}{(b(a))^2} + \frac{1 - (\theta, \xi)^2 - (\theta, \xi^\perp)^2}{a^2} \right)^{-n+1} d\theta \\ &\leq a^2 (1 - \delta)^{-n+1} C_1(n, K) |S^{2n-1}|. \end{aligned}$$

where

$$C_1(n, K) = 2C(n) \max_{x \in S^{2n-1}} \|x\|_K^{-2}.$$

Now we can choose a so small that $I_2 \leq \epsilon/2$. \square

Lemma 7. *If μ is a finite measure on S^{2n-1} and $a, b > 0$, then the function*

$$f(\xi) = \int_{S^{n-1}} \|\theta\|_{E_{a,b}(\xi)}^{-2} d\mu(\theta)$$

can be approximated in $C_c(S^{2n-1})$ by the sums of the form

$$\sum_{i=1}^m \|\xi\|_{E_i}^{-2},$$

where E_1, \dots, E_m are complex ellipsoids.

Proof : Let $\sigma > 0$ be a small number and choose a finite covering of the sphere by spherical σ -balls $B_\sigma(\eta_i) = \{\eta \in S^{n-1} : |\eta - \eta_i| < \sigma\}$, $\eta_i \in S^{2n-1}$, $i = 1, \dots, m = m(\delta)$. Define

$$\tilde{B}_\sigma(\xi_1) = B_\sigma(\xi_1)$$

and

$$\tilde{B}_\sigma(\xi_i) = B_\sigma(\xi_i) \setminus \bigcup_{j=1}^{i-1} B_\sigma(\xi_j), \quad \text{for } i = 2, \dots, m.$$

Let $1/p_i = \mu(\tilde{B}_\sigma(\xi_i))$. Clearly, $1/p_1 + \dots + 1/p_m = \mu(S^{2n-1})$.

Let $\rho(E_{a,b}(\xi), x)$ be the value of the radial function of the ellipsoid $E_{a,b}(\xi)$ at the point x , that is

$$\rho(E_{a,b}(\xi), x) = \|x\|_{E_{a,b}(\xi)}^{-1}.$$

Note that $\rho(E_{a,b}(\xi), x) = \rho(E_{a,b}(x), \xi)$, because both depend only on the modulus of the complex scalar product of x and ξ , therefore

$$|\rho^2(E_{a,b}(\xi), x) - \rho^2(E_{a,b}(\theta), x)| \leq C_{a,b} |\xi - \theta|,$$

with a constant $C_{a,b}$ that depends only on a and b .

Then,

$$\begin{aligned}
 & \left| \int_{S^{2n-1}} \rho^2(E_{a,b}(\xi), x) d\mu(\xi) - \sum_{i=1}^m \frac{1}{p_i} \rho^2(E_{a,b}(\xi_i), x) \right| \\
 &= \left| \sum_{i=1}^m \left(\int_{\tilde{B}_\sigma(\xi_i)} \rho^2(E_{a,b}(\xi), x) d\mu(\xi) - \int_{\tilde{B}_\sigma(\xi_i)} \rho^2(E_{a,b}(\xi_i), x) d\mu(\xi) \right) \right| \\
 &\leq \sum_{i=1}^m \int_{\tilde{B}_\sigma(\xi_i)} |\rho^2(E_{a,b}(\xi), x) - \rho^2(E_{a,b}(\xi_i), x)| d\mu(\xi) \\
 &\leq \sum_{i=1}^m \int_{\tilde{B}_\sigma(\xi_i)} C_{a,b} |\xi - \xi_i| d\mu(\xi) \leq C_{a,b} \sigma \mu(S^{2n-1}).
 \end{aligned}$$

Since σ is arbitrarily small, the result follows after we define ellipsoids E_i by

$$\|x\|_{E_i}^{-2} = \frac{1}{p_i} \rho^2(E_{a,b}(\xi_i), x). \quad \square$$

Proof of Theorem 3. The “if” part immediately follows from Theorem 2, since for any ellipsoid E the distribution $\|\cdot\|_E^{-2}$ is positive definite, as the linear perturbation of the the same function for the Euclidean ball.

To prove the converse, suppose that K is a complex intersection body and μ is the measure on S^{2n-1} corresponding to K by the definition of complex intersection body. By Lemma 6, $\|\xi\|_K^{-2}$ can be uniformly approximated by the integrals of the form

$$\frac{C(n)}{a^{2n-4}} \int_{S^{2n-1}} \|\theta\|_K^{-2} \|\theta\|_{E_{b,a}(\xi)}^{-2n+2} d\theta, \quad (15)$$

as $a \rightarrow 0$.

By Lemma 5 and Lemma 4,

$$\frac{C(n)}{a^{2n-4}} \mathcal{R}_c(\|\cdot\|_{E_{b,a}(\xi)}^{-2n+2}) = (2\pi)^{2n-1} \|\cdot\|_{E_{a,b}(\xi)}^{-2},$$

and by the definition of complex intersection body, (15) is equal to

$$(2\pi)^{2n-1} \int_{S^{2n-1}} \|\theta\|_{E_{a,b}(\xi)}^{-2} d\mu(\theta).$$

Now, by Lemma 7, $\int_{S^{2n-1}} \|\theta\|_{E_{a,b}(\xi)}^{-2} d\mu(\theta)$ can be uniformly approximated by sums of the form $\sum_{i=1}^m \|\xi\|_{E_i}^{-1}$, where E_i are complex ellipsoids. \square

6. STABILITY IN THE BUSEMANN-PETTY PROBLEM AND HYPERPLANE INEQUALITIES

Intersection bodies played an important role in the solution of the Busemann-Petty problem posed in [9] in 1956. Suppose that K and L are origin symmetric convex bodies in \mathbb{R}^n so that, for every $\xi \in S^{n-1}$,

$$|K \cap \xi^\perp| \leq |L \cap \xi^\perp|.$$

Does it follow that $|K| \leq |L|$? The problem was completely solved at the end of 1990's, and the answer is affirmative if $n \leq 4$ and negative if $n \geq 5$. The solution appeared as the result of a sequence of papers [36], [3], [16], [7], [37], [40], [11], [12], [44], [45], [26], [30], [47], [13] (see [10, Chapter 8] or [22, Chapter 5] for details). One of the main ingredients of the solution was a connection between intersection bodies and the Busemann-Petty problem established by Lutwak [37]: if K is an intersection body then the answer to the Busemann-Petty problem is affirmative for any star body L . On the other hand, if L is a symmetric convex body that is not an intersection body then one can construct K giving together with L a counterexample.

The complex Busemann-Petty problem can be formulated as follows. Suppose that K, L are origin symmetric complex convex bodies in \mathbb{R}^{2n} and, for every $\xi \in S^{2n-1}$, we have $|K \cap H_\xi| \leq |L \cap H_\xi|$. Does it follow that $|K| \leq |L|$? As proved in [31], the answer is affirmative if $n \leq 3$, and it is negative if $n \geq 4$. The proof is based on a connection with intersection bodies, similar to Lutwak's connection in the real case (see [31, Theorem 2]):

- (i) If K is a complex intersection body in \mathbb{R}^{2n} and L is any origin symmetric complex star body in \mathbb{R}^{2n} , then the answer to the question of the complex Busemann-Petty problem is affirmative;
- (ii) if there exists an origin symmetric complex convex body in \mathbb{R}^{2n} that is not a complex intersection body, then one can construct a counterexample to the complex Busemann-Petty problem.

These connections were formulated in [31] in terms of positive definite distributions, so one has to use Theorem 2 to get the statements in terms of convex intersection bodies.

Zvavitch [48] found a generalization of the Busemann-Petty problem to arbitrary measures, namely, one can replace volume by any measure γ with even continuous density in \mathbb{R}^n . In particular, if $n \leq 4$, then for any origin symmetric convex bodies K and L in \mathbb{R}^n the inequalities

$$\gamma(K \cap \xi^\perp) \leq \gamma(L \cap \xi^\perp), \quad \forall \xi \in S^{n-1}$$

imply

$$\gamma(K) \leq \gamma(L).$$

Zvavitch also proved that this is generally not true if $n \geq 5$, namely, for any γ with strictly positive even continuous density there exist K and L providing a counterexample. In [49] the result of Zvavitch was extended to complex convex bodies.

In this section we are going to prove stability in the affirmative part of the result from [49]; see Theorem 4 below. Note that stability in the original Busemann-Petty problem was established in [27], and for the complex Busemann-Petty problem it was done in [28]. Stability in Zvavitch's result was proved in [29], and in [32] the result of [29] was extended to sections of lower dimensions in place of hyperplane sections.

Let f be an even continuous non-negative function on \mathbb{R}^{2n} , and denote by γ the measure on \mathbb{R}^{2n} with density f so that for every closed bounded set $B \subset \mathbb{R}^{2n}$

$$\gamma(B) = \int_B f(x) dx.$$

Since we apply γ only to complex star bodies, we can assume without loss of generality that the measure γ and the function f are R_θ -invariant.

We need a polar formula for the measure of a complex star body K :

$$\gamma(K) = \int_K f(x) dx = \int_{S^{2n-1}} \left(\int_0^{\|\theta\|_K^{-1}} r^{2n-1} f(r\theta) dr \right) d\theta. \quad (16)$$

For every $\xi \in S^{2n-1}$,

$$\begin{aligned} \gamma(K \cap H_\xi) &= \int_{K \cap H_\xi} f(x) dx \\ &= \int_{S^{2n-1} \cap H_\xi} \left(\int_0^{\|\theta\|_K^{-1}} r^{2n-3} f(r\theta) dr \right) d\theta \\ &= \mathcal{R}_c \left(\int_0^{\|\cdot\|_K^{-1}} r^{2n-3} f(r\cdot) dr \right) (\xi), \end{aligned} \quad (17)$$

We need the following elementary lemma, which was also used by Zvavitch in [48].

Lemma 8. *Let $a, b \in [0, \infty)$, $n \in \mathbb{N}, n \geq 2$, and let g be a non-negative integrable function on $[0, \max\{a, b\}]$. Then*

$$\int_0^a r^{2n-1} g(r) dt - a^2 \int_0^a r^{2n-3} g(r) dr$$

$$\leq \int_0^b r^{2n-1} g(r) dr - a^2 \int_0^b r^{2n-3} g(r) dr.$$

Denote by

$$d_n = \frac{|B_2^{2n}|^{\frac{n-1}{n}}}{|B_2^{2n-2}|},$$

where B_2^n stands for the unit Euclidean ball in \mathbb{R}^n , and

$$|B_2^n| = \frac{\pi^{n/2}}{\Gamma(1 + \frac{n}{2})}.$$

Note that $d_n < 1$ for every $n \in \mathbb{N}$; this easily follows from the log-convexity of the Γ -function.

It is well-known (see for example [22, p.32]) that the surface area of the sphere S^{n-1} in \mathbb{R}^n is equal to

$$|S^{n-1}| = n|B_2^n|. \quad (18)$$

The following stability result extends [29, Theorem 2] to the complex case. The proof is similar to that of [32, Theorem 2].

Theorem 4. *Let K and L be origin symmetric complex star bodies in \mathbb{R}^{2n} , let $\varepsilon > 0$ and let γ be a measure on \mathbb{R}^{2n} with even continuous non-negative density f . Suppose that K is a complex intersection body, and that for every $\xi \in S^{2n-1}$*

$$\gamma(K \cap H_\xi) \leq \gamma(L \cap H_\xi) + \varepsilon. \quad (19)$$

Then

$$\gamma(K) \leq \gamma(L) + \frac{n}{n-1} d_n \varepsilon |K|^{\frac{1}{n}}.$$

Proof : By (17), the condition (19) can be written as

$$\begin{aligned} & \mathcal{R}_c \left(\int_0^{\|\cdot\|_K^{-1}} r^{2n-3} f(r\cdot) dr \right) (\xi) \\ & \leq \mathcal{R}_c \left(\int_0^{\|\cdot\|_L^{-1}} r^{2n-3} f(r\cdot) dr \right) (\xi) + \varepsilon, \quad \forall \xi \in S^{2n-1}. \end{aligned}$$

Integrating the latter inequality with respect to the measure μ on S^{2n-1} , corresponding to the body K by Definition 2, and then using the equality of Definition 2, we get

$$\int_{S^{n-1}} \|\theta\|_K^{-2} \left(\int_0^{\|\theta\|_K^{-1}} r^{2n-3} f(r\theta) dr \right) d\theta \quad (20)$$

$$\leq \int_{S^{2n-1}} \|\theta\|_L^{-2} \left(\int_0^{\|\theta\|_L^{-1}} r^{2n-3} f(r\theta) dr \right) d\theta + \varepsilon \int_{S^{2n-1}} d\mu(\xi).$$

Applying Lemma 8 with $a = \|\theta\|_K^{-1}$, $b = \|\theta\|_L^{-1}$ and $g(r) = f(r\theta)$ and then integrating over the sphere, we get

$$\begin{aligned} & \int_{S^{2n-1}} \left(\int_0^{\|\theta\|_K^{-1}} r^{2n-1} f(r\theta) dr \right) d\theta \\ & - \int_{S^{2n-1}} \|\theta\|_K^{-k} \left(\int_0^{\|\theta\|_K^{-1}} r^{2n-3} f(r\theta) dr \right) d\theta \\ & \leq \int_{S^{2n-1}} \left(\int_0^{\|\theta\|_L^{-1}} r^{2n-1} f(r\theta) dr \right) d\theta \\ & - \int_{S^{2n-1}} \|\theta\|_K^{-k} \left(\int_0^{\|\theta\|_L^{-1}} r^{2n-3} f(r\theta) dr \right) d\theta. \end{aligned} \quad (21)$$

Adding (20) and (21) and using (16) we get

$$\gamma(K) \leq \gamma(L) + \varepsilon \int_{S^{2n-1}} d\mu(\xi).$$

Since $\mathcal{R}_c 1 = |S^{2n-3}|1$, where $1(\xi) \equiv 1$, we again apply Definition 2, Hölder's inequality, the polar formula for volume and (18):

$$\begin{aligned} & \int_{S^{2n-1}} d\mu(\xi) = \frac{1}{|S^{2n-3}|} \int_{S^{2n-1}} \mathcal{R}_c 1(\xi) d\mu(\xi) \\ & = \frac{1}{|S^{2n-3}|} \int_{S^{2n-1}} \|x\|_K^{-2} dx \leq \frac{1}{|S^{2n-3}|} \left(\int_{S^{2n-1}} \|x\|_K^{-2n} dx \right)^{\frac{1}{n}} |S^{2n-1}|^{\frac{n-1}{n}} \\ & = \frac{(2n)^{\frac{1}{n}} |S^{2n-1}|^{\frac{n-1}{n}}}{|S^{2n-3}|} |K|^{\frac{1}{n}} = \frac{n}{n-1} d_n |K|^{\frac{1}{n}}. \end{aligned}$$

□

Interchanging K and L in Theorem 4, we get a complex version of [29, Corollary 1].

Corollary 4. *If K and L are complex intersection bodies in \mathbb{R}^{2n} , then*

$$\begin{aligned} & |\gamma(K) - \gamma(L)| \\ & \leq \frac{n}{n-1} d_n \max_{\xi \in S^{2n-1}} |\gamma(K \cap H_\xi) - \gamma(L \cap H_\xi)| \max \left\{ |K|^{\frac{1}{n}}, |L|^{\frac{1}{n}} \right\}. \end{aligned}$$

Putting $L = \emptyset$ in the latter inequality, we extend to the complex case the hyperplane inequality for real intersection bodies from [29, Theorem 1].

Theorem 5. *If K is a complex intersection body in \mathbb{R}^{2n} , and γ is an arbitrary measure on \mathbb{R}^{2n} with even continuous density, then*

$$\gamma(K) \leq \frac{n}{n-1} d_n \max_{\xi \in S^{2n-1}} \gamma(K \cap H_\xi) |K|^{\frac{1}{n}}.$$

By Corollary 3, this inequality holds for any origin symmetric complex convex body K in \mathbb{R}^4 or \mathbb{R}^6 .

The constant in Theorem 5 is optimal, as can be easily seen from the same example as in [29]. Let $K = B_2^n$ and, for every $j \in N$, let f_j be a non-negative continuous function on $[0, 1]$ supported in $(1 - \frac{1}{j}, 1)$ and such that $\int_0^1 f_j(t) dt = 1$. Let γ_j be the measure on \mathbb{R}^{2n} with density $f_j(|x|_2)$, where $|x|_2$ is the Euclidean norm in \mathbb{R}^{2n} . Then a simple computation shows that

$$\lim_{j \rightarrow \infty} \frac{\gamma_j(B_2^{2n})}{\max_{\xi \in S^{2n-1}} \gamma_j(B_2^{2n} \cap H_\xi) |B_2^{2n}|^{1/n}} = \frac{n}{n-1} d_n.$$

Note that in the case of volume (when the density $f \equiv 1$) the inequality of Theorem 5 follows from [28, Corollary 1] and the constant is just d_n without the term $n/(n-1)$. One has to follow the proof of Theorem 1 from [28] to restore the constant d_n which is estimated by 1 everywhere in [28].

The result of Theorem 5 is related to the famous hyperplane problem asking whether there exists an absolute constant C so that for any origin symmetric convex body K in \mathbb{R}^n

$$\text{Vol}_n(K)^{\frac{n-1}{n}} \leq C \max_{\xi \in S^{n-1}} \text{Vol}_{n-1}(K \cap \xi^\perp), \quad (22)$$

where ξ^\perp is the central hyperplane in \mathbb{R}^n perpendicular to ξ . The problem is still open, with the best-to-date estimate $C \sim n^{1/4}$ established by Klartag [21], who slightly improved the previous estimate of Bourgain [6].

7. COMPLEX INTERSECTION BODIES OF CONVEX COMPLEX BODIES

In this section we extend two classical results about intersection bodies of convex bodies to the complex setting. The well-known result of Busemann [8] is that the intersection body of a symmetric convex body is also symmetric convex. We prove a complex version of this result.

Theorem 6. *Let K be an origin symmetric convex body in \mathbb{C}^n and $I_c(K)$ the complex intersection body of K . Then $I_c(K)$ is also an origin symmetric convex body in \mathbb{C}^n .*

Before we prove Theorem 6 we need some preparations. We write O_2 for the set of all rotations in \mathbb{R}^2 as described in (3) and H_0 for the (real) 2-dimensional subspace of \mathbb{R}^{2n} , spanned by the standard unit vectors e_1, e_2 . If V_{H_0} is any orthogonal transformation in H_0 ($V_{H_0} \in O_2$), we define $U_{V_{H_0}} \in O_{2n}$ to be the orthogonal transformation in \mathbb{R}^{2n} that has (as a matrix) n copies of V_{H_0} on its diagonal. Then (3) implies that $U_{V_{H_0}}K = K$. We will use this property in the following form:

$$\mathbf{1}_K(U_{V_{H_0}}x) = \mathbf{1}_K(x), \quad \forall x \in \mathbb{R}^{2n}, \quad \forall V_{H_0} \in O_2. \quad (23)$$

Assume that $n \geq 3$. Let $u_1, u_2 \in \mathbb{C}^n$, $|u_1|_2 = |u_2|_2 = 1$, with $H_{u_i}^\perp = \text{span}\{u_i, u_i^\perp\}$, as introduced earlier, and $\theta_i \in S_{H_{u_i}^\perp}$, $i = 1, 2$. We define $u_3 := \frac{u_1 + u_2}{|u_1 + u_2|_2}$, with $H_{u_3}^\perp$ and $\theta_3 := \frac{\theta_1 + \theta_2}{|\theta_1 + \theta_2|_2} \in S_{H_{u_3}^\perp}$ such that $|\theta_1 + \theta_2|_2 = |u_1 + u_2|_2$. We can assume that $H_{u_1}^\perp \cap H_{u_2}^\perp = \{0\}$.

Now, let $r_1, r_2 > 0$. We define r_3, t (as functions of r_1, r_2) such that

$$t := \frac{r_1}{r_1 + r_2}, \quad r_3 := \frac{1}{\frac{1}{r_1} + \frac{1}{r_2}} |u_1 + u_2|_2. \quad (24)$$

If $S := (H_{u_1}^\perp + H_{u_2}^\perp)^\perp$, we write $E_i := \text{span}\{H_{u_i}^\perp, S\}$, $i = 1, 2, 3$. We define the functions $g_i : H_{u_i}^\perp \rightarrow \mathbb{R}$, $h_i : [0, \infty) \rightarrow [0, \infty)$, $i = 1, 2, 3$ to be

$$g_i(x) := \int_{S+x} \mathbf{1}_K(y) dy = |K \cap (S + x)|, \quad h_i(r) := g_i(r\theta_i). \quad (25)$$

In the following we exploit the fact that K satisfies (23).

Lemma 9. *For $i = 1, 2, 3$, we have that*

$$|K \cap E_i| = 2\pi \int_0^\infty r h_i(r) dr. \quad (26)$$

Proof: First we will show that the functions g_i are rotation invariant. Let $S_0 := \text{span}\{e_5, \dots, e_{2n}\}$ and $\theta_i^{(1)}, \theta_i^{(2)} \in S_{H_{u_i}^\perp}$, $i = 1, 2, 3$. There exists $U_i \in O_{2n}$ such that $U_i H_{u_i}^\perp = H_0$ and $U_i S = S_0$, and $\phi_i^{(1)}, \phi_i^{(2)} \in S_{H_0}$, such that $U_i^\top(\phi_i^{(1)}) = \theta_i^{(1)}$ and $U_i^\top(\phi_i^{(2)}) = \theta_i^{(2)}$, $i = 1, 2, 3$. Moreover, there exists $V \in O_2$ such that $V(\phi_i^{(2)}) = \phi_i^{(1)}$. Let $V_0 \in O_{2n}$ be the diagonal operator with V on its diagonal entries. Then, it is clear that $V_0(\phi_i^{(2)}) = \phi_i^{(1)}$ and $V_0 S_0 = S_0$. Then, by (23), we have that for every $r > 0$,

$$g_i(r\theta_i^{(1)}) = \int_{S+r\theta_i^{(1)}} \mathbf{1}_K(y) dy = \int_{U_i^\top S_0 + r U_i^\top(\phi_i^{(1)})} \mathbf{1}_K(y) dy$$

$$\begin{aligned}
&= \int_{U_i^\top(S_0+r\phi_i^{(1)})} \mathbf{1}_K(y) dy = \int_{S_0+r\phi_i^{(1)}} \mathbf{1}_K(U_i y) dy \\
&= \int_{V_0 S_0+rV_0(\phi_i^{(2)})} \mathbf{1}_K(U_i y) dy = \int_{V_0(S_0+r\phi_i^{(2)})} \mathbf{1}_K(U_i y) dy \\
&= \int_{S_0+r\phi_i^{(2)}} \mathbf{1}_K(V_0^\top U_i y) dy = \int_{S_0+r\phi_i^{(2)}} \mathbf{1}_K(U_i y) dy \\
&= \int_{U_i^\top S_0+rU_i^\top(\phi_i^{(2)})} \mathbf{1}_K(y) dy = \int_{S+r\theta_i^{(2)}} \mathbf{1}_K(y) dy = g_i(r\theta_i^{(2)}).
\end{aligned}$$

So, by Fubini's theorem we have that for $i = 1, 2, 3$,

$$\begin{aligned}
|K \cap E_i| &= \int_{H_{u_i}^\perp} \int_{S+x} \mathbf{1}_K(y) dy dx = \int_{H_{u_i}^\perp} g_i(x) dx \\
&= \int_{S_{H_{u_i}^\perp}} \int_0^\infty r g_i(r\theta) dr d\theta = 2\pi \int_0^\infty r g_i(r\theta_i) dr = 2\pi \int_0^\infty r h_i(r) dr.
\end{aligned}$$

This finishes the proof. \square

The convexity of K is exploited in the following

Lemma 10. *With the above notation we have that*

$$h_3(r_3) \geq h_1^{(1-t)}(r_1) h_2^t(r_2). \quad (27)$$

Proof : Note that $h_i(r) := |K \cap (S + r\theta_i)|$, for $i = 1, 2, 3$. Also observe that the sets $K \cap (S + r_i\theta_i)$ all lie in the same hyperplane, and by convexity,

$$(1-t)(K \cap (S + r_1\theta_1)) + t(K \cap (S + r_2\theta_2)) \subseteq (K \cap (S + r_3\theta_3)).$$

The result follows from the Brunn-Minkowski inequality. \square

Finally we need the following result of K. Ball [2]. It can be seen as an extension of the inequality of Busemann (see also [5]). In [2] this proposition has been proved but has not been stated in this form. In this form it can be found in [21].

Proposition 6. *Let $r_1, r_2 > 0$. Define t, r_3 as follows:*

$$t := \frac{r_1}{r_1 + r_2}, \quad r_3 := \frac{2}{\frac{1}{r_1} + \frac{1}{r_2}}. \quad (28)$$

Assume that $h_1, h_2, h_3 : [0, \infty) \rightarrow [0, \infty)$ such that

$$h_3(r_3) \geq h_1^{(1-t)}(r_1) h_2^t(r_2), \quad \forall r_1, r_2 > 0. \quad (29)$$

Let $p \geq 1$ and denote

$$\begin{aligned} A &:= \left(\int_0^\infty r^{p-1} h_1(r) dr \right)^{\frac{1}{p}}, \\ B &:= \left(\int_0^\infty r^{p-1} h_2(r) dr \right)^{\frac{1}{p}}, \\ C &:= \left(\int_0^\infty r^{p-1} h_3(r) dr \right)^{\frac{1}{p}}. \end{aligned}$$

Then,

$$C \geq \frac{2}{\frac{1}{A} + \frac{1}{B}}. \quad (30)$$

We rewrite the previous proposition in a form that fits our setting:

Corollary 5. *Let $r_1, r_2 > 0$ and let $\alpha > 0$. Define t, r_3 as follows:*

$$t := \frac{r_1}{r_1 + r_2}, \quad r_3 := \frac{\alpha}{\frac{1}{r_1} + \frac{1}{r_2}}. \quad (31)$$

Assume that $h_1, h_2, h_3 : [0, \infty) \rightarrow [0, \infty)$ such that

$$h_3(r_3) \geq h_1^{(1-t)}(r_1) h_2^t(r_2), \quad \forall r_1, r_2 > 0. \quad (32)$$

Let $p \geq 1$ and denote

$$\begin{aligned} A &:= \left(\int_0^\infty r^{p-1} h_1(r) dr \right)^{\frac{1}{p}}, \\ B &:= \left(\int_0^\infty r^{p-1} h_2(r) dr \right)^{\frac{1}{p}}, \\ C &:= \left(\int_0^\infty r^{p-1} h_3(r) dr \right)^{\frac{1}{p}}. \end{aligned}$$

Then,

$$C \geq \frac{\alpha}{\frac{1}{A} + \frac{1}{B}}. \quad (33)$$

Proof : Let $H_3 : [0, \infty) \rightarrow [0, \infty)$ be such that $H_3(r) = h_3\left(\frac{\alpha}{2}r\right)$, and let $C' := \int_0^\infty r^{p-1} H_3(r) dr$. Let $r'_3 := \frac{2}{\alpha}r_3$. Then r_1, r_2, r'_3 satisfy (28) and

$$H_3(r'_3) = h_3(r_3) \geq h_1^{(1-t)}(r_1) h_2^t(r_2),$$

so h_1, h_2, H_3 satisfy also (29). So by Proposition 6 we have that

$$\frac{2}{\frac{1}{A} + \frac{1}{B}} \leq C' = \left(\int_0^\infty r^{p-1} h_3\left(\frac{\alpha}{2}r\right) dr \right)^{\frac{1}{p}} = \frac{2}{\alpha} \left(\int_0^\infty s^{p-1} h_3(s) ds \right)^{\frac{1}{p}}$$

$$\text{or } C \geq \frac{\alpha}{\frac{1}{A} + \frac{1}{B}}.$$

This completes the proof. \square

Proposition 7. *In the notation introduced above, if $\alpha := |\theta_1 + \theta_2|_2$ then*

$$\frac{\alpha}{|K \cap E_3|^{\frac{1}{2}}} \leq \frac{1}{|K \cap E_1|^{\frac{1}{2}}} + \frac{1}{|K \cap E_2|^{\frac{1}{2}}}. \quad (34)$$

Proof : By Lemma 10 we have that $h_1, h_2, h_3, r_1, r_2, r_3, t$ satisfy (31) and (32). By Corollary 5, applied to $p = 2$, we have that for A, B, C as in Corollary 5,

$$\frac{\alpha}{C} \leq \frac{1}{A} + \frac{1}{B}. \quad (35)$$

Note that by Lemma 9,

$$A^2 := \int_0^\infty r h_1(r) dr = \frac{|K \cap E_1|}{2\pi}, \quad B^2 := \int_0^\infty r h_2(r) dr = \frac{|K \cap E_2|}{2\pi} \text{ and}$$

$$C^2 := \int_0^\infty r^{p-1} h_3(r) dr = \frac{|K \cap E_3|}{2\pi}. \quad (36)$$

By (35) and (36), we complete the proof. \square

Corollary 6. *Let K be a symmetric complex convex body in \mathbb{R}^n . Let H be an $(n - 2)$ -dimensional subspace of \mathbb{C}^n . Let $u \in H^\perp$ complex unit vector and let $H_u := \text{span}\{H, u\}$ and $r(u) := |K \cap H_u|^{\frac{1}{2}}$. Then $r : H^\perp \cap S^{2n-1} \rightarrow (0, \infty)$ is the boundary of a complex convex body in H^\perp .*

Proof : In order to show that the curve r is the boundary of a convex body in \mathbb{C}^n it is enough to show that r^{-1} is the restriction of a norm to H^\perp . So if u_1, u_2 are two non-parallel unit vectors in H^\perp , let $u_3 := \frac{u_1 + u_2}{|u_1 + u_2|} =: \frac{1}{\alpha}(u_1 + u_2)$. It is enough to show that

$$\frac{\alpha}{r(u_3)} \leq \frac{1}{r(u_1)} + \frac{1}{r(u_2)}.$$

In the notation of this section we have that $H_{u_i} = E_i$ and $r(u_i) = |K \cap E_i|^{\frac{1}{2}}$. The result follows from Proposition 7. \square

Proof of Theorem 6: In the case where $n = 2$ the body $I_c(K)$ is simply a rotation of K , so the result is obvious. Let $n \geq 3$. Then Corollary 6 implies that $I_c(K) \cap H^\perp$ is convex for every $(n - 2)$ -dimensional subspace H of \mathbb{C}^n . This implies that $I_c(K)$ is convex. The symmetry of

$I_c(K)$ is obvious from the definition. Finally, it is not difficult to see that $I_c(K)$ satisfies (3). This implies that $I_c(K)$ is a complex convex body. \square

In the case where $K \subset \mathbb{R}^n$ is convex, by results of Hensley [20] and Borell [4], one has that the intersection body of K , $I(K)$, is isomorphic to an ellipsoid; i.e. $d_{BM}(I(K), B_2^n) \leq c$ where d_{BM} stands for the Banach-Mazur distance and $c > 0$ is a universal constant. Recall that the Banach-Mazur distance of two symmetric convex bodies K_1, K_2 in \mathbb{R}^n as

$$d_{BM}(K_1, K_2) := \inf_{T \in GL_n} \inf\{a > 0 : K_1 \subseteq TK_2 \subseteq aK_1\}.$$

One can show that the same result holds also in the complex case by using a result of K. Ball [2]. However, we can immediately deduce the “complex Hensley” theorem by using a more general fact (where the result of K. Ball has been used) proved in [33]:

Proposition 8. *Let K be an origin symmetric convex body in \mathbb{R}^n and assume that the 2-intersection body, $I_2(K)$, exists and it is convex. Then*

$$d_{BM}(I_2(K), B_2^n) \leq c,$$

where $c > 0$ is an absolute constant.

Combining the above result with Proposition 2 and Theorem 7, we immediately get the following

Theorem 7. *Let K be an origin symmetric convex body in \mathbb{C}^n . Then*

$$d_{BM}(I_c(K), B_2^n(\mathbb{C})) \leq c,$$

where $c > 0$ is an absolute constant and $B_2^n(\mathbb{C}) := B_2^{2n}$ is the Euclidean ball in \mathbb{C}^n .

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