MEASURE COMPARISON AND DISTANCE INEQUALITIES FOR CONVEX BODIES

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ABSTRACT. We prove new versions of the isomorphic Busemann-Petty problem for two different measures and show how these results can be used to recover slicing and distance inequalities. We also prove a sharp upper estimate for the outer volume ratio distance from an arbitrary convex body to the unit balls of subspaces of L_p.

1. INTRODUCTION

Let f be a non-negative locally integrable function on $\mathbb{R}^n$, and let μ be the measure in $\mathbb{R}^n$ with density f, i.e. for every compact set $B \subset \mathbb{R}^n$

$$\mu(B) = \int_B f(x)dx.$$

For $\xi \in S^{n-1}$, let $\xi^\perp$ be the central hyperplane orthogonal to ξ, i.e. $\xi^\perp = \{x \in \mathbb{R}^n : \langle x, \xi \rangle = 0\}$. We write

$$\mu(B \cap \xi^\perp) = \int_{B \cap \xi^\perp} f(x) \, dx,$$

where in the right-hand side we integrate the restriction of f to $\xi^\perp$ with respect to Lebesgue measure on $\xi^\perp$.

The following Busemann-Petty problem for general measures (BPGM) was solved in [41]. Fix $n \geq 2$. Given two convex origin-symmetric bodies K and L in $\mathbb{R}^n$ such that

$$\mu(K \cap \xi^\perp) \leq \mu(L \cap \xi^\perp)$$

for every $\xi \in S^{n-1}$, does it follow that

$$\mu(K) \leq \mu(L)?$$

It was proved in [41], that for every strictly positive even continuous density f the answer to BPGM is affirmative if $n \leq 4$ and negative if $n \geq 5$.

The BPGM problem is a generalization of the original Busemann-Petty problem, posed in 1956 in [9] and asking the same question for Lebesgue measure $\mu(K) = |K|$, where the density $f \equiv 1$; see [39, 12, 11, 22] for the last steps of the solution and historical details. Throughout the paper we denote by $|K|$ the volume of K in proper dimension.

Since the answer to BPGM is negative in most dimensions, it is natural to consider the following isomorphic version. Let C_n be the smallest constant C such that for any measure μ with continuous non-negative even density f and any two origin-symmetric convex bodies K and L in $\mathbb{R}^n$ satisfying

$$\mu(K \cap \xi^\perp) \leq \mu(L \cap \xi^\perp)$$

for every $\xi \in S^{n-1}$, one necessarily has

$$\mu(K) \leq C\mu(L). \tag{1}$$

It was shown in [27] that $C_n \leq \sqrt{n}$. It is not known whether this estimate is optimal. The above result is a particular case of a more general statement connecting the BPGM with the Banach-Mazur distance between two convex bodies. Let

$$d_{BM}(K, \mathcal{I}_n) = \inf\{\frac{b}{a} : \ aD \subseteq K \subseteq bD, \text{ for some } D \in \mathcal{I}_n\}$$

be the Banach-Mazur distance between symmetric bodies $K \subset \mathbb{R}^n$ and the class of intersection bodies (see equality (10), below, for the definition). It was shown in [27] that the constant in (1) is actually less or equal to $d_{BM}(K, \mathcal{I}_n)$. It is an open question whether the Banach-Mazur distance can be replaced here by the outer volume ratio distance $d_{\text{ovr}}(K, \mathcal{I}_n)$; see the definition below. This would improve the $\sqrt{n}$ estimate for certain classes of bodies, for example we could get an absolute constant for unconditional convex bodies; see [23].

In this note we consider a version of the isomorphic Busemann-Petty problem with two different measures, as follows. Define the outer volume ratio distance from a star body K in $\mathbb{R}^n$ to the class of generalized k-intersection bodies (see equality (10), below, for the definition) by

$$d_{\text{ovr}}(K, \mathcal{BP}_k^n) = \inf\left\{\left(\frac{|D|}{|K|}\right)^{\frac{1}{n}} : \ K \subset D, \ D \in \mathcal{BP}_k^n\right\}.$$

We prove

Theorem 1. *Let K, L be star bodies in $\mathbb{R}^n$, let $0 < k < n$, and let μ_1, μ_2 be measures on $\mathbb{R}^n$ with non-negative locally integrable densities f, g so that $\|g\|_\infty = g(0) = 1$ and*

$$\mu_1(K \cap H) \leq \mu_2(L \cap H), \qquad \forall H \in Gr_{n-k}. \tag{2}$$

Then

$$\mu_1(K) \leq (d_{\mathrm{ovr}}(K, BP_k^n))^k \frac{n}{n-k} |K|^{\frac{k}{n}} (\mu_2(L))^{\frac{n-k}{n}}.$$

Note that by John's theorem [18], if K is origin-symmetric convex then $d_{\mathrm{ovr}}(K, BP_k^n) \leq \sqrt{n}$. Also the distance $d_{\mathrm{ovr}}(K, BP_k^n)$ is smaller than the Banach-Mazur distance. In particular, $d_{\mathrm{ovr}}(K, BP_k^n)$ is bounded by an absolute constant for many classes of convex bodies K. For example, for unconditional convex bodies K this distance is bounded by $\sqrt{e}$; see [23].

A version of the Busemann-Petty problem for the moments of a measure on convex bodies was established in [6]. Similarly to the case of sections, we extend this result to the case of two different measures.

For $p > 0$ denote by L_p^n the class of the unit balls of n-dimensional subspaces of L_p. In other words (see for example [22, p.117]), L_p^n is the class of origin-symmetric convex bodies D in $\mathbb{R}^n$ such that there exists a finite Borel measure ν_D on S^{n-1} satisfying

$$\|x\|_D^p = \int_{S^{n-1}} |\langle x, \theta \rangle|^p \, d\nu_D(\theta), \qquad \forall x \in \mathbb{R}^n. \tag{3}$$

For a convex body K in $\mathbb{R}^n$, denote by

$$d_{\mathrm{ovr}}(K, L_p^n) = \inf \left\{ \left(\frac{|D|}{|K|} \right)^{1/n} : K \subset D, \ D \in L_p^n \right\} \tag{4}$$

the outer volume ratio distance from K to the class of unit balls of subspaces of L_p.

Theorem 2. *Let $K, M \subset \mathbb{R}^n$ be two star bodies, let $p > 0$, and let f, g be two non-negative measurable functions on $\mathbb{R}^n$, such that $\|g\|_\infty = g(0) = 1$. Suppose that for every $\xi \in S^{n-1}$*

$$\int_K |\langle x, \xi \rangle|^p g(x) dx \leq \int_M |\langle x, \xi \rangle|^p f(x) \, dx. \tag{5}$$

Then

$$\left(\int_K g(x) dx \right)^{(n+p)/n} \leq \frac{n+p}{n} d_{\mathrm{ovr}}(M, L_p^n)^p \, |M|^{\frac{p}{n}} \int_M f(x) dx.$$

We show how Theorems 1 and 2 can be used to recover slicing inequalities for sections and moments from [23] and [6] and distance inequalities from [19] and [6, 20]. In particular, it was shown in [6, 20] that

$$c \sqrt{\frac{n}{p}} \leq \sup_K \ d_{\mathrm{ovr}}(K, L_p^n) \leq \sqrt{n}.$$

We prove a sharp estimate

$$\sup_K \ d_{\mathrm{ovr}}(K, L_p^n) \leq C \sqrt{\frac{n+p}{p}}.$$

Here c, C are absolute constants.

2. Sections

We need several definitions and facts. A closed bounded set K in $\mathbb{R}^n$ is called a *star body* if every straight line passing through the origin crosses the boundary of K at exactly two points different from the origin, the origin is an interior point of K, and the Minkowski functional of K defined by

$$\|x\|_K = \min\{a \geq 0 : x \in aK\}$$

is a continuous function on $\mathbb{R}^n$. We will denote by $\mathcal{K}_n$ the class of convex bodies in $\mathbb{R}^n$. By $|K|_m$, or simply $|K|$ when there is no ambiguity, we denote the m-dimensional Lebesgue measure of a set K. We use the polar formula for the volume $|K|$ of a star body K :

$$|K| = \frac{1}{n} \int_{S^{n-1}} \|\theta\|_K^{-n} d\theta. \tag{6}$$

If μ is a measure on $\mathbb{R}^n$ with continuous density f, we have

$$\mu(K) = \int_K f = \int_{S^{n-1}} \left(\int_0^{\|\theta\|_K^{-1}} r^{n-1} f(r\theta) dr \right) d\theta. \tag{7}$$

Let Gr_{n-k} be the Grassmanian of $(n-k)$-dimensional subspaces of $\mathbb{R}^n$. For $1 \leq k \leq n-1$, the $(n-k)$-dimensional spherical Radon transform $\mathcal{R}_{n-k} : C(S^{n-1}) \to C(Gr_{n-k})$ is a linear operator defined by

$$\mathcal{R}_{n-k} g(H) = \int_{S^{n-1} \cap H} g(x) \, dx, \quad \forall H \in Gr_{n-k}$$

for every function $g \in C(S^{n-1})$.

For every $H \in Gr_{n-k}$, the $(n-k)$-dimensional volume of the section of a star body K by H can be written as

$$|K \cap H| = \frac{1}{n-k} \mathcal{R}_{n-k}(\|\cdot\|_K^{-n+k})(H). \tag{8}$$

More generally, for a measure μ with continuous density f and any $H \in Gr_{n-k}$, we write

$$\mu(K \cap H) = \int_{K \cap H} f = \mathcal{R}_{n-k} \left(\int_0^{\|\cdot\|_K^{-1}} r^{n-k-1} f(r \, \cdot) \, dr \right)(H). \tag{9}$$

A generalization of the concept of an intersection body was introduced by Zhang [40]. We say that an origin symmetric star body D in $\mathbb{R}^n$ is a generalized k-intersection body, and write $D \in \mathcal{BP}_k^n$, if there exists a finite Borel non-negative measure ν_D on Gr_{n-k} so that for every $g \in C(S^{n-1})$

$$\int_{S^{n-1}} \|x\|_D^{-k} g(x) \, dx = \int_{Gr_{n-k}} R_{n-k} g(H) \, d\nu_D(H). \tag{10}$$

When $k = 1$ we get the original class of intersection bodies $\mathcal{BP}_1^n = \mathcal{I}_n$ introduced by Lutwak [28].

Proof of Theorem 1. For a small $\delta > 0$, let $D \in \mathcal{BP}_k^n$ be a body such that $K \subset D$ and

$$|D|^{\frac{1}{n}} \leq (1 + \delta) \, d_{\mathrm{ovr}}(K, BP_k^n) \, |K|^{\frac{1}{n}}, \tag{11}$$

and let ν_D be the measure on Gr_{n-k} corresponding to D by the definition (10).

By (8) and (9), the condition (2) of the theorem can be written as

$$\mathcal{R}_{n-k}\left(\int_0^{\|\cdot\|_K^{-1}} r^{n-k-1} f(r \, \cdot) \, dr\right)(H) \leq \mathcal{R}_{n-k}\left(\int_0^{\|\cdot\|_L^{-1}} r^{n-k-1} g(r \, \cdot) \, dr\right)(H)$$

for every $H \in Gr_{n-k}$. Integrating both sides of the latter inequality with respect to ν_D and using the definition (10), we get

$$\int_{S^{n-1}} \|x\|_D^{-k} \left(\int_0^{\|x\|_K^{-1}} r^{n-k-1} f(rx) \, dr\right) dx \tag{12}$$

$$\leq \int_{S^{n-1}} \|x\|_D^{-k} \left(\int_0^{\|x\|_L^{-1}} r^{n-k-1} g(rx) \, dr\right) dx,$$

which is equivalent to

$$\int_K \|x\|_D^{-k} f(x) dx \leq \int_L \|x\|_D^{-k} g(x) dx. \tag{13}$$

Since $K \subset D$, we have $1 \geq \|x\|_K \geq \|x\|_D$ for every $x \in K$. Therefore,

$$\int_K \|x\|_D^{-k} f(x) dx \geq \int_K \|x\|_K^{-k} f(x) dx \geq \int_K f(x) dx = \mu_1(K).$$

On the other hand, by the Lemma from section 2.1 from Milman-Pajor [33, p.76],

$$\left(\frac{\int_L \|x\|_D^{-k} g(x) dx}{\int_D \|x\|_D^{-k} dx}\right)^{1/(n-k)} \leq \left(\frac{\int_L g(x) dx}{\int_D dx}\right)^{1/n}.$$

Since $\int_D \|x\|_D^{-k} dx = \frac{n}{n-k}|D|$, (see [33]) we can estimate the right-hand side of (13) by

$$\int_L \|x\|_D^{-k} g(x) dx \leq \frac{n}{n-k}(\mu_2(L))^{\frac{n-k}{n}} |D|^{\frac{k}{n}}.$$

Now apply (11) and send δ to zero to get the result.

$\square$

In the case where μ_2 is volume, i.e. $g \equiv 1$, we get the following.

Corollary 1. *Let K, L be star bodies in $\mathbb{R}^n$, let $0 < k < n$, and let μ be a measure on $\mathbb{R}^n$ with density f so that*

$$\mu(K \cap H) \leq |L \cap H|, \qquad \forall H \in Gr_{n-k}. \tag{14}$$

Then

$$\mu(K) \leq (d_{\mathrm{ovr}}(K, BP_k^n))^k \frac{n}{n-k}|L|^{\frac{n-k}{n}} |K|^{\frac{k}{n}}.$$

Corollary 1 provides a new proof of the slicing inequality for measures established earlier in [23], as follows. Let $c_{n,k} = |B_2^n|^{\frac{n-k}{n}}/|B_2^{n-k}|$, where B_2^n is the unit Euclidean ball in $\mathbb{R}^n$. Note that $c_{n,k} \in (e^{-k/2}, 1)$ (see for example [24, Lemma 2.1]).

Corollary 2. *([23]) Let K be a star body in $\mathbb{R}^n$. Then for any measure μ with density f on K we have*

$$\mu(K) \leq (d_{\mathrm{ovr}}(K, \mathcal{BP}_k^n))^k \; \frac{n}{n-k} c_{n,k} \max_{H \in Gr_{n-k}} \mu(K \cap H) \; |K|^{k/n}.$$

Proof :   Apply Corollary 1 to the bodies K and $L = cB_2^n$, where

$$c = \left(\frac{\max_{H \in Gr_{n-k}} \mu(K \cap H)}{|B_2^{n-k}|} \right)^{\frac{1}{n-k}}.$$

$\square$

We note that using a similar argument one can provide a sharp estimate for the constant in the following isomorphic version of an Busemann-Petty problem

Corollary 3. *Consider a constant $C_n > 0$ such that for any convex, symmetric bodies K, L in $\mathbb{R}^n$, and any μ be a measure on $\mathbb{R}^n$ with even density f if*

$$\mu(K \cap \xi^\perp) \leq |L \cap \xi^\perp|, \qquad \forall \xi \in S^{n-1},$$

one necessarily has

$$\mu(K) \leq C_n |L|^{\frac{n-1}{n}} |K|^{\frac{1}{n}}.$$

Then $c_1\sqrt{n} \leq C_n \leq c_2\sqrt{n}$, where c_1, c_2 are positive absolute constants.

Proof :   The fact that $C_n \leq c_2\sqrt{n}$ follows immediately from $d_{\mathrm{ovr}}(K, BP_1^n) \leq \sqrt{n}$ and Corollary 1. To prove the lower bound on C_n one can use an argument similar to that in the proof of Corollary 2 and use the bounds provided in [19, 20].

$\square$

As it was mentioned in the proof of Corollary 3, in the case $k = 1$, where $\mathcal{BP}_1^n = \mathcal{I}_n$ is the class of intersection bodies, it was proved in [19] (with an extra logarithmic term, which can be removed due to the result from [20]) that there exists an absolute constant $c > 0$ so that

$$c\sqrt{n} \leq \max_{K \in \mathcal{K}_n} d_{\mathrm{ovr}}(K, \mathcal{I}_n) \leq \sqrt{n}, \tag{15}$$

where maximum is taken over all origin-symmetric convex bodies, and the upper bound follows from John's theorem [18], see also [1].

On the other hand, for many classes of bodies, the outer volume ratio distance $d_{\mathrm{ovr}}(K, \mathcal{BP}_k^n)$ admits better bounds. For example, this distance is bounded by an absolute constant for unconditional convex bodies [23], and for the unit balls of subspaces of L_p it is bounded by $\sqrt{p}$; see [31, 25].

A weaker version of the left-hand side inequality in (15) has been proved in [[26], Proposition 6.1] where it was shown that

$$\max_{K \in \mathcal{K}_n} d_{BM}(K, \mathcal{I}_n) \geq d_{BM}(B_\infty^n, \mathcal{I}_n) \geq c\sqrt{n},$$

where d_{BM} is the Banach-Mazur distance. Actually the same proof provides bounds for the volume ratio, that complements (15). In particular we have the following

$$c\sqrt{n} \leq \max_{K \in \mathcal{K}_n} d_{\mathrm{vr}}(K, \mathcal{I}_n) \leq \sqrt{n}. \tag{16}$$

Here

$$d_{\mathrm{vr}}(K, \mathcal{I}_n) = \inf \left\{ \left(\frac{|K|}{|D|} \right)^{\frac{1}{n}} : \ D \subset K, \ D \in \mathcal{I}_n \right\}.$$

For completeness we briefly sketch the proof. Recall that the volume ratio v.r.(K, L) of two convex bodies K and L is defined by v.r.$(K, L) = \inf\{(\frac{|K|}{|T(L)|})^{1/n}\}$, where the infimum is over all affine transformations T of $\mathbb{R}^n$ for which $T(L) \subset K$. We write v.r.$(K, B_2^n) = $ v.r.(K). Let $\mathcal{VR}_n(a)$ be the class of all symmetric convex bodies in $\mathbb{R}^n$ that have volume ratio less than a. Then, there exists a universal constant c such that for every n, $\mathcal{I}_n \subseteq \mathcal{VR}_n(c)$ (see Proposition 6.2 in [26]). So, $d_{\mathrm{vr}}(K, \mathcal{I}_n) \geq d_{\mathrm{vr}}(K, \mathcal{VR}_n(c))$. By the sub-additivity of log(v.r.), we get that for all K_1, K_2, K_3 centrally symmetric convex bodies in $\mathbb{R}^n$,

$$\mathrm{v.r.}(K_1, K_2) \leq \mathrm{v.r.}(K_1, K_3) \, \mathrm{v.r.}(K_3, K_2).$$

We choose $K_1 := B_\infty^n$, $K_2 := B_2^n$ and $K_3 \in \mathcal{VR}_n(c)$ and we get that $d_{\mathrm{vr}}(B_\infty^n, \mathcal{VR}_n(c)) \geq \frac{c'\sqrt{n}}{c}$. This proves the left-hand side inequality in (16). The right-hand side follows from John's theorem [18].

It was proved in [26] that for any origin-symmetric convex body K in $\mathbb{R}^n$ and for all $1 \leq k < n$,

$$d_{\mathrm{ovr}}(K, \mathcal{BP}_k^n) \leq \sqrt{\frac{n}{k}} \left(\log \left(\frac{en}{k} \right) \right)^{3/2}.$$

Problem 1. *Does there exist an absolute constant $c > 0$ so that for all $2 \leq k < n$*

$$c\sqrt{\frac{n}{k}} \leq \max_{K \in \mathcal{K}_n} d_{\mathrm{ovr}}(K, \mathcal{BP}_k^n),$$

where maximum is taken over all origin-symmetric convex bodies K?

Actually, a weaker version of the above question was asked in [26] with d_{ovr} replaced by d_{BM}. This question is also open.

3. Moments

Proof of Theorem 2. For $\delta > 0$, let D be an origin-symmetric convex body such that $D \in L_p^n$, $M \subset D$, and

$$|D|^{\frac{1}{n}} \leq (1 + \delta) \, d_{\mathrm{ovr}}(M, L_p^n) \, |M|^{\frac{1}{n}}. \qquad (17)$$

Let ν_D be the measure on S^{n-1} corresponding to D by (17). Integrating both sides of (19) over S^{n-1} with the measure ν_D we get

$$\int_K \|x\|_D^p \, g(x) \, dx \leq \int_M \|x\|_D^p \, f(x) \, dx. \qquad (18)$$

Since $M \subset D$, we have $\|x\|_D \leq \|x\|_M \leq 1$ for every $x \in M$, so the right-hand side of (18) can be estimated from above by

$$\int_M \|x\|_D^p \, f(x) dx \leq \int_M f(x) \, dx.$$

To estimate the left-hand side of (18) from below, we use the Lemma from section 2.1 from Milman-Pajor [33, p.76], which asserts that if $\tilde{g} : \mathbb{R}^n \to \mathbb{R}$ a measurable function such that $\|\tilde{g}\|_\infty = 1$ and D is a symmetric convex body in $\mathbb{R}^n$, then

$$\left(\frac{\int_{\mathbb{R}^n} \|x\|_D^p \tilde{g}(x) dx}{\int_D \|x\|_D^p dx} \right)^{1/(n+p)} \geq \left(\frac{\int_{\mathbb{R}^n} \tilde{g}(x) dx}{\int_D dx} \right)^{1/n}.$$

Using that $\int_D \|x\|_D^p dx = \frac{n}{n+p} |D|$ we get

$$\int_{\mathbb{R}^n} \|x\|_D^p \tilde{g}(x) dx \geq \frac{n}{n+p} |D|^{-p/n} \left(\int_{\mathbb{R}^n} \tilde{g}(x) dx \right)^{(n+p)/n}.$$

applying this inequality for $\tilde{g} = g \cdot \chi_K$ we get

$$\int_K \|x\|_D^p \, g(x) dx \geq \frac{n}{n+p} |D|^{-p/n} \left(\int_K g(x) dx \right)^{(n+p)/n}.$$

Combining these inequalities with (17), we get

$$\left(\int_K g(x) dx \right)^{(n+p)/n} \leq \frac{n+p}{n} |D|^{p/n} \int_M \|x\|_D^p f(x) \, dx$$

$$\left(\int_K g(x) dx \right)^{(n+p)/n} \leq \frac{n+p}{n} d_{\mathrm{ovr}}(M, L_p^n)^p \, |M|^{\frac{p}{n}} \int_M \|x\|_D^p f(x) dx$$

$\square$

Taking $g \equiv 1$ we get the following corollary of Theorem 2:

Corollary 4. *Let K, M be origin-symmetric convex bodies in $\mathbb{R}^n$, let $p > 0$, and let f be a non-negative continuous function on $\mathbb{R}^n$. Suppose that for every $\xi \in S^{n-1}$*

$$\int_K |\langle x, \xi \rangle|^p dx \leq \int_M |\langle x, \xi \rangle|^p f(x) \, dx. \qquad (19)$$

Then

$$|K|^{\frac{n+p}{n}} \leq \frac{n+p}{n} \left(d_{\mathrm{ovr}}(M, L_p^n)\right)^p |M|^{\frac{p}{n}} \int_M f.$$

4. THE DISTANCE TO SUBSPACES OF L_p

Similarly to methods used in Section 2, choosing K in Corollary 4 as a proper multiple of the Euclidean ball, one gets the following inequality established earlier in [6]: there exists an absolute constant C so that for every origin-symmetric convex body M and every $p \geq 1$,

$$\min_{\xi \in S^{n-1}} \int_M |\langle x, \xi \rangle|^p f(x) \, dx \leq (Cp)^{p/2} d_{\mathrm{ovr}}^p(M, L_p^n) |K|^{p/n} \int_K f(x) \, dx.$$

Using this inequality, it was shown in [6, 20] (in [6] there was an extra logarithmic term in the left-hand side, which was removed in [20]) that there exists an absolute constant $c > 0$ so that for every n and every $p \geq 1$

$$c\sqrt{\frac{n}{p}} \leq \max_M \ d_{\mathrm{ovr}}(M, L_p^n) \leq \sqrt{n}, \tag{20}$$

where maximum is taken over all origin-symmetric convex bodies M in $\mathbb{R}^n$.

The goal of this section is to provide an optimal estimate in the right-hand side of (20) with respect to p. Actually we will show that the upper bound can be achieved by considering a mush smaller class of convex bodies. Recall that

$$K^\circ = \{x \in \mathbb{R}^n : \langle x, y \rangle \leq 1, \text{ for all } y \in K\}$$

is the polar body of K (we refer to [38, 1] for more information). Let $p \geq 1$ and C be a symmetric convex body in $\mathbb{R}^n$ of volume 1. We define a convex body $Z_p(C)$ via its support function

$$h_{Z_p(C)}(\theta) := \left(\int_C |\langle x, \theta \rangle|^p dx\right)^{\frac{1}{p}}, \theta \in S^{n-1}, \tag{21}$$

These bodies has been introduced by the second-named author in [34], [35] in order to investigate concentration properties of the random vector X that is uniformly distributed in C. For results of this type see [36], [37], [10]. The $Z_p(C)$-bodies is a family of convex bodies that interpolate (in an increasing matter) between a multiple of a ball ($p = 2$) and the body C, ($p = \infty$), with a no-more than linear "speed" in the distance (See [35]). A more detailed collection of properties of these bodies can be found in [36], [37] or in Chapter 5 in [8]. We define $\mathcal{M}_p^n$ be the set of symmetric convex bodies in $\mathbb{R}^n$ that are polar to some $Z_p(C)$, i.e.

$$\mathcal{M}_p^n := \left\{L \in \mathcal{K}_n : \exists C \in K_n, |C| = 1, L = Z_p^\circ(C)\right\}.$$

We have that

$$\mathcal{M}_p^n \subseteq L_p^n.$$

Indeed, writing (21) in polar coordinates

$$\|\theta\|_L^p = h_{Z_p(C)}^p(\theta) \quad = \quad \int_{S^{n-1}} |\langle x, \theta \rangle|^p \frac{\delta_n}{\|x\|_C^{n+p}} d\sigma(x)$$

$$= \quad \int_{S^{n-1}} |\langle x, \theta \rangle|^p d\mu(x),$$

where $d\mu(x) := \frac{\delta_n}{\|x\|_C^{n+p}} d\sigma(x)$ is a finite Borel measure on S^{n-1} and δ_n a constant depending only on n. It follows that $(\mathbb{R}^n, \|\cdot\|_L)$ can be isometrically embedded into L_p (see Lemma 6.4 in [22]).

At this point we have to emphasize that $\mathcal{M}_p^n$ is a much smaller class than L_p^n. For example we have that if $L \in \mathcal{M}_p^n$, $L = Z_p^\circ(C)$,

$$d_{BM}(L, B_2^n) = d_{BM}(Z_p(C), B_2^n) \leq cp,$$

while

$$\max_{K \in L_p^n} d_{BM}(K, B_2^n) \geq d_{BM}(B_p^n, B_2^n) \geq n^{\frac{1}{2} - \frac{1}{p}}.$$

In the above we have used the following well known application of Brunn-Minkowski inequality,

$$Z_p(C) \subseteq cp Z_2(C), \tag{22}$$

where $c > 0$ is an absolute constant (see, for example, [8], Proposition 5.1.2).

Another way to measure how different the two classes are is to consider the volume ratio. We will need the following result of Lutwak and Zhang [29] stating that for every star body C of volume 1 and every $p > 0$,

$$|Z_p^\circ(C)|^{\frac{1}{n}} \leq |Z_p^\circ(D_n)|^{\frac{1}{n}} \leq c_1 \frac{1}{n} \sqrt{\frac{n+p}{p}}, \tag{23}$$

where $D_n = B_2^n / |B_2^n|$ is the Euclidean ball of volume one and $c_1 > 0$ an absolute constant. So, using (22) and (29) we get that if $L = Z_p^\circ(C) \in \mathcal{M}_p^n$,

$$\text{v.r.}(L) \leq \left(\frac{|Z_p^\circ(C)|}{|\frac{1}{cp} Z_2^\circ(C)|} \right)^{\frac{1}{n}} \leq c' \min\{\sqrt{p}, \sqrt{n}\} L_C,$$

where $L_C := \left(\frac{|Z_2(C)|}{|B_2^n|} \right)^{\frac{1}{n}}$ is the isotropic constant of C (we refer to [8] for a definition and properties).

Moreover, by a result of K. Ball [3] (see also [30], Theorem 3) we have that

$$\max_{K \in L_p^n} \text{v.r.}(K) = \text{v.r.}(B_p^n) \simeq n^{\frac{1}{2} - \frac{1}{p}}.$$

We should also notice that for $p \simeq n$ the two classes are isomorphically the same and (isomorphically) equal to $\mathcal{K}_n$ (the class of centrally symmetric convex bodies in $\mathbb{R}^n$). Indeed, for $p \geq n$ an application of Brunn-Minkowski inequality (see Lemma 3.6 in [37]) implies that for every $C \in \mathcal{K}_n$,

$$c_0 C \subseteq Z_p(C) \subseteq C, \tag{24}$$

where $c_0 > 0$ is an absolute constant.

Theorem 3. *There exists $c > 0$ such that for every $n \geq 2$, $p \geq 1$,*

$$\sup_{K \in \mathcal{K}_n} d_{\mathrm{ovr}}\left(K, L_p^n\right) \leq \sup_{K \in \mathcal{K}_n} d_{\mathrm{ovr}}\left(K, \mathcal{M}_p^n\right) \leq c\sqrt{\frac{n+p}{p}} \qquad (25)$$

Proof : Let $K \in \mathcal{K}_n$, and let $C := K^\circ/|K^\circ|^{\frac{1}{n}}$. Note that

$$|C| = 1 \text{ and } C^\circ = |K^\circ|^{\frac{1}{n}} K. \qquad (26)$$

Let $L \in \mathcal{M}_p^n$ be defined as

$$L := |K^\circ|^{-\frac{1}{n}} Z_p^\circ(C) \text{ and } L^\circ = |K^\circ|^{\frac{1}{n}} Z_p(C). \qquad (27)$$

Note that, since $|C| = 1$, (21) implies that $Z_p(C) \subseteq C$. So, by (26) and (27),

$$C^\circ \subseteq Z_p^\circ(C) \text{ or } \frac{C^\circ}{|K^\circ|^{\frac{1}{n}}} \subseteq \frac{Z_p^\circ(C)}{|K^\circ|^{\frac{1}{n}}} \text{ or } K \subseteq L.$$

We will also need the Bourgain-Milman theorem ([7], see also [1]) on lower bounds of the volume product of convex bodies. It states that that for every $C \subseteq \mathbb{R}^n$ symmetric convex body of volume 1,

$$|C^\circ|^{\frac{1}{n}} \geq \frac{c_0}{n}, \qquad (28)$$

where $c_0 > 0$ is an absolute constant. The last ingredient of the proof is an inequality of Lutwak and Zhang [29] stating that for every star body C of volume 1 and every $p > 0$,

$$|Z_p^\circ(C)|^{\frac{1}{n}} \leq |Z_p^\circ(D_n)|^{\frac{1}{n}} \leq c_1 \frac{1}{n}\sqrt{\frac{n+p}{p}}, \qquad (29)$$

where $D_n = B_2^n/|B_2^n|$ is the Euclidean ball of volume one and $c_1 > 0$ an absolute constant. Using (26), (27), (28) and (29), we conclude that

$$\left(\frac{|L|}{|K|}\right)^{\frac{1}{n}} = \left(\frac{|Z_p^\circ(C)|}{|C^\circ|}\right)^{\frac{1}{n}} \leq c\sqrt{\frac{n+p}{p}},$$

where $c := \frac{c_1}{c_0} > 0$ is an absolute constant.

$\square$

Note that using (24), (25) and (20) we have that for all $p \geq 1$, $n \geq 2$,

$$\sup_{K \in \mathcal{K}_n} d_{\mathrm{ovr}}\left(K, L_p^n\right) \simeq \sqrt{\frac{n+p}{p}}.$$

In the case where $1 \leq p \leq \sqrt{n}$ there is an improvement in the inequality of Lutwak and Zhang due to Klartag and Milman [21]. Their result states that for every C symmetric convex body in $\mathbb{R}^n$ of volume 1 and every $1 \leq p \leq \sqrt{n}$,

$$|Z_p^\circ(C)|^{\frac{1}{n}} \leq \frac{c_2}{nL_C}\sqrt{\frac{n}{p}}, \qquad (30)$$

where $c_2 > 0$ is an absolute constant and L_C is the isotropic constant of C. If in the previous proof we will use (30) instead of (29) we get the following

Theorem 4. *Let $n \geq 1$, $K \in \mathcal{K}_n$ and $1 \leq p \leq \sqrt{n}$. Then*

$$d_{\mathrm{ovr}}\left(K, L_p^n\right) \leq \frac{c_3}{L_{K^\circ}}\sqrt{\frac{n}{p}} \tag{31}$$

Let us also mention that one may define M_p^n as the class of the polar bodies of $Z_p(\mu)$-bodies, where μ is a log-concave measure. This class is isomorphically equivalent to the class that we used, by a result of Ball [2] (see Proposition 3.4 in [37]).

Remark. In the case of projections the connection between comparison and distance inequalities was established by Ball [4, 5]. In particular, it was proved in [5, Example 2] that

$$c\sqrt{n} \leq \max_L d_{\mathrm{vr}}(L, \Pi_n) \leq \sqrt{n}, \tag{32}$$

where Π_n is the class of projection bodies (origin-symmetric zonoids; see [38]) in $\mathbb{R}^n$, c is an absolute constant, and maximum is taken over all origin-symmetric convex bodies in $\mathbb{R}^n$.

The distance $d_{\mathrm{vr}}(L, \Pi_n)$ has been studied by several authors. It was introduced in [5] and was proved to be equivalent to the *weak-right-hand-Gordon-Lewis* constant of L. Also it was connected to the random unconditional constant of the dual space (see Theorem 5 and Proposition 6 in [5]). In [13] this distance was called zonoid ratio, and it was proved that it is bounded above by the projection constant of the space. In the same paper the zonoid ratio was computed for several classical spaces. We refer the interested reader to [13], [14], [15], [16] for more information.

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