Negative moments of the support function and applications to isotropic convex bodies

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Abstract

We study lower tails of support functions of isotropic convex bodies. An endpoint negative moment estimate, combined with a spherical cap argument, gives polynomial deviation estimates for the inradius of random projections in every dimension; in the unconditional class this yields the optimal order of the median inradius and shows that the cube is extremal. The same argument gives estimates for convex hulls of independent rotations and, after an additional projection, a mixed rotation–projection theorem. We also prove weighted lower and upper estimates for Minkowski averages of rotated polars and their random projections, as well as an inradius dependent estimate for the geometric distance of global averages from the Euclidean ball. A family of isotropic product cylinders shows that the latter estimate is sharp, up to absolute constants, throughout the possible range of the inradius. For the cube, an explicit negative moment computation gives a quantitative projected (m, k, n)-profile and, in full dimension, recovers the classical order $n/\ln(en)$ for bounded geometric distance. Finally, we determine the sharp volume profile of intersections of independent rotations in the unconditional isotropic class and show that the cube is extremal.

1 Introduction

The geometry of an isotropic convex body is often studied through the distribution of its norm, support function and radial function in random directions. Random projections and random rotations provide two natural ways of revealing Euclidean structure, while negative moments measure the lower tail which is not seen by the usual concentration inequalities. The purpose of this paper is to combine these points of view. Our main results concern small values of the support function, inradii of random projections, intersections of independently rotated isotropic convex bodies, Minkowski averages and the geometric distance of global averages from the Euclidean ball in the isotropic position.

Let $K \subseteq \mathbb{R}^n$ be a convex body with the origin in its interior. We denote by h_K and p_K its support function and Minkowski functional. For $q \neq 0$ we set

$$M_q(K) = \left(\int_{S^{n-1}} p_K(\theta)^q \, d\sigma(\theta)\right)^{1/q}, \qquad w_q(K) = \left(\int_{S^{n-1}} h_K(\theta)^q \, d\sigma(\theta)\right)^{1/q},$$

and write $M(K) = M_1(K)$, $w(K) = w_1(K)$ and $b(K) = \max_{S^{n-1}} p_K$.

Polar integration gives

$$w_{-n}(K)^{-n} = \frac{|K^\circ|}{\omega_n}, \qquad \mathrm{vrad}(K)^n = \frac{|K|}{\omega_n},$$

where $\mathrm{vrad}(K)$ is the volume radius of K, and therefore

$$\left(\frac{w_{-n}(K)}{\mathrm{vrad}(K)}\right)^n = \frac{\omega_n^2}{|K|\,|K^\circ|}.$$

If K is centered, the Blaschke–Santaló inequality [11] gives

$$w_{-n}(K) \geqslant \mathrm{vrad}(K), \qquad \sigma\{\theta : h_K(\theta) \leqslant t\,\mathrm{vrad}(K)\} \leqslant \min\{1, t^n\}, \quad t > 0. \tag{1.1}$$

Equality in the first inequality holds exactly for ellipsoids centered at the origin. In particular, if $|K| = 1$, then

$$\sigma\{\theta : h_K(\theta) \leqslant t\sqrt{n}\} \leqslant \min\{1, (Ct)^n\}, \qquad t > 0. \tag{1.2}$$

This is the endpoint power mean form of the volume product inequality; related formulations appear in Santaló [48] and Lutwak [38].

For $0 < \vartheta < 1$ put

$$d_*(K, \vartheta) = \min\{-\ln \sigma\{h_K \leqslant \vartheta w(K)\}, n\}.$$

The parameter $d_*(K, 1/2)$ was introduced by Klartag and Vershynin [32]. The endpoint estimate has the following consequence.

Theorem 1.1. *Let $K \subseteq \mathbb{R}^n$ be a centered convex body of volume one. Then, for every $0 < \vartheta < 1$,*

$$d_*(K, \vartheta) \geqslant \min\left\{n, n \max\left\{0, \ln\left(\frac{c_1\sqrt{n}}{\vartheta w(K)}\right)\right\}\right\}.$$

In particular, $d_*(K, \vartheta) = n$ whenever

$$\vartheta \leqslant c_2 \frac{\sqrt{n}}{w(K)}.$$

For isotropic convex bodies, the optimal mean-width estimate of Bizeul [6], after rescaling from covariance-one normalization, gives $w(K) \leqslant CL_K\sqrt{n\ln(en)}$. Together with the slicing theorem of Klartag and Lehec [30], this implies $w(K) \leqslant C\sqrt{n\ln(en)}$, and hence $d_*(K, \vartheta) = n$ whenever $\vartheta \leqslant c/\sqrt{\ln(en)}$.

Our first projection result follows from an estimate of Klartag and Vershynin [32] relating negative moments of projection inradii to negative moments of the support function, together with the estimate of Pajor and Tomczak-Jaegermann [42] for the low-M^* theorem.

Theorem 1.2. *There exist absolute constants $c, c' > 0$ such that the following holds. Let $K \subseteq \mathbb{R}^n$ be a symmetric convex body of volume one and let $1 \leqslant k < (n-1)/4$. Then a Haar-distributed $F \in G_{n,k}$ satisfies*

$$r(P_F K) \geqslant c \max\left\{\frac{n}{w(K)}, \frac{1}{M(K)}\right\}$$

with probability at least $1 - e^{-c'k}$, where $r(P_F K)$ denotes the inradius of $P_F K$.

In particular, if K is symmetric and isotropic, then (2.3) gives

$$r(P_F K) \geqslant c\sqrt{\frac{n}{\ln(en)}}$$

with probability at least $1 - e^{-c'k}$ in the range of Theorem 1.2.

We also obtain two all dimensional projection results. Random sections of isotropic convex bodies were studied in [21]; high-probability regular positions and random Gelfand numbers were developed in [40]. Let $K \subseteq \mathbb{R}^n$ be isotropic, let $1 \leqslant k < n$, let F be Haar-distributed on $G_{n,k}$ and put $\Delta = n-k+1$.

Theorem 1.3. *There exist absolute constants $c, C > 0$ such that, for every $\tau \geqslant 1$,*

$$\mathbb{P}\left\{r(P_F K) < \frac{c\sqrt{n}}{\tau}\exp\left(-\frac{Ck}{\Delta}\right)\right\} \leqslant \tau^{-\Delta}. \tag{1.3}$$

In particular, writing $d = n - k$, for every $0 < \delta < 1$ there is $c_\delta > 0$ such that, whenever $d \geqslant \delta n$,

$$\mathbb{P}\left\{P_F K \not\supseteq \frac{c_\delta \sqrt{n}}{\tau} B_F\right\} \leqslant \tau^{-(d+1)}.$$

For $d \geqslant n/2$, the constant c_δ may be chosen absolute.

A second argument, based on Gordon's theorem [28], gives, with probability at least $1 - Ce^{-cd}$, the complementary bound

$$r(P_F K) \geqslant c\sqrt{\frac{d}{\ln(en)}}, \qquad d = n - k.$$

In the unconditional class the optimal order of the median inradius can be determined. Let $\mathcal{I}_n^{\mathrm{unc}}$ be the class of volume-one unconditional isotropic bodies and, for $1 \leqslant d < n$ and Haar-distributed $F \in G_{n,n-d}$, set

$$\mathfrak{r}_{n,d}^{\mathrm{unc}} = \sup\left\{s > 0 : \inf_{K \in \mathcal{I}_n^{\mathrm{unc}}} \mathbb{P}\{r(P_F K) \geqslant s\} \geqslant \frac{1}{2}\right\}.$$

Theorem 1.4. *For every $1 \leqslant d < n$,*

$$\mathfrak{r}_{n,d}^{\mathrm{unc}} \simeq \max\left\{1, \sqrt{\frac{d}{\ln(en/d)}}\right\}. \tag{1.4}$$

In particular, the volume-one cube is extremal, up to absolute constants, among unconditional isotropic bodies in every projection dimension.

We next turn to independent random rotations. The comparison between sections and intersections of a few rotations originates in [41, 22]; quantitative local-to-global forms were obtained in [49]. The theorem below follows from a mixed rotation–projection statement proved in Section 4.

Theorem 1.5. *There exist absolute constants $c, C > 0$ such that the following holds. Let $n \geqslant 2$ and $m \geqslant 2$, let $K_1, \ldots, K_m \subseteq \mathbb{R}^n$ be isotropic convex bodies and let $U_1 = I_n, U_2, \ldots, U_m$ be independent Haar-distributed orthogonal transformations. Put $q_m = (m-1)n + 1$. Then, for every $\tau \geqslant 1$,*

$$\mathbb{P}\left\{R\left(\bigcap_{j=1}^{m} U_j K_j^\circ\right) > \frac{C\tau}{\sqrt{n}}\right\} \leqslant \tau^{-q_m}.$$

Equivalently, with probability at least $1 - \tau^{-q_m}$,

$$\mathrm{conv}\{U_1 K_1, \ldots, U_m K_m\} \supseteq \frac{c\sqrt{n}}{\tau} B_2^n. \tag{1.5}$$

For every fixed $\tau_0 > 1$ the exceptional probability is exponentially small in $(m-1)n + 1$. The factor $\sqrt{n}$ in this inclusion is optimal: for the volume-one cube $Q_n = [-1/2, 1/2]^n$,

$$\mathrm{conv}\{U_1 Q_n, \ldots, U_m Q_n\} \subseteq \frac{\sqrt{n}}{2} B_2^n$$

for every choice of the rotations. More generally, if an independent $F \in G_{n,k}$ is introduced, the tail exponent becomes $mn - k + 1$.

The next result is obtained as the equal body consequence of a weighted estimate for heterogeneous bodies.

Theorem 1.6. *Let $K \subseteq \mathbb{R}^n$ be a symmetric convex body and let $U_1, \ldots, U_m$ be independent Haar-*

distributed orthogonal transformations. Put

$$A_m = \frac{1}{m}\sum_{j=1}^{m} U_j^*(K^\circ), \qquad M = M(K), \qquad b = b(K).$$

Assume that, for some $p_0 > 0$,

$$M_{-p}(K) \geqslant c_0 M \qquad (0 < p \leqslant p_0).$$

Let $2n \leqslant q \leqslant mp_0$ and put $\Delta = q - n + 1$. Then, for every $u \geqslant 0$, with probability at least $1 - e^{-u}$,

$$A_m \supseteq cM\left(\frac{cM}{b}\right)^{\frac{n-1}{\Delta}} e^{-u/\Delta} B_2^n. \tag{1.6}$$

Moreover, with probability at least $1 - e^{-cn}$,

$$A_m \subseteq C\left(M + \frac{b}{\sqrt{m}}\right) B_2^n. \tag{1.7}$$

Consequently, with probability at least $1 - e^{-u} - e^{-cn}$,

$$d_{\mathrm{G}}(A_m, B_2^n) \leqslant C\left(1 + \frac{b}{M\sqrt{m}}\right)\left(\frac{Cb}{M}\right)^{\frac{n-1}{\Delta}} e^{u/\Delta}. \tag{1.8}$$

The same method also applies after an independent random projection. In Section 5 we prove a projected weighted estimate in which the residual exponent is $q - k + 1$ and the upper deviation term is reduced by the factor $\sqrt{k/n}$. In particular, if F is an independent Haar-distributed element of $G_{n,k}$ and

$$A_m = \frac{1}{m}\sum_{j=1}^{m} U_j^*(Q_n^\circ), \qquad L_{m,k} = \ln\left(e + \min\left\{n, \frac{mn}{k\ln(en)}\right\}\right),$$

then, with probability at least $1 - Ce^{-ck}$,

$$d_{\mathrm{G}}(P_F A_m, B_F) \leqslant C\left(\sqrt{\frac{\ln(en)}{L_{m,k}}} + \sqrt{\frac{k}{mL_{m,k}}}\right).$$

Next, we consider the global averaging problem. Bourgain, Lindenstrauss and V. Milman [10] proved that sufficiently many random rotations of a symmetric convex body give an almost Euclidean average, while Chasapis and Giannopoulos [14] obtained an isomorphic version for fixed number of rotations of bodies in John's position. Litvak, V. D. Milman and Schechtman [35, 36] determined, up to absolute constants, the number of rotations needed for a uniformly Euclidean average in terms of $(b(K)/M(K))^2$. Combining the Bourgain–Lindenstrauss–Milman theorem with the Euclidean regularization scheme of Fresen [18] and Chasapis–Giannopoulos [14], adapted to the isotropic position, gives the following inradius dependent form.

Theorem 1.7. *There exist an absolute integer $k_0 \geqslant 2$ and absolute constants $c, C > 0$ such that the following holds. Let $K \subseteq \mathbb{R}^n$ be a symmetric isotropic convex body. For every integer $k \geqslant k_0$, independent Haar-distributed orthogonal transformations $U_1, \dots, U_k \in O(n)$ satisfy*

$$d_{\mathrm{G}}\left(\frac{1}{k}\sum_{i=1}^{k} U_i^*(K^\circ), B_2^n\right) \leqslant C\max\left\{1, \frac{1}{r(K)}\sqrt{\frac{n}{k}}\right\} \tag{1.9}$$

with probability at least $1 - e^{-cn}$.

Since K has volume one, $r(K) \leqslant \omega_n^{-1/n} \simeq \sqrt{n}$, while every isotropic body satisfies $r(K) \geqslant c$. Thus (1.9) gives bounded geometric distance as soon as $k \geqslant Cn/r(K)^2$. In particular, the number of rotations ranges from order n when $r(K) \simeq 1$ to an absolute number when $r(K) \simeq \sqrt{n}$. Proposition 6.3 shows that this dependence on the inradius is optimal, up to absolute constants, throughout this range.

Finally, we consider the volume of intersections of independent rotations. General estimates for the volume and radius of $C \cap U(C)$ were obtained by Brazitikos and Stavrakakis [12]. In the unconditional isotropic class we determine the correct order for every number of rotations and show that the cube is extremal. Put

$$Q_n = [-1/2, 1/2]^n, \qquad \Lambda_{m,n} = \min\{n, \ln(e+m)\}.$$

The lower estimate below is deterministic, while the matching upper estimate for the cube holds with exponentially high probability.

Theorem 1.8. *There exist absolute constants $c, C, c' > 0$ with the following property. Let $n, m \geqslant 2$, let $K_1, \dots, K_m \subseteq \mathbb{R}^n$ be unconditional isotropic convex bodies, and let $U_1, \dots, U_m \in O(n)$. Then*

$$\left|\bigcap_{i=1}^m U_i K_i\right|^{1/n} \geqslant \frac{c}{\sqrt{\Lambda_{m,n}}}. \tag{1.10}$$

On the other hand, if $U_1 = I_n$ and $U_2, \dots, U_m$ are independent Haar-distributed orthogonal transformations, then

$$\mathbb{E}\left|\bigcap_{i=1}^m U_i Q_n\right| \leqslant \left(\frac{C}{\sqrt{\Lambda_{m,n}}}\right)^n. \tag{1.11}$$

Consequently,

$$\mathbb{P}\left\{\frac{c}{\sqrt{\Lambda_{m,n}}} \leqslant \left|\bigcap_{i=1}^m U_i Q_n\right|^{1/n} \leqslant \frac{C}{\sqrt{\Lambda_{m,n}}}\right\} \geqslant 1 - e^{-c'n}. \tag{1.12}$$

In particular, the volume-one cube is extremal, up to absolute constants, for the median volume radius of intersections of m independent rotations in the unconditional isotropic class.

The paper is organized as follows. Section 2 recalls notation and preliminary facts. Section 3 contains the endpoint estimates and their applications to projections. Section 4 treats independent rotations and the mixed rotation–projection theorem. Section 5 studies small deviations, weighted Minkowski averages and their random projections. Section 6 proves Theorem 1.7, its application to the cube and the sharpness of its dependence on the inradius. Section 7 determines the sharp volume profile of random intersections in the unconditional isotropic class. The calculations for the cube and the estimates for the weak projection profile introduced in Section 3 are collected in Appendix A.

2 Notation and preliminary facts

We work in $\mathbb{R}^n$ with its standard Euclidean structure. The Euclidean norm, unit ball and unit sphere are denoted by $|\cdot|$, B_2^n and S^{n-1}, respectively. Lebesgue measure in $\mathbb{R}^n$ is also denoted by $|\cdot|$, and $\omega_n = |B_2^n|$. For $1 \leqslant p \leqslant \infty$, B_p^n denotes the unit ball of ℓ_p^n; in particular, $B_\infty^n = [-1,1]^n$. We write σ for the rotationally invariant probability measure on S^{n-1} and ν for Haar probability measure on the orthogonal group $O(n)$. For $1 \leqslant k \leqslant n$, $G_{n,k}$ denotes the set of k-dimensional linear subspaces of $\mathbb{R}^n$, and $\nu_{n,k}$ denotes its rotationally invariant probability measure. If $E \subseteq \mathbb{R}^n$ is a nonzero linear subspace, we write $B_E = B_2^n \cap E$, $S_E = S^{n-1} \cap E$ and σ_E for the rotationally invariant probability measure on S_E. If $F \in G_{n,k}$, then P_F is the orthogonal projection onto F. For $I \subseteq \{1, \dots, n\}$, let $E_I = \mathrm{span}\{e_i : i \in I\}$, let $P_I = P_{E_I}$ and write $B_2^I = B_2^n \cap E_I$, where $e_1, \dots, e_n$ are the standard coordinate vectors. We write I_n for the identity operator on $\mathbb{R}^n$ and U^* for the adjoint of U; in particular, $U^* = U^{-1}$ for $U \in O(n)$. The symbols $\mathbb{P}$ and $\mathbb{E}$ denote probability and expectation with respect to all random objects under consideration. Throughout, G denotes a standard Gaussian vector in $\mathbb{R}^n$. The symbols $c, C, c_1, C_1, \dots$ denote positive absolute constants whose values may change from line to line; $a \simeq b$ means that $c_1 a \leqslant b \leqslant C_1 a$ for absolute constants $c_1, C_1 > 0$, and a subscript indicates the parameters on which these constants may depend. For a set A, $\mathbb{1}_A$ denotes its indicator function. For $x \in \mathbb{R}^n$ and a nonempty set $A \subseteq \mathbb{R}^n$ put $\mathrm{dist}(x, A) = \inf\{|x - y| : y \in A\}$.

A convex body is a compact convex set with nonempty interior. It is symmetric if $K = -K$ and centered if its barycenter is the origin. It is unconditional if $(x_1, \dots, x_n) \in K$ implies $(\varepsilon_1 x_1, \dots, \varepsilon_n x_n) \in K$ for every choice of $\varepsilon_i \in \{-1, 1\}$. For a body K containing the origin in its interior, its radial function, support function and Minkowski functional are defined by

$$\rho_K(\theta) = \sup\{t \geqslant 0 : t\theta \in K\}, \qquad h_K(x) = \sup_{y \in K} \langle x, y \rangle, \qquad p_K(x) = \inf\{t > 0 : x \in tK\}.$$

Its polar body is

$$K^\circ = \{x \in \mathbb{R}^n : \langle x, y \rangle \leqslant 1 \text{ for all } y \in K\}.$$

If K is symmetric, then $p_K = \| \cdot \|_K$. On the sphere, $p_K = \rho_K^{-1}$, $\rho_{K^\circ} = h_K^{-1}$ and $h_{K^\circ} = p_K$. We shall also use without further comment that $(UK)^\circ = U(K^\circ)$, that $(P_F K)_F^\circ = K^\circ \cap F$ (the polar being taken in F), and that

$$\left(\bigcap_{j=1}^m K_j \right)^\circ = \mathrm{conv} \left(\bigcup_{j=1}^m K_j^\circ \right).$$

Moreover, $h_{K+L} = h_K + h_L$, $h_{\mathrm{conv}(K \cup L)} = \max\{h_K, h_L\}$ and $h_{U^*(K^\circ)}(x) = p_K(Ux)$ for $U \in O(n)$.

We write

$$R(K) = \max_{x \in K} |x|, \qquad r(K) = \max\{r > 0 : rB_2^n \subseteq K\}, \qquad \mathrm{vrad}(K) = \left(\frac{|K|}{\omega_n} \right)^{1/n}.$$

Thus $r(K) = 1/R(K^\circ)$ and $r(K) = \min_{S^{n-1}} h_K$, while $b(K) = \max_{S^{n-1}} p_K = 1/r(K)$. Polar integration gives

$$|K| = \omega_n \int_{S^{n-1}} \rho_K^n \, d\sigma = \omega_n \int_{S^{n-1}} p_K^{-n} \, d\sigma, \qquad |K^\circ| = \omega_n \int_{S^{n-1}} h_K^{-n} \, d\sigma, \tag{2.1}$$

and $\omega_n^{1/n} \simeq n^{-1/2}$. We use the notation $M_q(K)$, $w_q(K)$, $M(K)$, $w(K)$, $b(K)$ and $d_*(K, \vartheta)$ introduced in the Introduction. Polarity gives $w_q(K) = M_q(K^\circ)$ and both $M_q(K)$ and $w_q(K)$ are nondecreasing functions of q. For a symmetric body K, the function p_K is $b(K)$-Lipschitz on S^{n-1}.

For a non-negative random variable Y and $q > 0$ we write

$$\|Y\|_{L_{q,\infty}} = \sup_{s > 0} s \, \mathbb{P}\{Y > s\}^{1/q}.$$

This is the usual weak L_q, or Lorentz $L_{q,\infty}$, quasi-norm. For $A > 0$, the inequality $\|Y\|_{L_{q,\infty}} \leqslant A$ is equivalent to

$$\mathbb{P}\{Y > A\tau\} \leqslant \tau^{-q} \qquad (\tau > 0).$$

For a symmetric convex body $C \subseteq \mathbb{R}^n$ and $0 \leqslant d < n$ we use the convention

$$c_d(C, \ell_2^n) = \inf\{R(C \cap F) : F \in G_{n,n-d}\}$$

for its d-th Gelfand width in ℓ_2^n. This is equivalent to the convention in [17], where the infimum is taken over the kernels of $d \times n$ matrices. For symmetric bodies

$$d_{\mathrm{G}}(K,L) = \inf\{ab : a^{-1}L \subseteq K \subseteq bL,\ a,b>0\}$$

is the geometric distance of K and L.

A convex body K is isotropic if $|K| = 1$, its barycenter is the origin and

$$\int_K \langle x,\theta\rangle^2 \, dx = L_K^2 \qquad (\theta \in S^{n-1})$$

for some $L_K > 0$. Every convex body has an isotropic affine image, unique up to orthogonal transformations; see [24]. The slicing problem was posed by Bourgain [9]. We shall use its solution by Klartag and Lehec [30], which gives

$$c \leqslant L_K \leqslant L_0 \tag{2.2}$$

for an absolute constant L_0. We also use the optimal mean-width estimate of Bizeul [6], which improves the earlier bound of E. Milman [39], and the gauge estimate of Bizeul and Klartag [7]. After rescaling from covariance-one normalization, and using the dimension-free bound for the third-moment parameter of Letwin [33] as recorded in [6], these give

$$w(K) \leqslant C L_K \sqrt{n\ln(en)}, \qquad M(K) \leqslant C\sqrt{\frac{\ln(en)}{n}}. \tag{2.3}$$

We refer to [11] for background on isotropic convex bodies and log-concave measures and to [2, 3] for the local theory of finite-dimensional normed spaces.

3 Small support values and random projection inradii

We first justify the endpoint estimate used in the Introduction. By (2.1) and the Blaschke–Santaló inequality [11], every centered convex body C satisfies

$$|C^\circ| = \omega_n \int_{S^{n-1}} h_C(\theta)^{-n} \, d\sigma(\theta) \leqslant \frac{\omega_n^2}{|C|}.$$

This is equivalent to the first inequality in (1.1). Markov's inequality yields the tail estimate

$$\sigma\{h_C \leqslant t\,\mathrm{vrad}(C)\} \leqslant t^n \mathrm{vrad}(C)^n \int_{S^{n-1}} h_C^{-n} \, d\sigma \leqslant t^n.$$

Finally, if $|C| = 1$, then $\mathrm{vrad}(C) = \omega_n^{-1/n} \simeq \sqrt{n}$, and the volume-one form (1.2) follows as well.

Proof of Theorem 1.1. Apply (1.2) with $t = \vartheta w(K)/\sqrt{n}$ and take minus logarithms. The last assertion follows by decreasing the absolute constant $c > 0$ so that the resulting probability is at most e^{-n}. $\square$

For the proof of Theorem 1.2 we recall two standard estimates. Klartag and Vershynin [32] proved that, if K is symmetric, $1 \leqslant q < n$ and $1 \leqslant k < q/4$, then

$$\left(\int_{G_{n,k}} r(P_F K)^{-k} \, d\nu_{n,k}(F)\right)^{1/k} \leqslant C\frac{w(K)}{w_{-q}(K)^2}.$$

We shall also use the low M^* estimate in the following form (see, for example, [2]): for a symmetric convex body C and $1 \leqslant k < n$, a Haar-distributed $F \in G_{n,k}$ satisfies

$$R(C \cap F) \leqslant C\sqrt{\frac{n}{n-k}}\, w(C)$$

with probability at least $1 - \exp(-c(n-k))$.

Proof of Theorem 1.2. Apply the first estimate with $q = n-1$. By (1.1) and monotonicity of the spherical moments,

$$w_{-(n-1)}(K) \geqslant w_{-n}(K) \geqslant \omega_n^{-1/n} \geqslant c\sqrt{n}.$$

Hence

$$\left(\int_{G_{n,k}} r(P_F K)^{-k}\, d\nu_{n,k}(F)\right)^{-1/k} \geqslant c\frac{n}{w(K)}.$$

Markov's inequality gives $r(P_F K) \geqslant cn/w(K)$ with probability at least $1 - e^{-k}$.

Applying the low M^* estimate to $C = K^\circ$ and using $w(K^\circ) = M(K)$, we get

$$R(K^\circ \cap F) \leqslant C\sqrt{\frac{n}{n-k}}\, M(K)$$

outside a set of probability at most $e^{-c(n-k)}$. Since $(P_F K)^\circ_F = K^\circ \cap F$,

$$P_F K \supseteq c\sqrt{1 - \frac{k}{n}}\,\frac{1}{M(K)} B_F.$$

Since $k < (n-1)/4$, the factor $\sqrt{1-k/n}$ is bounded below by an absolute constant. Intersecting the two good events completes the proof. □

Let X be uniformly distributed on an isotropic convex body K. Paouris' deviation inequality [43] gives

$$\mathbb{P}\{|X| > Ct\sqrt{n}L_K\} \leqslant e^{-ct\sqrt{n}}, \qquad t \geqslant 1. \tag{3.1}$$

Lemma 3.1. *There exist absolute constants $c, C > 0$ such that every isotropic convex body $K \subseteq \mathbb{R}^n$, $n \geqslant 2$, admits a centered convex body $\widehat{K}$ of volume one satisfying*

$$R(\widehat{K}) \leqslant C\sqrt{n}, \tag{3.2}$$

$$\widehat{K} \subseteq \lambda_n K, \qquad 1 \leqslant \lambda_n \leqslant 1 + Ce^{-c\sqrt{n}}, \tag{3.3}$$

and

$$K^\circ \subseteq \lambda_n \widehat{K}^\circ, \qquad \widehat{K}^\circ \supseteq \frac{c}{\sqrt{n}} B_2^n. \tag{3.4}$$

Moreover, for every $t > 0$,

$$\sigma\{\rho_{\widehat{K}^\circ} \geqslant t\} \leqslant \min\left\{1, \left(\frac{C}{t\sqrt{n}}\right)^n\right\}. \tag{3.5}$$

Proof. Choose a sufficiently large absolute A and put

$$K_0 = K \cap A\sqrt{n}L_K B_2^n, \qquad v = |K_0|, \qquad R_0 = A\sqrt{n}L_K.$$

By (3.1), $1 - v \leqslant e^{-c\sqrt{n}}$, and in particular $v \geqslant 1/2$. If z is the barycenter of K_0, the fact that K is centered gives

$$z = \frac{1}{v}\int_{K_0} x\, dx = -\frac{1}{v}\int_{K\setminus K_0} x\, dx.$$

Consequently,

$$|z| \leqslant \frac{1}{v}\mathbb{E}\left[|X|\mathbb{1}_{\{|X|>R_0\}}\right].$$

Integration of the tail gives

$$\mathbb{E}\left[|X|\mathbb{1}_{\{|X|>R_0\}}\right] = R_0\mathbb{P}\{|X| > R_0\} + \int_{R_0}^{\infty} \mathbb{P}\{|X| > s\}\, ds \leqslant CL_K e^{-c\sqrt{n}},$$

and therefore $|z| \leqslant CL_K e^{-c\sqrt{n}}$. On the other hand, for every $\theta \in S^{n-1}$, the one-dimensional reverse Hölder inequality for log-concave random variables (see, for example, [11]) applied to $\langle X, \theta\rangle$ gives

$$h_K(\theta) \geqslant \mathbb{E}\langle X, \theta\rangle_+ = \frac{1}{2}\mathbb{E}|\langle X, \theta\rangle| \geqslant cL_K.$$

Here the equality follows from $\mathbb{E}\langle X, \theta\rangle = 0$ and the identity $y_+ = (|y| + y)/2$. Thus $r(K) \geqslant cL_K$ and $-z \in Ce^{-c\sqrt{n}}K$. Therefore

$$\widehat{K} = v^{-1/n}(K_0 - z) \subseteq (1 + Ce^{-c\sqrt{n}})K.$$

The body $\widehat{K}$ is centered, has volume one and satisfies $R(\widehat{K}) \leqslant C\sqrt{n}L_K \leqslant C\sqrt{n}$ by (2.2). Polarity gives (3.4), while centered Santaló and polar integration yield (3.5). Consequently, every monotone positively homogeneous operation applied to K° is controlled, up to the factor λ_n, by the same operation applied to $\widehat{K}^\circ$. $\square$

The following lemma is the spherical cap step underlying the low-M^* and escape through a mesh methods of Pajor and Tomczak-Jaegermann [42] and Gordon [28].

Lemma 3.2. *Let D be a convex body in a k-dimensional Euclidean space E, $k \geqslant 1$, and assume that $rB_E \subseteq D$. If $R \geqslant r$ and $Ru \in D$ for some $u \in S_E$, then*

$$\rho_D(\theta) \geqslant \frac{R}{2} \qquad \textit{whenever} \qquad |\theta - u| \leqslant \frac{r}{R}.$$

Moreover,

$$\sigma_E\left\{\theta : \rho_D(\theta) \geqslant \frac{R}{2}\right\} \geqslant \frac{1}{2}\left(\frac{r}{8R}\right)^{k-1}.$$

Proof. If $|\theta - u| \leqslant r/R$, then $R(\theta - u) \in rB_E \subseteq D$. Since $Ru \in D$ and D is convex,

$$\frac{R}{2}\theta = \frac{1}{2}Ru + \frac{1}{2}R(\theta - u) \in D.$$

For $k \geqslant 2$, the usual estimate for the measure of a spherical cap, applied with Euclidean radius $r/(2R)$, gives

$$\sigma_E\left\{\theta : |\theta - u| \leqslant \frac{r}{R}\right\} \geqslant \frac{1}{2}\left(\frac{r}{8R}\right)^{k-1};$$

see, for example, [4, pp. 10–12]. For $k = 1$, the sphere consists of two points of mass $1/2$, and one of them is the extremal direction u. $\square$

Theorem 3.3. *Let (C, F) be a random pair such that C is a convex body in $\mathbb{R}^n$ and $F \in G_{n,k}$, and assume that*

$$rB_2^n \subseteq C$$

for every realization. Put $d = k-1$ and

$$\overline{\alpha}(t) = \mathbb{E}\left(\sigma_F\{\theta \in S_F : \rho_C(\theta) \geqslant t\}\right).$$

Then, for every $s > 0$,

$$\mathbb{P}\{R(C \cap F) > s\} \leqslant 2 \sum_{\ell=0}^{\infty} \left(\frac{2^{\ell+4}s}{r}\right)^{d} \overline{\alpha}(2^{\ell-1}s). \tag{3.6}$$

Proof. For $\ell \geqslant 0$ put

$$\mathcal{E}_\ell = \{2^\ell s < R(C \cap F) \leqslant 2^{\ell+1}s\}.$$

On $\mathcal{E}_\ell$, Lemma 3.2 gives

$$\sigma_F\{\theta \in S_F : \rho_C(\theta) \geqslant 2^{\ell-1}s\} \geqslant \frac{1}{2}\left(\frac{r}{2^{\ell+4}s}\right)^{d}.$$

Markov's inequality therefore yields

$$\mathbb{P}(\mathcal{E}_\ell) \leqslant 2\left(\frac{2^{\ell+4}s}{r}\right)^{d} \overline{\alpha}(2^{\ell-1}s).$$

Summing over $\ell \geqslant 0$ proves (3.6). □

Corollary 3.4. *Assume in addition that, for some $a, \beta > 0$,*

$$\overline{\alpha}(t) \leqslant \min\left\{1, \left(\frac{a}{t}\right)^{\beta}\right\}, \qquad t > 0.$$

If $\Delta = \beta - d > 0$, then

$$\mathbb{P}\{R(C \cap F) > s\} \leqslant \frac{2^{4d+\beta+1}}{1 - 2^{-\Delta}} \left(\frac{a}{r}\right)^{d} \left(\frac{a}{s}\right)^{\Delta}. \tag{3.7}$$

If moreover $\beta \geqslant 2d$ and $\Delta \geqslant 1$, then, for every $\tau \geqslant 1$,

$$\mathbb{P}\left\{R(C \cap F) > Ca\left(\frac{a}{r}\right)^{d/\Delta}\tau\right\} \leqslant \tau^{-\Delta}, \tag{3.8}$$

where $C > 0$ is absolute.

Proof. Insert the assumed bound into (3.6). The ℓ-th term is at most

$$2^{4d+\beta+1}\left(\frac{a}{r}\right)^{d}\left(\frac{a}{s}\right)^{\Delta} 2^{-\ell\Delta}.$$

Summation gives (3.7). If $\beta \geqslant 2d$, then $d/\Delta \leqslant 1$ and $\beta/\Delta \leqslant 2$, so all numerical factors in (3.7) are absorbed by an absolute dilation of s. □

The use of negative moments to pass from one-dimensional small-ball information to diameters of random sections goes back to Klartag and Vershynin [32]; the formulation above records explicitly the residual exponent $\Delta = \beta - k + 1$ for a random pair (C, F).

The loss of $k-1$ is unavoidable: for $D = \mathrm{conv}(rB_E, Re_1)$, the directions on which ρ_D is comparable to R form a cap of measure comparable to $(r/R)^{k-1}$.

We now apply Theorem 3.3 to random sections. Let $C \subseteq \mathbb{R}^n$ be a convex body and $F \in G_{n,k}$. For $t > 0$ put

$$Z_t(F) = \sigma_F\{\theta \in S_F : \rho_C(\theta) \geqslant t\}.$$

The invariant probability measure on the incidence space gives

$$\int_{G_{n,k}} Z_t(F)\,d\nu_{n,k}(F) = \sigma\{\theta \in S^{n-1} : \rho_C(\theta) \geqslant t\}. \tag{3.9}$$

Proof of Theorem 1.3. Let $\widehat{K}$ be the body from Lemma 3.1 and set $C = \widehat{K}^\circ$. Then

$$C \supseteq \frac{c}{\sqrt{n}} B_2^n$$

and

$$\sigma\{\rho_C \geqslant t\} \leqslant \min\left\{1, \left(\frac{C}{t\sqrt{n}}\right)^n\right\}.$$

Apply (3.9) and Corollary 3.4 to the pair (C, F) with $a \simeq r \simeq n^{-1/2}$, $\beta = n$ and $d = k-1$. Then $\Delta = \beta - d = n - k + 1$. If $n \geqslant 2(k-1)$, the estimate (3.8) gives the stronger bound without the exponential factor. In general, (3.7) gives

$$\mathbb{P}\left\{R(C \cap F) > \frac{C}{\sqrt{n}} \exp\left(\frac{Ck}{\Delta}\right) \tau\right\} \leqslant \tau^{-\Delta}.$$

Indeed, $n = (k-1) + \Delta$, so the factor $2^{4(k-1)+n+1}$ in (3.7) is absorbed after taking the Δ-th root by $C\exp(Ck/\Delta)$. Since $K^\circ \subseteq \lambda_n C$,

$$R(K^\circ \cap F) \leqslant \lambda_n R(C \cap F).$$

Using $(P_F K)^\circ_F = K^\circ \cap F$ and $r(P_F K) = R(K^\circ \cap F)^{-1}$ proves (1.3). Writing $d = n - k$, the remaining assertions follow because $k/(d+1)$ is bounded in the stated ranges. $\square$

A second estimate follows by applying Gordon's theorem to the set of directions on which the support function is small. For a closed set $T \subseteq S^{n-1}$ put

$$\omega(T) = \mathbb{E} \sup_{\theta \in T} \langle G, \theta \rangle,$$

and, for $t > 0$,

$$T_t(K) = \{\theta \in S^{n-1} : h_K(\theta) \leqslant t\sqrt{n}\}, \qquad \omega_K(t) = \omega(T_t(K)).$$

Proposition 3.5. *Let $K \subseteq \mathbb{R}^n$ contain the origin, let $1 \leqslant d < n$, and let F be Haar-distributed on $G_{n,n-d}$. If $\omega_K(t) \leqslant c_0\sqrt{d}$ for a sufficiently small absolute constant $c_0 > 0$, then*

$$P_F K \supseteq t\sqrt{n} B_F$$

with probability at least $1 - Ce^{-cd}$. If K is isotropic, then

$$\omega_K(t) \leqslant CtnM(K) \leqslant Ct\sqrt{n \ln(en)}, \tag{3.10}$$

and consequently

$$P_F K \supseteq c\sqrt{\frac{d}{\ln(en)}} B_F$$

with the same probability estimate.

Proof. Gordon's theorem [28, Corollary 3.4] implies that $F \cap T_t(K) = \varnothing$ outside a set of probability at most Ce^{-cd} whenever $\omega_K(t) \leqslant c_0\sqrt{d}$. On this event, $h_{P_F K} = h_K$ on F and hence $P_F K \supseteq t\sqrt{n} B_F$.

For isotropic K, put $L_t = t\sqrt{n} K^\circ \cap B_2^n$. Then $T_t(K) \subseteq L_t$ and

$$\omega_K(t) \leqslant \mathbb{E} h_{L_t}(G) \leqslant t\sqrt{n}\, \mathbb{E} p_K(G) \leqslant CtnM(K).$$

The estimate $M(K) \leqslant C\sqrt{\ln(en)/n}$ gives (3.10); take $t = c\sqrt{d/(n\ln(en))}$. $\square$

Combining Theorem 1.3 and Proposition 3.5, a Haar-distributed $F \in G_{n,n-d}$ satisfies

$$r(P_F K) \geqslant c\sqrt{n}\max\left\{\exp\left(-\frac{C(n-d)}{d+1}\right), \sqrt{\frac{d}{n\ln(en)}}\right\} \tag{3.11}$$

with probability at least $1 - Ce^{-cd}$.

For the cube, the Gaussian width of the set $T_t(Q_n)$ is computed in Lemma A.1 in Appendix A.

Proposition 3.6. *Let $Q_n = [-1/2, 1/2]^n$. For $\frac{1}{2\sqrt{n}} \leqslant t \leqslant \frac{1}{2}$, one has*

$$\omega_{Q_n}(t) \simeq t\sqrt{n\ln\left(e + \frac{1}{t^2}\right)}.$$

For $0 < t < 1/(2\sqrt{n})$, the set $T_t(Q_n)$ is empty.

The proof is given in Appendix A.

Corollary 3.7. *Let $Q_n = [-1/2, 1/2]^n$, let $1 \leqslant d < n$, and let F be Haar-distributed on $G_{n,n-d}$. Then*

$$P_F Q_n \supseteq c\sqrt{\frac{d}{\ln(en/d)}}B_F$$

with probability at least $1 - Ce^{-cd}$.

Proof. Put

$$t = c_1\sqrt{\frac{d/n}{\ln(en/d)}}.$$

If $t < 1/(2\sqrt{n})$, then $T_t(Q_n) = \varnothing$ and the conclusion is immediate. Otherwise, Proposition 3.6 gives

$$\omega_{Q_n}(t)^2 \leqslant Ct^2 n\ln\left(e + \frac{1}{t^2}\right) \leqslant Cc_1^2 d.$$

Choose c_1 sufficiently small and apply Proposition 3.5. $\square$

The same argument applies to every body which contains a coordinate cube.

Proposition 3.8. *Let $K \subseteq \mathbb{R}^n$ be a convex body containing the origin, assume that*

$$K \supseteq aB_\infty^n$$

for some $a > 0$, let $1 \leqslant d < n$, and let F be Haar-distributed on $G_{n,n-d}$. Then

$$r(P_F K) \geqslant ca\max\left\{1, \sqrt{\frac{d}{\ln(en/d)}}\right\} \tag{3.12}$$

with probability at least $1 - Ce^{-cd}$.

The proof is given in Appendix A.

For a volume-one isotropic body K and Haar-distributed $F \in G_{n,n-d}$ put

$$Y_K(F) = \sqrt{n}\,R(K^\circ \cap F).$$

For $1 \leqslant d < n$ and $q > 0$ define

$$\mathfrak{A}_{n,d}(q) = \sup_K \|Y_K\|_{L_{q,\infty}},$$

where the $L_{q,\infty}$ quasi-norm is taken with respect to the randomness of F and the supremum is over all volume-one isotropic bodies.

The cube gives a deterministic lower bound for $\mathfrak{A}_{n,d}(q)$. The relevant width estimate goes back to Kashin [29] and Garnaev–Gluskin [19]; we use the sharp formulation [17, Theorem 1.1].

Proposition 3.9. *For every $1 \leqslant d < n$, every $q > 0$ and every subspace $F \subseteq \mathbb{R}^n$ of codimension d,*

$$\sqrt{n}\, R(Q_n^\circ \cap F) \geqslant c\sqrt{n} \min\left\{1, \sqrt{\frac{\ln(en/d)}{d}}\right\}. \tag{3.13}$$

Consequently,

$$\mathfrak{A}_{n,d}(q) \geqslant c\sqrt{n} \min\left\{1, \sqrt{\frac{\ln(en/d)}{d}}\right\}. \tag{3.14}$$

The proof is given in Appendix A.

We next prove Theorem 1.4. The cube containment used below goes back to Bobkov and Nazarov [8]; we use the formulation in [20, Proposition 2.3].

Proof of Theorem 1.4. Let $K \in \mathcal{I}_n^{\mathrm{unc}}$. By [20, Proposition 2.3],

$$K \supseteq cB_\infty^n. \tag{3.15}$$

Proposition 3.8 now gives

$$r(P_F K) \geqslant c \max\left\{1, \sqrt{\frac{d}{\ln(en/d)}}\right\}$$

with probability at least $1 - Ce^{-cd}$. For all sufficiently large d this probability is at least $1/2$; for bounded d, the deterministic inclusion (3.15) gives the same conclusion after changing the absolute constant. This proves the lower estimate in (1.4).

For the reverse estimate, take $K = Q_n$. Proposition 3.9 gives, for every codimension-d subspace F,

$$R(Q_n^\circ \cap F) \geqslant c \min\left\{1, \sqrt{\frac{\ln(en/d)}{d}}\right\}.$$

Since $r(P_F Q_n) = R(Q_n^\circ \cap F)^{-1}$, we obtain

$$r(P_F Q_n) \leqslant C \max\left\{1, \sqrt{\frac{d}{\ln(en/d)}}\right\}$$

for every F. This proves the upper estimate and completes the proof. □

The comparison between $\mathfrak{A}_{n,d}(d)$ and $\mathfrak{A}_{n,d}(d+1)$ is given in Appendix A.

4 Random intersections and independent rotations

Comparisons between sections and intersections of a few rotations go back to V. D. Milman [41] and Giannopoulos and V. D. Milman [22]. We first recall two known estimates for the intersection of two random rotations. A result of Klartag and E. Milman [31], together with [24, Theorem 3.3], implies that for every symmetric isotropic convex body K a Haar rotation U satisfies

$$K^\circ \cap U(K^\circ) \subseteq \frac{C}{\sqrt{n}} B_2^n$$

with probability at least $1-\exp(-cn)$. The local-to-global theorem of Rudelson and Vershynin [49, Appendix], combined with the low M^* estimate, gives the related bound

$$R(K^\circ \cap U(K^\circ)) \leqslant CM(K) \leqslant C\sqrt{\frac{\ln(en)}{n}}$$

with the same type of probability, where the last inequality follows from (2.3). Related estimates may be found in [41, 23, 37, 12]. Our proof of Theorem 1.5 is different and starts from an exact radial-tail identity.

A star body is a compact set, star shaped about the origin, with positive continuous radial function. For such a body C put

$$\alpha_C(t) = \sigma\{\theta : \rho_C(\theta) \geqslant t\}.$$

Lemma 4.1. *Let $C_1, \dots, C_m$ be star bodies, let $U_1, \dots, U_m$ be independent Haar rotations and put $D_m = \bigcap_{j=1}^m U_j C_j$. If*

$$\alpha_{C_j}(t) = \sigma\{\theta : \rho_{C_j}(\theta) \geqslant t\},$$

then, for every $p > 0$,

$$\mathbb{E}\int_{S^{n-1}} \rho_{D_m}(\theta)^p \, d\sigma(\theta) = p\int_0^\infty t^{p-1}\prod_{j=1}^m \alpha_{C_j}(t)\, dt. \tag{4.1}$$

The same identity holds with $U_1 = I_n$. In particular, if the C_j are polars of centered volume-one bodies, then, for $0 < p < mn$,

$$\left(\mathbb{E}\int_{S^{n-1}} \rho_{D_m}^p \, d\sigma\right)^{1/p} \leqslant \frac{C}{\sqrt{n}}\left(\frac{mn}{mn-p}\right)^{1/p}.$$

Proof. If all rotations are random, then for fixed θ, independence and rotational invariance give

$$\mathbb{P}\{\rho_{D_m}(\theta) \geqslant t\} = \prod_{j=1}^m \alpha_{C_j}(t).$$

Layer cake and Tonelli prove (4.1). If $U_1 = I_n$, the pointwise probability equals

$$\mathbb{1}_{\{\rho_{C_1}(\theta) \geqslant t\}} \prod_{j=2}^m \alpha_{C_j}(t),$$

and integration in θ gives the same product. For centered volume-one bodies, (1.2) gives $\alpha_{C_j}(t) \leqslant \min\{1, (C/(t\sqrt{n}))^n\}$, and integration yields the last estimate. $\square$

Theorem 4.2. *There exist absolute constants $c, C > 0$ such that the following holds. Let $n \geqslant 2$, $m \geqslant 2$, let $K_1, \dots, K_m \subseteq \mathbb{R}^n$ be isotropic convex bodies, and let $U_1 = I_n, U_2, \dots, U_m$ be independent Haar rotations. Independently, let F be Haar-distributed on $G_{n,k}$, where $1 \leqslant k \leqslant n$. Put*

$$H_m = \mathrm{conv}\{U_1K_1, \dots, U_mK_m\}, \qquad q_{m,k} = mn - k + 1.$$

Then, for every $\tau \geqslant 1$,

$$\mathbb{P}\left\{P_F H_m \not\supseteq \frac{c\sqrt{n}}{\tau} B_F\right\} \leqslant \tau^{-q_{m,k}}. \tag{4.2}$$

Equivalently,

$$\mathbb{P}\left\{R\left(F \cap \bigcap_{j=1}^{m} U_j K_j^{\circ}\right) > \frac{C\tau}{\sqrt{n}}\right\} \leqslant \tau^{-q_{m,k}}.$$

Proof. For every j, let $\widehat{K}_j$ be the body given by Lemma 3.1, and put

$$C_j = \widehat{K}_j^{\circ}, \qquad \widehat{D}_m = \bigcap_{j=1}^{m} U_j C_j.$$

Then

$$\widehat{D}_m \supseteq \frac{c}{\sqrt{n}} B_2^n$$

and

$$\alpha_{C_j}(t) \leqslant \min\left\{1, \left(\frac{C}{t\sqrt{n}}\right)^n\right\}.$$

For $t > 0$ define

$$Z_t(U, F) = \sigma_F\{\theta \in S_F : \rho_{\widehat{D}_m}(\theta) \geqslant t\}.$$

The incidence identity (3.9), independence and rotational invariance give

$$\mathbb{E}_{U,F} Z_t(U, F) = \prod_{j=1}^{m} \alpha_{C_j}(t) \leqslant \min\left\{1, \left(\frac{C}{t\sqrt{n}}\right)^{mn}\right\}.$$

The same formula is valid with $U_1 = I_n$, because the averaging over F makes the first sampled direction uniform on S^{n-1}.

Apply Corollary 3.4 to the pair $(\widehat{D}_m, F)$ with $a \simeq r \simeq n^{-1/2}$, $\beta = mn$ and $d = k-1$. Then $\Delta = q_{m,k}$. Since $m \geqslant 2$ and $k \leqslant n$ then $mn \geqslant 2(k-1)$, so the absolute-constant conclusion (3.8) applies. Therefore

$$\mathbb{P}\left\{R(\widehat{D}_m \cap F) > \frac{C\tau}{\sqrt{n}}\right\} \leqslant \tau^{-q_{m,k}}.$$

By (3.4),

$$\bigcap_{j=1}^{m} U_j K_j^{\circ} \subseteq \lambda_n \widehat{D}_m,$$

so the same estimate holds, after changing the absolute constant, for $F \cap \bigcap_j U_j K_j^{\circ}$.

Finally,

$$H_m^{\circ} = \bigcap_{j=1}^{m} U_j K_j^{\circ}, \qquad (P_F H_m)_F^{\circ} = H_m^{\circ} \cap F.$$

Taking polars in F proves (4.2). □

Taking $k = n$ in Theorem 4.2 proves Theorem 1.5. In particular, for every fixed $\tau_0 > 1$ the exceptional probability is at most $\exp[-c_{\tau_0}((m-1)n+1)]$, where $c_{\tau_0} > 0$ depends only on τ_0. Integration of the tail gives the corresponding circumradius moments for every $0 < p < mn - k + 1$.

The following isotropic cylinder shows that the cap-incidence exponent has the correct dimensional order. Put

$$Z_n = [-\sqrt{3}, \sqrt{3}] \times \sqrt{n+1}\, B_2^{n-1}, \qquad K_n = |Z_n|^{-1/n} Z_n.$$

Then K_n is isotropic, $|Z_n|^{1/n} \simeq 1$, and, writing $\theta = (\theta_1, \theta')$,

$$\rho_{K_n^\circ}(\theta) \simeq \frac{1}{|\theta_1| + \sqrt{n}\,|\theta'|}. \tag{4.3}$$

Consequently, there exist absolute constants $A, c, C > 0$ such that, for $A \leqslant \tau \leqslant c\sqrt{n}$,

$$\left\{\operatorname{dist}(\theta, \{e_1, -e_1\}) \leqslant \frac{c}{\tau}\right\} \subseteq \left\{\rho_{K_n^\circ}(\theta) \geqslant \frac{c\tau}{\sqrt{n}}\right\} \subseteq \left\{\operatorname{dist}(\theta, \{e_1, -e_1\}) \leqslant \frac{C}{\tau}\right\}. \tag{4.4}$$

Proposition 4.3. *Let $m \geqslant 2$, let $U_1 = I_n, U_2, \dots, U_m$ be independent Haar rotations, and put*

$$D_{m,n} = \bigcap_{j=1}^{m} U_j K_n^\circ.$$

There are absolute constants $A, c, c_0, C > 0$ such that, for $A \leqslant \tau \leqslant c_0\sqrt{n}$,

$$\left(\frac{c}{\tau}\right)^{(m-1)(n-1)} \leqslant \mathbb{P}\left\{R(D_{m,n}) \geqslant \frac{c\tau}{\sqrt{n}}\right\} \leqslant \left(\frac{C}{\tau}\right)^{(m-1)(n-1)}.$$

Proof. For the lower estimate, require $U_j^{-1}e_1$ to lie within c/τ of e_1 for every $j \geqslant 2$ and use the first inclusion in (4.4). For the upper estimate, if $R(D_{m,n}) \geqslant c\tau/\sqrt{n}$, then some direction lies simultaneously in the corresponding high-radius caps, so $U_j e_1$ lies within C/τ of $\{e_1, -e_1\}$ for every $j \geqslant 2$. For $0 < a \leqslant c_0$, a spherical cap of Euclidean radius a has measure between $(ca)^{n-1}$ and $(Ca)^{n-1}$; independence completes the proof. □

The endpoint argument gives the exponent $(m-1)n+1$, while the cylinder has exponent $(m-1)(n-1)$; for fixed m, the two have the same leading dimensional order.

5 Small deviations, negative moments and Minkowski averages

We begin by recalling the small-deviation information needed below. For a symmetric convex body K put

$$d(K) = d_*(K^\circ, 1/2) = \min\{-\ln\sigma\{\theta : 2p_K(\theta) \leqslant M(K)\}, n\},$$

and

$$\beta(K) = \frac{\operatorname{Var} p_K(G)}{(\mathbb{E}p_K(G))^2}, \qquad \beta_*(K) = \beta(K^\circ) = \frac{\operatorname{Var} h_K(G)}{(\mathbb{E}h_K(G))^2}.$$

We shall also use the critical dimension

$$k(K) = n\left(\frac{M(K)}{b(K)}\right)^2.$$

Paouris and Valettas [46, Proposition 4.3] proved the general comparison

$$\ln\frac{1}{\beta(K)} \leqslant Ck(K) \leqslant \frac{C}{\beta(K)}. \tag{5.1}$$

The estimates of Klartag–Vershynin [32], Paouris–Valettas [45] and Paouris–Pivovarov–Valettas [44] imply, for absolute constants $c, C > 0$,

$$M_{-q}(K) \geqslant cM(K) \qquad (0 < q < cd(K)), \tag{5.2}$$

and $d(K) \geqslant c/\beta(K)$. More precisely, for $0 < q < c/\beta(K)$,

$$M_{-q}(K) \geqslant \left[1 - C\min\left\{\frac{q}{k(K)}, \max\{\sqrt{\beta(K)}, q\beta(K)\}\right\}\right] M(K).$$

In particular,

$$M_{-q}(K) \geqslant cM(K) \qquad (0 < q < c/\beta(K)). \tag{5.3}$$

Equivalently, $w_{-q}(C) \geqslant cw(C)$ in the ranges $0 < q < cd_*(C, 1/2)$ and $0 < q < c/\beta_*(C)$.

The use of random Minkowski averages as an empirical regularization goes back to Bourgain, Lindenstrauss and V. Milman [10]. Litvak, V. D. Milman and Schechtman [35, 36] studied q-averages of norms and identified families of projective caps as a central geometric mechanism, while concentration and net implementations were developed by Artstein-Avidan, Friedland and V. D. Milman [1]. In particular, the bounded-distance threshold corresponding to $(b(K)/M(K))^2$ is classical. The new point below is the quantitative estimate below this threshold: weighted negative moments provide the lower inclusion, while the upper inclusion follows the classical empirical method.

It is nevertheless false that $w_{-cn}(C)$ is always comparable to $w(C)$ for isotropic convex bodies. The standard cross-polytope gives a simple example.

Example 5.1. *Let $C_n = a_n B_1^n$, where*

$$a_n = |B_1^n|^{-1/n} = \left(\frac{n!}{2^n}\right)^{1/n} \simeq n.$$

Then C_n is a symmetric isotropic convex body of volume one and

$$w(C_n) \simeq \sqrt{n\ln(n+1)}. \tag{5.4}$$

For every fixed $\alpha \in (0, 1)$,

$$w_{-\alpha n}(C_n) \leqslant C_\alpha\sqrt{n}. \tag{5.5}$$

Here $C_\alpha > 0$ depends only on α. Consequently, $w_{-\alpha n}(C_n)/w(C_n) \to 0$ as $n \to \infty$.

Indeed, if $G = (g_1, \dots, g_n)$ is standard Gaussian and $\Theta = G/|G|$, then Θ is uniform on S^{n-1}, independent of $|G|$, and

$$\mathbb{E}|G| \simeq \sqrt{n}, \qquad \mathbb{E}\|G\|_\infty \simeq \sqrt{\ln(n+1)}.$$

Hence

$$\mathbb{E}\|\Theta\|_\infty \simeq \sqrt{\frac{\ln(n+1)}{n}},$$

and (5.4) follows from $h_{C_n}(\theta) = a_n\|\theta\|_\infty$. For (5.5), use the event $1 \leqslant |g_i| \leqslant 2$ for all i. It has probability p_0^n for an absolute $p_0 \in (0, 1)$ and implies $\|G\|_\infty/|G| \leqslant 2/\sqrt{n}$. Therefore

$$\sigma\left\{\theta : \|\theta\|_\infty \leqslant \frac{2}{\sqrt{n}}\right\} \geqslant p_0^n,$$

which gives $w_{-\alpha n}(C_n) \leqslant C_\alpha\sqrt{n}$.

Let $K_1, \dots, K_m$ contain the origin, let $U_1, \dots, U_m$ be independent Haar rotations and let $\lambda_j \geqslant 0$, $\sum_j \lambda_j = 1$. Put

$$A_\lambda = \sum_{j=1}^m \lambda_j U_j^*(K_j^\circ), \qquad b_j = b(K_j), \qquad b_\lambda = \sum_{j=1}^m \lambda_j b_j,$$

and, for $q > 0$,

$$a_q(\lambda) = \prod_{j=1}^{m} M_{-q\lambda_j}(K_j)^{\lambda_j},$$

with the usual interpretation when $\lambda_j = 0$.

Theorem 5.2. *Let the notation be as above. Assume that $q \geqslant 2n$ and put*

$$\Delta = q - n + 1.$$

Then, for every $u \geqslant 0$, with probability at least $1 - e^{-u}$,

$$A_\lambda \supseteq c\, a_q(\lambda) \left(\frac{c\, a_q(\lambda)}{b_\lambda} \right)^{\frac{n-1}{\Delta}} e^{-u/\Delta} B_2^n. \tag{5.6}$$

Moreover, if

$$\overline{M}_\lambda = \sum_{j=1}^{m} \lambda_j M(K_j), \qquad B_\lambda = \left(\sum_{j=1}^{m} \lambda_j^2 b_j^2 \right)^{1/2},$$

then, with probability at least $1 - e^{-cn}$,

$$A_\lambda \subseteq C(\overline{M}_\lambda + B_\lambda) B_2^n. \tag{5.7}$$

Proof. For fixed $\theta \in S^{n-1}$ put $X_j = p_{K_j}(U_j\theta)$. Then

$$h_{A_\lambda}(\theta) = \sum_{j=1}^{m} \lambda_j X_j.$$

By weighted arithmetic-geometric mean,

$$h_{A_\lambda}(\theta)^{-q} \leqslant \prod_{j=1}^{m} X_j^{-q\lambda_j}.$$

Independence and rotational invariance give

$$\mathbb{E} h_{A_\lambda}(\theta)^{-q} \leqslant \prod_{j=1}^{m} \left(\int_{S^{n-1}} p_{K_j}^{-q\lambda_j}\, d\sigma \right) = a_q(\lambda)^{-q}.$$

Let $L = A_\lambda^\circ$. Since $h_{A_\lambda} \leqslant b_\lambda$, we have $L \supseteq b_\lambda^{-1} B_2^n$. Also,

$$\mathbb{E}\sigma\{\rho_L \geqslant t\} = \mathbb{E}\sigma\{h_{A_\lambda} \leqslant t^{-1}\} \leqslant (t a_q(\lambda))^{-q}.$$

Apply Corollary 3.4 to the pair $(L, \mathbb{R}^n)$ with $r = b_\lambda^{-1}$, $a = a_q(\lambda)^{-1}$, $\beta = q$ and $d = n-1$. Since $q \geqslant 2n > 2(n-1)$, the absolute-constant form applies and yields

$$\mathbb{P}\left\{ R(L) > \frac{C}{a_q(\lambda)} \left(\frac{C b_\lambda}{a_q(\lambda)} \right)^{\frac{n-1}{\Delta}} e^{u/\Delta} \right\} \leqslant e^{-u}.$$

Taking polars proves (5.6).

For the upper inclusion we follow the standard empirical concentration-and-net scheme; see [10, 1].

Each p_{K_j} is b_j-Lipschitz on the sphere. For fixed θ, tensorized spherical concentration gives

$$\mathbb{P}\{|h_{A_\lambda}(\theta) - \overline{M}_\lambda| > t\} \leqslant 2\exp\left(-\frac{cnt^2}{B_\lambda^2}\right).$$

Take $t = C_0 B_\lambda$ and use a $1/2$-net $\mathcal{N} \subseteq S^{n-1}$ with $|\mathcal{N}| \leqslant 5^n$. Outside a set of probability at most e^{-cn},

$$h_{A_\lambda}(\theta) \leqslant \overline{M}_\lambda + C_0 B_\lambda \qquad (\theta \in \mathcal{N}).$$

For every convex body A containing the origin,

$$R(A) \leqslant 2 \max_{\theta \in \mathcal{N}} h_A(\theta).$$

This proves (5.7). □

For a single body and equal weights put

$$A_m = \frac{1}{m} \sum_{j=1}^{m} U_j^*(K^\circ).$$

Proof of Theorem 1.6. Apply Theorem 5.2 with $K_j = K$, $\lambda_j = 1/m$ and the order q from the statement. Then

$$a_q(\lambda) = M_{-q/m}(K) \geqslant c_0 M(K), \qquad b_\lambda = b(K),$$

and the lower inclusion is exactly (1.6), after changing the absolute constant. Moreover,

$$\overline{M}_\lambda = M(K), \qquad B_\lambda = \frac{b(K)}{\sqrt{m}},$$

so (5.7) gives (1.7). Combining the two inclusions proves (1.8). □

Corollary 5.3. *Let K, $U_1, \ldots, U_m$ and A_m be as in Theorem* 1.6*. If $md(K) \geqslant Cn$, then, with probability at least $1 - e^{-cn}$,*

$$d_{\mathrm{G}}(A_m, B_2^n) \leqslant C\left(1 + \frac{b(K)}{M(K)\sqrt{m}}\right)\left(\frac{Cb(K)}{M(K)}\right)^{\frac{Cn}{md(K)}}. \tag{5.8}$$

The same conclusion holds with the last factor replaced by

$$\left(\frac{Cb(K)}{M(K)}\right)^{Cn\beta(K)/m}$$

under the assumption $m/\beta(K) \geqslant Cn$.

Proof. Use (5.2), respectively (5.3), in Theorem 1.6, choose the order to be a sufficiently small multiple of $md(K)$, respectively $m/\beta(K)$, and take $u = cn$. □

The same argument gives a projected estimate without changing the negative-moment input.

Theorem 5.4. *Let $K_1, \ldots, K_m$ contain the origin, let $U_1, \ldots, U_m$ be independent Haar rotations and let $\lambda_j \geqslant 0$, $\sum_j \lambda_j = 1$. Put*

$$A_\lambda = \sum_{j=1}^{m} \lambda_j U_j^*(K_j^\circ), \qquad b_\lambda = \sum_{j=1}^{m} \lambda_j b(K_j), \qquad a_q(\lambda) = \prod_{j=1}^{m} M_{-q\lambda_j}(K_j)^{\lambda_j}.$$

As before, the last expression has the usual interpretation when $\lambda_j = 0$. Independently, let F be Haar-distributed on $G_{n,k}$, where $1 \leqslant k \leqslant n$. Assume that $q \geqslant 2k$ and put $\Delta = q - k + 1$. Then, for every $u \geqslant 0$, with probability at least $1 - e^{-u}$,

$$P_F A_\lambda \supseteq c\, a_q(\lambda) \left(\frac{c\, a_q(\lambda)}{b_\lambda} \right)^{\frac{k-1}{\Delta}} e^{-u/\Delta} B_F. \tag{5.9}$$

Moreover, if

$$\overline{M}_\lambda = \sum_{j=1}^{m} \lambda_j M(K_j), \qquad B_\lambda = \left(\sum_{j=1}^{m} \lambda_j^2 b(K_j)^2 \right)^{1/2},$$

then, with probability at least $1 - e^{-ck}$,

$$P_F A_\lambda \subseteq C \left(\overline{M}_\lambda + \sqrt{\frac{k}{n}}\, B_\lambda \right) B_F. \tag{5.10}$$

Proof. The calculation in the proof of Theorem 5.2 gives, for every fixed $\theta \in S^{n-1}$,

$$\mathbb{E}_U h_{A_\lambda}(\theta)^{-q} \leqslant a_q(\lambda)^{-q}.$$

Let $L = A_\lambda^\circ$. Since $h_{A_\lambda} \leqslant b_\lambda$, we have $L \supseteq b_\lambda^{-1} B_2^n$. For $t > 0$ put

$$Z_t(U, F) = \sigma_F\{\theta \in S_F : \rho_L(\theta) \geqslant t\}.$$

Since F is independent of the rotations, the preceding negative-moment estimate and the incidence identity give

$$\mathbb{E}_{U,F} Z_t(U, F) = \mathbb{E}_F \int_{S_F} \mathbb{P}_U\{h_{A_\lambda}(\theta) \leqslant t^{-1}\}\, d\sigma_F(\theta) \leqslant (t a_q(\lambda))^{-q}.$$

Apply Corollary 3.4 to the pair (L, F) with $r = b_\lambda^{-1}$, $a = a_q(\lambda)^{-1}$, $\beta = q$ and $d = k - 1$. Since $q \geqslant 2k > 2(k-1)$, we obtain

$$\mathbb{P}\left\{ R(L \cap F) > \frac{C}{a_q(\lambda)} \left(\frac{C b_\lambda}{a_q(\lambda)} \right)^{\frac{k-1}{\Delta}} e^{u/\Delta} \right\} \leqslant e^{-u}.$$

Since $(P_F A_\lambda)_F^\circ = L \cap F$, taking polars in F proves (5.9).

For the upper inclusion, condition on F and let $\mathcal{N}_F$ be a 1/2-net of S_F with $|\mathcal{N}_F| \leqslant 5^k$. The concentration estimate used above, with $t = C_0 \sqrt{k/n}\, B_\lambda$, and a union bound give, outside a set of probability at most e^{-ck},

$$h_{A_\lambda}(\theta) \leqslant \overline{M}_\lambda + C_0 \sqrt{\frac{k}{n}}\, B_\lambda \qquad (\theta \in \mathcal{N}_F).$$

Since

$$R(P_F A_\lambda) \leqslant 2 \max_{\theta \in \mathcal{N}_F} h_{A_\lambda}(\theta),$$

this proves (5.10). $\square$

For equal bodies and equal weights we obtain the following consequence.

Corollary 5.5. *Let $K \subseteq \mathbb{R}^n$ be a symmetric convex body, let $U_1, \ldots, U_m$ be independent Haar rotations and put*

$$A_m = \frac{1}{m} \sum_{j=1}^{m} U_j^*(K^\circ), \qquad M = M(K), \qquad b = b(K).$$

Independently, let F be Haar-distributed on $G_{n,k}$, where $1 \leqslant k \leqslant n$. Assume that, for some $p_0 > 0$,

$$M_{-p}(K) \geqslant c_0 M \qquad (0 < p \leqslant p_0).$$

Let $2k \leqslant q \leqslant mp_0$ and put $\Delta = q - k + 1$. Then, for every $u \geqslant 0$, with probability at least $1 - e^{-u}$,

$$P_F A_m \supseteq cM \left(\frac{cM}{b}\right)^{\frac{k-1}{\Delta}} e^{-u/\Delta} B_F. \tag{5.11}$$

Moreover, with probability at least $1 - e^{-ck}$,

$$P_F A_m \subseteq C \left(M + b\sqrt{\frac{k}{mn}}\right) B_F. \tag{5.12}$$

Consequently, with probability at least $1 - e^{-u} - e^{-ck}$,

$$d_{\mathrm{G}}(P_F A_m, B_F) \leqslant C \left(1 + \frac{b}{M}\sqrt{\frac{k}{mn}}\right) \left(\frac{Cb}{M}\right)^{\frac{k-1}{\Delta}} e^{u/\Delta}. \tag{5.13}$$

Proof. Apply Theorem 5.4 with $K_j = K$ and $\lambda_j = 1/m$. Then

$$a_q(\lambda) = M_{-q/m}(K) \geqslant c_0 M(K), \qquad b_\lambda = b(K), \qquad \overline{M}_\lambda = M(K), \qquad B_\lambda = \frac{b(K)}{\sqrt{m}}.$$

The two inclusions and their combination give the assertion. □

Corollary 5.6. *Let K, $U_1, \ldots, U_m$, F and A_m be as in Corollary 5.5. If $md(K) \geqslant Ck$, then, with probability at least $1 - e^{-ck}$,*

$$d_{\mathrm{G}}(P_F A_m, B_F) \leqslant C \left(1 + \frac{b(K)}{M(K)}\sqrt{\frac{k}{mn}}\right) \left(\frac{Cb(K)}{M(K)}\right)^{\frac{Ck}{md(K)}}. \tag{5.14}$$

The same conclusion holds with the last factor replaced by

$$\left(\frac{Cb(K)}{M(K)}\right)^{Ck\beta(K)/m}$$

under the assumption $m/\beta(K) \geqslant Ck$.

Proof. Use (5.2), respectively (5.3), in Corollary 5.5, choose q to be a sufficiently small multiple of $md(K)$, respectively $m/\beta(K)$, and take $u = ck$. □

The projected theorem also accepts inputs from L_p-centroid bodies. Let $T \subseteq \mathbb{R}^n$ be isotropic and let $Z_p(T)$ denote its L_p-centroid body. Letwin and Mikulincer [34, Proposition 2.4] proved that $w_{-2p}(Z_p(T)) \simeq \sqrt{p}$ for $1 \leqslant p \leqslant 2c_0 n$. Moreover, $w(Z_p(T)) \leqslant C\sqrt{p\ln(ep)}$ by [25, Lemma 3.4]. Since $p_{Z_p(T)^\circ} = h_{Z_p(T)}$ and $R(Z_p(T)) \leqslant Cp$, put $A_{m,p} = m^{-1}\sum_j U_j^* Z_p(T)$. The two parts of Theorem 5.4, applied with $K_j = Z_p(T)^\circ$, equal weights and $q = 2mp$, show, with U_j and F as above, that if $mp \geqslant k$, then, for every $u \geqslant 0$,

$$P_F A_{m,p} \supseteq c\sqrt{p}\, p^{-\frac{k-1}{2(2mp-k+1)}} e^{-\frac{u}{2mp-k+1}} B_F,$$
$$P_F A_{m,p} \subseteq C \left(\sqrt{p\ln(ep)} + p\sqrt{\frac{k}{mn}}\right) B_F,$$

with probability at least $1 - e^{-u}$ and $1 - e^{-ck}$, respectively.

For the cube, $M_{-p}(B_\infty^n)$ can be estimated for every $p > 0$. Chasapis, König and Tkocz [15] obtained sharp negative-moment estimates for one-dimensional sums associated with cube sections. The proposition below concerns instead the spherical ℓ_∞ gauge.

Proposition 5.7. *There exist absolute constants $c, C > 0$ such that, for every $n \geqslant 2$ and every $p > 0$,*

$$c\sqrt{\frac{\ln\left(e+\frac{n}{1+p}\right)}{n}} \leqslant M_{-p}(B_\infty^n) \leqslant C\sqrt{\frac{\ln\left(e+\frac{n}{1+p}\right)}{n}}. \tag{5.15}$$

The proof is given in Appendix A.

The preceding proposition makes the projected estimate explicit for the cube.

Corollary 5.8. *Let $n, m \geqslant 2$ and $1 \leqslant k \leqslant n$. Let $U_1, \dots, U_m$ be independent Haar-distributed orthogonal transformations, let F be an independent Haar-distributed element of $G_{n,k}$, and put*

$$A_m = \frac{1}{m}\sum_{j=1}^m U_j^*(Q_n^\circ), \qquad Q_n = [-1/2, 1/2]^n.$$

For $p \geqslant 2k/m$ set

$$\ell_p = \ln\left(e + \frac{n}{1+p}\right), \qquad \Delta_p = mp - k + 1.$$

Then, with probability at least $1 - Ce^{-ck}$,

$$d_{\mathrm{G}}(P_F A_m, B_F) \leqslant C\left(\sqrt{\frac{\ln(en)}{\ell_p}} + \sqrt{\frac{k}{m\ell_p}}\right)\left(C\sqrt{\frac{n}{\ell_p}}\right)^{\frac{k-1}{\Delta_p}} \exp\left(\frac{Ck}{\Delta_p}\right). \tag{5.16}$$

In particular, if

$$L_{m,k} = \ln\left(e + \min\left\{n, \frac{mn}{k\ln(en)}\right\}\right), \tag{5.17}$$

then

$$d_{\mathrm{G}}(P_F A_m, B_F) \leqslant C\left(\sqrt{\frac{\ln(en)}{L_{m,k}}} + \sqrt{\frac{k}{mL_{m,k}}}\right) \tag{5.18}$$

with probability at least $1 - Ce^{-ck}$.

The proof is given in Appendix A.

6 Isomorphic global Dvoretzky theorem for isotropic convex bodies

We use two standard estimates. Let $\overline{B}_2^n = \omega_n^{-1/n} B_2^n$ be the Euclidean ball of volume one. The reverse Brunn–Minkowski estimate for isotropic convex bodies gives

$$|K + s\overline{B}_2^n|^{1/n} \leqslant C(1+s), \qquad s \geqslant 0, \tag{6.1}$$

for every isotropic K; see [24, Theorem 3.9]. We shall also use the theorem of Bourgain, Lindenstrauss and V. Milman [10] in the Chernoff form of Artstein-Avidan, Friedland and V. D. Milman [1]: if K is symmetric, $M = M(K)$, $b = b(K)$, $0 < \varepsilon < 1/2$ and

$$k \geqslant \frac{c_1}{\varepsilon^2}\left(\frac{b}{M}\right)^2,$$

then independent Haar rotations $U_1, \dots, U_k$ satisfy

$$\frac{M}{1+\varepsilon}|x| \leqslant \frac{1}{k}\sum_{i=1}^{k}\|U_i x\|_K \leqslant (1+\varepsilon)M|x| \qquad (x \in \mathbb{R}^n) \tag{6.2}$$

with probability at least

$$1 - \exp\left(-c_2\varepsilon^2 nk\left(\frac{M}{b}\right)^2\right).$$

The regularization by adjoining a Euclidean ball follows Fresen [18] and Chasapis–Giannopoulos [14]. We compare K with a body obtained by adjoining a Euclidean ball. In the isotropic setting the inradius determines the value of the parameter t used in the comparison below.

Proof of Theorem 1.7. Let $r = r(K)$, so $rB_2^n \subseteq K$. For $t \geqslant 1$ set

$$K_t = \mathrm{conv}(K \cup trB_2^n), \qquad M_t = \int_{S^{n-1}} \|x\|_{K_t}\, d\sigma(x), \qquad b_t = \max_{S^{n-1}} \|x\|_{K_t}.$$

Since $rB_2^n \subseteq K$,

$$K \subseteq K_t \subseteq tK, \qquad \frac{1}{t}\|x\|_K \leqslant \|x\|_{K_t} \leqslant \|x\|_K. \tag{6.3}$$

Also, $h_{K_t} = \max\{h_K, tr\}$ and $\min_{S^{n-1}} h_K = r$, so $r(K_t) = tr$ and

$$b_t = \frac{1}{tr}.$$

By (2.1) and Hölder's inequality,

$$M_t \geqslant \left(\int_{S^{n-1}} \|x\|_{K_t}^{-n}\, d\sigma(x)\right)^{-1/n} = \frac{\omega_n^{1/n}}{|K_t|^{1/n}} \geqslant \frac{c}{\sqrt{n}\,|K_t|^{1/n}}.$$

Moreover,

$$K_t \subseteq K + trB_2^n \subseteq K + C\frac{tr}{\sqrt{n}}\overline{B}_2^n,$$

and (6.1) gives

$$|K_t|^{1/n} \leqslant C\max\left\{1, \frac{tr}{\sqrt{n}}\right\}.$$

Consequently,

$$\left(\frac{b_t}{M_t}\right)^2 \leqslant C\max\left\{\frac{n}{t^2r^2}, 1\right\}. \tag{6.4}$$

Fix $\varepsilon = 1/4$ and set

$$t_k = \max\left\{1, \frac{A}{r}\sqrt{\frac{n}{k}}\right\},$$

where A is a sufficiently large absolute constant. Then $n/(t_k^2r^2) \leqslant k/A^2$, and (6.4) shows, after choosing A and then an absolute k_0, that the hypothesis of the Bourgain–Lindenstrauss–Milman theorem is satisfied for every $k \geqslant k_0$. Thus (6.2) applied to K_{t_k} gives, with probability at least $1 - e^{-c'n}$,

$$\frac{4}{5} M_{t_k} |x| \leqslant \frac{1}{k} \sum_{i=1}^{k} \|U_i x\|_{K_{t_k}} \leqslant \frac{5}{4} M_{t_k} |x|.$$

Using (6.3),

$$\frac{4}{5} M_{t_k} |x| \leqslant \frac{1}{k} \sum_{i=1}^{k} \|U_i x\|_{K} \leqslant \frac{5t_k}{4} M_{t_k} |x|. \tag{6.5}$$

Finally,

$$h_{\frac{1}{k}\sum_{i=1}^{k} U_i^*(K^\circ)}(x) = \frac{1}{k} \sum_{i=1}^{k} \|U_i x\|_K.$$

It follows from (6.5) that

$$d_{\mathrm{G}}\left(\frac{1}{k} \sum_{i=1}^{k} U_i^*(K^\circ), B_2^n \right) \leqslant 2t_k \leqslant C \max\left\{1, \frac{1}{r(K)} \sqrt{\frac{n}{k}} \right\},$$

which proves (1.9). □

Let $K \subseteq \mathbb{R}^n$ be a symmetric isotropic convex body, let $m \geqslant k_0$, and let $U_1, \ldots, U_m$ be independent Haar rotations. Put

$$A_m = \frac{1}{m} \sum_{j=1}^{m} U_j^*(K^\circ).$$

Combining Theorem 1.7 with the estimate of Section 5, we obtain, for every $p \geqslant 2n/m$, outside an event of probability at most $2e^{-cn}$,

$$d_{\mathrm{G}}(A_m, B_2^n) \leqslant C \min\left\{ \max\left\{1, \frac{1}{r(K)} \sqrt{\frac{n}{m}} \right\}, \frac{M(K) + 1/(r(K)\sqrt{m})}{M_{-p}(K)\left(cr(K)M_{-p}(K)\right)^{\frac{n-1}{mp-n+1}} e^{-cn/(mp-n+1)}} \right\}.$$

For the cube this gives an improvement over the inradius-dependent estimate. Put

$$Q_n = [-1/2, 1/2]^n, \qquad L_m = \ln\left(e + \min\left\{ n, \frac{m}{\ln(en)} \right\} \right).$$

Corollary 6.1. *Let $m \geqslant 2$, let $U_1, \ldots, U_m$ be independent Haar-distributed orthogonal transformations and put*

$$A_m = \frac{1}{m} \sum_{j=1}^{m} U_j^*(Q_n^\circ).$$

Then, with probability at least $1 - e^{-cn}$,

$$d_{\mathrm{G}}(A_m, B_2^n) \leqslant C \left(\sqrt{\frac{\ln(en)}{L_m}} + \sqrt{\frac{n}{mL_m}} \right). \tag{6.6}$$

In particular, if $m \geqslant Cn/\ln(en)$, then

$$d_{\mathrm{G}}(A_m, B_2^n) \leqslant C.$$

Proof. This is also the case $k = n$ of Corollary 5.8; we retain the direct argument for the full-dimensional statement. Since $p_{Q_n} = 2\|\cdot\|_\infty$, Proposition 5.7 gives

$$M_{-p}(Q_n) \simeq \sqrt{\frac{\ln\left(e + \frac{n}{1+p}\right)}{n}}, \qquad M(Q_n) \simeq \sqrt{\frac{\ln(en)}{n}}, \qquad b(Q_n) = 2.$$

Choose $p = An\ln(en)/m$, with A a sufficiently large absolute constant. Then $mp \geqslant 2n$ and

$$M_{-p}(Q_n) \geqslant c\sqrt{\frac{L_m}{n}}.$$

Moreover, $(n-1)/(mp-n+1) \leqslant C/\ln(en)$, and hence

$$(cr(Q_n)M_{-p}(Q_n))^{\frac{n-1}{mp-n+1}} \geqslant c, \qquad e^{-cn/(mp-n+1)} \geqslant c.$$

Substitution gives (6.6); the last assertion follows from $L_m \simeq \ln(en)$ in the stated range. $\square$

Since $r(Q_n) = 1/2$, Theorem 1.7 alone gives bounded geometric distance only for $m \geqslant Cn$. Thus Corollary 6.1 improves the number of rotations by the factor $\ln(en)$.

The following deterministic observation gives a lower estimate for every number of rotations.

Proposition 6.2. *Let $K \subseteq \mathbb{R}^n$ be a symmetric convex body containing the origin, let $m \geqslant 1$, let $U_1, \ldots, U_m \in O(n)$ and put*

$$A_m = \frac{1}{m}\sum_{j=1}^m U_j^*(K^\circ).$$

Then

$$d_{\mathrm{G}}(A_m, B_2^n) \geqslant \max\left\{1, \frac{b(K)}{M(K)\sqrt{m}}\right\}. \tag{6.7}$$

In particular, for the cube,

$$d_{\mathrm{G}}(A_m, B_2^n) \geqslant c\max\left\{1, \sqrt{\frac{n}{m\ln(en)}}\right\}. \tag{6.8}$$

Proof. For every j choose $x_j \in U_j^*(K^\circ)$ with $|x_j| = b(K) = R(K^\circ)$. Since K° is symmetric,

$$\frac{1}{m}\sum_{j=1}^m \varepsilon_j x_j \in A_m$$

for every choice of signs $\varepsilon_j \in \{-1, 1\}$. On averaging over the signs,

$$\mathbb{E}_\varepsilon \left|\sum_{j=1}^m \varepsilon_j x_j\right|^2 = \sum_{j=1}^m |x_j|^2 = mb(K)^2,$$

and hence $R(A_m) \geqslant b(K)/\sqrt{m}$. On the other hand, rotation invariance and additivity of the mean width give $w(A_m) = w(K^\circ) = M(K)$, so $r(A_m) \leqslant M(K)$. Since A_m is symmetric, $d_{\mathrm{G}}(A_m, B_2^n) = R(A_m)/r(A_m)$, and (6.7) follows. The cube estimate follows from $b(Q_n) = 2$ and $M(Q_n) \simeq \sqrt{\ln(en)/n}$. $\square$

It follows that bounded geometric distance for the cube requires $m \geqslant cn/\ln(en)$. The matching sufficiency also follows from the classical theorem of Litvak–Milman–Schechtman [35, 36], since $(b(Q_n)/M(Q_n))^2 \simeq n/\ln(en)$. Thus the additional content of Corollary 6.1 is the quantitative estimate (6.6) below this threshold.

We finally show that the dependence on the inradius in Theorem 1.7 is optimal throughout its possible range.

Proposition 6.3. *Let $n \geqslant 2$ and let $1 \leqslant s \leqslant n/2$ be an integer. Put*

$$Z_{n,s} = \sqrt{s+2}\, B_2^s \times \sqrt{n-s+2}\, B_2^{n-s}, \qquad K_{n,s} = |Z_{n,s}|^{-1/n} Z_{n,s}.$$

Then $K_{n,s}$ is symmetric and isotropic, and

$$r(K_{n,s}) \simeq \sqrt{s}, \qquad b(K_{n,s}) \simeq \frac{1}{\sqrt{s}}, \qquad M(K_{n,s}) \simeq \frac{1}{\sqrt{n}}. \tag{6.9}$$

Consequently, for every $m \geqslant 1$ and every $U_1, \dots, U_m \in O(n)$,

$$d_{\mathrm{G}}\left(\frac{1}{m}\sum_{j=1}^m U_j^*(K_{n,s}^\circ), B_2^n\right) \geqslant c \max\left\{1, \sqrt{\frac{n}{sm}}\right\}. \tag{6.10}$$

If $m \geqslant k_0$ and the rotations are independent and Haar-distributed, then, with probability at least $1-e^{-cn}$,

$$d_{\mathrm{G}}\left(\frac{1}{m}\sum_{j=1}^m U_j^*(K_{n,s}^\circ), B_2^n\right) \simeq \max\left\{1, \sqrt{\frac{n}{sm}}\right\}. \tag{6.11}$$

In particular, the threshold $m \simeq n/r(K)^2$ in Theorem 1.7 is optimal, up to absolute constants, throughout the possible range of the inradius.

Proof. The uniform probability measure on $\sqrt{d+2}\, B_2^d$ has covariance matrix I_d. It follows that the uniform probability measure on $Z_{n,s}$ has covariance matrix I_n, and hence its volume-one homothetic image $K_{n,s}$ is isotropic. Moreover,

$$|Z_{n,s}|^{1/n} = \Big[\omega_s (s+2)^{s/2} \omega_{n-s} (n-s+2)^{(n-s)/2}\Big]^{1/n} \simeq 1.$$

Writing $\theta = (\theta', \theta'') \in S^{n-1} \subseteq \mathbb{R}^s \times \mathbb{R}^{n-s}$, we have

$$p_{Z_{n,s}}(\theta) = \max\left\{\frac{|\theta'|}{\sqrt{s+2}}, \frac{|\theta''|}{\sqrt{n-s+2}}\right\}.$$

This proves the first two estimates in (6.9). The same formula gives $p_{Z_{n,s}}(\theta) \geqslant 1/\sqrt{n+4}$, while

$$\int_{S^{n-1}} |\theta'|\, d\sigma(\theta) \leqslant \sqrt{\frac{s}{n}}, \qquad \int_{S^{n-1}} |\theta''|\, d\sigma(\theta) \leqslant \sqrt{\frac{n-s}{n}}.$$

Since the maximum is bounded by the sum, we obtain $M(K_{n,s}) \simeq n^{-1/2}$. Now (6.10) follows from Proposition 6.2, while the reverse estimate in (6.11) follows from Theorem 1.7. □

7 Volumes of random intersections

We finally turn to the proof of Theorem 1.8. The deterministic part is a consequence of polarity and the classical estimates for polytopes with few vertices. The random upper estimate uses the complete radial profile of the cube. We isolate the only estimate that is needed.

Lemma 7.1. *There exist absolute constants $A, C > 0$ such that the following holds. Let $n, m \geqslant 2$, put $\Lambda = \Lambda_{m,n}$ and*

$$a = A\sqrt{\frac{n}{\Lambda}}, \qquad \alpha_n(t) = \sigma\left\{\theta \in S^{n-1} : \|\theta\|_\infty \leqslant \frac{1}{2t}\right\}.$$

Then

$$\alpha_n(a)^{m-1} \leqslant \left(\frac{C}{\sqrt{\Lambda}}\right)^n. \tag{7.1}$$

Proof. Let $G = (g_1, \ldots, g_n)$ be a standard Gaussian vector and write $G = R\Theta$, where $R = |G|$ and Θ is uniform on S^{n-1} and independent of R. Put

$$s = \frac{1}{2a} = \frac{\sqrt{\Lambda}}{2A\sqrt{n}}, \qquad x = 2s\sqrt{n} = \frac{\sqrt{\Lambda}}{A}.$$

On the event $\{R \leqslant 2\sqrt{n}\}$, the inequality $\|\Theta\|_\infty \leqslant s$ implies $\max_i |g_i| \leqslant x$. Thus, if $q = \mathbb{P}\{|g_1| > x\}$, then

$$\alpha_n(a) \leqslant (1-q)^n + \mathbb{P}\{R > 2\sqrt{n}\} \leqslant e^{-nq} + e^{-c_0 n} \tag{7.2}$$

for an absolute constant $c_0 > 0$. The standard Gaussian tail estimate gives

$$q \geqslant \frac{c}{1+x} e^{-x^2/2} \geqslant \frac{c}{\sqrt{\Lambda}} \exp\left(-\frac{\Lambda}{2A^2}\right). \tag{7.3}$$

Choose A sufficiently large and put $\eta = 1/(2A^2)$. Since $\Lambda \leqslant n$, after changing the absolute constants in the bounded range of n we may combine (7.2) and (7.3) to obtain

$$\alpha_n(a) \leqslant e^{-cnq}.$$

For all sufficiently large Λ, the definition of Λ gives $m - 1 \geqslant ce^{\Lambda}$. Hence

$$(m-1)nq \geqslant cn\frac{e^{(1-\eta)\Lambda}}{\sqrt{\Lambda}} \geqslant C_1 n \ln \Lambda,$$

and the last quantity dominates $n \ln \Lambda$ for all sufficiently large Λ. It follows that

$$\alpha_n(a)^{m-1} \leqslant \Lambda^{-n}.$$

For bounded Λ the assertion follows after increasing C. □

Proof of Theorem 1.8. We first prove the deterministic lower estimate. By [20, Proposition 2.3], every unconditional isotropic convex body K satisfies $K \supseteq cB_\infty^n$. Since $Q_n = \frac{1}{2}B_\infty^n$, it is enough to prove (1.10) for $K_1 = \cdots = K_m = Q_n$. Put

$$P_{m,n} = \mathrm{conv}\{\pm U_i e_j : 1 \leqslant i \leqslant m,\ 1 \leqslant j \leqslant n\}.$$

Since $Q_n^\circ = 2B_1^n$, polarity gives

$$\left(\bigcap_{i=1}^m U_i Q_n\right)^\circ = 2P_{m,n}. \tag{7.4}$$

The body $P_{m,n}$ is contained in B_2^n and has at most $2mn$ vertices. The classical few-vertex estimates of

Carl and Pajor [13] and Bárány and Füredi [5] imply

$$\mathrm{vrad}(P_{m,n}) \leqslant C \min\left\{1, \sqrt{\frac{\ln(e+m)}{n}}\right\} \leqslant C\sqrt{\frac{\Lambda_{m,n}}{n}}.$$

By the reverse Santaló inequality and (7.4),

$$\left|\bigcap_{i=1}^m U_i Q_n\right|^{1/n} \geqslant c\frac{\omega_n^{1/n}}{\mathrm{vrad}(P_{m,n})} \geqslant \frac{c}{\sqrt{\Lambda_{m,n}}}.$$

This proves (1.10).

We next prove (1.11). Since $\rho_{Q_n}(\theta) = (2\|\theta\|_\infty)^{-1}$, the exact radial-tail identity (4.1) with $p = n$ gives

$$\mathbb{E}\left|\bigcap_{i=1}^m U_i Q_n\right| = n\omega_n \int_0^\infty t^{n-1}\alpha_n(t)^m \, dt.$$

Let a be as in Lemma 7.1. Since α_n is non-increasing and $|Q_n| = 1$,

$$\begin{aligned}\mathbb{E}\left|\bigcap_{i=1}^m U_i Q_n\right| &\leqslant \omega_n a^n + \alpha_n(a)^{m-1} n\omega_n \int_a^\infty t^{n-1}\alpha_n(t)\, dt \\ &\leqslant \omega_n a^n + \alpha_n(a)^{m-1} \\ &\leqslant \left(\frac{C}{\sqrt{\Lambda_{m,n}}}\right)^n.\end{aligned}$$

This proves (1.11). Finally, Markov's inequality gives

$$\mathbb{P}\left\{\left|\bigcap_{i=1}^m U_i Q_n\right|^{1/n} > \frac{A_0}{\sqrt{\Lambda_{m,n}}}\right\} \leqslant \left(\frac{C}{A_0}\right)^n.$$

Choosing A_0 sufficiently large and using the deterministic lower estimate proves (1.12). □

We finish by recording some more general consequences of the same point of view, without pursuing them further here. For symmetric convex bodies $C_1, \ldots, C_m$ of volume one and every non-empty proper subset $S \subsetneq \{1, \ldots, m\}$, the exact radial-tail identity gives

$$\mathbb{E}\left|\bigcap_{i=1}^m U_i C_i\right| \leqslant \min\left\{1, \omega_n\left(\frac{2}{\min_{i\in S} M(C_i)}\right)^n + \exp\left(-\sum_{i\in S} d(C_i)\right)\right\}. \tag{7.5}$$

Indeed, one splits the radial integral at $2/\min_{i\in S} M(C_i)$, uses the definition of $d(C_i)$ on the tail factors corresponding to $i \in S$, and keeps one remaining radial factor in order to integrate the tail. The Euclidean term in (7.5) is necessary, as is seen from the volume-one Euclidean ball. More generally, for $\overline{B}_p^n = |B_p^n|^{-1/n} B_p^n$ and $p > 2$ one has $\sqrt{n}\, M(\overline{B}_p^n) \simeq \sqrt{\min\{p, \ln(en)\}}$; compare [12, Remark 4.7]. If $\sqrt{n}\, M(C_i) \geqslant B_0$ on a set of r indices, (7.5) gives an exponential estimate involving the sum of the corresponding $d(C_i)$'s, with one index omitted when $r = m$. Since $d(C) \geqslant c\sqrt{n}$ for symmetric isotropic bodies by [12, Section 4], this yields

$$\mathbb{E}\left|\bigcap_{i=1}^m U_i C_i\right| \leqslant C \exp\left[-c \min\left\{n, \min\{r, m-1\}\sqrt{n}\right\}\right]$$

whenever r of the bodies satisfy the above non-Euclideanity condition.

There is also a general deterministic lower estimate which contains all positive spherical moments simultaneously. If $K_1, \dots, K_m$ contain the origin in their interiors, then for every $q > 0$ and every $U_1, \dots, U_m \in O(n)$,

$$\left| \bigcap_{i=1}^{m} U_i K_i \right| \geqslant \omega_n \left(\sum_{i=1}^{m} M_q(K_i)^q \right)^{-n/q}. \tag{7.6}$$

This follows immediately from $p_{\cap_i U_i K_i} = \max_i p_{K_i} \circ U_i^*$, the power-mean inequality and polar integration. For isotropic bodies, the standard spherical concentration inequality applied to the gauge gives $M_q(K) \leqslant M(K) + C\sqrt{q/n}$ for $1 \leqslant q \leqslant n$. Combining this with (2.3) and choosing $q \simeq \ln(em)$ in (7.6) gives

$$\left| \bigcap_{i=1}^{m} U_i K_i \right| \geqslant \left(\frac{c}{\sqrt{\min\{n, \ln(enm)\}}} \right)^n.$$

For a fixed number of isotropic bodies one can do better and obtain $|\cap_i U_i K_i| \geqslant c^{n(m-1)}$. One way to see this is to represent the volume of the intersection as the value at the origin of a convolution-type log-concave density, use Prékopa's theorem [47] and Fradelizi's barycentric estimate [16], and then apply the reverse Brunn–Minkowski estimate in isotropic M-position [26, Theorem 4.5 and Remark 4.6]. We do not develop these extensions here, since Theorem 1.8 gives a complete sharp answer in the unconditional class.

A Calculations for the cube and estimates for $\mathfrak{A}_{n,d}(q)$

We collect here the calculations for the cube, including the optimization in Corollary 5.8, and the comparison of $\mathfrak{A}_{n,d}(d)$ with $\mathfrak{A}_{n,d}(d+1)$.

Lemma A.1. *For $1 \leqslant a \leqslant \sqrt{n}$,*

$$\omega\{\theta \in S^{n-1} : \|\theta\|_1 \leqslant a\} \simeq a\sqrt{\ln\left(e + \frac{n}{a^2}\right)}.$$

Proof. Let $m \simeq a^2$. The set of m-sparse unit vectors is contained in $B_2^n \cap aB_1^n$, while the usual block decomposition gives

$$B_2^n \cap aB_1^n \subseteq 2\,\mathrm{conv} \bigcup_{|I| \leqslant m} B_2^I.$$

The assertion follows from the sparse Gaussian-width estimate

$$\mathbb{E} \max_{|I|=m} |P_I G| \simeq \sqrt{m \ln(e + n/m)};$$

see [50, Exercise 9.27]. $\square$

Proof of Proposition 3.6. Since $h_{Q_n}(\theta) = \frac{1}{2}\|\theta\|_1$, we have

$$T_t(Q_n) = \{\theta \in S^{n-1} : \|\theta\|_1 \leqslant 2t\sqrt{n}\}.$$

Apply Lemma A.1 with $a = 2t\sqrt{n}$. If $t < 1/(2\sqrt{n})$, the assertion follows from $\|\theta\|_1 \geqslant 1$ on the sphere. $\square$

Proof of Proposition 3.8. Since $h_K(\theta) \geqslant a\|\theta\|_1$, for every $t > 0$,

$$T_t(K) \subseteq \left\{\theta \in S^{n-1} : \|\theta\|_1 \leqslant \frac{t\sqrt{n}}{a}\right\}.$$

Put

$$L_d = \ln \frac{en}{d}, \qquad t = c_0 a \sqrt{\frac{d}{nL_d}}, \qquad A = \frac{t\sqrt{n}}{a} = c_0 \sqrt{\frac{d}{L_d}}.$$

If $A < 1$, then $T_t(K)$ is empty. Otherwise, Lemma A.1 gives

$$\omega_K(t) \leqslant CA\sqrt{\ln\left(e + \frac{n}{A^2}\right)} \leqslant Cc_0\sqrt{\frac{d}{L_d}}\sqrt{\ln\left(e + \frac{nL_d}{c_0^2 d}\right)} \leqslant C' c_0 \sqrt{d}.$$

Choose $c_0 > 0$ sufficiently small and apply Proposition 3.5. This gives

$$r(P_F K) \geqslant ca\sqrt{\frac{d}{\ln(en/d)}}$$

with the stated probability. Finally, $K \supseteq aB_\infty^n \supseteq aB_2^n$, so $r(P_F K) \geqslant a$ deterministically. Combining the two estimates proves (3.12). $\square$

Proof of Proposition 3.9. Since $Q_n^\circ = 2B_1^n$,

$$R(Q_n^\circ \cap F) = 2\sup\{|x| : x \in F,\ \|x\|_1 \leqslant 1\}.$$

The right-hand side is bounded below by twice the d-th Gelfand width of B_1^n in ℓ_2^n. By [17, Theorem 1.1],

$$c_d(B_1^n, \ell_2^n) \simeq \min\left\{1, \sqrt{\frac{1 + \ln(n/d)}{d}}\right\},$$

which is equivalent to the right-hand side of (3.13). Since the estimate holds for every F, it gives (3.14) for every $q > 0$. $\square$

We next compare the two quantities $\mathfrak{A}_{n,d}(d)$ and $\mathfrak{A}_{n,d}(d+1)$.

Theorem A.2. *For every* $1 \leqslant d < n$,

$$\mathfrak{A}_{n,d}(d) \leqslant \mathfrak{A}_{n,d}(d+1) \leqslant C\mathfrak{A}_{n,d}(d). \tag{A.1}$$

If $\mathfrak{A}_{n,d} = \mathfrak{A}_{n,d}(d)$, *then*

$$c\sqrt{n}\min\left\{1, \sqrt{\frac{\ln(en/d)}{d}}\right\} \leqslant \mathfrak{A}_{n,d} \leqslant C\min\left\{\sqrt{n}, \exp\left(\frac{C(n-d)}{d+1}\right)\right\}. \tag{A.2}$$

Proof. Monotonicity gives the first inequality in (A.1). Moreover, $r(K) \geqslant cL_K \geqslant c$, so

$$Y_K(F) = \sqrt{n}\, R(K^\circ \cap F) \leqslant C\sqrt{n}$$

for every $F \in G_{n,n-d}$. If a non-negative random variable $Y \leqslant M$ satisfies $\mathbb{P}\{Y > A\tau\} \leqslant \tau^{-d}$, then

$$\|Y\|_{L_{d+1,\infty}} \leqslant A^{d/(d+1)} M^{1/(d+1)}.$$

Applying this with $Y = Y_K$ and using Proposition 3.9, which gives $\mathfrak{A}_{n,d}(d) \geqslant c\sqrt{n/d}$, proves the second inequality in (A.1). The lower bound in (A.2) is Proposition 3.9; the upper bound follows from Theorem 1.3 and the deterministic estimate $Y_K \leqslant C\sqrt{n}$. $\square$

Consequently, $\mathfrak{A}_{n,d} \simeq \sqrt{n}$ when $d \leqslant c\ln(en/d)$, while $\mathfrak{A}_{n,d} \simeq_\delta 1$ when $d \geqslant \delta n$.

Proof of Proposition 5.7. Let $G = (g_1, \ldots, g_n)$ be a standard Gaussian vector and write $G = R\Theta$, $R = |G|$, where Θ is uniform on S^{n-1} and independent of R. Put $Z = \|G\|_\infty$. We first assume $0 < p \leqslant n/2$. Since $Z = R\|\Theta\|_\infty$,

$$M_{-p}(B_\infty^n) = (\mathbb{E}R^{-p})^{1/p}(\mathbb{E}Z^{-p})^{-1/p}. \tag{A.3}$$

Since R^2 has the chi-square distribution with n degrees of freedom,

$$\mathbb{E}R^{-p} = 2^{-p/2}\frac{\Gamma((n-p)/2)}{\Gamma(n/2)}.$$

The standard gamma-ratio estimates give

$$(\mathbb{E}R^{-p})^{1/p} \simeq \frac{1}{\sqrt{n}}, \qquad 0 < p \leqslant n/2. \tag{A.4}$$

We claim that

$$(\mathbb{E}Z^{-p})^{-1/p} \simeq \sqrt{\ln\left(e + \frac{n}{1+p}\right)}, \qquad 0 < p \leqslant n/2. \tag{A.5}$$

Let $H(t) = \mathbb{P}\{|g_1| \leqslant t\}$. For $1 \leqslant p \leqslant n/2$ define t_p by $H(t_p) = e^{-p/n}$. The two-sided Gaussian tail estimates imply

$$t_p \simeq \sqrt{\ln\left(e + \frac{n}{p}\right)}.$$

Since $\mathbb{P}\{Z \leqslant t_p\} = e^{-p}$ then $\mathbb{E}Z^{-p} \geqslant e^{-p}t_p^{-p}$, which gives $(\mathbb{E}Z^{-p})^{-1/p} \leqslant et_p$.

For the reverse inequality, the variables $H(|g_i|)$ are independent and uniform on $[0,1]$. Hence $S = -n\ln H(Z)$ has the standard exponential distribution. If $f_n(s) = H^{-1}(e^{-s/n})$, then

$$\mathbb{E}Z^{-p} = \int_0^\infty f_n(s)^{-p}e^{-s}\,ds. \tag{A.6}$$

Uniform Gaussian estimates give

$$f_n(s) \simeq \sqrt{\ln\frac{en}{s}} \quad (0 < s \leqslant n/2), \qquad f_n(s) \simeq e^{-s/n} \quad (s \geqslant n/2). \tag{A.7}$$

Put $L = \ln(en/p)$ and $s_0 = pe^{L/2} = \sqrt{enp}$. On $0 < s \leqslant p$ we have $f_n(s) \geqslant f_n(p) = t_p$. On $p \leqslant s \leqslant \min\{s_0, n/2\}$, we have $f_n(s) \geqslant c\sqrt{L} \geqslant ct_p$. If $s_0 < n/2$, then the contribution of $[s_0, n/2]$ is at most $C^pe^{-s_0}$, which is bounded by $(C/t_p)^p$ because $s_0 \geqslant cpL$. Finally, by the second estimate in (A.7) and $p \leqslant n/2$,

$$\int_{n/2}^\infty f_n(s)^{-p}e^{-s}\,ds \leqslant C^p\int_{n/2}^\infty e^{ps/n-s}\,ds \leqslant 2C^pe^{-n/4} \leqslant (C/t_p)^p.$$

Together with (A.6), this proves $\mathbb{E}Z^{-p} \leqslant (C/t_p)^p$. Thus (A.5) holds for $1 \leqslant p \leqslant n/2$. If $0 < p < 1$, monotonicity of power means gives $(\mathbb{E}Z^{-1})^{-1} \leqslant (\mathbb{E}Z^{-p})^{-1/p} \leqslant \mathbb{E}Z$, and both endpoints are comparable to $\sqrt{\ln(en)}$.

Combining (A.3), (A.4) and (A.5) proves (5.15) for $p \leqslant n/2$. If $p > n/2$, monotonicity and $\min_{S^{n-1}}\|\theta\|_\infty = 1/\sqrt{n}$ give

$$\frac{1}{\sqrt{n}} \leqslant M_{-p}(B_\infty^n) \leqslant M_{-n/2}(B_\infty^n) \leqslant \frac{C}{\sqrt{n}}.$$

Since $\ln(e + n/(1+p)) \simeq 1$ in this range, the proof is complete. □

Proof of Corollary 5.8. Apply Theorem 5.4 with $K_j = Q_n$, $\lambda_j = 1/m$, $q = mp$ and $u = k$. Since $p_{Q_n} = 2\|\cdot\|_\infty$, Proposition 5.7 gives

$$a_q(\lambda) = M_{-p}(Q_n) \simeq \sqrt{\frac{\ell_p}{n}}, \qquad M(Q_n) \simeq \sqrt{\frac{\ln(en)}{n}}, \qquad b(Q_n) = 2.$$

Moreover, $\overline{M}_\lambda = M(Q_n)$ and $B_\lambda = 2/\sqrt{m}$. Combining the two inclusions in Theorem 5.4 proves (5.16).

For the second assertion choose $p = (Ak\ln(en))/m$, where $A > 0$ is a sufficiently large absolute constant. Then $mp \geqslant 2k$ and $\Delta_p \geqslant ck\ln(en)$. Moreover,

$$1 + \frac{Ak\ln(en)}{m} \simeq \max\left\{1, \frac{k\ln(en)}{m}\right\},$$

and hence $\ell_p \simeq L_{m,k}$. Since

$$\frac{k-1}{\Delta_p}\ln\left(C\sqrt{\frac{n}{\ell_p}}\right) \leqslant C, \qquad \frac{k}{\Delta_p} \leqslant \frac{C}{\ln(en)},$$

the last two factors in (5.16) are bounded by an absolute constant. This proves (5.18). □

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