# SMALL BALL PROBABILITIES FOR SIMPLE RANDOM TENSORS 

XUEHAN HU ${ }^{1}$ GRIGORIS PAOURIS ${ }^{2}$


#### Abstract

We study the small ball probability of an order- $\ell$ simple random tensor $X=X^{(1)} \otimes \cdots \otimes X^{(\ell)}$ where $X^{(i)}, 1 \leq i \leq \ell$ are independent random vectors in $\mathbb{R}^{n}$ that are log-concave or have independent coordinates with bounded densities. We show that the probability that the projection of $X$ onto an $m$-dimensional subspace $F$ falls within an Euclidean ball of length $\varepsilon$ is upper bounded by $\frac{\varepsilon}{(\ell-1)!}\left(C \log \left(\frac{1}{\varepsilon}\right)\right)^{\ell}$ and also this upper bound is sharp when $m$ is small. We also established that a much better estimate holds true for a random subspace.


## 1. Introduction

Tensor decomposition is a crucial problem in learning many latent variable models in data science, such as mixture models (see [4, [11), hidden Markov models (see [4], 19, 30), phylogenetic reconstruction (see [21, [46]) and so on. Many results and algorithms have been developed in the framework where the component vectors of the tensors are noisy, known as smoothed analysis model (see 57). Bhaskara et al 11 established algorithms to decompose fixed-rank random tensors based on Chang's lemma whose efficiency and robustness have been reduced to finding the small ball estimate for simple random tensors. The study of anti-concentration of random tensors is primarily inspired by this problem.

In recent years, there has been a plethora of results on the concentration of random tensors. Several papers discussed the concentration inequalities for the projection of simple random tensor onto any fixed direction. Latala 38 gives an estimate on the case where the component vectors of the simple tensor are independent standard Guassian random vectors, of which Lehec [40] provides another proof based on Talagrand's majorizing theorem. Later there are papers that considered more general distributions for the component vectors: Adamczak and Latala [2] considered log-concave distribution; Adamczak and Wolff [3] considered distribution that satisfies a Sobolev inequality; Götze, Sambale, and Sinuli 28] considered $\alpha$-subexponential distribution; Vershynin [60] considered subgaussian distribution with optimal dependence on the degree of the tensors, of which Bamberger, Krahmer, Ward [9] gave improved dependence on the dimension of the component vectors; and Adamczak, Latala, and Melle [1] extended the result to values in some Banach spaces. There are more applications of concentration inequalities for simple tensors: Jin, Kolda, Ward 31] introduced a faster Johnson-Lindenstrauss projection

[^0]for embedding vectors with Kronecker product structure; Bamberger, Krahmer, Ward [10] gave optimal Johnson-Lindenstrauss property for the aforementioned embeddings.

On the other hand, the anti-concentration is also an important part of contemporary probability and has strong connections with the metric-algebraic structure of the underlying space (see for example the survey 42 for processes or 58 50 for the Littwood-Offord problems). While anti-concentration of random variables and random vectors (see for example [52] and [56]) have been extensively studied, that of random tensors remains unclear in many aspects. Small ball probabilities for simple random tensors have been established by Bhaskara et al [11], Anari et al [6], Glazer and Miklincer [26]. Our main result builds on this direction by providing a sharp, small ball probability for random tensors under minimal assumptions on the randomness.

In this paper, we constructed anti-concentration results for simple random tensors under minimal assumptions. It is well-known that for continuous distributions, being anti-concentrated is equivalent to having bounded densities. We provided sharp small ball estimates for the projection of the tensor product of independent log-concave distributions onto a subspace of fixed dimension. And we used symmetrization techniques to extend the results to distributions with independent coordinates of bounded densities, which is the minimal assumption we may have. We also established that a much better estimate holds true for a random projected subspace, i.e. the random subspace in the Grassmanian with respect to the Haar measure on the real orthogonal group.

We refer to 44, [37] for detailed definition of tensors. A tensor $X \in \mathbb{R}^{n_{1} \otimes \cdots \otimes n_{\ell}}$ can be represented as a multidimensional array, that is

$$
X=\left(X_{i_{1} \cdots i_{\ell}}\right)_{i_{1} \cdots i_{\ell}}
$$

In particular, $X$ is a simple tensor if there exist vectors $X^{(j)}=\left(X_{1}^{(j)}, \cdots, X_{n}^{(j)}\right) \in$ $\mathbb{R}^{n_{j}}, 1 \leq j \leq \ell$, such that

$$
X=X^{(1)} \otimes \cdots \otimes X^{(\ell)}=\otimes_{j=1}^{\ell} X^{(j)}=\left(X_{i_{1}}^{(1)} \cdots X_{i_{\ell}}^{(\ell)}\right)_{i_{1} \cdots i_{\ell}}
$$

$X \in \mathbb{R}^{n_{1} \otimes \cdots \otimes n_{\ell}}$ is also a multilinear map $X: \mathbb{R}^{n_{1} \times \cdots \times n_{\ell}} \rightarrow \mathbb{R}$, such that for $Y \in \mathbb{R}^{n_{1} \otimes \cdots \otimes n_{\ell}}$,

$$
\langle X, Y\rangle:=\sum_{i_{1} \cdots i_{\ell}} X_{i_{1} \cdots i_{\ell}} Y_{i_{1} \cdots i_{\ell}}
$$

which is usually refered to as the Frobenius inner product of tensors. And

$$
\|X\|_{2}:=\sqrt{\langle X, X\rangle}
$$

is the Frobenius norm of a tensor, which aligns with the Euclidean norm when we view the tensor in $\mathbb{R}^{n_{1} \otimes \cdots \otimes n_{\ell}}$ as a flattened vector in $\mathbb{R}^{n_{1} \times \cdots \times n_{\ell}}$. In particular, if

$$
X=\otimes_{j=1}^{\ell} X^{(j)}, Y=\otimes_{j=1}^{\ell} Y^{(j)}
$$

then

$$
\langle X, Y\rangle:=\prod_{j=1}^{\ell}\left\langle X^{(j)}, Y^{(j)}\right\rangle
$$

and

$$
\|X\|_{2}=\prod_{j=1}^{\ell}\left\|X^{(j)}\right\|_{2}
$$

Recall that a vector has a unique orthogonal decomposition with respect to a linear subspace. For any tensor $X \in \mathbb{R}^{n_{1} \otimes \cdots \otimes n_{\ell}}$ and subspace $F \subset \mathbb{R}^{n_{1} \otimes \cdots \otimes n_{\ell}}$, there exists a unique tensor $\Pi_{F} X$ in $F$ such that $\left\|X-\Pi_{F} X\right\|$ is minimized. $\Pi_{F} X$ is called the orthogonal projection of $X$ onto $F$.

In the paper, we use the letters $C, C^{\prime}, C_{1}, C_{2}, C_{3}, \ldots$ to represent universal constants that may be different from line to line. We haven't tried to optimize these constants. One can compute all the constants explicitly, which is not the main purpose of this paper. Our main result reads as follows:

Theorem 1.1. Let $X^{(j)} \in \mathbb{R}^{n_{j}}, 1 \leq j \leq \ell$ be independent random vectors with independent coordinates whose densities have uniform norms bounded by some constant $M>0$. Suppose $F$ is a subspace in $\mathbb{R}^{n_{1} \otimes \cdots \otimes n_{\ell}}$ with dimension $m$ and suppose $z_{j} \in \mathbb{R}^{n_{j}}, 1 \leq j \leq \ell$ are arbitrary vectors, then for $0<\varepsilon<e^{-c \ell}$,
$\mathbb{P}\left(\left\|\Pi_{F} \otimes_{j=1}^{\ell}\left(X^{(j)}-z_{j}\right)\right\|_{2} \leq \frac{1}{(C M)^{\ell}} \varepsilon \sqrt{m}\right) \leq \min \left\{m, C^{\prime \ell} \log \frac{1}{\varepsilon}\right\} \frac{\varepsilon}{(\ell-1)!}\left(C^{\prime \prime} \log \frac{1}{\varepsilon}\right)^{\ell-1}$.
The above estimate cannot be improved (see Appendix A for the details). In fact, let $X^{(1)}, \cdots, X^{(\ell)}$ be independent uniform distributions on $[-\sqrt{3}, \sqrt{3}]^{n}$ such that $\mathbb{E}\left[X_{i}^{(j)}\right]=0$ and $\operatorname{Var}\left(X_{i}^{(j)}\right)=1$ for $1 \leq i \leq n, 1 \leq j \leq \ell$. Then for any $1 \leq m \leq n$, there exists a subspace $F$ of dimension $m$ such that

$$
\mathbb{P}\left(\left\|\Pi_{F} X^{(1)} \otimes \cdots \otimes X^{(\ell)}\right\|_{2} \leq \varepsilon \sqrt{m}\right) \geq \frac{C \varepsilon}{(\ell-2)!}\left(\log \frac{1}{\varepsilon}\right)^{\ell-2}
$$

It is easy to check (see Section 1 in [56]) that for continuous distributions, to be anti-concentrated is equivalent to have bounded densities. In this sense, for continuous distributions, the densities to be bounded are practically the minimal assumptions that we can pose to the problem.

We observe that the orthogonal projection of $\otimes_{j=1}^{\ell} X^{(j)}$ onto subspaces with the same dimension can have quite different small ball behaviors. When the subspace is generic, then with high probability we are able to obtain much better estimates. Consider the orthogonal group $\mathbb{O}\left(n^{\ell}\right)$ equipped with the unique Haar probability measure invariant under the action of the group (See Chapter 1 in [45]). In the rest of the paper we denote by $G_{n, m}$ the Grassmannian manifold of $m$-dimensional subspaces in $\mathbb{R}^{n}$ equipped with the Haar probability measure invariant under the action of orthogonal group. We have the following theorem.
Theorem 1.2. There exists a subset $\mathcal{S}_{\mathcal{F}}$ in $G_{n^{\ell}, m}$ with Haar measure at least $1-e^{-c \max \{m, n\}}$. Let $X^{(1)}, \cdots, X^{(\ell)} \in \mathbb{R}^{n}$ be independent random vectors with independent coordinates whose densities have uniform norms bounded by some constant $M>0$. Let $z_{1}, \cdots, z_{\ell} \in \mathbb{R}^{n}$ be arbitrary vectors and let $m \leq n^{\ell}$. Then for every subspace $F \in \mathcal{S}_{\mathcal{F}}$ with dimension $m$, then for $0<\varepsilon<1$,

$$
\mathbb{P}\left(\left\|\Pi_{F} \otimes_{j=1}^{\ell}\left(X^{(j)}-z_{j}\right)\right\|_{2} \leq \frac{1}{(C M)^{\ell}} \varepsilon \sqrt{m}\right) \leq\left(C^{\prime} \varepsilon\right)^{C^{\prime \prime} \min \{m, n\}}+e^{-C^{\prime \prime \prime} n}
$$

The proof of Theorem 1.1 relies on a stochastic dominance argument. We show that the small ball behavior of tensor product of independent random vectors
with independent coordinates and bounded densities is dominated by that of tensor product of uniform random vectors on the cube. In fact we have the following more general theorem. This line of research builds on a series of papers on empirical isoperimetric inequalities and applications to small ball estimates. See Paouris, Pivovarov [54, [53] and Dann, Paouris, Pivovarov [23].

Theorem 1.3. Let $X^{(j)} \in \mathbb{R}^{n_{j}}, 1 \leq j \leq \ell$ be independent random vectors with independent coordinates whose densities have uniform norms bounded by 1, then for any symmetric convex body $K \subset \mathbb{R}^{n_{1} \otimes \cdots \otimes n_{\ell}}$, we have

$$
\mathbb{P}\left(\otimes_{j=1}^{\ell} X^{(j)} \in K\right) \leq \mathbb{P}\left(\otimes_{j=1}^{\ell} \mathbf{1}_{\left[-\frac{1}{2}, \frac{1}{2}\right]^{n_{j}}} \in K\right)
$$

Here by an abuse of notation, we let $\mathbf{1}_{\left[-\frac{1}{2}, \frac{1}{2}\right]^{n_{j}}}$ denotes the uniform distribution on $\left[-\frac{1}{2}, \frac{1}{2}\right]^{n_{j}}$.

With Theorem 1.2 it suffices to study small ball probabilities for tensor product of random vectors with independent coordinates uniformly distributed on a centered interval. Instead of working with these particular distributions, we will work with a general log-concave isotropic distribution. (See Section 2.2 for precise definition).
Theorem 1.4. Let $X^{(j)} \in \mathbb{R}^{n_{j}}, 1 \leq j \leq \ell$ be independent log-concave isotropic random vectors. Suppose $F$ is a subspace in $\mathbb{R}^{n_{1} \otimes \cdots \otimes n_{\ell}}$ with dimension $m$ and suppose $f^{(1)}, \cdots, f^{(m)}$ is an orthonormal basis for $F$. Then for $0<\varepsilon<e^{-c \ell}$,

$$
\mathbb{P}\left(\left|\left\langle X^{(1)} \otimes \cdots \otimes X^{(\ell)}, f^{(k)}\right\rangle\right| \leq \varepsilon\right) \leq \frac{\varepsilon}{(\ell-1)!}\left(C \log \frac{1}{\varepsilon}\right)^{\ell-1}
$$

and
$\mathbb{P}\left(\left\|\Pi_{F} X^{(1)} \otimes \cdots \otimes X^{(\ell)}\right\|_{2} \leq \varepsilon \sqrt{m}\right) \leq \min \left\{m, C^{\prime \ell} \log \frac{1}{\varepsilon}\right\} \frac{\varepsilon}{(\ell-1)!}\left(C \log \frac{1}{\varepsilon}\right)^{\ell-1}$.
Note that in Theorem 1.4, in contrast to Theorem 1.1. we do not require the coordinates of each component vector be independent.

Theorem 1.5. Let $X^{(1)}, \cdots, X^{(\ell)} \in \mathbb{R}^{n}$ be independent isotropic log-concave random vectors. Let $z_{1}, \cdots, z_{\ell} \in \mathbb{R}^{n}$ be arbitrary vectors and let $m \leq n^{\ell}$. Then there exists a subset $\mathcal{S}_{\mathcal{F}}$ in $G_{n^{\ell}, m}$ with Haar measure at least $1-e^{-c \max \{m, n\}}$. For every subspace $F \in \mathcal{S}_{\mathcal{F}}$ with dimension $m$, then for $0<\varepsilon<1$,

$$
\mathbb{P}\left(\left\|\Pi_{F} \otimes_{j=1}^{\ell}\left(X^{(j)}-z_{j}\right)\right\|_{2} \leq \varepsilon \sqrt{m}\right) \leq\left(C \varepsilon \mathcal{L}_{C^{\prime} \min \{m, n\}}\right)^{C^{\prime} \min \{m, n\}}+e^{-\frac{C^{\prime \prime} \sqrt{n}}{C_{P}}}
$$

where the isotropic constant

$$
\mathcal{L}_{r}:=\sup _{F} \mathcal{L}_{\Pi_{F}}(\mu)=\sup _{F}\left\|f_{\Pi_{F} X}(x)\right\|_{\infty}^{1 / r},
$$

where $F$ is an $r$-dimensional subspace of $\mathbb{R}^{n}$ and $f_{\Pi_{F} X}(x)$ denotes the density of the marginal distribution $\Pi_{F} X$. And $C_{P}$ denotes the maximum over the Poincaré constants of $X^{(1)}, \cdots, X^{(\ell)}$.

Acknowledgments: The second named author is grateful to J.M. Landsberg and Liza Rebrova for several interesting discussions and references. The second named author is grateful to the Mathematics Department of Princeton for its hospitality, where part of this work was carried out. And we are grateful to Dan Mikluncer and Alperen A. Ergür for many helpful comments.

## 2. Background and Related Results

2.1. Motivation. The study of small ball probabilities for random tensors is primarily inspired by the tensor decomposition problem. Tensor decomposition is an important question in many latent variable models, such as multi-view model, Gaussian mixture model ([11]), Hidden Markov model ([20], [47, [5], [30]) and assembly of neurons ([6]). The tensor rank of $X$ is the smallest number $r$ such that

$$
X=\sum_{i=1}^{r} X_{i}^{(1)} \otimes \cdots \otimes X_{i}^{(\ell)}
$$

is a sum of $r$ simple tensors. For a fixed rank- $r$ tensor $X$, retrieving all the component vectors $X_{i}^{(j)}$,s is an NP-hard problem in the worst case scenario. One method is to apply the smoothed analysis model introduced by Spielman and Teng (see [57]). That is to say we introduce random noises to $X_{k}^{(j)}$ 's. The following is the smoothed analysis model used by Bhaskara et al in [11. Consider random vectors

$$
\tilde{X}_{i}^{(j)}=X_{i}^{(j)}+G_{i}^{(j)} \in \mathbb{R}^{n}, 1 \leq i \leq r, 1 \leq j \leq l
$$

where

$$
\left\|X_{i}^{(j)}\right\|_{2} \leq C, \quad G_{i}^{(j)} \sim N\left(0, \frac{\rho^{2}}{n} \mathbb{I}_{n}\right)
$$

Here $\widetilde{X}_{i}^{(j)}$ is the noisy version of $X_{i}^{(j)}$, and we have normalized it such that in expectation its length has not been increased. Define an $n \times r$ matrix $\widetilde{A}_{j}=$ $\left[\widetilde{X}_{1}^{(j)}, \cdots, \widetilde{X}_{r}^{(j)}\right]$ for $1 \leq j \leq \ell$, then the Khatri-Rao product of $\widetilde{A}_{j}$ 's is defined to be an $n^{\ell} \times r$ matrix

$$
\widetilde{A}=\widetilde{A}_{1} \odot \widetilde{A}_{2} \odot \cdots \odot \widetilde{A}_{\ell}
$$

where the $i$-th column of $\widetilde{A}$ is the $n^{\ell}$-dimensional flattened vector $\otimes_{j=1}^{\ell} \widetilde{X}_{k}^{(i)}$. Then we can rewrite the rank- $r$ tensor $X$ using Katri-Rao product as

$$
X=\sum_{i=1}^{r}\left(\widetilde{A}_{1} \odot \cdots \odot \widetilde{A}_{\left\lfloor\frac{\ell-1}{2}\right\rfloor}\right) \otimes\left(\widetilde{A}_{\left\lfloor\frac{\ell-1}{2}\right\rfloor+1} \odot \cdots \odot \widetilde{A}_{\ell-1}\right) \otimes \widetilde{A}_{\ell}
$$

If the columns of the matrices $\left(\widetilde{A}_{1} \odot \cdots \odot \widetilde{A}_{\left\lfloor\frac{\ell-1}{2}\right\rfloor}\right)$ and $\left(\widetilde{A}_{\left\lfloor\frac{\ell-1}{2}\right\rfloor+1} \odot \cdots \odot \widetilde{A}_{\ell-1}\right)$ are robustly linearly independent respectively, then Bhaskara et al show that there is an efficient algorithm to retrieve all the $\widetilde{X}_{i}^{(j)}$ s. The algorithm is known as simultaneous diagonalization ([4]) or Chang's lemma ([22]).

We want to show that columns of $\widetilde{A}$ are robustly linearly independent, or $\widetilde{A}$ is well "invertible". This is equivalent to bounding the smallest singular value of the Khatri-Rao product. And by Lemma 7.1 the smallest singular value is closely related to the orthogonal projections of a column vector onto the orthogonal complement of the rest of the column vectors. The investigation of the papers Bhaskara et al [11] and Anari et al [6] reduces the quantification of the algorithm (running time, probability of failure, and size of the noise compared to the size of data) to the problem of small ball probabilities for orthogonal projection of simple tensors.
2.2. Preliminaries. To study the small ball probabilities for the orthogonal projection of simple random tensors, we first need to decide what randomness we care about. In [11, the authors considered Gaussian perturbations. We want to extend the independent Gaussian random vectors to independent random vectors with independent coordinates whose densities are universally bounded. Another setting we care about is when the random vectors are isotropic and log-concave.

A random vector in $\mathbb{R}^{n}$ is log-concave if its density $f$ is log-concave, i.e. for $x, y$ in the support of $f$ and $\theta \in(0,1)$, we have

$$
f(\theta x+(1-\theta) y) \geq f(x)^{\theta} f(y)^{1-\theta}
$$

Prékopa-Leindler inequality (see for example [7]) implies sum of log-concave random vectors is log-concave. And affine linear map of log-concave random vector is also log-concave (see Lemma 2.1 in [24]). Examples of log-concave vectors are the Gaussians, exponentials and uniform measures on convex bodies. A random vector in $X \in \mathbb{R}^{n}$ is isotropic if

$$
\mathbb{E}\left[X X^{T}\right]=I d
$$

Any random vector with second moments has a linear image that is isotropic. Log-concave measure has been extensively investigated in the last decades. Concentration questions for these measures were (and still are) some of the most major open questions in asymptotic geometric analysis (see [7], [8, [15]). Two of the most important open questions are Bourgain's hyperplane conjecture (see [13, [12]) and Kannan Lovasz Simonovitch's Question (see [32). We refer to Klartag and Milman [35] and Klartag [34] for the history of the problem and the latest developments.

If $X$ is an isotropic log-concave random vector in $\mathbb{R}^{n}$ then the isotropic constant is

$$
\mathcal{L}_{X}:=\left\|f_{\mu}\right\|_{\infty}^{\frac{1}{n}}
$$

The KLS constant (or Poincaré constant) of $X$ (or of $\mu$ that $X$ is distributed) is the smallest $C_{P}$ such that for every $f$ smooth function on $\mathbb{R}^{n}$

$$
\operatorname{Var}(f(X)) \leq C_{P} \mathbb{E}\|\nabla f(X)\|_{2}^{2}
$$

Bourgain's slicing problem asks (equivalently) if $\mathcal{L}_{X}$ is uniformly bounded by a constant for every $X$ isotropic log-concave (independent of the dimension). KLS question can be expressed (equivalently) as if $C_{P}$ can be uniformly bounded by a constant for every $X$ isotropic log-concave. The best up-to-date bounds are

$$
\max \left\{\mathcal{L}_{X}, \sqrt{C_{P}(X)}\right\} \leq C \sqrt{\log n}
$$

due to Klartag [34. It is well known that Poincaré inequality implies exponential concentration (see [39]), i.e. for every $f: \mathbb{R}^{n} \rightarrow \mathbb{R} 1$-Lipschitz, then for $0<\varepsilon<1$,

$$
\mathbb{P}(|f(X)-\mathbb{E}[f(X)]| \geq t) \leq 2 e^{-\frac{C_{t}}{C_{P}}}
$$

Estimates for large deviations and small ball probabilities for the Euclidean norm are also known as follows (see 51, [52]).
Theorem 2.1. Let $X \in \mathbb{R}^{n}$ be isotropic log-concave random vector and let $q \geq 1$, then

$$
\left(\mathbb{E}\|X\|_{2}^{q}\right)^{\frac{1}{q}} \leq C(\sqrt{n}+q)
$$

Theorem 2.2. Let $X \in \mathbb{R}^{n}$ be isotropic log-concave random vector, then for $0<$ $\varepsilon<1$,

$$
\mathbb{P}\left(\|X\|_{2} \leq \varepsilon \sqrt{n}\right) \leq \varepsilon^{c \sqrt{n}}
$$

Here we are mostly interested in small ball probabilities. Small ball probabilities for log-concave measures have been investigated by Carbery and Wright [18], Guedon [29, Fradelizi [25], Nazarov, Sodin and Volberg 48] 49] and Glazer and Mikulincer [26.
Theorem 2.3. (Carbery-Wright, see [18]) Let $p: \mathbb{R}^{n} \longrightarrow X$ be a polynomial of degree at most $d$, let $\mu$ be an isotropic log-concave measure on $\mathbb{R}^{n}$ and let $0 \leq q \leq \infty$. Define the functional $p^{\#}(x)=\|p(x)\|^{\frac{1}{d}}$. Then there exists an absolute constant $C$ independent of $p, d, \mu, n, q$ and $X$ so that for any $\alpha>0$,
(a) if $n \leq q$ then

$$
\left\|p^{\#}\right\|_{q} \alpha^{-1} \mu\left(x \in K: p^{\#}(x) \leq \alpha\right) \leq C n
$$

(b) if $q \leq n$ then

$$
\left\|p^{\#}\right\|_{q} \alpha^{-1} \mu\left(x \in K: p^{\#}(x) \leq \alpha\right) \leq C \max (q, 1)
$$

Theorem 2.4. (Guédon, see [29]) Let $A$ be a symmetric convex body, $\mu$ a logconcave probability measure over $\mathbb{R}^{n}$, then for all $0<\varepsilon \leq 1$

$$
\frac{\mu(\varepsilon A)}{\varepsilon} \leq 2 \log \left(\frac{1}{1-\mu(A)}\right)
$$

Finally we state the concentration inequalities for random orthogonal matrices (see [45]). The orthogonal group $\mathbb{O}(n)$ is the group of orthogonal $n \times n$ matrices. Then there exists a unique translation invariant probability measure on $\mathbb{O}(n)$, called Haar measure.

Theorem 2.5. (See Theorem 5.5 in [45]) Suppose $X \in \mathbb{O}(n)$ is a random orthogonal matrix distributed according to Haar measure. If $\mathcal{A}:\left(\operatorname{Mat}_{n \times n}(\mathbb{R}),\|\cdot\|_{H S}\right) \rightarrow$ $(\mathbb{R},|\cdot|)$ is 1 -Lipschitz with $\mathbb{E}[\mathcal{A}(X)]<\infty$, then for every $t>0$,

$$
\mathbb{P}(|\mathcal{A}(X)-\mathbb{E}[\mathcal{A}(X)]| \geq t) \leq 2 e^{-C n t^{2}}
$$

The following is a direct corollary of Theorem 2.5.
Corollary 2.6. Suppose $X \in \mathbb{O}(n)$ is a random orthogonal matrix distributed according to Haar measure. If $\mathcal{A}:\left(\operatorname{Mat}_{n \times n}(\mathbb{R}),\|\cdot\|_{H S}\right) \rightarrow(\mathbb{R},|\cdot|)$ is 1-Lipschitz with $\mathbb{E}[\mathcal{A}(X)]<\infty$, then for every $q \geq 1$,

$$
\mathbb{E}|\mathcal{A}(X)-\mathbb{E}[\mathcal{A}(X)]|^{q} \leq\left(\frac{C q}{n}\right)^{\frac{q}{2}}
$$

and

$$
\mathbb{E}\left[\mathcal{A}(X)^{2}\right] \leq(\mathbb{E}[\mathcal{A}(X)])^{2}+\frac{C}{n}
$$

where $C>1$ is universal constant.
2.3. Stochastic Dominance. We will use stochastic dominance to prove the anticoncentration of simple random tensors when the component vectors are independent with independent coordinates whose densities are universally bounded. We introduce the following notions and definitions from [43], [17, and [54].

Definition 2.7. Let $A$ be a Borel subset of $\mathbb{R}^{n}$ with finite Lebsegue measure. Then the symmetric rearrangement $A^{*}$ of $A$ is the open ball centered at the origin, whose volume is equal to the measure of $A$. Let $f: \mathbb{R}^{n} \rightarrow \mathbb{R}_{+}$be a Borel measurable
function such that $\left\{x \in \mathbb{R}^{n}: f(x)>t\right\}$ has finite measure for every $t>0$. Then the symmetric decreasing rearrangement $f^{*}$ of $f$ is defined by

$$
f^{*}(x)=\int_{0}^{\infty} \mathbf{1}_{\{f>t\}^{*}}(x) d t
$$

Definition 2.8. Let $X_{1}, X_{2}$ be random vectors in $\mathbb{R}^{n}$. Then we say that $X_{1}$ is less peaked than $X_{2}$, or $X_{1}$ is stochasically dominated by $X_{2}$, denoted by

$$
X_{1} \prec X_{2}
$$

if

$$
\mathbb{P}\left(X_{1} \in K\right) \leq \mathbb{P}\left(X_{2} \in K\right)
$$

for any symmetric convex body $K \subset \mathbb{R}^{n}$.
Lemma 2.9. Suppose $X$ is a random variable and $f: \mathbb{R} \longrightarrow \mathbb{R}_{+}$is its density, such that

$$
\|f\|_{1}=1,\|f\|_{\infty} \leq 1
$$

and $f$ is even, then

$$
X^{*} \prec \mathbf{1}_{\left[-\frac{1}{2}, \frac{1}{2}\right]} .
$$

Here by an abuse of notation, we let $1_{\left[-\frac{1}{2}, \frac{1}{2}\right]}$ denotes the uniform distribution on $\left[-\frac{1}{2}, \frac{1}{2}\right]$.

Remark 2.10. Here the evenness of $f$ is not necessary. In fact, suppose $f$ satisfies that

$$
\|f\|_{1}=1,\|f\|_{\infty} \leq 1
$$

and note that

$$
\|f\|_{p}=\left\|f^{*}\right\|_{p}, \quad 1 \leq p \leq \infty
$$

and $f^{*}$ is even, then

$$
f^{* *}=f^{*} \prec \mathbf{1}_{\mathbb{Q}} .
$$

Theorem 2.11. (Rogers-Brascamp-Lieb-Luttinger inequality, see 55 [14) Let $\mu$ be a quasi-concave measure in $\mathbb{R}^{n}$ supported in a symmetric convex set $K$. Let $u_{1}, \ldots, u_{m}$ be non-zero vectors in $\mathbb{R}^{n}$. Let $f_{1}, \ldots, f_{m}$ be measurable non-negative integrable functions on $\mathbb{R}$. Then

$$
\int_{\mathcal{R}^{n}} \prod_{i=1}^{m} f_{i}\left(\left\langle x, u_{i}\right\rangle\right) d \mu(x) \leq \int_{\mathcal{R}^{n}} \prod_{i=1}^{m} f_{i}^{*}\left(\left\langle x, u_{i}\right\rangle\right) d \mu(x)
$$

A function $f$ is unimodal if f it is the increasing limit of a sequence of functions of the form,

$$
\sum_{i=1}^{m} t_{i} \mathbf{1}_{K_{i}}
$$

where $t_{i}>0$ and $K_{i}$ are symmetric convex bodies in $\mathbb{R}^{n}$. For every integrable function $f: \mathbb{R}^{n} \rightarrow \mathbb{R}$, its symmetric decreasing rearrangement $f^{*}$ is unimodal.

Theorem 2.12. (Kanter, see [33]) Let $f_{1}, g_{1}$ be functions on $\mathbb{R}^{n_{1}}$ such that $f_{1} \prec g_{1}$ and $f$ a unimodal function on $\mathbb{R}^{n_{2}}$, then

$$
f f_{1} \prec f g_{1}
$$

In particular. if $f_{i}, g_{i}$ are unimodal functions on $\mathbb{R}^{n_{i}}, i=1, \cdots, M$ and $f_{i} \prec g_{i}$ for all $i$, then

$$
\prod_{i=1}^{M} f_{i} \prec \prod_{i=1}^{M} g_{i} .
$$

2.4. Related Results. Since there is no good way to characterize the span of $m$ simple random tensors, most known results consider a general $m$-dimensional space (see [11, 6]). Let $F$ be an $m$-dimensional space in $\mathbb{R}^{n^{\ell}}$. To study the small ball behavior of $\Pi_{F} \otimes_{j=1}^{\ell} X^{(j)}$, we choose $f^{(1)}, \cdots, f^{(m)}$ to be an orthonormal basis for $F$. Then

$$
\begin{aligned}
\left\|\Pi_{F} X^{(1)} \otimes \cdots \otimes X^{(\ell)}\right\|_{2}^{2} & =\sum_{k=1}^{m}\left|\left\langle X^{(1)} \otimes \cdots \otimes X^{(\ell)}, f^{(k)}\right\rangle_{F}\right|^{2} \\
& =\sum_{k=1}^{m}\left|\sum_{i_{1}, \ldots, i_{\ell}} X_{i_{1}}^{(1)} \ldots X_{i_{\ell}}^{(\ell)} f_{i_{1} \ldots i_{\ell}}^{k}\right|^{2}
\end{aligned}
$$

Note that $p=\left\|\Pi_{F} X^{(1)} \otimes \ldots X^{(\ell)}\right\|_{2}^{2}$ is a polynomial of degree $2 \ell$. By CarberyWright inequality, we have

$$
\mathbb{P}\left(\left\|\Pi_{F} X^{(1)} \otimes \cdots \otimes X^{(\ell)}\right\|_{2}<\varepsilon \sqrt{m}\right) \leq C \ell \varepsilon^{1 / l}
$$

The following result is from Anari et al. It applies to a very broad setting of random vectors including discrete ones. The result reads as follows (see Lemma 6 in (6]).

Theorem 2.13. A random vector $X \in \mathbb{R}^{n}, 1 \leq i \leq \ell$ is drawn from $(\varepsilon, p)$ nondeterministic distribution if for every $j \in[n]$ and any interval of the form $(t-\varepsilon, t+\varepsilon)$, we have

$$
\mathbb{P}\left(X_{j} \in(t-\delta, t-\delta) \mid X_{-j}\right) \leq p
$$

where $X_{j}$ represents the projection of $X$ onto the coordinates $[n]-j$. Assume the $n$ dimensional vectors $X^{(1)}, \ldots, X^{(\ell)}$ are drawn according to an $(\varepsilon, p)$-nondeterministic distribution. Further assume that $V \subset \mathbb{R}^{n^{\otimes l}}$ is a subspace of dimension at most $(c n)^{\ell}$. Then

$$
\mathbb{P}\left(\operatorname{dist}\left(X^{(1)} \otimes \cdots \otimes X^{(\ell)}, V\right)<\left(\frac{\varepsilon}{\sqrt{n}}\right)^{\ell}\right) \leq 2 n^{\ell-1} p^{(1-c) n}
$$

Note that if a random vector has independent coordinates with bounded densities (by 1) then it is $(\varepsilon, c \varepsilon)$-nondeterministic for any $\varepsilon$ in $(0,1)$. Since $\left\|\Pi_{F} X^{(1)} \otimes \cdots \otimes X^{(\ell)}\right\|_{2}=$ $\operatorname{dist}\left(X^{(1)} \otimes \cdots \otimes X^{(\ell)}, F^{\perp}\right)$. Then their result under our assumptions implies the following estimate

$$
\mathbb{P}\left(\left\|\Pi_{F}\left(X^{(1)} \otimes \cdots \otimes X^{(\ell)}\right)\right\|_{2}<\varepsilon\right) \leq 2 n^{\ell-1}(c \sqrt{n})^{n-\left(n^{\ell}-m\right)^{1 / \ell}} \varepsilon^{\frac{n-\left(n^{\ell}-m\right)^{1 / \ell}}{\ell}}
$$

If $n<\ell$, or if $n \geq \ell$ and $m<n^{\ell}-(n-\ell)^{\ell}$, then $\frac{n-\left(n^{\ell}-m\right)^{1 / \ell}}{\ell}<1$.
Small ball estimates can also come from concentration inequalities of simple random tensors. The following results are from Vershynin 60] and Bamberger, Krahmer, Ward 9].

Theorem 2.14. (See Corollary 6.1 in [60]) Let $X^{(1)}, \cdots, X^{(\ell)} \in \mathbb{R}^{n}$ be independent random vectors whose coordinates are independent, mean zero, unit variance with subgaussian norm bounded by $L \geq 1$. Suppose $F$ is a subspace in $\mathbb{R}^{n^{\ell}}$ of dimension $m$. Then for $0 \leq \varepsilon \leq 1$,

$$
\mathbb{P}\left(\left\|\Pi_{F} \otimes_{j=1}^{\ell} X^{(j)}\right\|_{2} \leq \varepsilon \sqrt{m}\right) \leq 2 \exp \left(-\frac{C(1-\varepsilon)^{2} m}{\ell n^{\ell-1}}\right)
$$

Theorem 2.15. (See Theorem 2.1 in [9]) Let $X^{(1)}, \cdots, X^{(\ell)} \in \mathbb{R}^{n}$ be independent random vectors whose coordinates are independent, mean zero, unit variance, subgaussian random variables. Suppose $F$ is a subspace in $\mathbb{R}^{n^{\ell}}$ of dimension $m$. Then for a constant $C_{\ell}$ that only depends on $\ell$ and for $0 \leq \varepsilon \leq 1$,

$$
\mathbb{P}\left(\left\|\Pi_{F} \otimes_{j=1}^{\ell} X^{(j)}\right\|_{2} \leq \varepsilon \sqrt{m}\right) \leq e^{2} \exp \left(-C_{\ell} \frac{(1-\varepsilon)^{2} m}{n^{\ell-1}}\right)
$$

Since the above estimates come from concentration inequalities, they do not approach to 0 as $\varepsilon$ approaches to 0 . Also, they only apply to the case when the dimension of the subspace $m \gg \mathcal{O}_{\ell}\left(n^{\ell-1}\right)$.

## 3. Stochastic Dominance

We are interested in tensor product of independent random vectors with independent coordinates and bounded densities. It is known that random vectors with independent coordinates whose densities are bounded by 1 are stochastically dominated by the uniform random vectors on the unit cube. In this section we will show that tensor product of random vectors preserves this property. It is well known (see for example section 4.1 in [16]) that we can approximate any symmetric convex body by circumscribed polytopes.

Lemma 3.1. (See [16]) Let $d \geq n+1$ and denote by $P_{d, o}^{n}(K)$ the set of polytopes in $\mathbb{R}^{n}$ that are circumscribed out of $K$ with at most $d$ vertices. Let $K \subset \mathbb{R}^{n}$ be $a$ symmetric convex body, then

$$
\inf _{V \in P_{d, o}^{n}(K)}|K \backslash V| \leq \frac{c(K)}{d^{\frac{2}{n-1}}},
$$

where $c(K)$ is a constant that depends only on $K$.
Theorem 3.2. Let $X^{(j)}=\left(X_{1}^{(j)}, \cdots, X_{n_{j}}^{(j)}\right) \in \mathbb{R}^{n_{j}}, 1 \leq j \leq \ell$ be independent random vectors with independent coordinates. Then

$$
\otimes_{j=1}^{\ell} X^{(j)} \prec \otimes_{j=1}^{\ell} X^{(j)^{*}}
$$

Proof. We will consider the tensor as a flattened vector:

$$
\otimes_{j=1}^{\ell} X^{(j)}=\left(\prod_{j=1}^{\ell} X_{i_{j}}^{(j)}\right)_{i_{1} \ldots i_{\ell}}
$$

Let $K$ be a compact and symmetric convex set in $\mathbb{R}^{n^{\ell}}$, then there exists a sequence of polytopes $\left\{K_{n}\right\}_{n=1}^{\infty}$ satisfying Lemma 3.1, where

$$
K_{n}=\bigcap_{k=1}^{N}\left\{x:\left|\left\langle x, u^{k}\right\rangle\right| \leq 1\right\}
$$

and

$$
x=\otimes_{j=1}^{\ell} X^{(j)}, \quad u^{k}=\left(u_{i_{1} \ldots i_{\ell}}^{k}\right)_{i_{1} \ldots i_{\ell}} .
$$

Without loss of generality, we can assume that

$$
K=\bigcap_{l=1}^{N}\left\{x:\left|\left\langle x, u^{k}\right\rangle\right| \leq 1\right\}
$$

This is to say

$$
\begin{equation*}
\otimes_{j=1}^{\ell} X^{(j)} \in K \Longleftrightarrow\left\langle\otimes_{j=1}^{\ell} X^{(j)}, u^{k}\right\rangle \leq 1, \quad \forall 1 \leq k \leq N \tag{1}
\end{equation*}
$$

For every $p=1, \cdots, \ell$ we can write

$$
\begin{align*}
\left\langle\otimes_{j=1}^{\ell} X^{(j)}, u^{k}\right\rangle & =\sum_{i_{1}, \ldots, i_{\ell}}\left(X_{i_{1}}^{(1)} \cdots X_{i_{\ell}}^{(\ell)}\right) u_{i_{1} \ldots i_{\ell}}^{k} \\
& =\sum_{i_{p}} X_{i_{p}}^{(p)} \sum_{i_{1}, \cdots, i_{p-1}, i_{p+1}, \cdots, i_{\ell}}\left(X_{i_{1}}^{(1)} \cdots X_{i_{p-1}}^{(p-1)} X_{i_{p+1}}^{(p+1)} \cdots X_{i_{\ell}}^{(\ell)}\right) u_{i_{1} \cdots i_{\ell}}^{k} \\
& =\left\langle X^{(p)}, \theta_{p}^{k}\right\rangle \tag{2}
\end{align*}
$$

where
$\theta_{p}^{k}:=\theta_{p}^{k}\left(X^{(j)}, j \neq p, u^{k}\right)=\left(\sum_{i_{1}, \cdots, i_{p-1}, i_{p+1}, \cdots, i_{\ell}}\left(X_{i_{1}}^{(1)} \cdots X_{i_{p-1}}^{(p-1)} X_{i_{p+1}}^{(p+1)} \cdots X_{i_{\ell}}^{(\ell)}\right) u_{i_{1} \cdots i_{p-1} i_{p} i_{p+1} \cdots i_{\ell}}^{k}\right)_{i_{p}=1}^{n_{p}}$
is an $n_{p}$-dimensional random vector independent of $X^{(p)}$. Here the index $i_{p}$ only appears in $u_{i_{1} \cdots i_{p-1} i_{p} i_{p+1} \cdots i_{\ell}}^{k}$. Combine (11) and (2) we have for every $p=1, \cdots, \ell$,

$$
\begin{equation*}
\mathbf{1}_{K}\left(\otimes_{j=1}^{\ell} X^{(j)}\right)=\prod_{k=1}^{N} \mathbf{1}_{[-1,1]}\left(\left\langle X^{(p)}, \theta_{p}^{k}\right\rangle\right) \tag{3}
\end{equation*}
$$

Therefore for every $p=1, \cdots, \ell$,

$$
\begin{aligned}
& \mathbb{P}\left(\otimes_{j=1}^{\ell} X^{(j)} \in K\right) \\
= & \int_{\mathbb{R}^{n_{1}}} \cdots \int_{\mathbb{R}^{n_{\ell}}} \mathbf{1}_{K}\left(\otimes_{j=1}^{\ell} X^{(j)}\right) \prod_{i=1}^{n_{1}} f_{i}^{(1)}\left(\left\langle X^{(1)}, e_{i}\right\rangle\right) \mathrm{d} X^{(1)} \cdots \prod_{i=1}^{n_{\ell}} f_{i}^{(\ell)}\left(\left\langle X^{(\ell)}, e_{i}\right\rangle\right) \mathrm{d} X^{(\ell)} \\
= & \int_{\mathbb{R}^{n_{1}}} \cdots \int_{\mathbb{R}^{n_{\ell}}} \prod_{k=1}^{N} \mathbf{1}_{[-1,1]}\left(\left\langle X^{(p)}, \theta_{p}^{k}\right\rangle\right) \prod_{i=1}^{n_{1}} f_{i}^{(1)}\left(\left\langle X^{(1)}, e_{i}\right\rangle\right) \mathrm{d} X^{(1)} \cdots \prod_{i=1}^{n_{\ell}} f_{i}^{(\ell)}\left(\left\langle X^{(\ell)}, e_{i}\right\rangle\right) \mathrm{d} X^{(\ell)}
\end{aligned}
$$

which is an $\ell$-fold integral. By Fubini's theorem, we first choose $p=1$ and consider the single integral about $X^{(1)}$

$$
\begin{aligned}
& \int_{\mathbb{R}^{n_{1}}} \prod_{k=1}^{N} \mathbf{1}_{[-1,1]}\left(\left\langle X^{(1)}, \theta_{1}\right\rangle\right) \prod_{i=1}^{n_{1}} f_{i}^{(1)}\left(\left\langle X^{(1)}, e_{i}\right\rangle\right) \mathrm{d} X^{(1)} \\
\leq & \int_{\mathbb{R}^{n_{1}}} \prod_{k=1}^{N} \mathbf{1}_{[-1,1]}\left(\left\langle X^{(1)}, \theta_{1}\right\rangle\right) \prod_{i=1}^{n_{1}} f_{i}^{(1)^{*}}\left(\left\langle X^{(1)}, e_{i}\right\rangle\right) \mathrm{d} X^{(1)} .
\end{aligned}
$$

The inequality follows from Rogers-Brascamp-Lieb-Luttinger inequality (THEOREM 2.11) since the indicator function $\mathbf{1}_{[-1,1]}\left(\left\langle X^{(1)}, \theta_{1}\right\rangle\right)$ is non-negative and even. Then

$$
\begin{aligned}
& \mathbb{P}\left(\otimes_{j=1}^{\ell} X^{(j)} \in K\right) \\
\leq & \int_{\mathbb{R}^{n_{1}}} \cdots \int_{\mathbb{R}^{n_{\ell}}} \prod_{k=1}^{N} \mathbf{1}_{[-1,1]}\left(\left\langle X^{(p)}, \theta_{p}^{k}\right\rangle\right) \prod_{i=1}^{n_{1}} f_{i}^{(1)^{*}}\left(\left\langle X^{(1)}, e_{i}\right\rangle\right) \mathrm{d} X^{(1)} \cdots \prod_{i=1}^{n_{\ell}} f_{i}^{(\ell)}\left(\left\langle X^{(\ell)}, e_{i}\right\rangle\right) \mathrm{d} X^{(\ell)}
\end{aligned}
$$

Now we repeat (3) for $p=2, \cdots, \ell$ and apply Fubini's theorem for the single integrals of $X^{(2)}, \cdots, X^{(\ell)}$, we have

$$
\begin{aligned}
& \mathbb{P}\left(\otimes_{j=1}^{\ell} X^{(j)} \in K\right) \\
\leq & \int_{\mathbb{R}^{n_{1}}} \ldots \int_{\mathbb{R}^{n_{\ell}}} \prod_{k=1}^{N} \mathbf{1}_{[-1,1]}\left(\left\langle X^{(p)}, u^{p}\right\rangle\right) \prod_{i=1}^{n_{1}} f_{i}^{(1)^{*}}\left(\left\langle X^{(1)}, e_{i}\right\rangle\right) \mathrm{d} X^{(1)} \cdots \prod_{i=1}^{n_{\ell}} f_{i}^{(\ell)^{*}}\left(\left\langle X^{(\ell)}, e_{i}\right\rangle\right) \mathrm{d} X^{(\ell)} \\
= & \mathbb{P}\left(\otimes_{j=1}^{\ell} X^{(j)^{*}} \in K\right)
\end{aligned}
$$

where $K$ is any polytope. For a general symmetric convex body $K$, recall from Lemma 3.1 that there exists a sequence of polytopes $V_{n}$ that are circumscribed out of $K$ such that

$$
\lim _{n \rightarrow \infty}\left|V_{n} \backslash K\right|=0
$$

Define $K_{n}=\cap_{i=1}^{n} V_{i}$, then $K_{n}$ is a polytope as well and $K_{n} \supset K_{n+1}$ and

$$
\lim _{n \rightarrow \infty}\left|K_{n} \backslash K\right|=0
$$

Hence $\lim _{n \rightarrow \infty} K_{n}=K$ almost everywhere. Define

$$
f_{n}(x)=\mathbf{1}_{K_{n}^{C}}(x) .
$$

Then $0 \leq f_{n}(x) \leq f_{n+1}(x)$, and for almost every $x$

$$
\mathbf{1}_{K^{C}}(x)=\lim _{n \rightarrow \infty} f_{n}(x)
$$

By monotone convergence theorem,

$$
\begin{aligned}
& \lim _{n \rightarrow \infty} \mathbb{P}\left(\otimes_{j=1}^{\ell} X^{(j)} \in K_{n}\right)=1-\lim _{n \rightarrow \infty} \mathbb{P}\left(\otimes_{j=1}^{\ell} X^{(j)} \in K_{n}^{C}\right) \\
= & 1-\lim _{n \rightarrow \infty} \mathbb{P}\left(\otimes_{j=1}^{\ell} X^{(j)} \in K^{C}\right)=\mathbb{P}\left(\otimes_{j=1}^{\ell} X^{(j)} \in K\right)
\end{aligned}
$$

Following similar strategy but applying Kanter's theorem instead of Rogers-Brascamp-Lieb-Luttinger inequality, we prove the following.

Theorem 3.3. Let $X^{(j)} \in \mathbb{R}^{n_{j}}, 1 \leq j \leq \ell$ be independent random vectors with independent coordinates whose densities have uniform norms bounded by 1, then

$$
\otimes_{j=1}^{\ell} X^{(j)^{*}} \prec \otimes_{j=1}^{\ell} \mathbf{1}_{\left[-\frac{1}{2}, \frac{1}{2}\right]^{n_{j}} .}
$$

Here by an abuse of notation, we let $\mathbf{1}_{\left[-\frac{1}{2}, \frac{1}{2}\right]^{n_{j}}}$ denotes the uniform distribution on $\left[-\frac{1}{2}, \frac{1}{2}\right]^{n_{j}}$.

Proof. As in the proof of the previous theorem, it suffices to approximate any symmetric convex body by a polytope. Let $K, u^{k}, \theta_{p}^{k}$ for $k=1, \cdots$, be defined as in the proof of the previous theorem. Then

$$
\begin{aligned}
& \mathbb{P}\left(\otimes_{j=1}^{\ell} X^{(j)^{*}} \in K\right) \\
= & \int_{\mathbb{R}^{n_{1}}} \cdots \int_{\mathbb{R}^{n_{\ell}}} \mathbf{1}_{K}\left(\otimes_{j=1}^{\ell} X^{(j)}\right) \prod_{i=1}^{n_{1}} f_{i}^{(1)^{*}}\left(\left\langle X^{(1)}, e_{i}\right\rangle\right) \mathrm{d} X^{(1)} \cdots \prod_{i=1}^{n_{\ell}} f_{i}^{(\ell)^{*}}\left(\left\langle X^{(\ell)}, e_{i}\right\rangle\right) \mathrm{d} X^{(\ell)} \\
= & \int_{\mathbb{R}^{n_{1}}} \cdots \int_{\mathbb{R}^{n_{\ell}}} \prod_{k=1}^{N} \mathbf{1}_{[-1,1]}\left(\left\langle X^{(p)}, \theta_{p}^{k}\right\rangle\right) \prod_{i=1}^{n_{1}} f_{i}^{(1)}\left(\left\langle X^{(1) *}, e_{i}\right\rangle\right) \mathrm{d} X^{(1)} \cdots \prod_{i=1}^{n_{\ell}} f_{i}^{(\ell)^{*}}\left(\left\langle X^{(\ell)}, e_{i}\right\rangle\right) \mathrm{d} X^{(\ell)}
\end{aligned}
$$

which is an $\ell$-fold integral. By Fubini's theorem, we first choose $p=1$ and consider the single integral

$$
\begin{aligned}
& \int_{\mathbb{R}^{n_{1}}} \prod_{k=1}^{N} \mathbf{1}_{[-1,1]}\left(\left\langle X^{(1)}, \theta_{1}\right\rangle\right) \prod_{i=1}^{n_{1}} f_{i}^{(1)^{*}}\left(\left\langle X^{(1)}, e_{i}\right\rangle\right) \mathrm{d} X^{(1)} \\
\leq & \int_{\mathbb{R}^{n_{1}}} \prod_{k=1}^{N} \mathbf{1}_{[-1,1]}\left(\left\langle X^{(1)}, \theta_{1}\right\rangle\right) \prod_{i=1}^{n_{1}} \mathbf{1}_{\left[-\frac{1}{2}, \frac{1}{2}\right]}\left(\left\langle X^{(1)}, e_{i}\right\rangle\right) \mathrm{d} X^{(1)}
\end{aligned}
$$

This follows from Kanter's theorem (Theorem 2.12). For the $\ell$-fold integral, by choosing $p=2, \ldots, \ell$ and using Fubini's theorem, we have

$$
\begin{aligned}
& \mathbb{P}\left(\otimes_{j=1}^{\ell} X^{(j)} \in K\right) \\
= & \int_{\mathbb{R}^{n_{1}}} \ldots \int_{\mathbb{R}^{n} \ell} \prod_{k=1}^{N} \mathbf{1}_{[-1,1]}\left(\left\langle X^{(p)}, u^{p}\right\rangle\right) \prod_{i=1}^{n_{1}} f_{i}^{(1)^{*}}\left(\left\langle X^{(1)}, e_{i}\right\rangle\right) \mathrm{d} X^{(1)} \cdots \prod_{i=1}^{n_{\ell}} f_{i}^{(\ell)^{*}}\left(\left\langle X^{(\ell)}, e_{i}\right\rangle\right) \mathrm{d} X^{(\ell)} \\
= & \mathbb{P}\left(\otimes_{j=1}^{\ell} X^{(j)^{*}} \in K\right) .
\end{aligned}
$$

By definition, stochastic dominance is transitive. Hence combining Theorem 3.2 and 3.3, we have the corrollary.

Corollary 3.4. Let $X^{(j)} \in \mathbb{R}^{n_{j}}, 1 \leq j \leq \ell$ be independent random vectors with independent coordinates whose densities have uniform norms bounded by 1, then

$$
\otimes_{j=1}^{\ell} X^{(j)} \prec \otimes_{j=1}^{\ell} X^{(j)^{*}} \prec \otimes_{j=1}^{\ell} \mathbf{1}_{\left[-\frac{1}{2}, \frac{1}{2}\right]^{n_{j}}}
$$

## 4. Anti-CONCENTRATION OF "LOG-CONCAVE" SIMPLE RANDOM TENSORS

In this section we will prove Theorem 1.4. Suppose $X^{(j)}=\left(X_{1}^{(j)}, \ldots, X_{n_{j}}^{(j)}\right), 1 \leq$ $j \leq \ell$ are random vectors in $\mathbb{R}^{n_{j}}$. Define the simple random tensor

$$
X:=X^{(1)} \otimes \cdots \otimes X^{(\ell)}=\left(X_{i_{1}}^{(1)} \cdots X_{i_{\ell}}^{(\ell)}\right)_{i_{1} \cdots i_{\ell}}
$$

Let $F$ be an $m$-dimensional subspace in $\mathbb{R}^{n_{1} \otimes \cdots \otimes n_{\ell}}$ and let $f^{(1)}, \cdots, f^{(m)}$ be an orthonormal basis for $F$. Denote by $\Pi_{F} X^{(1)} \otimes \cdots \otimes X^{(\ell)}$ the orthogonal projection of $X^{(1)} \otimes \cdots \otimes X^{(\ell)}$ onto $F$. Then by definition we have

$$
\left\|\Pi_{F} X^{(1)} \otimes \cdots \otimes X^{(\ell)}\right\|_{2}^{2}=\sum_{k=1}^{m}\left|\left\langle X^{(1)} \otimes \cdots \otimes X^{(\ell)}, f^{(k)}\right\rangle\right|^{2}
$$

Here we apply the following version of Guedon's theorem, which is an application of Theorem 2.4

Lemma 4.1. Let $X \in \mathbb{R}^{n}$ be a log-concave random vector and let $\|\cdot\|$ be a seminorm. Then for every $\varepsilon \in(0,1)$ we have

$$
\mathbb{P}(\|X\| \leq \varepsilon \mathbb{E}\|X\|) \leq C \varepsilon
$$

We start the proof of Theorem 1.4 for the case $\ell=2$. We would like to show that if $X^{(1)} \in \mathbb{R}^{n_{1}}, X^{(2)} \in \mathbb{R}^{n_{2}}$ are independent isotropic log-concave random vectors and $f$ is in the orthonormal basis for subspace $F \in \mathbb{R}^{n_{1} \times n_{2}}$, then

$$
\mathbb{P}\left(\left|\left\langle X^{(1)} \otimes X^{(2)}, f\right\rangle\right| \leq \varepsilon\right) \leq C \varepsilon \log \frac{1}{\varepsilon}
$$

Note that for log-concave random vector $X, \mathbb{E}\|X\|_{2}$ and $\sqrt{\mathbb{E}\|X\|_{2}^{2}}$ are equivalent by Borell's lemma (see Theorem 1.5.7 in [7]). Set $\theta_{X^{(2)}}=\left(\sum_{i_{2}=1}^{n_{2}} X_{i_{2}}^{(2)} f_{i_{1} i_{2}}\right)_{i_{1}=1}^{n_{1}}$, which is linear image of $X^{(2)}$, therefore also log-concave. Note that

$$
\begin{gathered}
\mathbb{E}\left\|\theta_{X^{(2)}}\right\|_{2}^{2}=\sum_{i_{1}=1}^{n_{1}} \mathbb{E}\left(\sum_{i_{2}=1}^{n_{2}} X_{i_{2}}^{(2)} f_{i_{1} i_{2}}\right)^{2}=\sum_{i_{1}=1}^{n_{1}} \sum_{i_{2}=1}^{n_{2}} f_{i_{1} i_{2}}^{2}=1, \\
\mathbb{E}\left\|\theta_{X^{(2)}}\right\|_{2} \geq c_{0},
\end{gathered}
$$

where $0<c_{0}<1$ is some universal constant. Then for $0<\varepsilon<c_{0}$, we apply Lemma 4.1, and we have

$$
\begin{gather*}
\mathbb{P}\left(\left|\left\langle X^{(1)}, \theta\right\rangle\right| \leq \varepsilon\right) \leq C_{1} \varepsilon  \tag{4}\\
\mathbb{P}\left(\left\|\theta_{X^{(2)}}\right\|_{2} \leq \varepsilon\right) \leq C_{1} \varepsilon \tag{5}
\end{gather*}
$$

Define a partition as follows

$$
\begin{gathered}
B_{\lambda}^{c}=\left\{\left\|\theta_{X^{(2)}}\right\|_{2}<\lambda\right\} \\
B_{\lambda, p}=\left\{\lambda 2^{p}<\left\|\theta_{X^{(2)}}\right\|_{2}<\lambda 2^{p+1}\right\}, \quad p=0,1, \cdots, N, \\
B_{\lambda, N+1}=\left\{\left\|\theta_{X^{(2)}}\right\|_{2}>\lambda 2^{N+1}\right\} .
\end{gathered}
$$

Let $c=\min \left\{c_{0}, e^{-1}\right\}$. Then for $0<\varepsilon<c^{2}$,

$$
\begin{aligned}
& \mathbb{P}\left(\sum_{i_{1}, i_{2}} X_{i_{1}}^{(1)} X_{i_{2}}^{(2)} f_{i_{1} i_{2}} \leq \varepsilon\right) \\
= & \mathbb{P}\left(\left|\left\langle X^{(1)}, \theta_{X^{(2)}}\right\rangle\right| \leq \varepsilon\right) \\
= & \mathbb{E}_{X^{(2)}} \mathbb{P}_{X^{(1)}}\left(\left|\left\langle X^{(1)}, \theta_{X^{(2)}}\right\rangle\right| \leq \varepsilon\right) \\
= & \mathbb{E}_{X^{(2)}} \mathbf{1}_{B_{\lambda}^{c}} \mathbb{P}_{X^{(1)}}\left(\left|\left\langle X^{(1)}, \theta_{X^{(2)}}\right\rangle\right| \leq \varepsilon\right)+\sum_{p=0}^{N} \mathbb{E}_{X^{(2)}} \mathbf{1}_{B_{\lambda, p}} \mathbb{P}_{X^{(1)}}\left(\left|\left\langle X^{(1)}, \theta_{X^{(2)}}\right\rangle\right| \leq \varepsilon\right) \\
+ & \mathbb{E}_{X^{(2)}} \mathbf{1}_{B_{\lambda, N+1}} \mathbb{P}_{X^{(1)}}\left(\left|\left\langle X^{(1)}, \theta_{X^{(2)}}\right\rangle\right| \leq \varepsilon\right) .
\end{aligned}
$$

Here we estimate each term by repeatedly applying (4) and (5),

$$
\mathbb{E}_{X^{(2)}} \mathbf{1}_{B_{\lambda}^{c}} \mathbb{P}_{X^{(1)}}\left(\left|\left\langle X^{(1)}, \theta_{X^{(2)}}\right\rangle\right| \leq \varepsilon\right) \leq \mathbb{P}\left(B_{\lambda}^{c}\right)=\mathbb{P}\left(\left\|\theta_{X^{(2)}}\right\|_{2}<\lambda\right) \leq C_{1} \lambda
$$

and

$$
\begin{aligned}
& \mathbb{E}_{X^{(2)}} \mathbf{1}_{B_{\lambda, p}} \mathbb{P}_{X^{(1)}}\left(\left|\left\langle X^{(1)}, \theta_{X^{(2)}}\right\rangle\right| \leq \varepsilon\right) \\
\leq & \mathbb{E}_{X^{(2)}} \mathbf{1}_{B_{\lambda, p}} \mathbb{P}_{X^{(1)}}\left(\left|\left\langle X^{(1)}, \frac{\theta_{X^{(2)}}}{\left\|\theta_{X^{(2)}}\right\|_{2}}\right\rangle\right| \leq \frac{\varepsilon}{\lambda 2^{p}}\right) \\
\leq & \frac{C_{1} \varepsilon}{\lambda 2^{p}} \mathbb{P}\left(B_{\lambda, p}\right) \\
\leq & \frac{C_{1} \varepsilon}{\lambda 2^{p}} \mathbb{P}\left(\left\|\theta_{X^{(2)}}\right\|_{2} \leq 2^{p+1} \lambda\right) \\
\leq & C_{2} \varepsilon
\end{aligned}
$$

and
$\mathbb{E}_{X^{(2)}} \mathbf{1}_{B_{\lambda, N+1}} \mathbb{P}_{X^{(1)}}\left(\left|\left\langle X^{(1)}, \theta_{X^{(2)}}\right\rangle\right| \leq \varepsilon\right) \leq \mathbb{E}_{X^{(2)}} \mathbb{P}_{X^{(1)}}\left(\left|\left\langle X^{(1)}, \frac{\theta_{X^{(2)}}}{\left\|\theta_{X^{(2)}}\right\|_{2}}\right\rangle\right| \leq \frac{\varepsilon}{\lambda 2^{N+1}}\right) \leq \frac{\varepsilon}{\lambda 2^{N+1}}$.
Hence

$$
\mathbb{P}\left(\left|\left\langle X^{(1)} \otimes X^{(2)}, f\right\rangle\right| \leq \varepsilon\right) \leq C_{1} \lambda+C_{2} N \varepsilon+\frac{\varepsilon}{\lambda 2^{N+1}}
$$

Choose $\lambda=\frac{\varepsilon}{c}$ and $N=\log _{2} \frac{1}{\varepsilon}$, then

$$
\mathbb{P}\left(\left|\left\langle X^{(1)} \otimes X^{(2)}, f\right\rangle\right| \leq \varepsilon\right) \leq\left(\frac{C_{1}}{c}+\frac{c}{2}\right) \varepsilon+C_{2} \varepsilon \log _{2} \frac{1}{\varepsilon} \leq C \varepsilon \log \frac{1}{\varepsilon}
$$

Now we consider the general case for $\ell \geq 3$ by induction. Suppose it's true that for independent log-concave random vectors $X^{(j)} \in \mathbb{R}^{n_{j}}, 1 \leq j \leq \ell-1$ whose coordinates are mean zero, uncorrelated and have variances bounded by 1 and $f$ in the orthonormal basis for subspace $F \in \mathbb{R}^{n_{1} \times \cdots \times n_{\ell-1}}$, for $0<\varepsilon<c^{\ell-1}$ we have

$$
\mathbb{P}\left(\left|\left\langle X^{(1)} \otimes \cdots \otimes X^{(\ell-1)}, f\right\rangle\right| \leq \varepsilon\right) \leq \frac{\varepsilon}{(\ell-2)!}\left(C \log \frac{1}{\varepsilon}\right)^{\ell-2}
$$

Then for independent log-concave random vectors $X^{(j)} \in \mathbb{R}^{n_{j}}, 1 \leq i \leq \ell$ whose coordinates are mean zero, uncorrelated and have variances bounded by 1 and $f$ in the orthonormal basis for subspace $F \in \mathbb{R}^{n_{1} \otimes \cdots \otimes n_{\ell}}$, for $0<\varepsilon<c^{\ell}$,

$$
\begin{aligned}
& \mathbb{P}\left(\left|\left\langle X^{(1)} \otimes \cdots \otimes X^{(\ell)}, f\right\rangle\right| \leq \varepsilon\right) \\
= & \mathbb{P}\left(\left|\sum_{i_{1} \cdots i_{\ell}} X_{i_{1}}^{(1)} \cdots X_{i_{\ell}}^{(\ell)} f_{i_{1} \cdots i_{\ell}}\right| \leq \varepsilon\right) \\
= & \mathbb{E}_{X^{(\ell)}} \mathbb{P}_{X^{(1)} \cdots X^{(\ell-1)}}\left(\left|\sum_{i_{1} \cdots i_{\ell-1}} X_{i_{1}}^{(1)} \cdots X_{i_{\ell-1}}^{(\ell-1)}\left(\sum_{i_{\ell}} X_{i_{\ell}}^{(\ell)} f_{i_{1} \cdots i_{\ell}}\right)\right| \leq \varepsilon\right) \\
= & \mathbb{E}_{X^{(\ell)}} \mathbb{P}_{X^{(1)} \cdots X^{(\ell-1)}}\left(\left|\left\langle X^{(1)} \otimes \cdots \otimes X^{(\ell-1)}, \theta_{X^{(\ell)}}\right\rangle\right| \leq \varepsilon\right)
\end{aligned}
$$

where

$$
\theta_{X^{(\ell)}}:=\left(\sum_{i_{\ell}} X_{i_{\ell}}^{(\ell)} f_{i_{1} \cdots i_{\ell-1} i_{\ell}}\right)_{i_{1} \cdots i_{\ell-1}}=A_{\ell} X^{(\ell)} \in \mathbb{R}^{n_{1} \cdots n_{\ell-1}}
$$

is a linear image of $X^{(\ell)}$ where

$$
A_{\ell}=\left(f_{i_{1} \cdots i_{\ell-1} i_{\ell}}\right)_{i_{1} \cdots i_{\ell-1}, i_{\ell}} \in \mathbb{R}^{n_{1} \cdots n_{\ell-1} \times n_{\ell}}
$$

is an $n_{1} \cdots n_{\ell-1} \times n_{\ell}$ matrix. Hence $\theta_{X^{(\ell)}}$ is also log-concave. Consider the Frobenius norm (Euclidean norm) $\|\cdot\|_{2}$, then

$$
\begin{gathered}
\left\|\theta_{X^{(\ell)}}\right\|_{2}=\left[\sum_{i_{1} \cdots i_{\ell-1}}\left(\sum_{i_{\ell}} X_{i_{\ell}}^{(\ell)} f_{i_{1} \cdots i_{\ell-1} i_{\ell}}\right)^{2}\right]^{1 / 2} \\
\mathbb{E}\left\|\theta_{X^{(\ell)}}\right\|_{2}^{2}=\left\|f_{i_{1} \cdots i_{\ell}}\right\|_{2}^{2}=1 \\
\mathbb{E}\left\|\theta_{X^{(\ell)}}\right\|_{2} \geq c
\end{gathered}
$$

where $0<c<1$ is some universal constant. Then for $0<\varepsilon<c_{0}$, we have

$$
\mathbb{P}\left(\left\|\theta_{X^{(\ell)}}\right\|_{2} \leq \varepsilon\right) \leq C_{1} \varepsilon .
$$

Let $c=\min \left\{c_{0}, e^{-1}\right\}$. Then for $0<\varepsilon<c^{\ell}$,

$$
\begin{aligned}
& \mathbb{P}\left(\left|\left\langle X^{(1)} \otimes \cdots \otimes X^{(\ell)}, f\right\rangle\right| \leq \varepsilon\right) \\
= & \mathbb{E}_{X^{(\ell)}} \mathbb{P}_{X^{(1)} \ldots X^{(\ell-1)}}\left(\left|\left\langle X^{(1)} \otimes \cdots \otimes X^{(\ell-1)}, \frac{\theta_{X^{(\ell)}}\left\|\theta_{X^{(\ell)}}\right\|_{2}}{}\right\rangle\right| \leq \frac{\varepsilon}{\left\|\theta_{X^{(\ell)}}\right\|_{2}}\right) \\
= & \mathbb{E}_{X^{(\ell)}} \mathbf{1}_{\left\{\left\|\theta_{X^{(\ell)}}\right\|_{2} \leq c^{1-\ell} \varepsilon\right\}} \mathbb{P}_{X^{(1)} \ldots X^{(\ell-1)}}\left(\left|\left\langle X^{(1)} \otimes \cdots \otimes X^{(\ell-1)}, \frac{\theta_{X^{(\ell)}}}{\left\|\theta_{X^{(\ell)}}\right\|_{2}}\right\rangle\right| \leq \frac{\varepsilon}{\left\|\theta_{X^{(\ell)}}\right\|_{2}}\right) \\
+ & \mathbb{E}_{X^{(\ell)}} \mathbf{1}_{\left\{\left\|\theta_{X^{(\ell)}}\right\|_{2}>c^{1-\ell} \varepsilon\right\}} \mathbb{P}_{X^{(1)} \ldots X^{(\ell-1)}}\left(\left|\left\langle X^{(1)} \otimes \cdots \otimes X^{(\ell-1)}, \frac{\theta_{X^{(\ell)}}}{\left\|\theta_{X^{(\ell)}}\right\|_{2}}\right\rangle\right| \leq \frac{\varepsilon}{\left\|\theta_{X^{(\ell)}}\right\|_{2}}\right) \\
\leq & \mathbb{P}_{X^{(\ell)}}\left(\left\|\theta_{X^{(\ell)}}\right\|_{2} \leq c^{1-\ell} \varepsilon\right)+\mathbb{E}_{X^{(\ell)}} \mathbf{1}_{\left\{\left\|\theta_{X^{(\ell)}}\right\|_{2}>c^{1-\ell} \varepsilon\right\}} \frac{\varepsilon \theta_{X^{(\ell)}} \|_{2}(\ell-2)!}{}\left(C \log \frac{\left\|\theta_{X^{(\ell)}}\right\|_{2}}{\varepsilon}\right)^{\ell-2} \\
\leq & \frac{C_{1}}{c^{\ell}} \varepsilon+\mathbb{E}_{X^{(\ell)}} \mathbf{1}_{\left\{\left\|\theta_{X^{(\ell)}}\right\|_{2}>c^{1-\ell} \varepsilon\right\}} \frac{\varepsilon}{\left\|\theta_{X^{(\ell)}}\right\|_{2}(\ell-2)!}\left(C \log \frac{\left\|\theta_{X^{(\ell)}}\right\|_{2}}{\varepsilon}\right)^{\ell-2} \\
= & \frac{C_{1}}{c^{\ell}} \varepsilon+\frac{C^{\ell-2}}{(\ell-2)!} \mathbb{E}_{X^{(\ell)}} \mathbf{1}_{\left\{\left\|\theta_{X^{(\ell)}}\right\|_{2}>c^{1-\ell} \varepsilon\right\}} \frac{\varepsilon}{\left\|\theta_{X^{(\ell)}}\right\|_{2}}\left(\log \frac{\left\|\theta_{X^{(\ell)}}\right\|_{2}}{\varepsilon}\right)^{\ell-2}
\end{aligned}
$$

It now suffices to show that

$$
\mathbb{E} \mathbf{1}_{\left\{\left\|\theta_{X^{(\ell)}}\right\|_{2}>c^{1-\ell} \varepsilon\right\}} \frac{\varepsilon}{\left\|\theta_{X^{(\ell)}}\right\|_{2}}\left(\log \frac{\left\|\theta_{X^{(\ell)}}\right\|_{2}}{\varepsilon}\right)^{\ell-2} \leq \frac{C}{\ell-1} \varepsilon\left(\log \frac{1}{\varepsilon}\right)^{\ell-1}
$$

which is proved in Lemma B. 1 This completes the proof of

$$
\mathbb{P}\left(\left|\left\langle X^{(1)} \otimes \cdots \otimes X^{(\ell)}, f\right\rangle\right| \leq \varepsilon\right) \leq \frac{\varepsilon}{(\ell-1)!}\left(C \log \frac{1}{\varepsilon}\right)^{\ell-1}
$$

On one hand, recall that

$$
\left\|\Pi_{F} X^{(1)} \otimes \cdots \otimes X^{(\ell)}\right\|_{2}^{2}=\sum_{k=1}^{m}\left|\left\langle\otimes_{j=1}^{\ell} X^{(j)}, f^{(k)}\right\rangle\right|^{2}
$$

Then by a union bound we have

$$
\mathbb{P}\left(\left\|\Pi_{F} X^{(1)} \otimes \cdots \otimes X^{(\ell)}\right\|_{2} \leq \varepsilon \sqrt{m}\right) \leq \frac{\varepsilon m}{(\ell-1)!}\left(C \log \frac{1}{\varepsilon}\right)^{\ell-1}
$$

On the other hand, let $0<q<1$, then

$$
\begin{aligned}
\mathbb{E} \frac{1}{\left\|\Pi_{F} \otimes_{j=1}^{\ell} X^{(j)}\right\|_{2}^{q}} & =\frac{1}{m^{\frac{q}{2}}} \mathbb{E} \frac{1}{\left(\frac{1}{m} \sum_{k=1}^{m}\left|\left\langle\otimes_{j=1}^{\ell} X^{(j)}, f^{(k)}\right\rangle\right|^{2}\right)^{\frac{q}{2}}} \\
& \leq \frac{1}{m^{\frac{q}{2}}} \mathbb{E} \frac{1}{\prod_{k=1}^{m}\left|\left\langle\otimes_{j=1}^{\ell} X^{(j)}, f^{(k)}\right\rangle\right|^{\frac{q}{m}}} \\
& \leq \frac{1}{m^{\frac{q}{2}}} \prod_{k=1}^{m}\left(\mathbb{E} \frac{1}{\left|\left\langle\otimes_{j=1}^{\ell} X^{(j)}, f^{(k)}\right\rangle\right|^{q}}\right)^{\frac{1}{m}}
\end{aligned}
$$

where the first inequality follows from arithmetic and geometric mean inequality, and the second inequality follows from Hölder's inequality. Note that for $1 \leq k \leq m$, we apply Lemma C. 1 with $K=\frac{C^{\ell-1}}{(\ell-1)!}$, then for $1-\frac{1}{C} \leq q<1$, we have

$$
\mathbb{E} \frac{1}{\left|\left\langle\otimes_{j=1}^{\ell} X^{(j)}, f^{(k)}\right\rangle\right|^{q}} \leq \frac{\left(K \ell^{\ell}\right)^{q}}{(1-q)^{q(\ell-1)+1}}
$$

By Lemma C.2, we take $0<\varepsilon<e^{-C \ell}$ and $q=1-\frac{\ell}{\ell+\log \frac{1}{\varepsilon}}$. Then by Markov's inequality,

$$
\begin{aligned}
\mathbb{P}\left(\left\|\Pi_{F} \otimes_{j=1}^{\ell} X^{(j)}\right\|_{2} \leq \varepsilon \sqrt{m}\right) & =\mathbb{P}\left(\frac{1}{\left\|\Pi_{F} \otimes_{j=1}^{\ell} X^{(j)}\right\|_{2}^{q}} \geq \frac{1}{\varepsilon^{q} m^{\frac{q}{2}}}\right) \\
& \leq \varepsilon^{q} \frac{\left(K \ell^{\ell}\right)^{q}}{(1-q)^{q(\ell-1)+1}} \\
& \leq K(3 e)^{\ell} \varepsilon\left(\log \frac{1}{\varepsilon}\right)^{\ell} \\
& =\frac{C^{\prime \ell} \varepsilon}{(\ell-1)!} \log \frac{1}{\varepsilon}\left(C \log \frac{1}{\varepsilon}\right)^{\ell-1}
\end{aligned}
$$

## 5. Proof of Theorem 1.5

We now consider generic subspace and prove Theorem 1.5 . We are working under the assumption that the random tensor $\otimes_{j=2}^{\ell} X^{(j)}$ is concentrated around the sphere $n^{\frac{\ell-1}{2}} \mathbb{S}^{n^{\ell-1}-1}$. The following two propositions show that the assumption holds with high probability.

Proposition 5.1. Let $X^{(1)}, \cdots, X^{(\ell)}$ be independent isotropic log-concave random vectors in $\mathbb{R}^{n}$, then for $t \geq 2$,

$$
\mathbb{P}\left(\left\|\otimes_{j=1}^{\ell} X^{(j)}\right\|_{2} \geq t^{\ell} n^{\frac{\ell}{2}}\right) \leq e^{-c t \sqrt{n} \ell}
$$

Proof. By Markov's inequality and Theorem 2.1,

$$
\begin{aligned}
\mathbb{P}\left(\left\|\otimes_{j=1}^{\ell} X^{(j)}\right\|_{2} \geq s^{\ell} C^{\ell}(\sqrt{n}+q)^{\ell}\right) & \leq \frac{\mathbb{E}\left\|\otimes_{j=1}^{\ell} X^{(j)}\right\|_{2}^{q}}{s^{q \ell} C^{q \ell}(\sqrt{n}+q)^{q \ell}} \\
& =\frac{\prod_{j=1}^{\ell} \mathbb{E}\left\|X^{(j)}\right\|_{2}^{q}}{s^{q \ell} C^{q \ell}(\sqrt{n}+q)^{q \ell}} \\
& \leq \frac{1}{s^{q \ell}} .
\end{aligned}
$$

Take $s=\frac{1}{e}$ and $q=\left(\frac{e}{C} t-1\right) \sqrt{n}$ for $t \geq 2$, then

$$
\mathbb{P}\left(\left\|\otimes_{j=1}^{\ell} X^{(j)}\right\|_{2} \geq(t \sqrt{n})^{\ell}\right) \leq e^{-(t-1) \sqrt{n} \ell} \leq e^{-c t \sqrt{n} \ell}
$$

Proposition 5.2. Let $X^{(1)}, \cdots, X^{(\ell)}$ be independent isotropic log-concave random vectors in $\mathbb{R}^{n}$ with Poincaré constant $C_{P}$, where $n$ is bounded from below and $\ell<$ $\frac{2 \sqrt{n}}{C_{P}}$ for some universal constant $C$. Then for $t>0$,
$\mathbb{P}\left(\left\|\otimes_{j=1}^{\ell} X^{(j)}\right\|_{2} \geq(1+t) n^{\frac{\ell}{2}}\right) \leq \exp \left\{-C \min \left(\frac{n(\log (1+t))^{2}}{C^{2} C_{P}^{2} \ell}, \frac{\sqrt{n}(\log (1+t))}{C C_{P}}\right)\right\}$,
in particular, if $0<t<1$,

$$
\mathbb{P}\left(\left\|\otimes_{j=1}^{\ell} X^{(j)}\right\|_{2} \geq(1+t) n^{\frac{\ell}{2}}\right) \leq \exp \left\{-C \min \left(\frac{n t^{2}}{C^{2} C_{P}^{2} \ell}, \frac{\sqrt{n} t}{C C_{P}}\right)\right\}
$$

And for $1-\frac{1}{\left(1+\frac{C}{\sqrt{n}}\right)^{\ell}} \leq t<1$,
$\mathbb{P}\left(\left\|\otimes_{j=1}^{\ell} X^{(j)}\right\|_{2} \leq(1-t) n^{\frac{\ell}{2}}\right) \leq \exp \left\{-C \min \left(\frac{n\left(\log \frac{\left(1-\frac{C_{P}}{2 \sqrt{n}}\right)^{\ell}}{1-t}\right)^{2}}{C^{2} C_{P}^{2} \ell}, \frac{\sqrt{n} \log \frac{\left(1-\frac{C_{P}}{2 \sqrt{n}}\right)^{\ell}}{1-t}}{C C_{P}}\right)\right\}$.
Proof. Define

$$
Y_{j}=\log \frac{\left\|X^{(j)}\right\|_{2}}{\sqrt{n}}
$$

Note that

$$
\left\|\left\|X^{(j)}\right\|_{2}\right\|_{L_{0}}=\lim _{p \rightarrow 0}\left(\mathbb{E}\|X\|_{2}^{p}\right)^{\frac{1}{p}}=e^{\mathbb{E} \log \|X\|_{2}}
$$

and by Hölder's inequality,

$$
\left\|\left\|X^{(j)}\right\|_{2}\right\|_{L_{-1}} \leq\| \| X^{(j)}\left\|_{2}\right\|_{L_{0}} \leq\| \| X^{(j)}\left\|_{2}\right\|_{L_{1}} \leq \sqrt{n}
$$

Also by distribution formula

$$
\begin{aligned}
\mathbb{E} \frac{1}{\left\|X^{(j)}\right\|_{2}} & =\int_{0}^{\infty} \mathbb{P}\left(\frac{1}{\left\|X_{2}^{(j)}\right\|} \geq s\right) \mathrm{d} s \\
& \leq \int_{0}^{\frac{1}{\sqrt{n}-C_{P}}} 1 \mathrm{~d} s+\int_{\frac{1}{\sqrt{n}-C_{P}}}^{\infty} \mathbb{P}\left(\left\|X_{2}^{(j)}\right\| \leq \frac{1}{s}\right) \mathrm{d} s \\
& =\frac{1}{\sqrt{n}-C_{P}}+\frac{1}{\sqrt{n}} \int_{0}^{1-\frac{C_{P}}{\sqrt{n}}} \frac{1}{t^{2}} \mathbb{P}\left(\left\|X_{2}^{(j)}\right\| \leq t \sqrt{n}\right) \mathrm{d} t \\
& \leq \frac{1}{\sqrt{n}-C_{P}}+\frac{1}{\sqrt{n}} \int_{0}^{\frac{1}{2}} \frac{1}{t^{2}} t^{c \sqrt{n}} \mathrm{~d} t+\frac{4}{\sqrt{n}} \int_{\frac{1}{2}}^{1-\frac{C_{P}}{\sqrt{n}}} e^{-\frac{C^{\prime}(1-t) \sqrt{n}}{C_{P}}} \mathrm{~d} t \\
& \leq \frac{1}{\sqrt{n}-C_{P}}+\left.\frac{1}{\sqrt{n}(c \sqrt{n}-1)} t^{c \sqrt{n}-1}\right|_{0} ^{\frac{1}{2}}+\left.\frac{4 C_{P}}{\sqrt{n}\left(C^{\prime} \sqrt{n}\right)} e^{-\frac{C^{\prime}(1-t) \sqrt{n}}{C_{P}}}\right|_{\frac{1}{2}} ^{1-\frac{C_{P}}{\sqrt{n}}} \\
& \leq \frac{1}{\sqrt{n}-C_{P}}+\frac{1}{2^{c \sqrt{n}} n}+\frac{4 C_{P} e^{-C^{\prime}}}{C^{\prime} n} \\
& \leq \frac{1}{\sqrt{n}-\frac{C_{P}}{2}}
\end{aligned}
$$

where we used Theorem 2.2. Then

$$
\sqrt{n}-\frac{C_{P}}{2} \leq\| \| X^{(j)}\left\|_{2}\right\|_{L_{0}} \leq \mathbb{E}\left\|X^{(j)}\right\|_{2} \leq \sqrt{n}
$$

$$
\begin{aligned}
\mathbb{E}\left[Y_{j}\right] & =\mathbb{E} \log \left\|X^{(j)}\right\|_{2}-\log \sqrt{n} \\
& =\log \| \| X^{(j)}\left\|_{2}\right\|_{L_{0}}-\log \sqrt{n} \\
& \in\left[\log \left(1-\frac{C_{P}}{2 \sqrt{n}}\right), 0\right] .
\end{aligned}
$$

Define

$$
\overline{Y_{j}}=Y_{j}-\mathbb{E}\left[Y_{j}\right]
$$

then for $s \geq 1$,

$$
\begin{aligned}
& \left\|\overline{Y_{1}}\right\|_{L_{s}}^{s} \leq \mathbb{E}\left|\log \frac{\left\|X^{(1)}\right\|_{2}}{\sqrt{n}}-\mathbb{E}\left[Y_{1}\right]\right|^{s} \\
& =\int_{0}^{\infty} s u^{s-1} \mathbb{P}\left(\left|\log \frac{\frac{\left\|X^{(1)}\right\|_{2}}{\sqrt{n}}}{e^{\mathbb{E}\left[Y_{1}\right]}}\right| \geq u\right) \mathrm{d} u \\
& \leq \int_{0}^{\infty} s u^{s-1} \mathbb{P}\left(\frac{\left\|X^{(1)}\right\|_{2}}{\sqrt{n}} \geq e^{\mathbb{E}\left[Y_{1}\right]} e^{u}\right) \mathrm{d} u \\
& +\int_{0}^{\infty} s u^{s-1} \mathbb{P}\left(\frac{\left\|X^{(1)}\right\|_{2}}{\sqrt{n}} \leq e^{\mathbb{E}\left[Y_{1}\right]} e^{-u}\right) \mathrm{d} u \\
& \leq \int_{0}^{\infty} s u^{s-1} \mathbb{P}\left(\frac{\left\|X^{(1)}\right\|_{2}}{\sqrt{n}} \geq\left(1-\frac{C_{P}}{2 \sqrt{n}}\right) e^{u}\right) \mathrm{d} u \\
& +\int_{0}^{\infty} s u^{s-1} \mathbb{P}\left(\frac{\left\|X^{(1)}\right\|_{2}}{\sqrt{n}} \leq e^{-u}\right) \mathrm{d} u \\
& :=I_{1}+I_{2} \text {. }
\end{aligned}
$$

For the first integral, we have

$$
I_{1} \leq \int_{0}^{a} s u^{s-1} \mathrm{~d} u+\int_{a}^{\infty} s u^{s-1} \mathbb{P}\left(\frac{\left\|X^{(1)}\right\|_{2}}{\sqrt{n}} \geq\left(1-\frac{C_{P}}{2 \sqrt{n}}\right) e^{u}\right) \mathrm{d} u
$$

where $a \geq-\log \left(1-\frac{C_{P}}{2 \sqrt{n}}\right)$. Consider $n$ sufficiently large so that $1-\frac{C_{P}}{2 \sqrt{n}} \geq \frac{1}{2}$. This is guaranteed by Klartag's result. Take $a=\frac{3 C_{P}}{\sqrt{n}}$, then for $u \geq a$, we have

$$
\begin{aligned}
\left(1-\frac{C_{P}}{2 \sqrt{n}}\right) e^{u} & \geq\left(1-\frac{C_{P}}{2 \sqrt{n}}\right)(1+u) \\
& =1+\left(1-\frac{C_{P}}{2 \sqrt{n}}\right) u-\frac{C_{P}}{2 \sqrt{n}} \\
& \geq 1+\frac{1}{3} u+\left(\frac{1}{6} u-\frac{C_{P}}{2 \sqrt{n}}\right) \\
& \geq 1+\frac{1}{3} u
\end{aligned}
$$

$$
\begin{aligned}
I_{1} & \leq \int_{0}^{\frac{3 C_{P}}{\sqrt{n}}} s u^{s-1} \mathrm{~d} u+\int_{\frac{3 C_{P}}{\sqrt{n}}}^{\infty} s u^{s-1} \mathbb{P}\left(\left\|X^{(1)}\right\|_{2} \geq\left(1+\frac{1}{3} u\right) \sqrt{n}\right) \mathrm{d} u \\
& \leq\left(\frac{3 C_{P}}{\sqrt{n}}\right)^{s}+\int_{\frac{3 C_{P}}{\sqrt{n}}}^{\infty} s u^{s-1} e^{-\frac{C^{\prime} u \sqrt{n}}{3 C_{P}}} \mathrm{~d} u \\
& \leq\left(\frac{3 C_{P}}{\sqrt{n}}\right)^{s}+\int_{C^{\prime}}^{\infty} s\left(\frac{3 C_{P} v}{C^{\prime} \sqrt{n}}\right)^{s-1} e^{-v} \frac{3 C_{P}}{C^{\prime} \sqrt{n}} \mathrm{~d} v \\
& \leq\left(\frac{3 C_{P}}{\sqrt{n}}\right)^{s}+\left(\frac{3 C_{P}}{C^{\prime} \sqrt{n}}\right)^{s} s \Gamma(s) \\
& \leq\left(\frac{C C_{P} s}{\sqrt{n}}\right)^{s} .
\end{aligned}
$$

For the second integral, notice that

$$
1-e^{-u} \geq \frac{u}{e}
$$

for $0 \leq u \leq 1$, then by Poincaré's inequality and THEOREM 2.2.

$$
\begin{aligned}
I_{2} & \leq \int_{0}^{1} s u^{s-1} e^{-\frac{C^{\prime} u \sqrt{n}}{e C_{P}}} \mathrm{~d} u+\int_{1}^{\infty} s u^{s-1}\left(e^{-u}\right)^{c \sqrt{n}} \mathrm{~d} u \\
& \leq\left(\frac{e C_{P}}{C^{\prime} \sqrt{n}}\right)^{s} s \Gamma(s)+\frac{1}{(c \sqrt{n})^{s}} s \Gamma(s) \\
& =\left(\frac{C C_{P} s}{\sqrt{n}}\right)^{s} .
\end{aligned}
$$

Hence

$$
\left\|Y_{j}\right\|_{L_{s}} \leq \frac{C C_{P}}{\sqrt{n}}
$$

By Berstein's inequality (see [59]), for $t \geq 0$,

$$
\mathbb{P}\left(\left|\sum_{j=1}^{\ell} \overline{Y_{j}}\right| \geq t\right) \leq 2 \exp \left\{-C \min \left(\frac{t^{2} n}{C^{2} C_{P}^{2} \ell}, \frac{t \sqrt{n}}{C C_{P}}\right)\right\}
$$

Hence for $t>0$,

$$
\begin{aligned}
& \mathbb{P}\left(\prod_{j=1}^{\ell}\left\|X^{(j)}\right\|_{2} \geq(1+t) n^{\frac{\ell}{2}}\right) \leq \mathbb{P}\left(\sum_{j=1}^{\ell} Y_{j} \geq \log (1+t)\right) \\
\leq & \mathbb{P}\left(\sum_{j=1}^{\ell} \overline{Y_{j}} \geq \log (1+t)\right) \leq \exp \left\{-C \min \left(\frac{n(\log (1+t))^{2}}{C^{2} C_{P}^{2} \ell}, \frac{\sqrt{n}(\log (1+t))}{C C_{P}}\right)\right\}
\end{aligned}
$$

and for $1-\frac{1}{\left(1+\frac{C}{\sqrt{n}}\right)^{\ell}} \leq t<1$,

$$
\begin{aligned}
& \mathbb{P}\left(\prod_{j=1}^{\ell}\left\|X^{(j)}\right\|_{2} \leq(1-t) n^{\frac{\ell}{2}}\right) \\
& \leq \mathbb{P}\left(\sum_{j=1}^{\ell} Y_{j} \leq \log (1-t)\right) \\
& \leq \mathbb{P}\left(\sum_{j=1}^{\ell} \overline{Y_{j}} \leq \log \left(\frac{1-t}{\left(1-\frac{C_{P}}{2 \sqrt{n}}\right)^{\ell}}\right]\right) \\
& \leq \exp \left\{-C \min \left(\frac{n\left(\log \frac{\left(1-\frac{C_{P}}{2 \sqrt{n}}\right)^{\ell}}{1-t}\right)^{2}}{C^{2} C_{P}^{2} \ell}, \frac{\sqrt{n} \log \frac{\left(1-\frac{C_{P}}{2 \sqrt{n}}\right)^{\ell}}{1-t}}{C C_{P}}\right)\right\}
\end{aligned}
$$

This completes the proof.
Remark 5.3. If $X^{(1)}, \cdots, X^{(\ell)}$ are independent sub-gaussian random vectors with sub-gaussian norm $C_{K}$, then for $t>0$,

$$
\mathbb{P}\left(\left\|\otimes_{j=1}^{\ell} X^{(j)}\right\|_{2} \geq(1+t) n^{\frac{\ell}{2}}\right) \leq \exp \left\{-C \frac{n(\log (1+t))^{2}}{C^{\prime 2} C_{K}^{2} \ell}\right\}
$$

and for $1-\frac{1}{\left(1+\frac{C}{\sqrt{n}}\right)^{\ell}} \leq t<1$,

$$
\mathbb{P}\left(\left\|\otimes_{j=1}^{\ell} X^{(j)}\right\|_{2} \leq(1-t) n^{\frac{\ell}{2}}\right) \leq \exp \left\{-C \frac{n\left(\log \frac{\left(1-\frac{C_{P}}{2 \sqrt{n}}\right)^{\ell}}{1-t}\right)^{2}}{C^{\prime 2} C_{K}^{2} \ell}\right\}
$$

For log-concave random vectors we have the following small ball property.
Proposition 5.4. Let $X \in \mathbb{R}^{n}$ be an isotropic log-concave random vector and let $A \in \mathbb{R}^{m \times n}$ such that $r=\operatorname{rank}(A)=\min \{m, n\}$. Let $s_{1} \geq s_{2} \geq \cdots \geq s_{r}$ be the singular values of $A$ and let $A=U \Sigma V$ be its singular value decomposition, then for $0<\varepsilon<1$,

$$
\mathbb{P}\left(\|A X\|_{2} \leq \varepsilon s_{r} \sqrt{r}\right) \leq\left(C \mathcal{L}_{r} \varepsilon\right)^{r}
$$

where the isotropic constant

$$
\mathcal{L}_{r}:=\sup _{F} \mathcal{L}_{\Pi_{F}}(\mu)=\sup _{F}\left\|f_{\Pi_{F} X}(x)\right\|_{\infty}^{1 / r}
$$

here $F$ is an $r$-dimensional subspace of $\mathbb{R}^{n}$ and $f_{\Pi_{F} X}(x)$ denotes the density of the marginal distribution $\Pi_{F} X$. In particular,

$$
\mathbb{P}\left(\|A X\|_{2} \leq \varepsilon s_{r} \sqrt{r}\right) \leq(C \varepsilon \sqrt{\log r})^{r}
$$

Proof. Denote by $f_{X}$ the deinisty function of $X$.
If $r=m$, then

$$
\mathbb{R}^{n}=C\left(A^{T}\right) \oplus \mathcal{N}(A)
$$

where $C\left(A^{T}\right)$ denotes the row space of $A$ and $\mathcal{N}(A)$ denotes the null space of $A$. For every $x \in \mathbb{R}^{n}$,

$$
x=\Pi_{C\left(A^{T}\right)} x+\Pi_{\mathcal{N}(A)} x
$$

Then

$$
s_{m}\left\|\Pi_{C\left(A^{T}\right)} x\right\|_{2} \leq\|A x\|_{2}=\left\|A \Pi_{C\left(A^{T}\right)} x\right\|_{2} \leq s_{1}\left\|\Pi_{C\left(A^{T}\right)} x\right\|_{2},
$$

and

$$
\begin{aligned}
\mathbb{P}\left(\|A X\|_{2} \leq \varepsilon s_{m} \sqrt{m}\right) & \leq \mathbb{P}\left(\left\|\Pi_{C\left(A^{T}\right)} X\right\|_{2} \leq \varepsilon \sqrt{m}\right) \\
& \leq\left\|f_{\Pi_{C\left(A^{T}\right) X}}(x)\right\|_{\infty} \varepsilon^{m} \operatorname{vol}\left(\mathbb{B}_{2}^{m}\right) \\
& \leq\left(C \mathcal{L}_{m} \varepsilon\right)^{m}
\end{aligned}
$$

By Prékopa-Leindler inequality, $\Pi_{C\left(A^{T}\right)} X$ is an isotropic log-concave random vector in $\mathbb{R}^{m}$. Hence

$$
\mathcal{L}_{m} \leq C \sqrt{\log m}
$$

Hence

$$
\mathbb{P}\left(\|A X\|_{2} \leq \varepsilon s_{m} \sqrt{m}\right) \leq(C \varepsilon \sqrt{\log m})^{m}
$$

If $r=n$, then

$$
s_{n}\|x\|_{2} \leq\|A x\|_{2} \leq s_{1}\|x\|_{2}
$$

Hence

$$
\begin{aligned}
\mathbb{P}\left(\|A X\|_{2} \leq \varepsilon s_{n} \sqrt{n}\right) & \leq \mathbb{P}\left(\|X\|_{2} \leq \varepsilon \sqrt{n}\right) \leq\left\|f_{X}(x)\right\|_{\infty} \varepsilon^{n} \operatorname{vol}\left(\mathbb{B}_{2}^{n}\right) \\
& \leq\left(\mathcal{L}_{n} \varepsilon\right)^{n} \leq(C \varepsilon \sqrt{\log n})^{n}
\end{aligned}
$$

We also introduce the following lemma for special orthogonal groups.
Lemma 5.5. (See Section 2.1 in [45]) Let $U=\left(U_{i j}\right)_{i j}$ be distributed according to Haar measure in $\mathbb{O}(n)$, then

$$
\mathbb{E}\left[U_{i j} U_{i^{\prime} j^{\prime}}\right]=\frac{1}{n} \delta_{i, j}^{i^{\prime}, j^{\prime}}
$$

Lemma 5.6. Let $\mathcal{F}$ be an event in $\mathbb{O}(n)$. Fix $F_{0}$ a subspace in $G_{n, m}$ and let $\mathcal{S}_{\mathcal{F}}:=\left\{F \in G_{n, m}: F=U F_{0}, U \in \mathcal{F}\right\}$. Then

$$
\mathbb{P}_{\mathbb{O}(n)}(\mathcal{F})=\mathbb{P}_{G_{n, m}}\left(\mathcal{S}_{\mathcal{F}}\right)
$$

Now we are able to prove Theorem 1.5 ,
Proof. Fix $t$ which will be chosen appropriately later on. We will consider several cases, and in each case, $t$ may be different. Denote

$$
E_{t}=\left\{X^{(2)}, \cdots, X^{(\ell)}:\left|\frac{\prod_{j=2}^{\ell}\left\|X^{(j)}\right\|_{2}}{n^{\frac{\ell-1}{2}}}-1\right| \leq t\right\}
$$

and

$$
\bar{X}=X \mathbf{1}_{E_{t}}
$$

$$
\begin{aligned}
& \mathcal{T}_{\varepsilon}=\left\{\left\|\Pi_{F} \otimes_{j=1}^{\ell} X^{(j)}\right\|_{2} \leq \varepsilon \sqrt{m}\right\} \\
& \overline{\mathcal{T}}_{\varepsilon}=\left\{\left\|\Pi_{F} \otimes_{j=1}^{\ell} \bar{X}^{(j)}\right\|_{2} \leq \varepsilon \sqrt{m}\right\}
\end{aligned}
$$

Notice that

$$
\mathbb{P}_{X}\left(\mathcal{T}_{\varepsilon}\right)=\mathbb{P}_{X}\left(\mathcal{T}_{\varepsilon} \cap E_{t}\right)+\mathbb{P}_{X}\left(\mathcal{T}_{\varepsilon} \cap E_{t}^{C}\right) \leq \mathbb{P}_{\bar{X}}\left(\overline{\mathcal{T}}_{\varepsilon}\right) \mathbb{P}_{X}\left(E_{t}\right)+\mathbb{P}_{X}\left(E_{t}^{C}\right)
$$

Hence it suffices to estimate $\mathbb{P}_{\bar{X}}\left(\mathcal{T}_{\varepsilon}\right)$. Let $U \in \mathbb{O}\left(n^{\ell}\right)$ be a random special orthogonal matrix in Haar measure. For $F \in \mathbf{G}_{n^{\ell}, m}$ in Haar measure, we can write its orthonormal basis as $f^{(k)}=U e_{k}, 1 \leq k \leq m$ which are the first $m$ columns of $U$. For any $A=\left(a_{i_{1} i_{2} \cdots i_{\ell}}\right)_{i_{1} i_{2} \cdots i_{\ell}} \in \mathbb{R}^{n^{\ell}}$, we denote by

$$
\begin{equation*}
A^{1,2 \cdots \ell}=\left(a_{i_{1} i_{2} \cdots i_{\ell}}\right)_{i_{1}, i_{2} \cdots i_{\ell}} \in \operatorname{Mat}_{n, n^{\ell-1}} \tag{R}
\end{equation*}
$$

an $n \times n^{\ell-1}$ matrix. And we consider a tensor as a flattened vector when there is no confusion. Define

$$
\begin{aligned}
\theta_{X^{(2)}, \cdots, X^{(\ell)}}(U) & =\left(\left(U e_{1}\right)^{1,2 \cdots \ell} \otimes_{j=2}^{\ell} X^{(j)}, \cdots,\left(U e_{m}\right)^{1,2 \cdots \ell} \otimes_{j=2}^{\ell} X^{(j)}\right)^{T} \\
& =\left(f^{(1)^{1,2 \cdots \ell}} \otimes_{j=2}^{\ell} X^{(j)}, \cdots, f^{(m)^{1,2 \cdots \ell}} \otimes_{j=2}^{\ell} X^{(j)}\right)^{T}
\end{aligned}
$$

Then

$$
\begin{aligned}
\left\|\Pi_{F} \otimes_{j=1}^{\ell} X^{(j)}\right\|_{2}^{2} & =\sum_{k=1}^{m}\left|\left\langle f^{(k)}, \otimes_{j=1}^{\ell} X^{(j)}\right\rangle\right|^{2}=\sum_{k=1}^{m}\left|\left\langle f^{(k)^{1,2 \cdots \ell}} \otimes_{j=2}^{\ell} X^{(j)}, X^{(1)}\right\rangle\right|^{2} \\
& =\left\|\theta_{X^{(2)}, \cdots, X^{(\ell)}}(U) X^{(1)}\right\|_{2}^{2}
\end{aligned}
$$

and

$$
\left\|\Pi_{F} \otimes_{j=1}^{\ell} \bar{X}^{(j)}\right\|_{2}^{2}=\left\|\theta_{\bar{X}^{(2)}, \ldots, \bar{X}^{(\ell)}}(U) \bar{X}^{(1)}\right\|_{2}^{2}
$$

Notice that $f^{(k)}$ is the $k$-th column of $U$, then by Lemma 5.5

$$
\mathbb{E}_{U}\left[f_{i_{1} \cdots i_{\ell}}^{(k)} f_{i_{1}^{\prime} \cdots i_{\ell}^{\prime}}^{\left(k^{\prime}\right)}\right]=\frac{1}{n^{\ell}} \delta_{k, i_{1} \cdots, i_{\ell}}^{k^{\prime}, i_{1}^{\prime}, \cdots, i_{\ell}^{\prime}}
$$

In the rest of the proof, we sometimes write $\theta=\theta_{\bar{X}^{(2)}, \ldots, \bar{X}^{(\ell)}}(U)$ if there is no confusion.

Case 1: Suppose $m \leq C_{1} n$, where $0<C_{1}<1$ is a constant that is determined later. Then for any $\phi \in \mathbb{S}^{m-1}$,

$$
\left\|\theta_{\bar{X}^{(2)}, \ldots, \bar{X}^{(\ell)}}(U)^{T} \phi\right\|_{2}^{2}=\sum_{i_{1}=1}^{n}\left(\sum_{k=1}^{m} \phi_{k} \sum_{i_{2} \cdots i_{\ell}} f_{i_{1} \cdots i_{\ell}}^{(k)} \bar{X}_{i_{2}}^{(2)} \cdots \bar{X}_{i_{\ell}}^{(\ell)}\right)^{2}
$$

Fix $X^{(2)}, \cdots, X^{(\ell)}$ and define $\mathcal{A}_{\phi}(U)=\left\|\theta_{\bar{X}^{(2)}, \ldots, \bar{X}^{(\ell)}}(U)^{T} \phi\right\|_{2}$. Then for $U, U^{\prime} \in$ $\mathbb{O}\left(n^{\ell}\right)$,

$$
\begin{aligned}
\left|\mathcal{A}_{\phi}(U)-\mathcal{A}_{\phi}\left(U^{\prime}\right)\right| & =\| \| \sum_{k=1}^{m} \phi_{k}\left(U e_{k}\right)^{1,2 \cdots \ell} \otimes_{j=2}^{\ell} \bar{X}^{(j)}\left\|_{2}-\right\| \sum_{k=1}^{m} \phi_{k}\left(U^{\prime} e_{k}\right)^{1,2 \cdots \ell} \otimes_{j=2}^{\ell} \bar{X}^{(j)} \|_{2} \mid \\
& \leq\left\|\left(\sum_{k=1}^{m} \phi_{k}\left[\left(U-U^{\prime}\right) e_{k}\right]^{1,2 \cdots \ell}\right) \otimes_{j=2}^{\ell} \bar{X}^{(j)}\right\|_{2}
\end{aligned}
$$

Notice that $\sum_{k=1}^{m} \phi_{k}\left[\left(U-U^{\prime}\right) e_{k}\right]^{1,2 \cdots \ell} \in$ Mat $_{n, n^{\ell-1}}$ and $\otimes_{j=2}^{\ell} \bar{X}^{(j)} \in \mathbb{R}^{\ell-1}$, then

$$
\begin{aligned}
\left\|\left(\sum_{k=1}^{m} \phi_{k}\left[\left(U-U^{\prime}\right) e_{k}\right]^{1,2 \cdots \ell}\right) \otimes_{j=2}^{\ell} \bar{X}^{(j)}\right\|_{2} & \leq\left\|\sum_{k=1}^{m} \phi_{k}\left[\left(U-U^{\prime}\right) e_{k}\right]^{1,2 \cdots \ell}\right\|_{H S} \cdot \prod_{j=2}^{\ell}\left\|\bar{X}^{(j)}\right\|_{2} \\
& =\left\|\sum_{k=1}^{m} \phi_{k}\left(U-U^{\prime}\right) e_{k}\right\|_{2} \cdot \prod_{j=2}^{\ell}\left\|\bar{X}^{(j)}\right\|_{2} \\
& \leq(1+t) n^{\frac{\ell-1}{2}}\left\|U-U^{\prime}\right\|_{H S} .
\end{aligned}
$$

Hence $\mathcal{A}_{\phi}$ has Lipschitz constant less than or equal to $(1+t) n^{\frac{\ell-1}{2}}$. And

$$
\mathbb{E}_{U}\left[\mathcal{A}_{\phi}(U)^{2}\right]=\sum_{i_{1}=1}^{n} \sum_{i_{2}, \cdots, i_{\ell}} \frac{1}{n^{\ell}} \bar{X}_{i_{2}}^{(2)^{2}} \cdots \bar{X}_{i_{\ell}}^{(\ell)^{2}}=\frac{\prod_{j=2}^{\ell}\left\|\bar{X}_{i_{j}}^{(j)}\right\|_{2}^{2}}{n^{\ell-1}} .
$$

Let $\mathcal{N}_{1}$ be a $\delta$-net on $\mathbb{R}^{m}, \forall \phi \in \mathbb{S}^{m-1}, \exists \phi^{\prime} \in \mathcal{N}_{1}$, such that

$$
\left\|\phi-\phi^{\prime}\right\| \leq \delta
$$

Recall $U \in \mathbb{O}\left(n^{\ell}\right)$. For every $\phi \in \mathcal{N}_{1}$, by Minkowski's inequality we have

$$
\begin{aligned}
& \left(\mathbb{E}_{\bar{X}, U}\left|\mathcal{A}_{\phi}(U)-1\right|^{q}\right)^{\frac{1}{q}} \\
\leq & \left(\mathbb{E}_{\bar{X}, U}\left|\mathcal{A}_{\phi}(U)-\frac{\prod_{j=2}^{\ell}\left\|\bar{X}^{(j)}\right\|_{2}}{n^{\frac{\ell-1}{2}}}\right|^{q}\right)^{\frac{1}{q}}+\left(\mathbb{E}_{\bar{X}, U}\left|\frac{\prod_{j=2}^{\ell}\left\|\bar{X}^{(j)}\right\|_{2}}{n^{\frac{\ell-1}{2}}}-1\right|^{q}\right)^{\frac{1}{q}} .
\end{aligned}
$$

For the first part, we apply Corollary 2.6 for the 1-Lipschitz function $\frac{\mathcal{A}_{\phi}}{(1+t) n^{\frac{\ell-1}{2}}}$

$$
\left(\mathbb{E}_{\bar{X}, U}\left|\mathcal{A}_{\phi}(U)-\frac{\prod_{j=2}^{\ell}\left\|\bar{X}^{(j)}\right\|_{2}}{n^{\frac{\ell-1}{2}}}\right|^{q}\right)^{\frac{1}{q}} \leq\left(\frac{C q}{n^{\ell}}\right)^{\frac{1}{2}}(1+t) n^{\frac{\ell-1}{2}} \leq C(1+t) \sqrt{\frac{q}{n}} .
$$

For the second part, recall the definition of $\bar{X}$,

$$
\left(\mathbb{E}_{\bar{X}, U}\left|\frac{\prod_{j=2}^{\ell}\left\|\bar{X}^{(j)}\right\|_{2}}{n^{\frac{\ell-1}{2}}}-1\right|^{q}\right)^{\frac{1}{q}}=t .
$$

Choose $q=\frac{(\eta-1)^{2} t^{2}}{C^{2}(1+t)^{2}} n$, where $\eta>1$. Then

$$
\mathbb{E}_{U} \mathbb{E}_{\bar{X}}\left|\mathcal{A}_{\phi}(U)-1\right|^{\frac{\left.(\eta-1)^{2}\right)^{2}}{C^{2}(1+t)^{2}} n}=\mathbb{E}_{\bar{X}, U}\left|\mathcal{A}_{\phi}(U)-1\right|^{\frac{(\eta-1)^{2} t^{2}}{C^{2}(1+t)^{2}} n} \leq(\eta t)^{\frac{(\eta-1)^{2} t^{2}}{C^{2}(1+t)^{2}} n} .
$$

Then by Markov's inequality, for $1<\sigma_{1}<\frac{e}{e-1}$,
$\mathbb{P}_{U}\left(\mathbb{E}_{\bar{X}}\left|\mathcal{A}_{\phi}(U)-1\right|^{\frac{(\eta-1)^{2} t^{2}}{C^{2}(1+t)^{2}} n} \geq\left(\sigma_{1} \eta t\right)^{\frac{(\eta-1)^{2} t^{2}}{C^{2}(1+t)^{2}} n}\right) \leq e^{-\log \left(\sigma_{1}\right) \frac{\left.(\eta-1) t^{2}\right)^{2}}{C^{2}(1+t)^{2}} n} \leq e^{-\log \left(\sigma_{1}\right) \frac{(\eta-1)^{2} t^{2}}{4 C^{2}} n}$.
Denote by $\mathcal{F}$ the event in $\mathbb{O}\left(n^{\ell}\right)$ that

$$
\mathbb{E}_{\bar{X}}\left|\mathcal{A}_{\phi}(U)-1\right|^{\frac{(\eta-1)^{2} t^{2}}{\left.C^{2}(1+t)\right)^{2}}} \leq\left(\sigma_{1} \eta t\right)^{\frac{(\eta-1)^{2} t^{2}}{C^{2}(1+t)^{2}} n}
$$

for every $\phi \in \mathcal{N}_{1}$, then

$$
\mathbb{P}_{U}(\mathcal{F}) \geq 1-\left(\frac{3}{\delta}\right)^{m} e^{-\log \left(\sigma_{1}\right) \frac{(\eta-1)^{2} t^{2}}{4 C^{2}} n}
$$

We now fix a $U$ in the event $\mathcal{F}$. For every $\phi \in \mathcal{N}_{1}$, and for $1<\sigma_{2}<\frac{e}{\sigma_{1}(e-1)}$,

$$
\mathbb{P}_{\bar{X}}\left(\left\{\left|\mathcal{A}_{\phi}(U)-1\right| \geq \sigma_{1} \sigma_{2} \eta t\right\}\right) \leq e^{-\log \left(\sigma_{2}\right) \frac{(\eta-1)^{2} t^{2}}{4 C^{2}} n}
$$

Note that for $0<\delta<1$,

$$
\begin{aligned}
s_{1}\left(\theta^{T}\right)= & \max _{\phi \in \mathbb{S}^{m-1}}\left\|\theta^{T} \phi\right\|_{2} \\
\leq & \left\|\theta^{T} \phi^{\prime}\right\|_{2}+\max _{\phi \in \mathbb{S}^{m-1}}\left\{\theta^{T}\left(\phi-\phi^{\prime}\right)\right\} \\
\leq & \left\|\theta^{T} \phi^{\prime}\right\|_{2}+\delta s_{1}\left(\theta^{T}\right), \\
& s_{1}\left(\theta^{T}\right) \leq \frac{\left\|\theta^{T} \phi^{\prime}\right\|_{2}}{1-\delta} .
\end{aligned}
$$

Hence

$$
\begin{aligned}
\mathbb{P}_{\bar{X}}\left(s_{1}\left(\theta^{T}\right) \geq 1+\sigma_{1} \sigma_{2} \eta t\right) & \leq \mathbb{P}_{\bar{X}}\left(\exists \phi^{\prime} \in \mathcal{N}_{1},\left\|\theta^{T} \phi^{\prime}\right\|_{2} \geq(1-\delta)\left(1+\sigma_{1} \sigma_{2} \eta t\right)\right) \\
& \leq\left(\frac{3}{\delta}\right)^{m} \exp \left\{-\log \left(\sigma_{2}\right) \frac{(\eta-1)^{2}\left(\sigma_{1} \sigma_{2} \eta t-\delta \sigma_{1} \sigma_{2} \eta t-\delta\right)^{2}}{4 C^{2} \sigma_{1}^{2} \sigma_{2}^{2} \eta^{2}} n\right\}
\end{aligned}
$$

On the other hand, $\forall \phi \in \mathbb{S}^{m-1}, \exists \phi^{\prime} \in \mathcal{N}_{1}$, such that

$$
s_{n}=\min _{\phi \in \mathbb{S}^{m}-1}\left\|\theta^{T} \phi\right\|_{2} \geq\left\|\theta^{T} \phi^{\prime}\right\|-\delta s_{1}\left(\theta^{T}\right)
$$

Hence

$$
\begin{aligned}
& \mathbb{P}_{\bar{X}}\left(s_{m}\left(\theta^{T}\right) \leq 1-\sigma_{1} \sigma_{2} \eta t\right) \\
\leq & \mathbb{P}_{\bar{X}}\left(\exists \phi^{\prime} \in \mathcal{N}_{1},\left\|\theta^{T} \phi^{\prime}\right\|_{2} \leq 1-\sigma_{1} \sigma_{2} \eta t+\delta\left(1+\sigma_{1} \sigma_{2} \eta t\right)\right)+\mathbb{P}_{\bar{X}}\left(s_{1}\left(\theta^{T}\right) \geq 1+\sigma_{1} \sigma_{2} \eta t\right) \\
\leq & 2\left(\frac{3}{\delta}\right)^{m} \exp \left\{-\log \left(\sigma_{2}\right) \frac{(\eta-1)^{2}\left(\sigma_{1} \sigma_{2} \eta t-\delta \sigma_{1} \sigma_{2} \eta t-\delta\right)^{2}}{4 C^{2} \sigma_{1}^{2} \sigma_{2}^{2} \eta^{2}} n\right\} .
\end{aligned}
$$

Pick $\delta=\frac{\sigma_{1} \sigma_{2} \eta t}{4}$ so that $\sigma_{1} \sigma_{2} \eta t-\delta \sigma_{1} \sigma_{2} \eta t-\delta \geq \frac{\sigma_{1} \sigma_{2} \eta t}{2}$, then

$$
\mathbb{P}_{\bar{X}}\left(s_{m}\left(\theta^{T}\right) \leq 1-\sigma_{1} \sigma_{2} \eta t\right) \leq 2 \exp \left\{m \log \left(\frac{12}{\sigma_{1} \sigma_{2} \eta t}\right)-\log \left(\sigma_{2}\right) \frac{(\eta-1)^{2} t^{2}}{16 C^{2}} n\right\}
$$

Fix $\sigma_{1}=\sigma_{2}=\eta=\left(\frac{e}{e-1}\right)^{\frac{1}{12}}$ and fix $t=\left(\frac{e-1}{e}\right)^{\frac{1}{2}}$. Recall that $m \leq C_{1} n$, then we take

$$
C_{1}=\frac{\left[\left(\frac{e}{e-1}\right)^{\frac{1}{12}}-1\right]^{2} \frac{e-1}{e} \log \frac{e}{e-1}}{384 C^{2} \log 12\left(\frac{e}{e-1}\right)^{\frac{1}{4}}}
$$

then

$$
\mathbb{P}_{\bar{X}}\left(s_{m}\left(\theta^{T}\right) \leq 1-\left(\frac{e-1}{e}\right)^{\frac{1}{4}}\right) \leq 2 \exp \left\{-\frac{\left[\left(\frac{e}{e-1}\right)^{\frac{1}{12}}-1\right]^{2} \frac{e-1}{e} \log \frac{e}{e-1}}{384 C^{2}} n\right\}
$$

By Lemma 5.6, the event $\mathcal{F}$ in $\mathbb{O}\left(n^{\ell}\right)$ determines a subset $S_{\mathcal{F}} \subset \mathbf{G}_{n^{\ell}, m}$ with measure at least $1-e^{-c n}$, for every $F \in S_{\mathcal{F}}$,

$$
\begin{aligned}
\mathbb{P}_{\bar{X}}\left(\overline{\mathcal{T}}_{\varepsilon}\right) & =\mathbb{P}_{\bar{X}}\left(\left\|\Pi_{F} \otimes_{j=1}^{\ell} \bar{X}^{(j)}\right\|_{2} \leq \varepsilon \sqrt{m}\right) \\
& \leq \mathbb{P}_{\bar{X}}\left(\left\|\Pi_{F} \otimes_{j=1}^{\ell} \bar{X}^{(j)}\right\|_{2} \leq \frac{\varepsilon}{1-\left(\frac{e-1}{e}\right)^{\frac{1}{4}}} s_{m} \sqrt{m}\right)+\mathbb{P}_{\bar{X}}\left(s_{m} \leq 1-\left(\frac{e-1}{e}\right)^{\frac{1}{4}}\right) \\
& \leq\left(C \varepsilon \mathcal{L}_{m}\right)^{m}+2 e^{-c n}
\end{aligned}
$$

and by Proposition 5.2 ,

$$
\mathbb{P}_{X}\left(\mathcal{T}_{\varepsilon}\right) \leq \mathbb{P}_{\bar{X}}\left(\overline{\mathcal{T}}_{\varepsilon}\right) \mathbb{P}_{X}\left(E_{t}\right)+\mathbb{P}_{X}\left(E_{t}^{C}\right) \leq\left(C \varepsilon \mathcal{L}_{m}\right)^{m}+e^{-\frac{c \sqrt{n}}{C_{P}}}
$$

Case 2: Suppose $m \geq C_{2} n$, where $C_{2}>1$ is a constant that is determined later. Then for any $\psi \in \mathbb{S}^{n-1}$,

$$
\left\|\theta_{\bar{X}^{(2)}, \ldots, \bar{X}^{(\ell)}} \psi\right\|_{2}^{2}=\sum_{k=1}^{m}\left|\left\langle f^{(k)^{1,2 \cdots \ell}} \otimes_{j=2}^{\ell} \bar{X}^{(j)}, \psi\right\rangle\right|^{2}=\sum_{k=1}^{m}\left(\sum_{i_{1}=1} \psi_{i_{1}} \sum_{i_{2}, \cdots, i_{\ell}} f_{i_{1} \cdots i_{\ell}}^{(k)} \bar{X}_{i_{2}}^{(2)} \cdots \bar{X}_{i_{\ell}}^{(\ell)}\right)^{2},
$$

and

$$
\mathbb{E}_{\bar{X}, U}\left\|\theta_{\bar{X}^{(2)}, \ldots, \bar{X}^{(\ell)}}(U) \psi\right\|_{2}^{2}=\sum_{k=1}^{m} \sum_{i_{1}=1}^{n} \psi_{i_{1}}^{2} \sum_{i_{2}, \cdots, i_{\ell}} \frac{1}{n^{\ell}}=\frac{m}{n} .
$$

Define $\mathcal{B}_{\psi}(U)=\left\|\theta_{\bar{X}^{(2)}, \ldots, \bar{X}^{(\ell)}}(U) \psi\right\|_{2}$, then for $U, U^{\prime} \in \mathbb{O}\left(n^{\ell}\right)$,

$$
\begin{aligned}
\left|\mathcal{B}_{\psi}(U)-\mathcal{B}_{\psi}\left(U^{\prime}\right)\right| & =\left|\|\theta(U) \psi\|_{2}-\left\|\theta\left(U^{\prime}\right) \psi\right\|_{2}\right| \\
& \leq\left\|\left(\theta(U)-\theta\left(U^{\prime}\right)\right) \psi\right\|_{2} \\
& =\left\|\theta(U)-\theta\left(U^{\prime}\right)\right\|_{2}\|\psi\|_{2} \\
& =\sqrt{\sum_{k=1}^{m}\left\|\left[\left(U-U^{\prime}\right) e_{k}\right]^{1,2 \ldots \ell} \otimes_{j=2}^{\ell} \bar{X}^{(j)}\right\|_{2}} \\
& \leq\left\|U-U^{\prime}\right\|_{2}\left\|\otimes_{j=2}^{\ell} \bar{X}^{(j)}\right\|_{2} \\
& \leq(1+t) n^{\frac{\ell-1}{2}}\left\|U-U^{\prime}\right\|_{H S}
\end{aligned}
$$

Hence $\mathcal{B}$ has Lipschitz constant less than or equal to $(1+t) n^{\frac{\ell-1}{2}}$. And

$$
\mathbb{E}_{U}\left[\mathcal{B}_{\psi}(U)^{2}\right]=\sum_{k=1}^{m} \sum_{i_{2}, \cdots, i_{\ell}} \frac{1}{n^{\ell}} \bar{X}_{i_{2}}^{(2)^{2}} \cdots \bar{X}_{i_{\ell}}^{(\ell)^{2}}=\frac{m \prod_{j=2}^{\ell}\left\|\bar{X}^{(j)}\right\|_{2}^{2}}{n^{\ell}}
$$

Let $\mathcal{N}_{2}$ be a $\delta$-net on $\mathbb{R}^{n}, \forall \psi \in \mathbb{S}^{n-1}, \exists \psi^{\prime} \in \mathcal{N}_{2}$, such that

$$
\left\|\psi-\psi^{\prime}\right\| \leq \delta
$$

Recall $U \in \mathbb{O}\left(n^{\ell}\right)$. For every $\psi \in \mathcal{N}_{2}$, by Minkowski's inequality we have

$$
\begin{aligned}
& \left(\mathbb{E}_{\bar{X}, U}\left|\mathcal{B}_{\psi}(U)-\sqrt{\frac{m}{n}}\right|^{q}\right)^{\frac{1}{q}} \\
\leq & \left(\mathbb{E}_{\bar{X}, U}\left|\mathcal{B}_{\psi}(U)-\frac{\sqrt{m} \prod_{j=2}^{\ell}\left\|\bar{X}^{(j)}\right\|_{2}}{n^{\frac{\ell}{2}}}\right|^{q}\right)^{\frac{1}{q}}+\left(\mathbb{E}_{\bar{X}, U}\left|\frac{\sqrt{m} \prod_{j=2}^{\ell}\left\|\bar{X}^{(j)}\right\|_{2}}{n^{\frac{\ell}{2}}}-\sqrt{\frac{m}{n}}\right|^{q}\right)^{\frac{1}{q}} .
\end{aligned}
$$

For the first part, we apply Corollary 2.6 for the 1-Lipschitz function $\frac{\mathcal{B}_{\psi}}{(1+t) n^{\frac{\ell-1}{2}}}$

$$
\left(\mathbb{E}_{\bar{X}, U}\left|\mathcal{B}_{\psi}(U)-\frac{\sqrt{m} \prod_{j=2}^{\ell}\left\|\bar{X}^{(j)}\right\|_{2}}{n^{\frac{\ell}{2}}}\right|^{q}\right)^{\frac{1}{q}} \leq\left(\frac{C q}{n^{\ell}}\right)^{\frac{1}{2}}(1+t) n^{\frac{\ell-1}{2}} \leq C(1+t) \sqrt{\frac{q}{n}}
$$

For the second part, recall the definition of $\bar{X}$,

$$
\left(\mathbb{E}_{\bar{X}, U}\left|\frac{\sqrt{m} \prod_{j=2}^{\ell}\left\|\bar{X}^{(j)}\right\|_{2}}{n^{\frac{\ell}{2}}}-\sqrt{\frac{m}{n}}\right|^{q}\right)^{\frac{1}{q}}=t \sqrt{\frac{m}{n}}
$$

Choose $q=\frac{(\eta-1)^{2} t^{2}}{C^{2}(1+t)^{2}} m$, where $\eta>1$. Then
$\mathbb{E}_{U} \mathbb{E}_{\bar{X}}\left|\mathcal{B}_{\psi}(U)-\sqrt{\frac{m}{n}}\right|^{\frac{(\eta-1)^{2} t^{2}}{C^{2}(1+t)^{2}} m}=\mathbb{E}_{\bar{X}, U} \left\lvert\, \mathcal{B}_{\psi}(U)-\sqrt{\left.\frac{m}{n}\right|^{\frac{(\eta-1)^{2} t^{2}}{C^{2}(1+t)^{2}} m}} \leq\left(\eta t \sqrt{\frac{m}{n}}\right)^{\frac{(\eta-1)^{2} t^{2}}{C^{2}(1+t)^{2}} m}\right.$.
Then by Markov's inequality, for $1<\sigma_{1}<\frac{e}{e-1}$,
$\mathbb{P}_{U}\left(\mathbb{E}_{\bar{X}}\left|\mathcal{B}_{\psi}(U)-\sqrt{\frac{m}{n}}\right|^{\frac{(\eta-1)^{2} t^{2}}{C^{2}(1+t)^{2}} m} \geq\left(\sigma_{1} \eta t \sqrt{\frac{m}{n}}\right)^{\frac{(\eta-1)^{2} t^{2}}{C^{2}(1+t)^{2}} m}\right) \leq e^{-\log \left(\sigma_{1}\right) \frac{(\eta-1)^{2} t^{2}}{C^{2}(1+t)^{2}} m} \leq e^{-\log \left(\sigma_{1}\right) \frac{(\eta-1)^{2} t^{2}}{4 C^{2}} m}$.
Denote by $\mathcal{F}$ the event in $\mathbb{O}\left(n^{\ell}\right)$ that

$$
\mathbb{E}_{\bar{X}}\left|\mathcal{B}_{\psi}(U)-\sqrt{\frac{m}{n}}\right|^{\frac{(\eta-1)^{2} t^{2}}{C^{2}(1+t)^{2}} m} \leq\left(\sigma_{1} \eta t \sqrt{\frac{m}{n}}\right)^{\frac{(\eta-1)^{2} t^{2}}{C^{2}(1+t)^{2}} m}
$$

for every $\psi \in \mathcal{N}_{2}$, then

$$
\mathbb{P}_{U}(\mathcal{F}) \geq 1-\left(\frac{3}{\delta}\right)^{n} e^{-\log \left(\sigma_{1}\right) \frac{(\eta-1)^{2} t^{2}}{4 C^{2}} m}
$$

We now fix a $U$ in the event $\mathcal{F}$. For every $\psi \in \mathcal{N}_{2}$, and for $1<\sigma_{2}<\frac{e}{\sigma_{1}(e-1)}$,

$$
\mathbb{P}_{\bar{X}}\left(\left\{\left|\mathcal{B}_{\psi}(U)-\sqrt{\frac{m}{n}}\right| \geq \sigma_{1} \sigma_{2} \eta t \sqrt{\frac{m}{n}}\right\}\right) \leq e^{-\log \left(\sigma_{2}\right) \frac{(\eta-1)^{2} t^{2}}{4 C^{2}} m}
$$

Note that for $0<\delta<1$,

$$
\begin{aligned}
s_{1}(\theta) & =\max _{\psi \in \mathbb{S}^{n}-1}\|\theta \psi\|_{2} \\
& \leq\left\|\theta \psi^{\prime}\right\|_{2}+\max _{\psi \in \mathbb{S}^{n}-1}\left\{\theta\left(\psi-\psi^{\prime}\right)\right\} \\
& \leq\left\|\theta \psi^{\prime}\right\|_{2}+\delta s_{1}(\theta)
\end{aligned}
$$

$$
s_{1}(\theta) \leq \frac{\left\|\theta \psi^{\prime}\right\|_{2}}{1-\delta}
$$

Hence

$$
\begin{aligned}
\mathbb{P}_{\bar{X}}\left(s_{1}(\theta) \geq\left(1+\sigma_{1} \sigma_{2} \eta t\right) \sqrt{\frac{m}{n}}\right) & \leq \mathbb{P}_{\bar{X}}\left(\exists \psi^{\prime} \in \mathcal{N}_{2},\left\|\theta \psi^{\prime}\right\|_{2} \geq(1-\delta)\left(1+\sigma_{1} \sigma_{2} \eta t\right) \sqrt{\frac{m}{n}}\right) \\
& \leq\left(\frac{3}{\delta}\right)^{n} \exp \left\{-\log \left(\sigma_{2}\right) \frac{(\eta-1)^{2}\left(\sigma_{1} \sigma_{2} \eta t-\delta \sigma_{1} \sigma_{2} \eta t-\delta\right)^{2}}{4 C^{2} \sigma_{1}^{2} \sigma_{2}^{2} \eta^{2}} m\right\}
\end{aligned}
$$

On the other hand, $\forall \psi \in \mathbb{S}^{n-1}, \exists \psi^{\prime} \in \mathcal{N}_{2}$, such that

$$
s_{n}=\min _{\psi \in \mathbb{S}^{n-1}}\|\theta \psi\|_{2} \geq\left\|\theta \psi^{\prime}\right\|-\delta s_{1}(\theta)
$$

Hence

$$
\begin{aligned}
& \mathbb{P}_{\bar{X}}\left(s_{n}(\theta) \leq\left(1-\sigma_{1} \sigma_{2} \eta t\right) \sqrt{\frac{m}{n}}\right) \\
\leq & \mathbb{P}_{\bar{X}}\left(\exists \psi^{\prime} \in \mathcal{N}_{2},\left\|\theta \psi^{\prime}\right\|_{2} \leq\left[1-\sigma_{1} \sigma_{2} \eta t+\delta\left(1+\sigma_{1} \sigma_{2} \eta t\right)\right] \sqrt{\frac{m}{n}}\right)+\mathbb{P}_{\bar{X}}\left(s_{1}(\theta) \geq\left(1+\sigma_{1} \sigma_{2} \eta t\right) \sqrt{\frac{m}{n}}\right) \\
\leq & 2\left(\frac{3}{\delta}\right)^{n} \exp \left\{-\log \left(\sigma_{2}\right) \frac{(\eta-1)^{2}\left(\sigma_{1} \sigma_{2} \eta t-\delta \sigma_{1} \sigma_{2} \eta t-\delta\right)^{2}}{4 C^{2} \sigma_{1}^{2} \sigma_{2}^{2} \eta^{2}} m\right\}
\end{aligned}
$$

Pick $\delta=\frac{\sigma_{1} \sigma_{2} \eta t}{4}$ so that $\sigma_{1} \sigma_{2} \eta t-\delta \sigma_{1} \sigma_{2} \eta t-\delta \geq \frac{\sigma_{1} \sigma_{2} \eta t}{4}$, then
$\mathbb{P}_{\bar{X}}\left(s_{n}(\theta) \leq\left(1-\sigma_{1} \sigma_{2} \eta t\right) \sqrt{\frac{m}{n}}\right) \leq 2 \exp \left\{n \log \left(\frac{12}{\sigma_{1} \sigma_{2} \eta t}\right)-\log \left(\sigma_{2}\right) \frac{(\eta-1)^{2} t^{2}}{16 C^{2}} m\right\}$.
Fix $\sigma_{1}=\sigma_{2}=\eta=\left(\frac{e}{e-1}\right)^{\frac{1}{12}}$ and fix $t=\left(\frac{e-1}{e}\right)^{\frac{1}{2}}$. Recall that $m>C_{2} n$, then we take

$$
C_{2}=\frac{384 C^{2} \log 12\left(\frac{e}{e-1}\right)^{\frac{1}{4}}}{\left[\left(\frac{e}{e-1}\right)^{\frac{1}{12}}-1\right]^{2} \frac{e-1}{e} \log \frac{e}{e-1}}
$$

then

$$
\mathbb{P}_{\bar{X}}\left(s_{n}(\theta) \leq\left[1-\left(\frac{e-1}{e}\right)^{\frac{1}{4}}\right] \sqrt{\frac{m}{n}}\right) \leq 2 \exp \left\{-\log 12\left(\frac{e}{e-1}\right)^{\frac{1}{4}} n\right\} .
$$

By Lemma [5.6, the event $\mathcal{F}$ in $\mathbb{O}\left(n^{\ell}\right)$ determines a subset $S_{\mathcal{F}} \subset \mathbf{G}_{n^{\ell}, m}$ with measure at least $1-e^{-c n}$, for every $F \in S_{\mathcal{F}}$,

$$
\begin{aligned}
\mathbb{P}_{\bar{X}}\left(\overline{\mathcal{T}}_{\varepsilon}\right) & =\mathbb{P}_{\bar{X}}\left(\left\|\Pi_{F} \otimes_{j=1}^{\ell} \bar{X}^{(j)}\right\|_{2} \leq \varepsilon \sqrt{m}\right) \\
& \leq \mathbb{P}_{\bar{X}}\left(\left\|\Pi_{F} \otimes_{j=1}^{\ell} \bar{X}^{(j)}\right\|_{2} \leq \frac{\varepsilon}{1-\left(\frac{e-1}{e}\right)^{\frac{1}{4}}} s_{n} \sqrt{n}\right)+\mathbb{P}_{\bar{X}}\left(s_{n} \leq\left[1-\left(\frac{e-1}{e}\right)^{\frac{1}{4}}\right] \sqrt{\frac{m}{n}}\right) \\
& \leq\left(C \varepsilon \mathcal{L}_{n}\right)^{n}+2 e^{-c n},
\end{aligned}
$$

and by Proposition 5.2.

$$
\mathbb{P}_{X}\left(\mathcal{T}_{\varepsilon}\right) \leq \mathbb{P}_{\bar{X}}\left(\overline{\mathcal{T}}_{\varepsilon}\right) \mathbb{P}_{X}\left(E_{t}\right)+\mathbb{P}_{X}\left(E_{t}^{C}\right) \leq\left(C \varepsilon \mathcal{L}_{n}\right)^{n}+e^{-\frac{c \sqrt{n}}{C_{P}}}
$$

Case 3: Suppose $C_{1} n \leq m \leq C_{2} n$. We partition the $\theta$ into $N=\frac{m}{C_{1} n} \leq \frac{C_{2}}{C_{1}}$ matrices $\theta_{1}, \cdots, \theta_{N}$ of dimension $C_{1} n \times n$ as follows

$$
\theta=\left(\theta_{1}^{T}, \cdots, \theta_{N}^{T}\right)^{T}
$$

Then

$$
\left\|\Pi_{F} \otimes_{j=1}^{\ell} \bar{X}^{(j)}\right\|_{2}^{2}=\left\|\theta X^{(1)}\right\|_{2}^{2}=\sum_{p=1}^{N}\left\|\theta_{p} X^{(1)}\right\|_{2}^{2}
$$

By a union bound, there exists a subset $S_{\mathcal{F}} \subset \mathbf{G}_{n^{\ell}, m}$, which is an intersection of subsets where $\mathbb{P}\left(\left\|\theta_{p} \bar{X}^{(1)}\right\|_{2} \leq \varepsilon \sqrt{C_{1} n}\right) \leq\left(C \varepsilon \mathcal{L}_{C_{1} n}\right)^{C_{1} n}+2 e^{-c C_{1} n}$ for $1 \leq p \leq N$, with measure at least $1-\frac{C_{2}}{C_{1}} e^{-c C_{1} n}$, such that for every $F \in S_{\mathcal{F}}$,

$$
\begin{aligned}
\mathbb{P}_{\bar{X}}\left(\overline{\mathcal{T}}_{\varepsilon}\right) & =\mathbb{P}\left(\left\|\Pi_{F} \otimes_{j=1}^{\ell} \overline{X^{(j)}} \leq \varepsilon \sqrt{m}\right\|\right) \\
& \leq \mathbb{P}\left(\sum_{p=1}^{N}\left\|\theta_{p} \bar{X}^{(1)}\right\|_{2}^{2} \leq \varepsilon^{2} m\right) \\
& \leq \sum_{p=1}^{N} \mathbb{P}\left(\left\|\theta_{p} \bar{X}^{(1)}\right\|_{2} \leq \varepsilon \sqrt{C_{1} n}\right) \\
& \leq \frac{C_{2}}{C_{1}}\left(\left(C \varepsilon \mathcal{L}_{C_{1} n}\right)^{C_{1} n}+2 e^{-c C_{1} n}\right)
\end{aligned}
$$

and by Proposition 5.2.

$$
\mathbb{P}_{X}\left(\mathcal{T}_{\varepsilon}\right) \leq \mathbb{P}_{\bar{X}}\left(\overline{\mathcal{T}}_{\varepsilon}\right) \mathbb{P}_{X}\left(E_{t}\right)+\mathbb{P}_{X}\left(E_{t}^{C}\right) \leq \frac{C_{2}}{C_{1}}\left(C \varepsilon \mathcal{L}_{C_{1} n}\right)^{C_{1} n}+e^{-\frac{c \sqrt{n}}{C_{P}}}
$$

That concludes the proof of Theorem 1.5 .

## 6. Proof of Theorem 1.1 and 1.2

Now we first prove Theorem 1.1 .
Proof. Let $X^{(j)} \in \mathbb{R}^{n_{j}}, 1 \leq j \leq \ell$ be independent random vectors with independent coordinates whose densities have uniform norms bounded by $M>0$ and let $z_{1}, \cdots, z_{\ell} \in \mathbb{R}^{n}$. Suppose $F$ is a subspace in $\mathbb{R}^{n_{1} \otimes \cdots \otimes n_{\ell}}$ with dimension $m$. Note that $2 \sqrt{3} \cdot \mathbf{1}_{\left[-\frac{1}{2}, \frac{1}{2}\right]^{n_{j}}}$ is isotropic log-concave. And $K_{r}=\left\{X \in \mathbb{R}^{n_{1} \otimes \cdots \otimes n_{\ell}}:\left\|\mathbf{P}_{F} X\right\|_{2} \leq r\right\}$ is a convex set in $\mathbb{R}^{n_{1} \otimes \cdots \otimes n_{\ell}}$. We apply Theorem 1.4 for independent uniform distributions $2 \sqrt{3} \cdot \mathbf{1}_{\left[-\frac{1}{2}, \frac{1}{2}\right]^{n_{j}}}$, and we have
$\mathbb{P}\left(\left\|\Pi_{F} \otimes_{j=1}^{\ell}\left(2 \sqrt{3} \cdot \mathbf{1}_{\left[-\frac{1}{2}, \frac{1}{2}\right]^{n_{j}}}\right)\right\|_{2} \leq \varepsilon \sqrt{m}\right) \leq \min \left\{m, C^{\prime \ell} \log \frac{1}{\varepsilon}\right\} \frac{\varepsilon}{(\ell-1)!}\left(C \log \frac{1}{\varepsilon}\right)^{\ell-1}$.
Notice that by Corollary 3.4, for every $M>0$,

$$
\otimes_{j=1}^{\ell} M\left(X^{(j)}-z_{j}\right) \prec \otimes_{j=1}^{\ell} \mathbf{1}_{\left[-\frac{1}{2}, \frac{1}{2}\right]^{n_{j}}} .
$$

Hence we have for $0<\varepsilon<e^{-c \ell}$,

$$
\begin{aligned}
& \mathbb{P}\left(\left\|\Pi_{F} \otimes_{j=1}^{\ell}\left(X^{(j)}-z_{j}\right)\right\|_{2} \leq \frac{1}{(2 \sqrt{3} M)^{\ell}} \varepsilon \sqrt{m}\right) \\
\leq & \mathbb{P}\left(\left\|\Pi_{F} \otimes_{j=1}^{\ell} \cdot \mathbf{1}_{\left[-\frac{1}{2}, \frac{1}{2}\right]^{n_{j}}}\right\|_{2} \leq \frac{1}{(2 \sqrt{3})^{\ell}} \varepsilon \sqrt{m}\right) \\
= & \mathbb{P}\left(\left\|\Pi_{F} \otimes_{j=1}^{\ell}\left(2 \sqrt{3} \cdot \mathbf{1}_{\left[-\frac{1}{2}, \frac{1}{2}\right]^{n_{j}}}\right)\right\|_{2} \leq \varepsilon \sqrt{m}\right) \\
\leq & \min \left\{m, C^{\prime \ell} \log \frac{1}{\varepsilon}\right\} \frac{\varepsilon}{(\ell-1)!}\left(C \log \frac{1}{\varepsilon}\right)^{\ell-1} .
\end{aligned}
$$

Then we prove Theorem 1.2 ,
Proof. Notice that $2 \sqrt{3} \cdot \mathbf{1}_{\left[-\frac{1}{2}, \frac{1}{2}\right]^{n}}$ is isotropic log-concave, whose isotropic constant and Poincaré constant are bounded from above by universal constants. And notice that

$$
\left\|\otimes_{j=1}^{\ell} 2 \sqrt{3} \cdot \mathbf{1}_{\left[-\frac{1}{2}, \frac{1}{2}\right]^{n}}\right\|_{2}=\prod_{j=1}^{\ell}\left\|2 \sqrt{3} \cdot \mathbf{1}_{\left[-\frac{1}{2}, \frac{1}{2}\right]^{n}}\right\|_{2}
$$

Also notice that $2 \sqrt{3} \cdot \mathbf{1}_{\left[-\frac{1}{2}, \frac{1}{2}\right]^{n}}$ is sub-gaussian by Hoeffding inequality (see THEorem 2.2.2 in [59]). We apply Theorem 1.5 and Remark 5.3 for independent uniform distributions $2 \sqrt{3} \cdot \mathbf{1}_{\left[-\frac{1}{2}, \frac{1}{2}\right]^{n}}$. Then there exists a subset $\mathcal{S}_{\mathcal{F}}$ in $G_{n^{\ell}, m}$ with Haar measure at least $1-e^{-c \max \{m, n\}}$, for every subspace $F \in \mathcal{S}_{\mathcal{F}}$, we have for $0<\varepsilon<1$,

$$
\mathbb{P}\left(\left\|\Pi_{F} \otimes_{j=1}^{\ell}\left(2 \sqrt{3} \cdot \mathbf{1}_{\left[-\frac{1}{2}, \frac{1}{2}\right]^{n}}\right)\right\|_{2} \leq \varepsilon \sqrt{m}\right) \leq(C \varepsilon)^{C^{\prime} \min \{m, n\}}+e^{-C^{\prime \prime} n}
$$

Let $X^{(1)}, \cdots, X^{(\ell)} \in \mathbb{R}^{n}$ be independent random vectors with independent coordinates whose densities have uniform norms bounded by $M$. Let $z_{1}, \cdots, z_{\ell} \in \mathbb{R}^{n}$ be arbitrary vectors and let $m \leq n^{\ell}$. Notice again by Corollary 3.4, for every $M>0$,

$$
\otimes_{j=1}^{\ell} M\left(X^{(j)}-z_{j}\right) \prec \otimes_{j=1}^{\ell} \mathbf{1}_{\left[-\frac{1}{2}, \frac{1}{2}\right]^{n_{j}}}
$$

Hence we have

$$
\begin{aligned}
& \mathbb{P}\left(\left\|\Pi_{F} \otimes_{j=1}^{\ell}\left(X^{(j)}-z_{j}\right)\right\|_{2} \leq \frac{1}{(2 \sqrt{3} M)^{\ell}} \varepsilon \sqrt{m}\right) \\
\leq & \mathbb{P}\left(\left\|\Pi_{F} \otimes_{j=1}^{\ell}\left(2 \sqrt{3} \cdot \mathbf{1}_{\left[-\frac{1}{2}, \frac{1}{2}\right]^{n}}\right)\right\|_{2} \leq \varepsilon \sqrt{m}\right) \\
\leq & (C \varepsilon)^{C^{\prime} \min \{m, n\}}+e^{-C^{\prime \prime} n} .
\end{aligned}
$$

## 7. Application

Suppose we want to retrieve the component vectors with Gaussian perturbation. We first introduce the following lemma. This has been essentially proved in 27. For completeness we provide a proof.

Lemma 7.1. Let $X$ be an $r \times d$ matrix with $\operatorname{rank}(X)=r$. Let $v_{1}^{T}, \cdots, v_{r}^{T} \in \mathbb{R}^{d}$ be the row vectors of $X$ and let

$$
V_{i}=\operatorname{span}\left\{v_{j}: 1 \leq j \leq r, j \neq i\right\}, \quad 1 \leq i \leq r
$$

Then

$$
\left\|X^{-1}\right\|_{H S}^{2}=\sum_{\tau \subset[r],|\tau|=r-1} \frac{\operatorname{det}\left(X_{\tau}\left(X_{\tau}^{T}\right)\right)}{\operatorname{det}\left(X X^{T}\right)}=\sum_{i=1}^{d} \frac{1}{\left\|\Pi_{V_{i}} v_{i}\right\|_{2}^{2}}
$$

where $X^{-1}$ is the Moore-Penrose inverse.
Proof. Since $\operatorname{rank}(X)=r, X X^{T}$ is invertible,

$$
X^{-1}=X^{T}\left(X X^{T}\right)^{-1}
$$

Denote by $d_{i}$ the $i$-th entry on the diagonal of $X X^{T}$ and denote by $X_{i}$ the submatrix obtained by removing the $i$-th row of $X$ for $1 \leq i \leq r$. Then

$$
d_{i}=\frac{\operatorname{det}\left(X_{i} X_{i}^{T}\right)}{\operatorname{det}\left(X X^{T}\right)}
$$

Hence

$$
\left\|X^{-1}\right\|_{2}^{2}=\operatorname{tr}\left(\left(X X^{T}\right)^{-1} X X^{T}\left(X X^{T}\right)^{-1}\right)=\operatorname{tr}\left(\left(X X^{T}\right)^{-1}\right)=\sum_{i=1}^{r} \frac{\operatorname{det}\left(X_{i} X_{i}^{T}\right)}{\operatorname{det}\left(X X^{T}\right)}
$$

Denote by $[0,1]^{r}$ the $r$-dimensional unit cube and denote by

$$
X[0,1]^{r}=\left\{\sum_{i=1}^{r} a_{i} v_{i}^{T}: 0 \leq a_{1}, \cdots, a_{r} \leq 1\right\}
$$

the parallelepiped generated by $X$. For $1 \leq i \leq r$ and $1 \leq j \leq r$, define

$$
W_{i}^{(k)}=\operatorname{span}\left\{v_{j}, 1 \leq j \leq k, j \neq i\right\}
$$

Then for $1 \leq i \leq r$ and $1 \leq j \leq r$,

$$
\begin{aligned}
\operatorname{vol}\left(X[0,1]^{r}\right) & =\operatorname{det}\left(X X^{T}\right)^{\frac{1}{2}} \\
& =\left\|v_{1}\right\|_{2}\left\|\Pi_{W_{i}^{(1)} \perp v_{2}}\right\|_{2} \cdots\left\|\Pi_{W_{i}^{(i-2)} \perp v_{i-1}}\right\|_{2}\left\|\Pi_{W_{i}^{(i)}} \perp v_{i+1}\right\|_{2} \cdots\left\|\Pi_{W_{i}^{(r-1)}} \perp v_{r}\right\|_{2}\left\|\Pi_{W_{i}^{(r)}} \perp v_{i}\right\|_{2}
\end{aligned}
$$

and

$$
\begin{aligned}
\operatorname{vol}\left(X_{i}[0,1]^{r}\right) & =\operatorname{det}\left(X_{i} X_{i}^{T}\right)^{\frac{1}{2}} \\
& =\left\|v_{1}\right\|_{2}\left\|\Pi_{W_{i}^{(1)}}^{\perp} v_{2}\right\|_{2} \cdots\left\|\Pi_{W_{i}^{(i-2)} \perp v_{i-1}}\right\|_{2}\left\|\Pi_{W_{i}^{(i)} \perp v_{i+1}}\right\|_{2} \ldots\left\|\Pi_{W_{i}^{(r-1)} \perp v_{r}}\right\|_{2} .
\end{aligned}
$$

Note that $W_{i}^{(r)}=V_{i}$, then

$$
\sum_{i=1}^{r} \frac{\operatorname{det}\left(X_{i} X_{i}^{T}\right)}{\operatorname{det}\left(X X^{T}\right)}=\sum_{i=1}^{r} \frac{1}{\left\|\Pi_{W_{i}^{(r)} \perp v_{i}}\right\|_{2}^{2}}=\sum_{i=1}^{r} \frac{1}{\left\|\Pi_{V_{i} \perp} v_{i}\right\|_{2}^{2}}
$$

Theorem 7.2. Consider random vectors

$$
\widetilde{X}_{i}^{(j)}=X_{i}^{(j)}+G_{i}^{(j)} \in \mathbb{R}^{n}, 1 \leq i \leq r, 1 \leq j \leq l
$$

where

$$
\left\|X_{i}^{(j)}\right\|_{2} \leq C, \quad G_{i}^{(j)} \sim N\left(0, \frac{\rho^{2}}{n} \mathbb{I}_{n}\right)
$$

Let $r \leq \frac{1}{2} n^{\ell}$. Define the $n^{\ell} \times r$ matrix $A$ where the $i-$ th column of $A$ is the flattened vector of $\otimes_{j=1}^{\ell} \widetilde{X}_{i}^{(j)}$ for $1 \leq i \leq r$. Then for $0<\epsilon<e^{-C \ell}$,

$$
\mathbb{P}\left(s_{\min }(A) \leq \sqrt{1-\frac{r}{n^{\ell}}}(c \rho)^{\ell} \varepsilon\right) \leq \frac{\varepsilon r}{(\ell-1)!}\left(C^{\prime} \log \frac{1}{\varepsilon}\right)^{\ell}
$$

Remark 7.3. The result is true if we replace the Gaussian random vectors $G_{i}^{(j)}$ 's with any random vectors $Y_{i}^{(j)}$ 's with independent coordinates whose densities are bounded by $\frac{C \sqrt{n}}{\rho}$. Moreover, a similar statement is true if we replace $G_{i}^{(j)}$ 's with centered log-concave random vectors $W_{i}^{(j)}$ 's with covariance matrix $\frac{\rho^{2}}{n} \mathbb{I}_{n}$. We omit the details.

Proof. Let $v_{i}=\otimes_{j=1}^{\ell} \widetilde{X}_{i}^{(j)}$ for $1 \leq i \leq r$ be simple random tensors. If we view $v_{i}$ 's as flattened vectors, then $A^{T}$ is $r \times n^{\ell}$ matrix and $v_{i}$ has independent coordinates with densities bounded by $\frac{\sqrt{n}}{\sqrt{2 \pi} \rho}$. Let

$$
V_{i}=\operatorname{span}\left\{v_{j}: 1 \leq j \leq r, j \neq i\right\}, \quad 1 \leq i \leq r
$$

Then $v_{i}$ is independent of $V_{i}$. Hence

$$
\left\|\left(A^{T}\right)^{-1}\right\|_{H S}^{2}=\sum_{i=1}^{r} \frac{1}{s_{i}\left(A^{T}\right)^{2}}=\sum_{i=1}^{r} \frac{1}{\left\|\Pi_{V_{i}} v_{i}\right\|_{2}^{2}}
$$

where $\operatorname{dim}\left(V_{i}^{\perp}\right)=n^{\ell}-\lambda n^{\ell}+1$. Let $0 \leq q \leq 1$, then

$$
\begin{aligned}
\mathbb{E} \frac{1}{s_{\min }^{q}(A)} & \leq \mathbb{E}\left[\sum_{i=1}^{r} \frac{1}{\left\|\Pi_{V_{i}} v_{i}\right\|_{2}^{2}}\right]^{\frac{q}{2}}=\mathbb{E}\left[\sum_{i=1}^{r}\left(\frac{1}{\left\|\Pi_{V_{i} \perp} v_{i}\right\|_{2}^{q}}\right)^{\frac{2}{q}}\right]^{\frac{q}{2}} \\
& \leq \mathbb{E} \sum_{i=1}^{r} \frac{1}{\left\|\Pi_{V_{i} \perp v_{i}}\right\|_{2}^{q}}=\sum_{i=1}^{r} \mathbb{E} \frac{1}{\left\|\Pi_{V_{i} v_{i}}\right\|_{2}^{q}} \\
& =\sum_{i=1}^{r} \mathbb{E}_{v_{j}, j \neq i} \mathbb{E}_{v_{i}}\left[\left.\frac{1}{\left\|\Pi_{V_{i}} v_{i}\right\|_{2}^{q}} \right\rvert\, v_{j}, j \neq i\right]
\end{aligned}
$$

where the second line follows from the fact that $\mathbb{B}_{p}^{r} \supset \mathbb{B}_{1}^{r}$ for $p>1$. For $0<\epsilon<$ $e^{-C \ell}$, by Theorem 1.4

$$
\begin{aligned}
& \mathbb{P}\left(\left\|\Pi_{V_{i}} v_{i}\right\|_{2} \leq\left(\frac{\sqrt{2 \pi} \rho}{2 \sqrt{3} \sqrt{n}}\right)^{\ell} \varepsilon \sqrt{n^{\ell}-\lambda n^{\ell}+1}\right) \\
= & \mathbb{P}\left(\left\|\Pi_{V_{i}} v_{i}\right\|_{2} \leq \sqrt{1-\frac{r}{n^{\ell}}}(c \rho)^{\ell} \varepsilon\right) \\
\leq & \frac{\varepsilon}{(\ell-1)!}\left(C^{\prime} \log \frac{1}{\varepsilon}\right)^{\ell} .
\end{aligned}
$$

or for $0<\varepsilon<\frac{e^{-C \ell}}{(c \rho)^{\ell} \sqrt{1-\frac{r}{n^{\ell}}}}$,

$$
\mathbb{P}\left(\left\|\Pi_{V_{i} \perp} v_{i}\right\|_{2} \leq \varepsilon\right) \leq \frac{\varepsilon}{(\ell-1)!(c \rho)^{\ell} \sqrt{1-\frac{r}{n^{\ell}}}}\left(C^{\prime \prime} \log \frac{1}{\varepsilon}\right)^{\ell}
$$

By Lemma C. 1 .

$$
\mathbb{E}_{v_{j}, j \neq i} \mathbb{E}_{v_{i}}\left[\left.\frac{1}{\left\|\Pi_{V_{i}^{\perp}} v_{i}\right\|_{2}^{q}} \right\rvert\, v_{j}, j \neq i\right] \leq\left(\frac{\ell^{\ell}}{(\ell-1)!(c \rho)^{\ell} \sqrt{1-\frac{r}{n^{\ell}}}}\right)^{q} \frac{1}{(1-q)^{q(\ell-1)+1}}
$$

And by Corollary C.2, for $0<\varepsilon<\frac{e^{-c \ell}}{(c \rho)^{\ell} \sqrt{1-\frac{r}{n^{\ell}}}}$,

$$
\mathbb{P}\left(s_{\min }(A) \leq \epsilon\right) \leq \frac{\varepsilon r}{(\ell-1)!(c \rho)^{\ell} \sqrt{1-\frac{r}{n^{\ell}}}}\left(C^{\prime \prime} \log \frac{1}{\varepsilon}\right)^{\ell}
$$

or for $0<\epsilon<e^{-C \ell}$,

$$
\mathbb{P}\left(s_{\min }(A) \leq \sqrt{1-\frac{r}{n^{\ell}}}(c \rho)^{\ell} \varepsilon\right) \leq \frac{\varepsilon r}{(\ell-1)!}\left(C^{\prime} \log \frac{1}{\varepsilon}\right)^{\ell}
$$

Recall the definition of Kahtri-Rao product. Let $U=\left(u_{1}, \cdots, u_{r}\right)$ be an $m \times r$ matrix and let $V=\left(v_{1}, \cdots, v_{r}\right)$ be an $n \times r$ matrix, then the Kahtri-Rao product of $U$ and $V$ is an $m n \times r$ matrix $U \odot V$ whose $i$-th column is the flattened vector of the tensor $u_{i} \otimes v_{i}$.

There exists an efficient algorithm (see [11, [36]) to decompose an order-3 tensor

$$
X=\sum_{i=1}^{r} U_{i} \otimes V_{i} \otimes W_{i}
$$

using simultaneous diagonalization when the rank $r$ is no more than the dimension $n$. For tensor with order more than 3 , we can write

$$
\widetilde{X}=\sum_{i=1}^{r} \otimes_{j=1}^{\ell} \widetilde{X}_{i}^{(j)}=\sum_{i=1}^{r} u_{i} \otimes v_{i} \otimes w_{k}
$$

where

$$
\begin{gathered}
u_{i}=\widetilde{X}^{(1)} \otimes \cdots \otimes \tilde{X}^{\left(\left\lfloor\frac{\ell-1}{2}\right\rfloor\right)} \\
v_{i}=\widetilde{X}^{\left(\left\lfloor\frac{\ell-1}{2}\right\rfloor+1\right)} \otimes \cdots \otimes \widetilde{X}^{(\ell-1)} \\
w_{i}=\widetilde{X}_{k}^{(\ell)}
\end{gathered}
$$

Then the algorithm succeeds with high probability if the smallest singular values of $U=\left(u_{1}, \cdots, u_{r}\right)$, $V=\left(v_{1}, \cdots, v_{r}\right)$ and $W=\left(w_{1}, \cdots, w_{r}\right)$ are sufficiently large with high probability. And the running time of the algorithm also depends on the smallest singular values. We refer to Section 2 in [11] and Appendix A in [6] for details. Our result in Theorem 7.2 improves the previous results since we conclude that the median of the singular value is of order $\mathcal{O}_{\ell}\left(\frac{\rho^{\ell}(\log e r)^{\ell}}{r}\right)$ where $r$ is the rank. In comparison, the best known result due to [6] is of order $\mathcal{O}_{\ell}\left(\frac{\rho^{\ell}}{n^{\ell} \sqrt{r}}\right)$. We want to emphasize that our result depends only on the rank and not on the dimension of the component vectors and allows $\rho$ (which measures how large the noise should be to guarantee that the algorithm works) to be significantly less. Our results combined with the known algorithms imply the following.
Corollary 7.4. For $r \leq \frac{n^{\frac{\ell-1}{2}}}{2}$, suppose we are given $\widetilde{T}+E$ where $\widetilde{T}$ and $\mathcal{F}$ are order $\ell$-tensors and $\widetilde{T}$ has rank $r$ and is obtained from the above smoothed analysis model. Moreover, for $0<\delta<e^{-C \ell}$ and $\varepsilon<1$, suppose the entries of $E$ are at most $\varepsilon \cdot \operatorname{poly}\left(\frac{(c \rho)^{\ell} \delta}{n^{\frac{\ell-1}{2}}}, \frac{\rho}{n}, \frac{1}{n^{\ell}}\right)$. Then there is an algorithm to recover the rank one terms of $\widetilde{T}$ up to an additive $\varepsilon$ error. The algorithm runs in time poly $\left(\frac{n^{\frac{\ell-1}{2}}}{(c \rho)^{\ell} \delta}, \frac{n}{\rho}, n^{\ell}\right)$ and succeeds with probability at least $1-C_{\ell} \delta r\left(\log \frac{e}{\delta}\right)^{\ell+1}$.

## References

[1] R. Adamczak, R. Latala, and R. Meller. Moments of gaussian chaoses in banach spaces. Electronic Journal of Probability, 26, 012021.
[2] R. Adamczak and R. Latała. Tail and moment estimates for chaoses generated by symmetric random variables with logarithmically concave tails. Annales de l'Institut Henri Poincaré, Probabilités et Statistiques, 48(4):1103-1136, 2012.
[3] R. Adamczak and P. Wolff. Concentration inequalities for non-lipschitz functions with bounded derivatives of higher order. Probability Theory and Related Fields, 162:531-586, 2013.
[4] A. Anandkumar, D. Hsu, and S. M. Kakade. A method of moments for mixture models and hidden markov models. In Conference on learning theory, pages 33-1. JMLR Workshop and Conference Proceedings, 2012.
[5] A. Anandkumar, D. Hsu, and S. M. Kakade. A method of moments for mixture models and hidden markov models. In S. Mannor, N. Srebro, and R. C. Williamson, editors, Proceedings of the 25th Annual Conference on Learning Theory, volume 23 of Proceedings of Machine Learning Research, pages 33.1-33.34, Edinburgh, Scotland, 25-27 Jun 2012. PMLR.
[6] N. Anari, C. Daskalakis, W. Maass, C. H. Papadimitriou, A. Saberi, and S. Vempala. Smoothed analysis of discrete tensor decomposition and assemblies of neurons. In Proceedings of the 32nd International Conference on Neural Information Processing Systems, NIPS'18, page 10880-10890, Red Hook, NY, USA, 2018. Curran Associates Inc.
[7] S. Artstein-Avidan, A. Giannopoulos, and V. Milman. Asymptotic geometric analysis. Number pt. 1 in Mathematical surveys and monographs. American Mathematical Society, 2015.
[8] S. Artstein-Avidan, A. Giannopoulos, and V. Milman. Asymptotic Geometric Analysis, Part II. Mathematical Surveys and Monographs. American Mathematical Society, 2021.
[9] S. Bamberger, F. Krahmer, and R. Ward. The hanson-wright inequality for random tensors. Sampling Theory Signal Processing and Data Analysis, 20, 2022.
[10] S. Bamberger, F. Krahmer, and R. Ward. Johnson-lindenstrauss embeddings with kronecker structure. SIAM Journal on Matrix Analysis and Applications, 43(4):1806-1850, 2022.
[11] A. Bhaskara, M. Charikar, A. Moitra, and A. Vijayaraghavan. Smoothed analysis of tensor decompositions. In Proceedings of the Forty-Sixth Annual ACM Symposium on Theory of Computing, STOC '14, page 594-603, New York, NY, USA, 2014. Association for Computing Machinery.
[12] J. Bourgain. Geometry of banach spaces and harmonic analysis. In Proceedings of the International Congress of Mathematicians, volume 1, page 2. Citeseer, 1986.
[13] J. Bourgain. On high dimensional maximal functions associated to convex bodies. American Journal of Mathematics, 108(6):1467-1476, 1986.
[14] H. Brascamp, E. Lieb, and J. Luttinger. A general rearrangement inequality for multiple integrals. Journal of Functional Analysis, 17(2):227-237, Oct. 1974. Funding Information: partially supported by National Science Foundation Grant GP-16147 A\#1 . partially supported by National Science Foundation Grant GP-31674 X. partially supported by a grant from the National Science Foundation.
[15] S. Brazitikos, A. Giannopoulos, P. Valettas, and B.-H. Vritsiou. Geometry of isotropic convex bodies, volume 196. American Mathematical Soc., 2014.
[16] E. M. Bronstein. Approximation of convex sets by polytopes. Journal of Mathematical Sciences, 153, 092008.
[17] A. Burchard. A short course on rearrangement inequalities. Lecture notes, IMDEA Winter School, Madrid, 2009.
[18] A. Carbery and J. Wright. Distributional and l q norm inequalities for polynomials over convex bodies in $\mathbb{R}^{n}$. Mathematical Research Letters, 8, 052001.
[19] J. T. Chang. Full reconstruction of markov models on evolutionary trees: identifiability and consistency. Mathematical biosciences, 137(1):51-73, 1996.
[20] J. T. Chang. Full reconstruction of markov models on evolutionary trees: identifiability and consistency. Mathematical biosciences, 137 1:51-73, 1996.
[21] P. Comon. Independent component analysis, a new concept? Signal processing, 36(3):287314, 1994.
[22] P. Comon. Independent component analysis, a new concept? Signal Process., 36:287-314, 1994.
[23] S. Dann, G. Paouris, and P. Pivovarov. Bounding marginal densities via affine isoperimetry. Proceedings of the London Mathematical Society, 113(2):140-162, 2016.
[24] S. Dharmadhikari and K. Joag-Dev. Unimodality, convexity, and applications. Elsevier, 1988.
[25] M. Fradelizi. Concentration inequalities for s-concave measures of dilations of borel sets and applications. Electronic Journal of Probability, 14, 082008.
[26] I. Glazer and D. Mikulincer. Anti-concentration of polynomials: Dimension-free covariance bounds and decay of fourier coefficients. Journal of Functional Analysis, 283(9):109639, 2022.
[27] E. Gluskin and A. Olevskii. Invertibility of sub-matrices and the octahedron width theorem. Israel Journal of Mathematics, 186, 112011.
[28] F. Gotze, H. Sambale, and A. Sinulis. Concentration inequalities for polynomials in $\alpha$-subexponential random variables. Electronic Journal of Probability, 2019.
[29] O. Guédon. Kahane-khinchine type inequalities for negative exponent. Mathematika, 46(1):165-173, 1999.
[30] D. Hsu, S. M. Kakade, and T. Zhang. A spectral algorithm for learning hidden markov models. Journal of Computer and System Sciences, 78(5):1460-1480, 2012.
[31] R. Jin, T. G. Kolda, and R. Ward. Faster Johnson-Lindenstrauss transforms via Kronecker products. Information and Inference: A Journal of the IMA, 10(4):1533-1562, 102020.
[32] R. Kannan, L. Lovász, and M. Simonovits. Isoperimetric problems for convex bodies and a localization lemma. Discrete $E_{\mathcal{G}}$ Computational Geometry, 13:541-559, 1995.
[33] M. Kanter. Unimodality and dominance for symmetric random vectors. Transactions of The American Mathematical Society - TRANS AMER MATH SOC, 229:65-65, 051977.
[34] B. Klartag. Logarithmic bounds for isoperimetry and slices of convex sets. arXiv preprint arXiv:2303.14938, 2023.
[35] B. Klartag and J. Lehec. Bourgain's slicing problem and kls isoperimetry up to polylog (2022). arXiv preprint arXiv:2203.15551.
[36] J. B. Kruskal. Three-way arrays: rank and uniqueness of trilinear decompositions, with application to arithmetic complexity and statistics. Linear Algebra and its Applications, 18(2):95138, 1977.
[37] J. M. Landsberg. Tensors: geometry and applications: geometry and applications, volume 128. American Mathematical Soc., 2011.
[38] R. Latała. Estimation of moments of sums of independent real random variables. The Annals of Probability, 25(3):1502-1513, 1997.
[39] M. Ledoux. The concentration of measure phenomenon. AMS Surveys and Monographs, 89, 012001.
[40] J. Lehec. Moments of the Gaussian Chaos, pages 327-340. Springer Berlin Heidelberg, Berlin, Heidelberg, 2011.
[41] S. E. Leurgans, R. T. Ross, and R. B. Abel. A decomposition for three-way arrays. SIAM Journal on Matrix Analysis and Applications, 14(4):1064-1083, 1993.
[42] W. Li and Q.-M. Shao. Gaussian processes: Inequalities, small ball probabilities and applications. In Stochastic Processes: Theory and Methods, volume 19 of Handbook of Statistics, pages 533-597. Elsevier, 2001.
[43] E. Lieb and M. Loss. Analysis. Crm Proceedings \& Lecture Notes. American Mathematical Society, 2001.
[44] L.-H. Lim. Tensors in computations. Acta Numerica, 30:555-764, 2021.
[45] E. S. Meckes. The Random Matrix Theory of the Classical Compact Groups. Cambridge Tracts in Mathematics. Cambridge University Press, 2019.
[46] E. Mossel and S. Roch. Learning nonsingular phylogenies and hidden markov models. In Proceedings of the thirty-seventh annual ACM symposium on Theory of computing, pages 366-375, 2005.
[47] E. Mossel and S. Roch. Learning nonsingular phylogenies and hidden Markov models. The Annals of Applied Probability, 16(2):583 - 614, 2006.
[48] F. Nazarov, M. Sodin, and A. Volberg. Local dimension-free estimates for volumes of sublevel sets of analytic functions. arXiv preprint math/0108213, 2001.
[49] F. L. Nazarov, M. L. Sodin, and A. L. Volberg. The geometric kannan-lovász-simonovits lemma, dimension-free estimates for volumes of sublevel sets of polynomials, and distribution of zeros of random analytic functions. Algebra i Analiz, 14(2):214-234, 2002.
[50] H. H. Nguyen and V. H. Vu. Small Ball Probability, Inverse Theorems, and Applications, pages 409-463. Springer Berlin Heidelberg, Berlin, Heidelberg, 2013.
[51] G. Paouris. Concentration of mass on convex bodies. Geometric \& Functional Analysis GAFA, 16:1021-1049, 2006.
[52] G. Paouris. Small ball probability estimates for log-concave measures. Transactions of the American Mathematical Society, 364(1):287-308, 2012.
[53] G. Paouris and P. Pivovarov. Small-ball probabilities for the volume of random convex sets. Discrete $\mathcal{E}$ Computational Geometry, 042013.
[54] G. Paouris and P. Pivovarov. Randomized isoperimetric inequalities. In E. Carlen, M. Madiman, and E. M. Werner, editors, Convexity and Concentration, pages 391-425, New York, NY, 2017. Springer New York.
[55] C. A. Rogers. A Single Integral Inequality. Journal of the London Mathematical Society, s1-32(1):102-108, 011957.
[56] M. Rudelson and R. Vershynin. Small Ball Probabilities for Linear Images of HighDimensional Distributions. International Mathematics Research Notices, 2015(19):95949617, 122014.
[57] D. A. Spielman and S.-H. Teng. Smoothed analysis of algorithms: Why the simplex algorithm usually takes polynomial time. J. ACM, 51(3):385-463, may 2004.
[58] T. Tao and V. Vu. The littlewood-offord problem in high dimensions and a conjecture of frankl and füredi. Combinatorica, 042012.
[59] R. Vershynin. High-Dimensional Probability: An Introduction with Applications in Data Science. Cambridge Series in Statistical and Probabilistic Mathematics. Cambridge University Press, 2018.
[60] R. Vershynin. Concentration inequalities for random tensors. Bernoulli, 2019.

## Appendix A.

We construct an example to show the sharpness of the estimate in Theorem 1.2

Lemma A.1. Let $X_{1}, \cdots, X_{\ell}$ be independent uniform random variables on $[-1,1]$. Let $Z_{\ell}=X_{1} X_{2} \cdots X_{\ell}$, then its distribution satisfies

$$
F_{Z_{\ell}}(z)=\frac{1}{2}+\frac{z}{2} \sum_{j=0}^{\ell} \frac{\left(\log \frac{1}{z}\right)^{j}}{j!}
$$

Proof. First consider $Z_{2}=X_{1} X_{2}$. Suppose $0<z \leq 1$, then

$$
\begin{aligned}
\mathbb{P}\left(Z_{2} \leq z\right) & =\mathbb{E}_{X_{2}} \mathbb{P}_{X_{1}}\left(X_{1} X_{2} \leq z \mid X_{2}\right) \\
& =\int_{0}^{1} \frac{1}{2} \mathrm{~d} y \int_{-1}^{\min \left\{\frac{z}{y}, 1\right\}} \frac{1}{2} \mathrm{~d} x+\int_{-1}^{0} \frac{1}{2} \mathrm{~d} y \int_{\max \left\{\frac{z}{y},-1\right\}}^{1} \frac{1}{2} \mathrm{~d} x \\
& =\frac{1}{4}\left(\int_{0}^{1}\left(\min \left\{\frac{z}{y}, 1\right\}+1\right) \mathrm{d} y+\int_{-1}^{0}\left(1-\max \left\{\frac{z}{y},-1\right\}\right) \mathrm{d} y\right) \\
& =\frac{1}{4}\left(2+\int_{0}^{z} \mathrm{~d} x+\int_{z}^{1} \frac{z}{x} \mathrm{~d} x+\int_{-1}^{-z}-\frac{z}{x} \mathrm{~d} x+\int_{-z}^{0} \mathrm{~d} x\right) \\
& =\frac{1}{2}+\frac{z}{2}-\frac{z}{2} \log z .
\end{aligned}
$$

If $-1 \leq z<0$, then by symmetry of uniform distribution,

$$
\begin{aligned}
\mathbb{P}\left(Z_{2} \leq z\right) & =1-\mathbb{P}\left(Z_{2}>z\right) \\
& =1-\mathbb{P}\left(Z_{2}<-z\right) \\
& =1-\left(\frac{1}{2}+\frac{z}{2}+\frac{z}{2} \log (-x)\right) \\
& =\frac{1}{2}+\frac{z}{2}-\frac{z}{2} \log (-z) .
\end{aligned}
$$

Hence for $-1 \leq z \leq 1$, the distribution function of $Z_{2}$ is

$$
F_{Z_{2}}(z)=\frac{1}{2}+\frac{z}{2}-\frac{z}{2} \log |z|
$$

Now we assume that for $k \geq 2$ and $Z_{k}=X_{1} X_{2} \cdots X_{k}$, its distribution function is

$$
F_{Z_{k}}(z)=\frac{1}{2}+\frac{z}{2} \sum_{j=0}^{k-1} \frac{\left(\log \frac{1}{|z|}\right)^{j}}{j!}
$$

and denote its density fuction by $f_{Z_{k}}$. Then for $Z_{k+1}=X_{1} X_{2} \cdots X_{k+1}$, suppose $0<z \leq 1$,

$$
\begin{aligned}
\mathbb{P}\left(Z_{k+1} \leq z\right) & =\mathbb{E}_{X_{k+1}} \mathbb{P}_{Z_{k}}\left(Z_{k} X_{k+1} \leq z \mid X_{k+1}\right) \\
& =\int_{0}^{1} \frac{1}{2} \mathrm{~d} y \int_{-1}^{\min \left\{\frac{z}{y}, 1\right\}} f_{Z_{k}}(x) \mathrm{d} x+\int_{-1}^{0} \frac{1}{2} \mathrm{~d} y \int_{\max \left\{\frac{z}{y},-1\right\}}^{1} f_{Z_{k}}(x) \mathrm{d} x \\
& =\frac{1}{2}\left(\int_{0}^{1} F_{Z_{k}}\left(\min \left\{\frac{z}{y}, 1\right\}\right) \mathrm{d} y+\int_{-1}^{0}\left(1-F_{Z_{k}}\left(\max \left\{\frac{z}{y},-1\right\}\right)\right) \mathrm{d} y\right) \\
& =\frac{1}{2}\left(\int_{0}^{z} \mathrm{~d} y+\int_{z}^{1} F_{Z_{k}}\left(\frac{z}{y}\right) \mathrm{d} y+\int_{-1}^{-z}\left(1-F_{Z_{k}}\left(\frac{z}{y}\right)\right) \mathrm{d} y+\int_{-z}^{0} \mathrm{~d} y\right) \\
& =\frac{1}{2}\left(2 z+\int_{z}^{1}\left(\frac{1}{2}+\frac{z}{2 y} \sum_{j=0}^{k-1} \frac{\left(\log \left(\frac{y}{z}\right)\right)^{j}}{j!}\right) \mathrm{d} y+\int_{-1}^{-z}\left(\frac{1}{2}-\frac{z}{2 y} \sum_{j=0}^{k-1} \frac{\left(\log \left(-\frac{y}{z}\right)\right)^{j}}{j!}\right) \mathrm{d} y\right) \\
& =\frac{1}{2}\left(2 z+z \int_{1}^{\frac{1}{z}}\left(\frac{1}{2}+\frac{1}{2 u} \sum_{j=0}^{k-1} \frac{(\log u)^{j}}{j!}\right) \mathrm{d} u+z \int_{-\frac{1}{z}}^{-1}\left(\frac{1}{2}+\frac{1}{2 u} \sum_{j=0}^{k-1} \frac{(\log (-u))^{j}}{j!}\right) \mathrm{d} u\right) \\
& =z+\left.\frac{z}{2}\left(\frac{u}{2}+\frac{1}{2} \sum_{j=1}^{k} \frac{(\log u)^{j}}{j!}\right)\right|_{u=1} ^{u=\frac{1}{z}}+\left.\frac{z}{2}\left(\frac{u}{2}-\frac{1}{2} \sum_{j=1}^{k} \frac{(\log (-u))^{j}}{j!}\right)\right|_{u=-\frac{1}{z}} ^{u=-1} \\
& =\frac{1}{2}+\frac{z}{2} \sum_{j=0}^{k} \frac{\left(\log \frac{1}{z}\right)^{j}}{j!}
\end{aligned}
$$

Lemma A.2. Let $X^{(1)}, \cdots, X^{(\ell)} \in \mathbb{R}^{n}$ be independent uniform random vectors on $[-\sqrt{3}, \sqrt{3}]^{n}$, where $X^{(k)}=\left(X_{1}^{(k)}, \cdots, X_{n}^{(k)}\right)$. Then

$$
\mathbb{P}\left(\sum_{i=1}^{m} X_{i}^{(1)^{2}} X_{1}^{(2)^{2}} \cdots X_{1}^{(\ell)^{2}} \leq \varepsilon^{2} m\right) \geq \frac{C \varepsilon}{(\ell-2)!}\left(\log \frac{1}{\varepsilon}\right)^{\ell-2}
$$

Proof. Note that

$$
\begin{aligned}
& \mathbb{P}\left(\sum_{i=1}^{m} X_{i}^{(1)^{2}} X_{1}^{(2)^{2}} \cdots X_{1}^{(\ell)^{2}} \leq \varepsilon^{2} m\right) \\
& \geq \mathbb{E}_{X^{(1)}} \mathbf{1}_{\left\{\sum_{i=1}^{m} X_{i}^{(1)^{2}} \leq \varepsilon^{2} m\right\}} \mathbb{P}_{X^{(2)}, \cdots, X^{(\ell)}}\left(\left|X_{1}^{(2)} \cdots X_{1}^{(\ell)}\right| \leq \frac{\varepsilon \sqrt{m}}{\left(\sum_{i=1}^{m} X_{i}^{(1)^{2}}\right)^{1 / 2}}\right) \\
& =\mathbb{E}_{X^{(1)}} \frac{\varepsilon \sqrt{m}}{\sqrt{3}\left(\sum_{i=1}^{m} X_{i}^{(1)^{2}}\right)^{1 / 2}} \sum_{j=0}^{\ell-2} \frac{\left(\log \frac{\sqrt{3}\left(\sum_{i=1}^{m} X_{i}^{(1)^{2}}\right)^{1 / 2}}{\varepsilon \sqrt{m}}\right)^{j}}{j!} \\
& \geq \mathbb{E}_{X^{(1)}} \frac{\varepsilon \sqrt{m}}{\sqrt{3}\left(\sum_{i=1}^{m} X_{i}^{(1)^{2}}\right)^{1 / 2}} \frac{\left(\log \frac{\sqrt{3}\left(\sum_{i=1}^{m} X_{i}^{(1)^{2}}\right)^{1 / 2}}{\varepsilon \sqrt{m}}\right)^{\ell-2}}{(\ell-2)!} .
\end{aligned}
$$

Denote $Y=\left(X_{1}^{(1)}, \cdots, X_{m}^{(1)}\right)$. Then $Y$ is subgaussian and $\|Y\|_{2}=\left(\sum_{i=1}^{m} X_{i}^{(1)^{2}}\right)^{1 / 2}$. By concentration of norms for subgaussian random vectors,

$$
\mathbb{P}\left(\left|\|Y\|_{2}-\mathbb{E}\|Y\|_{2}\right| \leq t \sqrt{m}\right) \geq 1-e^{-C t^{2} m}
$$

Note that $\mathbb{E}\|Y\|_{2}^{2}=m$. Since $Y$ is log-concave, then by Borell's lemma

$$
\frac{1}{6} \sqrt{m} \leq \mathbb{E}\|Y\|_{2} \leq \sqrt{m}
$$

Then

$$
\begin{aligned}
& \mathbb{P}\left(\sum_{i=1}^{m} X_{i}^{(1)^{2}} X_{1}^{(2)^{2}} \cdots X_{1}^{(\ell)^{2}} \leq \varepsilon^{2} m\right) \\
\geq & \mathbb{E}_{X^{(1)}} \mathbf{1}_{\left\{\frac{1}{\sqrt{3}} \sqrt{m} \leq\|Y\|_{2} \leq\left(\frac{7}{6}-\frac{1}{\sqrt{3}}\right) \sqrt{m}\right\}} \frac{\varepsilon \sqrt{m}}{\sqrt{3}\|Y\|_{2}} \frac{\left(\log \frac{\sqrt{3}\|Y\|_{2}}{\varepsilon \sqrt{m}}\right)^{\ell-2}}{(\ell-2)!} \\
\geq & \left(1-e^{-\left(1-\frac{1}{\sqrt{3}}\right)^{2} C m}\right) \frac{\varepsilon}{\sqrt{3}\left(\frac{7}{6}-\frac{1}{\sqrt{3}}\right)(\ell-2)!}\left(\log \frac{1}{\varepsilon}\right)^{\ell-2} \\
= & \frac{C \varepsilon}{(\ell-2)!}\left(\log \frac{1}{\varepsilon}\right)^{\ell-2} \cdot
\end{aligned}
$$

Let $X_{1}, \cdots, X_{\ell}$ be independent uniform distributions on $[-\sqrt{3}, \sqrt{3}]$ such that $\mathbb{E}\left[X_{j}\right]=0$ and $\operatorname{Var}\left[X_{j}\right]=1$ for $1 \leq j \leq \ell$. Define $Z=\prod_{j=1}^{\ell} X_{j}$. By Lemma A. 1 ,
the cumulative distribution function of $Z$ is

$$
F_{Z}(z)=\frac{1}{2}+\frac{z}{2 \sqrt{3}} \sum_{j=0}^{\ell-1} \frac{\left(\log \frac{\sqrt{3}}{|z|}\right)^{j}}{j!}, \quad-\sqrt{3} \leq z \leq \sqrt{3} .
$$

Choose unit vector $f \in \mathbb{R}^{n_{1} \otimes \cdots \otimes n_{\ell}}$ such that

$$
f_{i_{1} \cdots i_{\ell}}=\left\{\begin{array}{cc}
1 & \text { if } i_{1}=\cdots=i_{\ell} \\
0 & \text { otherwise }
\end{array}\right.
$$

so that $\left\langle X^{(1)} \otimes \cdots \otimes X^{(\ell)}, f\right\rangle=X_{1}^{(1)} \cdots X_{1}^{(\ell)}$. For $0<\varepsilon<1$,
$\mathbb{P}\left[\left|\left\langle X^{(1)} \otimes \cdots \otimes X^{(\ell)}, f\right\rangle\right| \leq \varepsilon\right]=F(\varepsilon)-F(-\varepsilon)=\frac{\varepsilon}{\sqrt{3}} \sum_{j=0}^{\ell-1} \frac{\left(\log \frac{\sqrt{3}}{\varepsilon}\right)^{j}}{j!} \geq \frac{1}{\sqrt{3}(\ell-1)!} \varepsilon\left(\log \frac{\sqrt{3}}{\varepsilon}\right)^{\ell-1}$.
Choose unit vectors $f^{k} \in \mathbb{R}^{n_{1} \otimes \cdots \otimes n_{\ell}}$ for $1 \leq k \leq m \leq n$ such that

$$
f_{i_{1} \cdots i_{\ell}}=\left\{\begin{array}{cc}
1 & \text { if } i_{1}=k, i_{2}=\cdots=i_{\ell}=1 \\
0 & \text { otherwise }
\end{array}\right.
$$

Then by Lemma A. 2 ,

$$
\begin{aligned}
& \mathbb{P}\left(\left\|\Pi_{F} X^{(1)} \otimes \cdots \otimes X^{(\ell)}\right\|_{2} \leq \varepsilon \sqrt{m}\right) \\
= & \mathbb{P}\left(\sum_{i=1}^{m} X_{i}^{(1)^{2}} X_{1}^{(2)^{2}} \cdots X_{1}^{(\ell)^{2}} \leq \varepsilon^{2} m\right) \\
\geq & \frac{C \varepsilon}{(\ell-2)!}\left(\log \frac{1}{\varepsilon}\right)^{\ell-2} .
\end{aligned}
$$

## Appendix B.

Lemma B.1. Let $\xi$ be a non-negative random variable, let $\ell \geq 2$ be an integer and let $C \geq 1$ be a universal constant. Assume for $0<\varepsilon<c_{0} \leq 1$,

$$
\mathbb{P}(\xi \leq \varepsilon) \leq C \varepsilon
$$

Let $c=\min \left\{c_{0}, e^{-1}\right\}$, for $0<\varepsilon<c^{\ell}$,

$$
\mathbb{E} \mathbf{1}_{\left\{\xi>c^{1-\ell} \varepsilon\right\}} \frac{\varepsilon}{\xi}\left(\log \frac{\xi}{\varepsilon}\right)^{\ell-2} \leq C \varepsilon\left(\log \frac{1}{\varepsilon}\right)^{\ell-1}
$$

Proof. Define $g(t)=t\left(\log \frac{1}{t}\right)^{\ell-2}$, then $g(t) \geq 0$ for $0<t<1$. Also

$$
g^{\prime}(t)=\left(\log \frac{1}{t}\right)^{\ell-3}\left(\log \frac{1}{t}+2-l\right) \geq 0
$$

for $0<\varepsilon<e^{2-\ell}$. By using L'Hospital's Rule $\ell-2$ times we have $\lim _{t \rightarrow 0} g(t)=0$. Notice that $c^{\ell-1} \leq e^{1-\ell}<e^{2-\ell}$. Hence we can write

$$
g(T) \mathbf{1}_{\left\{0<T<c^{\ell-1}\right\}}=\int_{0}^{T} g^{\prime}(t) \mathbf{1}_{\left\{0<t<c^{\ell-1}\right\}} \mathrm{d} t=\int_{0}^{\infty} g^{\prime}(t) \mathbf{1}_{\left\{0<t<c^{\ell-1}\right\}} \mathbf{1}_{\{T \geq t\}} \mathrm{d} t
$$

and by Fubini-Tonelli's theorem,

$$
\mathbb{E}\left[g(T) \mathbf{1}_{\left\{0<T<c^{\ell-1}\right\}}\right]=\int_{0}^{\infty} g^{\prime}(t) \mathbf{1}_{\left\{0<t<c^{\ell-1}\right\}} \mathbb{P}(T \geq t) \mathrm{d} t=\int_{0}^{c^{\ell-1}} g^{\prime}(t) \mathbb{P}(T \geq t) \mathrm{d} t
$$

where $T$ is a non-negative random variable. Then for $0<\varepsilon<c^{\ell}$,

$$
\begin{aligned}
& \mathbb{E} \mathbf{1}_{\left\{\xi>c^{1-\ell} \varepsilon\right\}} \frac{\varepsilon}{\xi}\left(\log \frac{\xi}{\varepsilon}\right)^{\ell-2} \\
&= \mathbb{E} \mathbf{1}_{\left\{0<\frac{\varepsilon}{\xi}<c^{\ell-1}\right\}^{\prime}} g\left(\frac{\varepsilon}{\xi}\right) \\
&= \int_{0}^{c^{\ell-1}} g^{\prime}(t) \mathbb{P}\left(\frac{\varepsilon}{\xi} \geq t\right) d t \\
&= \int_{0}^{\frac{\varepsilon}{c}}\left(\log \frac{1}{t}\right)^{\ell-3}\left(\log \frac{1}{t}+2-l\right) \mathbb{P}\left(\frac{\varepsilon}{\xi} \geq t\right) d t+\int_{\frac{\varepsilon}{c}}^{c^{\ell-1}}\left(\log \frac{1}{t}\right)^{\ell-3}\left(\log \frac{1}{t}+2-l\right) \mathbb{P}\left(\frac{\varepsilon}{\xi} \geq t\right) d t \\
& \leq \int_{0}^{\frac{\varepsilon}{c}}\left(\log \frac{1}{t}\right)^{\ell-3}\left(\log \frac{1}{t}+2-l\right) d t+\int_{\frac{\varepsilon}{c}}^{c^{\ell-1}}\left(\log \frac{1}{t}\right)^{\ell-3}\left(\log \frac{1}{t}+2-l\right) \frac{C \varepsilon}{t} d t \\
&=\left.t\left(\log \frac{1}{t}\right)^{\ell-2}\right|_{0} ^{\frac{\varepsilon}{c}}+\left.C \varepsilon\left[\left(\log \frac{1}{t}\right)^{\ell-2}-\frac{1}{\ell-1}\left(\log \frac{1}{t}\right)^{\ell-1}\right]\right|_{\frac{\varepsilon}{c}} ^{c^{\ell-1}} \\
& \leq \frac{\varepsilon}{c}\left(\log \frac{c}{\varepsilon}\right)^{\ell-2}+C \varepsilon\left[\frac{1}{\ell-1}\left(\log \frac{c}{\varepsilon}\right)^{\ell-1}-\left(\log \frac{c}{\varepsilon}\right)^{\ell-2}\right] \\
& \leq C \varepsilon \\
& \ell-1 \\
& \ell\left.\log \frac{1}{\varepsilon}\right)^{\ell-1} \cdot
\end{aligned}
$$

## Appendix C.

Lemma C.1. Let $\xi$ be non-negative random variable, let $\ell \geq 2$ and let $K \leq\left(\frac{M}{\ell}\right)^{\ell}$ be a positive parameter for some positive universal constant $M \geq 1$. If there exists $c>\frac{1}{M}$, such that for $0 \leq \varepsilon \leq c^{\ell}$, we have

$$
\mathbb{P}(\xi \leq \varepsilon) \leq K \varepsilon\left(\log \frac{1}{\varepsilon}\right)^{\ell-1}
$$

then for $1-\frac{1}{M_{0}} \leq q<1$, where $M_{0} \geq \frac{3}{2}$ is a constant that only depends on $M$,

$$
\mathbb{E} \frac{1}{\xi^{q}} \leq \frac{\left(K \ell^{\ell}\right)^{q}}{(1-q)^{q(\ell-1)+1}}
$$

Proof. Note that for $0<a<c^{\ell}$,

$$
\begin{aligned}
\mathbb{E} \frac{1}{\xi^{q}} & =\int_{0}^{\infty} \mathbb{P}\left(\frac{1}{\xi^{q}}>u\right) \mathrm{d} u \\
& =q \int_{0}^{\infty} \frac{1}{t^{q+1}} \mathbb{P}(\xi<t) \mathrm{d} t \\
& \leq q K \int_{0}^{a} \frac{1}{t^{q}}\left(\log \frac{1}{t}\right)^{\ell-1} \mathrm{~d} t+q \int_{a}^{\infty} \frac{1}{t^{q+1}} \mathrm{~d} t \\
& =q K \int_{0}^{a} \frac{1}{t^{q}}\left(\log \frac{1}{t}\right)^{\ell-1} \mathrm{~d} t+\frac{1}{a^{q}}
\end{aligned}
$$

Denote $I_{p}=\int_{0}^{a} t^{-q}\left(\log \frac{1}{t}\right)^{p} \mathrm{~d} t$, then

$$
I_{0}=\int_{0}^{a} t^{-q} \mathrm{~d} t=\frac{a^{1-q}}{1-q}
$$

and by integration by parts

$$
\begin{aligned}
I_{p} & =\frac{1}{1-q} \int_{0}^{a}\left(\log \frac{1}{t}\right)^{p} \mathrm{~d} t^{1-q} \\
& =\frac{a^{1-q}}{1-q}\left(\log \frac{1}{a}\right)^{p}+\frac{p}{1-q} \int_{0}^{a} t^{-q}\left(\log \frac{1}{t}\right)^{p-1} \mathrm{~d} t \\
& =\frac{a^{1-q}}{1-q}\left(\log \frac{1}{a}\right)^{p}+\frac{p}{1-q} I_{p-1}
\end{aligned}
$$

Hence

$$
\begin{aligned}
I_{\ell-1} & =\frac{a^{1-q}}{1-q}\left(\log \frac{1}{a}\right)^{\ell-1}+\frac{\ell-1}{1-q} I_{l-2} \\
& =a^{1-q} \sum_{i=0}^{\ell-1} \frac{(\ell-1)!}{(\ell-1-i)!(1-q)^{i+1}}\left(\log \frac{1}{a}\right)^{\ell-1-i}
\end{aligned}
$$

and

$$
\begin{aligned}
\mathbb{E} \frac{1}{\xi^{q}} \leq q K I_{\ell-1}+\frac{1}{a^{q}} & =a^{-q}\left(\frac{q K a}{1-q}\left(\log \frac{1}{a}\right)^{\ell-1} \sum_{i=0}^{\ell-1} \frac{(\ell-1)!}{(\ell-1-i)!}\left(\frac{1}{(1-q) \log \frac{1}{a}}\right)^{i}+1\right) \\
& \leq a^{-q}\left(\frac{q K a}{1-q}\left(\log \frac{1}{a}\right)^{\ell-1} \sum_{i=0}^{\ell-1}\left(\frac{\ell}{(1-q) \log \frac{1}{a}}\right)^{i}+1\right)
\end{aligned}
$$

Note that $0<q<1$, and note that $\frac{\ell}{(1-q) \log \frac{1}{a}}=1$ when $a=e^{-\frac{\ell}{1-q}}$. Then

$$
\sum_{i=0}^{\ell-1}\left(\frac{\ell}{(1-q) \log \frac{e}{a}}\right)^{i} \leq \begin{cases}\ell\left(\frac{1}{(1-q) \log \frac{1}{a}}\right)^{\ell-1} & \text { if } a \geq e^{-\frac{\ell}{1-q}} \\ \ell & \text { if } a<e^{-\frac{\ell}{1-q}}\end{cases}
$$

Then

$$
\mathbb{E} \frac{1}{\xi^{q}} \leq \begin{cases}a^{-q}\left(\frac{q K \ell^{\ell} a}{(1-q)^{\ell}}+1\right) & \text { if } a \geq e^{-\frac{\ell}{1-q}} \\ a^{-q}\left[\frac{q K \ell a}{1-q}\left(\log \frac{1}{a}\right)^{\ell-1}+1\right] & \text { if } a<e^{-\frac{\ell}{1-q}}\end{cases}
$$

Take

$$
f(a):=a^{-q}\left(\frac{q K \ell^{\ell} a}{(1-q)^{\ell}}+1\right)
$$

and take $a_{0}=\frac{(1-q)^{\ell-1}}{K \ell^{\ell}}$, then we claim that $a_{0} \geq e^{-\frac{\ell}{1-q}}$ if $q \geq 1-\frac{1}{M_{0}}$ for some constant $M_{0} \geq \frac{3}{2}$. In fact, since $K \leq\left(\frac{M}{\ell}\right)^{\ell}$, there exists constant $M_{0} \geq \frac{3}{2}$ such that

$$
\left(K \ell^{\ell}\right)^{\frac{1}{\ell-1}} \leq\left(M^{\ell}\right)^{\frac{1}{\ell-1}} \leq M_{0}^{e-1}
$$

If $q \geq 1-\frac{1}{M_{0}}$, then

$$
\begin{gathered}
\frac{1}{1-q} \geq M_{0} \geq\left(K \ell^{\ell}\right)^{\frac{1}{(e-1)(\ell-1)}} \\
(e-1)(\ell-1) \log \frac{1}{1-q} \geq \log \left(K \ell^{\ell}\right)
\end{gathered}
$$

Observe that

$$
\frac{1}{1-q} \geq e \log \frac{1}{1-q}
$$

therefore

$$
\begin{aligned}
\log K \ell^{\ell} & \leq(e-1)(\ell-1) \log \frac{1}{1-q} \\
& \leq(\ell-1)\left(\frac{1}{1-q}-\log \frac{1}{1-q}\right) \\
& \leq \frac{\ell}{1-q}-(\ell-1) \log \frac{1}{1-q}, \\
-\frac{\ell}{1-q} & \leq(\ell-1) \log (1-q)-\log \left(K \ell^{\ell}\right), \\
e^{-\frac{\ell}{1-q}} & \leq \frac{(1-q)^{\ell-1}}{K \ell^{\ell}}=a_{0}
\end{aligned}
$$

Hence

$$
\mathbb{E} \frac{1}{\xi^{q}} \leq f\left(a_{0}\right)=\frac{\left(K \ell^{\ell}\right)^{q}}{(1-q)^{q(\ell-1)+1}}
$$

Remark C.2. Lemma C.1 can be reversed for sufficiently small $\varepsilon$ at a price. Let $\xi$ be non-negative random variable, let $\ell \geq 2$ and let $K \leq\left(\frac{M}{\ell}\right)^{\ell}$ be a positive parameter for some positive universal constant $M$ as assumed in LEmma C.1. If for $1-\frac{1}{M_{0}} \leq q<1$, where $M_{0} \geq \frac{3}{2}$ is a constant that only depends on $M$, we have

$$
\mathbb{E} \frac{1}{\xi^{q}} \leq \frac{\left(K \ell^{\ell}\right)^{q}}{(1-q)^{q(\ell-1)+1}}
$$

then for $0<\varepsilon \leq \frac{1}{e^{\left(M_{0}-1\right) \ell-1}}$,

$$
\mathbb{P}(\xi \leq \varepsilon) \leq K C^{\ell} \varepsilon\left(\log \frac{1}{\varepsilon}\right)^{\ell}
$$

Take $\varepsilon \leq \frac{1}{e^{\left(M_{0}-1\right) \ell-1}}$ and $q=1-\frac{\ell}{\ell+\log \frac{1}{\varepsilon}}$. Note that

$$
\frac{\ell}{\ell+\log \frac{1}{\varepsilon}} \leq \frac{\ell}{\ell+\left(M_{0}-1\right) \ell}=\frac{1}{M_{0}}
$$

then $1-\frac{1}{M_{0}} \leq q<1$. By Markov's inequality,

$$
\begin{aligned}
\mathbb{P}(\xi \leq \varepsilon) & \leq \varepsilon^{q} \frac{\left(K \ell^{\ell}\right)^{q}}{(1-q)^{q(\ell-1)+1}} \\
& =\left(K \ell^{\ell}\right)^{q} \varepsilon^{q}\left(\frac{1}{1-q}\right)^{q(\ell-1)+1} \\
& \leq K \ell^{\ell} \cdot \varepsilon \cdot \exp \left\{\frac{\ell \log \frac{1}{\varepsilon}}{\ell+\log \frac{1}{\varepsilon}}\right\}\left(\frac{\ell+\log \frac{1}{\varepsilon}}{\ell}\right)^{\ell} \\
& \leq K e^{\ell} \varepsilon\left(\ell+\log \frac{1}{\varepsilon}\right)^{\ell} .
\end{aligned}
$$

Note that $\log \frac{1}{\varepsilon} \geq\left(M_{0}-1\right) \ell$, then

$$
\mathbb{P}(\xi \leq \varepsilon) \leq K\left(\frac{e M_{0}}{M_{0}-1}\right)^{\ell} \varepsilon\left(\log \frac{1}{\varepsilon}\right)^{\ell} \leq K(3 e)^{\ell} \varepsilon\left(\log \frac{1}{\varepsilon}\right)^{\ell} .
$$


[^0]:    1 Texas A\&M University. Email: huxuehan@tamu.edu. This work is partially supported by NSF grant CCF-1900881 and Simon's grant 964286.

    2 Texas A\&M University. Email: grigoris@tamu.edu. This work is partially supported by NSF grant CCF-1900881 and Simon's grant 964286.

