A MAXIMAL THEOREM WITH FUNCTION-THEORETIC APPLICATIONS.

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I.

Introduction.

1. The kernel of this paper is 'elementary', but it originated in attempts, ultimately successful, to solve a problem in the theory of functions. We begin by stating this problem in its apparently most simple form.

Suppose that $\lambda > 0$, that

$$f(z) = f(re^{i\theta})$$

is an analytic function regular for $r \leqq 1$, and that

$$F(\theta) = \operatorname*{Max}_{0 \leqq r \leqq 1} |f(re^{i\theta})|$$

is the maximum of $|f|$ on the radius θ. Is it true that

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} F^{\lambda}(\theta)\,d\theta \leqq A(\lambda)\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(e^{i\theta})|^{\lambda}\,d\theta,$$

where $A(\lambda)$ is a function of λ only? The problem is very interesting in itself, and the theorem suggested may be expected, if it is true, to have important applications to the theory of functions.

The answer to the question is affirmative, and is contained in Theorems 17 and 24—27 below (where the problem is considered in various more general forms). There seems to be no really easy proof; the one we have found depends entirely on the difficult, though strictly elementary, argument of section II.[1] Here we solve a curious, and at first sight rather artificial, maximal problem. The key theorem is Theorem 2; a certain sum, defined by means of averages of a given finite set of positive numbers, is greatest when the numbers are arranged in descending order. It is this theorem which contains the essential novelty of our analysis; once it is proved, the rest of our work is comparatively a matter of routine. We have therefore written out the proof with the maximum of attention to detail.

It is noteworthy that the central idea of the solution is appropriate only for sums. What is required for the function-theoretic applications is not Theorem 2 itself but its analogue for integrals, Theorem 5. The proof for integrals, however, *cannot* run parallel to that for sums, a peculiarity very unusual in inequality theorems, and one which makes the final function-theoretic results rest on foundations very alien to their own content. It seems here to be essential to deduce the integral theorem from the sum theorem by a limiting process, and this transition is set out in section III. The argument is not quite trivial, but it is comparatively straightforward and involves no novel idea; we have therefore treated it less expansively, and have omitted a certain amount of detail which an experienced reader will easily supply for himself.

In section IV we deduce some inequalities for real integrals which are required later. The typical theorem is Theorem 10; if $f(x)$ is positive and belongs to the Lebesgue class L^k, where $k > 1$, in (a, b), and $\Theta(x)$ is 'the maximum average of $f(x)$ about the point x', then $\Theta(x)$ also belongs to L^k and

$$\int \Theta^k dx \leqq A(k) \int f^k dx,$$

where $A(k)$ depends only on k. This is false when $k = 1$, and we investigate also the theorem which then replaces it.

Finally, in section V we make some applications of the theorems which precede to the theory of functions. We have other such applications in view; here we go so far only as is necessary to solve the problem from which we started, the analogous problems for harmonic and sub-harmonic functions, and a similar problem which naturally suggests itself concerning the Cesàro means of a Fourier series.

[1] Another proof has since been found by Mr. R. E. A. C. Paley, and will be published in the *Proceedings of the London Mathematical Society*.

II.

The maximal problem.

2. The problem is most easily grasped when stated in the language of cricket, or any other game in which a player compiles a series of scores of which an average is recorded.[1] It will be convenient to begin by giving the solution of a much simpler problem.

Suppose that a batsman plays, in a given season, a given 'stock' of innings

$$a_1, a_2, \ldots, a_n \tag{2. 1}$$

(determined in everything except arrangement). Let α_ν be his average after the ν-th innings, so that

$$\alpha_\nu = \frac{A_\nu}{\nu} = \frac{a_1 + a_2 + \cdots + a_\nu}{\nu}. \tag{2. 2}$$

Let $s(x)$ be a positive function which increases (in the wide sense) with x, and let his 'satisfaction' after the ν-th innings be measured by

$$s_\nu = s(\alpha_\nu). \tag{2. 3}$$

Finally, let his total satisfaction for the season be measured by

$$S = \Sigma s_\nu = \Sigma s(\alpha_\nu). \tag{2. 4}$$

It is then easily verified that S *is a maximum, for a given stock of innings, when the innings are played in decreasing order.* For suppose that

$$a_\mu < a_\nu, \quad \mu < \nu.$$

If we interchange a_μ and a_ν, then $s_1, s_2, \ldots, s_{\mu-1}$ and $s_\nu, s_{\nu+1}, \ldots, s_n$ are unaltered, and $s_\mu, s_{\mu+1}, \ldots, s_{\nu-1}$ are increased, so that S is increased.

This problem is trivial and the result well known. We state it, for convenience of reference, as a formal theorem.

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[1] The arguments used in §§ 5—6 are indeed mostly of the type which are intuitive to a student of cricket averages. A batsman's average is increased by his playing an innings greater than his present average; if his average is increased by playing an innings x, it is further increased by playing next an innings $y > x$; and so forth.

Theorem 1. *If* $a_1, a_2, \ldots, a_n$ *are positive, in the wide sense, and given except in arrangement,* $s(x)$ *is any increasing function of* x, *and* α_ν, s_ν, S *are defined by* (2. 2), (2. 3), *and* (2. 4), *then* S *is a maximum when the* a_ν *are arranged in descending order.*

3. We obtain a non-trivial problem by a slight change in our definitions of α_ν and s_ν. Suppose now that α_ν is not the batsman's average for the complete season to date, but *his maximum average for any consecutive series of innings ending at the* ν*-th, so that*

$$(3.1) \qquad \alpha_\nu = \frac{a_{\nu^*} + a_{\nu^*+1} + \cdots + a_\nu}{\nu - \nu^* + 1} = \operatorname*{Max}_{\mu \leqq \nu} \frac{a_\mu + a_{\mu+1} + \cdots + a_\nu}{\nu - \mu + 1};$$

we may agree that, in case of ambiguity, ν^* is to be chosen as small as possible.[1] Let s_ν and S be then defined by (2. 3) and (2. 4) as before. The same maximal problem presents itself, and its solution is now much less obvious. Theorem 2, however, shows that *S is still a maximum when the innings are played in descending order.*

Theorem 2. *If* $a_1, a_2, \ldots, a_n$ *are positive, in the wide sense, and given except in arrangement,* $s(x)$ *is any increasing function of* x, *and* α_ν, s_ν, S *are defined by* (3. 1), (2. 3), *and* (2. 4), *then* S *is a maximum when the* a_ν *are arranged in descending order.*

Preliminary notes and definitions.

4. We suppose that $a_1, a_2, \ldots, a_n$ form N descending *pieces* C_i, where $i = 1, 2, \ldots, N$, C_i containing n_i terms a_ν $(\nu_i \leqq \nu < \nu_i + n_i)$ such that

$$a_{\nu_i} \geqq a_{\nu_i+1} \geqq \cdots \geqq a_{\nu_i+n_i-1}.$$

A piece may contain one term only; thus, if the a_ν *increase* strictly, each a_ν constitutes a piece and $N = n$. In any case

$$n_1 + n_2 + \cdots + n_N = n.$$

We shall prove that, *if* $N > 1$, *we can rearrange the* a_ν *so that* N *is decreased*

[1] If the innings to date are 82, 4, 133, 0, 43, 58, 65, 53, 86, 30, the batsman says to himself 'at any rate my average for my last 8 innings is 58.5' (a not uncommon psychology).

and with advantage, i. e. with increase of S. Here 'advantage', 'increase' are used widely (so that 'with increase' means 'without decrease') but 'N is decreased' is to be interpreted in the strict sense. It is plain that this will prove the theorem.

We denote the average of the terms of C_i by γ_i and write

$$\gamma_i = Av(C_i).$$

We use this notation systematically for averages; thus $Av(C_i, C_{i+1})$ means the average of all the terms of the two successive pieces C_i, C_{i+1}, and $Av(a_{\nu_i-1}, C_i)$ means the average of the terms of C_i and the immediately preceding term.

We call a_{ν_i}, the left hand end of C_i, the *summit* of C_i; it is, in the wide sense, the greatest term of C_i.

The set of terms

$$a_{\nu^*}, a_{\nu^*+1}, \ldots, a_\nu$$

which defines the α_ν associated with a_ν will be called the *stretch* σ_ν of a_ν, a_ν the *source* of σ_ν, and a_{ν^*} the *end* of σ_ν.

It is almost obvious (see Lemma 1 below) that any end of a stretch is a summit of a piece, so that any stretch which *enters* a piece (contains at least one term of the piece), other than the piece to which its source belongs, *passes through* that piece (contains all its terms). Here again 'passes through' is used widely; the stretch extends at least to the summit of the piece.

To *combine* two consecutive pieces C_i, C_{i+1} is to rearrange the aggregate of their terms as a descending sequence, thereby replacing the two pieces by a single piece and decreasing N. Similarly we may combine C_i with part of C_{i+1}, or C_{i+1} with part of C_i; this will not in general decrease N.

Lemmas for Theorem 2.

5. 1. **Lemma 1.** *Any end of a stretch is a summit of a piece.*

Suppose, if possible, that $a_\mu, a_{\mu+1}$ belong to the same piece, and that a stretch σ_ν ends at $a_{\mu+1}$. Then

$$a_{\mu+1} \geqq Av(a_{\mu+2}, a_{\mu+3}, \ldots, a_\nu)$$

and so

$$a_\mu \geqq a_{\mu+1} \geqq Av(a_{\mu+1}, a_{\mu+2}, \ldots, a_\nu).$$

Hence σ_ν goes on to include a_μ, a contradiction.

5.2. **Lemma 2.** *If a_ν and $a_{\nu+1}$ belong to the same piece C_i, then $\sigma_{\nu+1}$ extends at least as far to the left as σ_ν (and so includes it).*

Suppose, if possible, that σ_ν ends at the summit of C_{i-q}, while $\sigma_{\nu+1}$ ends at that of C_{i-p}, with $0 \leqq p < q$. Call

$$\begin{array}{ll} C_{i-q} + C_{i-q+1} + \cdots + C_{i-p-1} & C^1, \\ C_{i-p} + C_{i-p+1} + \cdots + C_{i-1} & C^2, \\ C_i \text{ (up to } a_\nu) & C^3. \end{array}$$

Since σ_ν goes back to C_{i-q}, instead of stopping in C_{i-p}, we have

$$Av(C^1) \geqq Av(C^2,\ C^3); \tag{5.21}$$

and since $\sigma_{\nu+1}$ does not do so

$$Av(C^1) < Av(C^2,\ C^3,\ a_{\nu+1}). \tag{5.22}$$

On the other hand, since $\sigma_{\nu+1}$ does go back to C_{i-p}, we have

$$Av(C^2) \geqq Av(C^3,\ a_{\nu+1}). \tag{5.23}$$

It follows from (5.22) and (5.23) that

$$Av(C^1) < Av(C^2), \tag{5.24}$$

and from (5.21) and (5.24) that

$$Av(C^3) < Av(C^2). \tag{5.25}$$

Also (5.21) and (5.22) show that

$$a_{\nu+1} > Av(C^2,\ C^3), \tag{5.26}$$

and then (5.25) and (5.26) show that

$$a_{\nu+1} > Av(C^3). \tag{5.27}$$

This is a contradiction, since $a_{\nu+1}$ cannot exceed any term of C^3.

5.3. **Lemma 3.** *Suppose that $\gamma_{i-1} \geqq \gamma_i$. Then any σ_ν whose source lies to the right of C_i (in $C_{i+1}, C_{i+2}, \ldots$), and which enters C_i, will pass through C_{i-1}.*

If not, it stops in (at the summit of) C_i. Let C' be the part of σ_ν between C_i and its source, and γ' the average of C'. Then $\gamma_i \geqq \gamma'$, and so $\gamma_{i-1} \geqq \gamma_i \geqq \gamma'$. Hence

$$\gamma_{i-1} \geqq Av(C_i, C');$$

which is a contradiction, since σ_ν does not enter C_{i-1}.

5.4. **Lemma 4.** *Suppose that C_{i-1} and C_i are consecutive pieces, and that the stretches of the terms of C_i all pass through C_{i-1}. Then combination of C_{i-1} and C_i increases the contribution of their terms to S.*

The contributions of $C_{i-2}, C_{i-3}, \ldots$ are obviously not changed. Those of $C_{i+1}, C_{i+2}, \ldots$ may be, but we are not concerned with that here.

It is easiest to prove *more.* Consider any arrangement of the terms of C_{i-1} and C_i, say

$$b_1, b_2, \ldots, b_\mu, \ldots, b_q,$$

and associate with each μ a stretch ζ_μ going back at least as far as b_1, and a corresponding average β_μ. *Among all possible arrangements of the b, that which makes*

$$s(\beta_1) + s(\beta_2) + \cdots + s(\beta_q)$$

greatest is the decreasing arrangement.

For suppose *(e.g.)* $b_{\mu+1} > b_\mu$. If we exchange b_μ and $b_{\mu+1}$, then plainly β_μ is increased and the remaining β are unaltered. It follows (going back to the state of affairs in the lemma) that, when we have combined C_{i-1} and C_i, there is *some* set of stretches σ'_ν, with corresponding averages α'_ν, which makes

$$\Sigma s(\alpha'_\nu)$$

at least as great as the original contribution of C_{i-1} and C_i. This set of σ'_ν is not necessarily identical with the set of σ''_ν actually associated with the piece replacing C_{i-1} and C_i; but

$$\Sigma s(\alpha''_\nu) \geqq \Sigma s(\alpha'_\nu),$$

by the definition of α''_ν, and is therefore at least as great as the original contribution. This proves the lemma.

5.5. **Lemma 5.** *If C_{i-1} and C_i are two consecutive pieces, and the stretches of the terms of each piece extend to the summits of these pieces only, then combination of C_{i-1} and C_i increases their contribution to S.*

This is the least trivial of our lemmas. It is obviously included in the following lemma.

Lemma 6. *Suppose that*

$$c_1 \geqq c_2 \geqq \cdots \geqq c_p, \quad d_1 \geqq d_2 \geqq \cdots \geqq d_q,$$

and that

$$e_1 \geqq e_2 \geqq \cdots \geqq e_{p+q}$$

is the set of c and d arranged in descending order. Let

$$C_\nu = c_1 + c_2 + \cdots + c_\nu, \tag{5.51}$$

and similarly with the other letters. Then

$$s(C_1)+s\left(\frac{C_2}{2}\right)+\cdots+s\left(\frac{C_p}{p}\right)+s(D_1)+s\left(\frac{D_2}{2}\right)+\cdots+s\left(\frac{D_q}{q}\right) \leqq s(E_1)+s\left(\frac{E_2}{2}\right)+\cdots+s\left(\frac{E_{p+q}}{p+q}\right). \tag{5.52}$$

We prove Lemma 6[1] by induction from $p+q-1$ to $p+q$. Suppose that it has been proved for $p+q-1$, but that (5.52) itself is false. Plainly

$$\frac{E_{p+q}}{p+q} \geqq \operatorname{Min}\left(\frac{C_p}{p}, \frac{D_q}{q}\right) = \frac{D_q}{q},$$

say. Writing (5.52) with '$>$' in place of '$\leqq$', and suppressing the last term on each side, we obtain

$$\sum_1^p s\left(\frac{C_\nu}{\nu}\right) + \sum_1^{q-1} s\left(\frac{D_\nu}{\nu}\right) > \sum_1^{p+q-1} s\left(\frac{E_\nu}{\nu}\right). \tag{5.53}$$

If $c_p \geqq d_q$, (5.53) contains the same c and d on its two sides, and accordingly contradicts our assumptions. We must therefore have $d_q > c_p$. Then d_q is missing from the left of (5.53) and c_p from the right. Let us suppose that

[1] Our original proof of this lemma was much less satisfactory; the present one is due in substance to Mr T. W. Chaundy.

$c_t \geqq d_q \geqq c_{t+1}$; the argument requires only an obvious modification when d_q is greater than every c. Then the terms involving d_q on the right of (5.53) are

$$(5.54)\quad s\left(\frac{c_1+\cdots+c_t+d_1+\cdots+d_q}{t+q}\right)+s\left(\frac{c_1+\cdots+c_{t+1}+d_1+\cdots+d_q}{t+q+1}\right)+\cdots+s\left(\frac{c_1+\cdots+c_{p-1}+d_1+\cdots+d_q}{p+q-1}\right).$$

If the E on the right of (5.53) were constructed in the manner of the theorem from $c_1, c_2, \ldots, c_p, d_1, \ldots, d_{q-1}$, then the right of (5.53) would contain, instead of the terms just written, the terms

$$s\left(\frac{c_1+\cdots+c_t+c_{t+1}+d_1+\cdots+d_{q-1}}{t+q}\right)+s\left(\frac{c_1+\cdots+c_{t+2}+d_1+\cdots+d_{q-1}}{t+q+1}\right)+\cdots+s\left(\frac{c_1+\cdots+c_p+d_1+\cdots+d_{q-1}}{p+q-1}\right).$$

Since $d_q \geqq c_{t+1} \geqq c_{t+2} \geqq \cdots$, none of these terms exceeds the corresponding term of (5.54). Hence (5.53) is true with the new interpretation of the E, when it contradicts our assumptions. We have thus arrived in any case at a contradiction which establishes the lemma.

Proof of Theorem 2.

6. We arrange the proof in three stages; (1) is a special case of (2), but is proved separately for the sake of clearness.

(1) *If $\gamma_1 \geqq \gamma_2$, then C_1 and C_2 may be combined with advantage.*

We begin with two preliminary observations.

(a) *A stretch from a source in $C_3, C_4, \ldots$ either never enters C_2 or passes through C_2 and C_1.*

This follows from Lemma 3.

(b) *Any rearrangement of C_1 and C_2, and in particular their combination, increases the contribution of $C_3, C_4, \ldots$*

For if a_ν belongs to $C_3, C_4, \ldots, \sigma_\nu$, by (a), stops short of C_2 or passes through C_2 and C_1. In either case the new α_ν is the maximum of a set of values which includes the value which determined the old maximum.

We observe next that C_2 consists of two parts C_2' and C_2'' (either of which may be nul) such that (i) C_2'' lies to the right of C_2' and (ii) the σ_ν of C_2' stop at the summit of C_2 while those of C_2'' pass through to the summit of C_1. All this follows from Lemma 2. We now prove (1) by operating in two stages, first combining C_1 and C_2' into a single piece C_1', and then combining C_1' and C_2'', and showing that each combination is advantageous.

First combine C_1 and C_2' into C_1'. The σ_ν of C_2'' continue to pass through C_1, and the contribution of C_2'' is unchanged. The contribution of $C_3, C_4, \ldots$ is increased, by (b) above; and that of C_1 and C_2' is increased, by Lemma 5. Hence the combination is advantageous.

Next combine C_1' and C_2''. The σ_ν of C_2'' pass through C_1', so that, by Lemma 4, the contribution of C_1' and C_2'' is increased. The contribution of $C_3, C_4, \ldots$ is increased, by (b) above. Hence the combination of C_1' and C_2'' is also advantageous. This completes the proof of (1).

(2) *If* $\gamma_1 < \gamma_2 < \cdots < \gamma_k$, $\gamma_k \geqq \gamma_{k+1}$, *then* C_k *and* C_{k+1} *may be combined with advantage.*

We note first that the σ_ν of C_k all stop at the summit of C_k.

(a) *A stretch from a source in* $C_{k+2}, C_{k+3}, \ldots$ *either never enters* C_{k+1} *or passes through* C_{k+1} *and* C_k.

This, like (a) under (1), follows from Lemma 3.

(b) *Any rearrangement of* C_k *and* C_{k+1} *increases the contribution of* C_{k+2}, $C_{k+3}, \ldots$

The proof is the same as that of (b) under (1).

We now argue as before. C_{k+1} divides into C'_{k+1} and C''_{k+1}, the σ_ν of C'_{k+1} stopping in C_{k+1} while those of C''_{k+1} pass through C_{k+1} and C_k. We first combine C_k and C'_{k+1} into C'_k. The σ_ν of C''_{k+1} continue to pass through C'_k, and the contribution of C''_{k+1} is unchanged. Since $\gamma_{k+1} \geqq \gamma_k > \gamma_{k-1} > \cdots$, the σ_ν of C'_k will still stop in C'_k. By Lemma 5, the contribution of C_k and C'_{k+1} will be increased. Finally the contribution of $C_{k+2}, C_{k+3}, \ldots$ is increased, by (b) above. Thus combination of C_k and C'_{k+1} is advantageous.

Next combine C'_k and C''_{k+1}. The σ_ν of each of these pass through at least to the summit of C'_k. Hence, by Lemma 4, the contribution of C'_k and C''_{k+1} is increased. That of $C_{k+2}, C_{k+3}, \ldots$ is increased, by (b) above. Hence combination of C'_k and C''_{k+1} is advantageous. This completes the proof of (2).

(3) It follows that we can decrease N by a combination of two pieces except perhaps when $\gamma_1 < \gamma_2 < \cdots < \gamma_N$. But in this case every stretch stops at the summit of the piece in which it originates; and this continues to be so when we combine any two consecutive pieces. By Lemma 5, any such

combination is advantageous. We can therefore in any case decrease N with advantage, and the theorem follows.

7. It is sometimes convenient to use Theorem 2 in a different but obviously equivalent form. Suppose that

$$A(\nu, \mu) = A(\nu, \mu, a) = \frac{a_\mu + a_{\mu+1} + \cdots + a_\nu}{\nu - \mu + 1},$$

and that $a_1^*, a_2^*, \ldots, a_n^*$ are $a_1, a_2, \ldots, a_n$ rearranged in descending order of magnitude. Then Theorem 2 may be restated as follows.

Theorem 3. *If $\mu = \mu(\nu)$ is any function of ν which is a positive integer for every positive integral ν, and never exceeds ν, then*

$$\sum_1^n s\{A(\nu, \mu, a)\} \leqq \sum_1^n s\{A(\nu, 1, a^*)\}.$$

III.

The maximal problem for integrals.

8. Suppose that $f(x)$ is positive, bounded, and measurable in $(0, a)$, and let $m(y)$ be the measure of the set in which $f(x) \geqq y$, so that $m(y)$ is a decreasing function of y which vanishes for sufficiently large y. We define $f^*(x)$, for $0 \leqq x \leqq a$, by

$$f^*\{m(y)\} = y \quad (0 \leqq m(y) \leqq a);$$

if $m(y)$ has a discontinuity, with a jump from μ_1 to μ_2, then $f^*(x)$ is constant in (μ_1, μ_2). It is plain that $f^*(x)$ is a decreasing function. We call $f^*(x)$ *the rearrangement of $f(x)$ in decreasing order*. Thus if $f(x)$ is $1 - x$ in $0 \leqq x < 1$, and $2 - x$ in $1 \leqq x \leqq 2$, then $f^*(x)$ is $\frac{1}{2}x$ in $0 \leqq x \leqq 2$.

Sets of zero measure are irrelevant in the definition of $f^*(x)$. The upper bound of $f^*(x)$ is the *effective* upper bound of $f(x)$, that is to say the least ξ such that $f(x) \leqq \xi$ except in a nul set.

The definition applies also to unbounded integrable functions, for which $m(y) \to 0$ when $y \to \infty$. If $f(x)$ is effectively unbounded, that is to say if $f(x) > G$, for every G, in a set of positive measure, then $f^*(x)$ is a decreasing function with an infinite peak at the origin.

It is evident that f and f^* are 'equimeasurable', *i.e.* the measures of the sets in which they assume values lying in any given interval are equal; and that

$$\int_0^a \psi(f)\,dx = \int_0^a \psi(f^*)\,dx,$$

for any positive function ψ, whenever either integral exists.

We write

(8. 1) $$A(x,\xi) = A(x,\xi,f) = \frac{1}{x-\xi}\int_\xi^x f(t)\,dt \quad (0 \leqq \xi < x), \quad A(x,x) = f(x).$$

If $f(x)$ is bounded, $A(x,\xi)$ is bounded; in any case it is continuous in ξ except perhaps for $\xi = x$. We define $\Theta(x)$ by

(8. 2) $$\Theta(x) = \Theta(x,f) = \operatorname*{Max}_{0 \leqq \xi \leqq x} A(x,\xi)^{1} = \overline{\operatorname*{bound}_{0 \leqq \xi \leqq x}} A(x,\xi).$$

When $f(x)$ decreases, $\Theta(x) = A(x, 0)$.

9. The theorems for integrals corresponding to Theorems 1, 2 and 3 are as follows.

Theorem 4. *If $s(x)$ is continuous and increasing, then*

$$\int_0^a s\{A(x,0,f)\}\,dx \leqq \int_0^a s\{A(x,0,f^*)\}\,dx.$$

Theorem 5. *If $s(x)$ is continuous and increasing, then*

$$\int_0^a s\{\Theta(x,f)\}\,dx \leqq \int_0^a s\{\Theta(x,f^*)\}\,dx.$$

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[1] In what follows the symbol 'Max', when it refers to an infinite aggregate of values, is always to be interpreted in the sense of upper bound.

Theorem 6. *If $\xi = \xi(x)$ is any measurable function such that $0 \leqq \xi \leqq x$, then*

$$\int_0^a s\{A(x, \xi, f)\}\, dx \leqq \int_0^a s\{A(x, 0, f^*)\}\, dx.$$

Of these theorems, Theorem 4 corresponds to the trivial Theorem 1, and is included in Theorem 6, which is an alternative form of Theorem 5, and corresponds to Theorem 3 as Theorem 5 corresponds to Theorem 2. The results are always to be interpreted as meaning 'if the integral on the right hand side is finite, then that on the left is finite and satisfies the inequality'.

10. We can deduce Theorems 5 and 6 from Theorems 2 and 3 by fairly straightforward processes, but a little care is required, since a change of f in a set of small measure may alter f^* throughout the whole interval. We begin by proving the theorems for *continuous* f. It is easy to see that f^* is continuous if f is continuous; for if f^* has a jump, say from $y-\delta$ to $y+\delta$, the measure of the set in which f lies between $y-\delta$ and $y+\delta$ is zero, and this is impossible, since f is continuous and assumes the value y.

We may take $a=1$. If Theorem 6 is not true for continuous f, there is a continuous f and an associated ξ such that

$$J(\xi, f) = \int_0^1 s\{A(x, \xi, f)\}\, dx > \int_0^1 s\{A(x, 0, f^*)\}\, dx = J(0, f^*).$$

Let

$$a_\nu = f\left(\frac{\nu}{n}\right) \qquad (\nu = 1, 2, \ldots, n);$$

and let μ be the integer such that

$$\frac{\mu - 1}{n} < \xi\left(\frac{\nu}{n}\right) \leqq \frac{\mu}{n},$$

and

$$A_\nu = A(\nu, \mu, a) = \frac{a_\mu + a_{\mu+1} + \cdots + a_\nu}{\nu - \mu + 1},$$

$$A_\nu^* = A(\nu, 1, a^*) = \frac{a_1^* + a_2^* + \cdots + a_\nu^*}{\nu}.$$

We can choose n so that $S = \Sigma s(A_\nu)$ and $S^* = \Sigma s(A_\nu^*)$ differ by as little as we please from $J(\xi, f)$ and $J(0, f^*)$[1], and therefore so that $S > S^*$, in contradiction to Theorem 3. Hence Theorems 5 and 6 are true for continuous f.

11. We prove next that the theorems are true for any bounded measurable f. We can approximate to f by a continuous f_n which (a) differs from f by less than ε_n except in a set of measure less than δ_n and (b) tends to f, when $n \to \infty$, for almost all x. Here ε_n and δ_n are positive and tend to zero when $n \to \infty$.

We consider

$$(11.1) \qquad J(f_n) = J(\xi, f_n) = \int_0^1 s\{A(x, \xi, f_n)\}\, dx,$$

and make $n \to \infty$ (keeping $\xi(x)$ the same function of x throughout). The integrand is uniformly bounded; and, whether $\xi < x$ or $\xi = x$, the functions $A(x, \xi, f_n)$ and $s(A)$ tend almost always to the corresponding functions with f in place of f_n. It follows that $J(f_n) \to J(f)$. It is therefore sufficient to prove that

$$(11.2) \qquad J(f_n) < J(f^*) + \eta$$

for any positive η and sufficiently large n.

We have $f_n < f + \varepsilon_n$ except in a set E of measure $\delta < \delta_n$. We define $g = g(x, n)$ as $f + \varepsilon_n$ except in E and as $M + 2\varepsilon_n$, where M is the upper bound of f, in E. Then $f_n \leqq g$ and so $f_n^* \leqq g^*$, so that

$$J(f_n) \leqq J(f_n^*) \leqq J(g^*).$$

A moment's consideration shows that

$$g^*(x) = M + 2\varepsilon_n \ (0 < x < \delta), \quad g^*(x) \leqq f^*(x - \delta) + \varepsilon_n \ (\delta < x < 1).$$

Since $f, A, s(A)$ are uniformly bounded, we can choose λ so that

$$(11.3) \qquad \int_0^\lambda s\{A(x, 0, g^*)\}\, dx < \frac{1}{2}\eta;$$

[1] We suppress the straightforward but tiresome details of the proof.

and we may suppose $\delta < \lambda$. For $x > \delta$ we have

$$A(x, 0, g^*) \leqq \frac{\delta(M + 2\varepsilon_n)}{x} + \frac{1}{x}\int_{\delta}^{x} f^*(t-\delta)\,dt + \varepsilon_n$$

$$= \frac{\delta(M + 2\varepsilon_n)}{x} + \frac{1}{x}\int_{0}^{x-\delta} f^*(t)\,dt + \varepsilon_n.$$

Hence

$$\overline{\lim_{n\to\infty}} A(x, 0, g^*) \leqq A(x, 0, f^*),$$

and so, by (11.3)

$$\overline{\lim} \int_0^1 s\{A(x, 0, g^*)\}\,dx \leqq \frac{1}{2}\eta + \overline{\lim} \int_\lambda^1 s\{A(x, 0, g^*)\}\,dx \leqq \frac{1}{2}\eta + \int_0^1 s\{A(x, 0, f^*)\}\,dx.$$

This is equivalent to (11.2), so that the theorem is proved for any bounded measurable f.

12. We have finally to make the transition to unbounded functions.. If $f_n = \mathrm{Min}(f, n)$, f_n increases with n and tends to f for almost all x. If $\xi = \xi(x)$ is independent of n, $A_n = A(x, \xi, f_n)$ and $s(A_n)$ also increase with n and tend to $A(x, \xi, f)$ and $s(A)$ for almost all x, and

$$J(f_n) \to J(f)$$

whenever the right hand side exists. Hence, for sufficiently large n,

$$J(f) < J(f_n) + \varepsilon \leqq J(f_n^*) + \varepsilon \leqq J(f^*) + \varepsilon,$$

and so

$$J(f) \leqq J(f^*).$$

13. We add a few supplementary theorems which are trivial corollaries of Theorem 5, but which are useful in applications.

Theorem 7. *If $\Theta(x)$ is the upper bound of*

$$A(x, \xi, f) = \frac{1}{x-\xi}\int_\xi^x f(t)\,dt$$

for $a \leqq \xi \leqq b$, *then*

$$\int_a^b s\{\Theta(x)\}\,dx \leqq 2\int_a^b s\left(\frac{1}{x-a}\int_a^x f^*(t)\,dt\right)dx.$$

The constant 2 *is the best possible constant.*

We may take $a=0$, $b=1$. Then

$$\Theta \leqq \operatorname{Max}(\Theta_1, \Theta_2),\quad s(\Theta) \leqq \operatorname{Max}(s(\Theta_1), s(\Theta_2)),$$

where Θ_1 and Θ_2 are the upper bounds of

$$\frac{1}{x-\xi}\int_\xi^x f\,dt\ (0 \leqq \xi \leqq x),\quad \frac{1}{\xi-x}\int_x^\xi f\,dt\ (x \leqq \xi \leqq 1)$$

(the averages being replaced by $f(x)$ when $\xi = x$). It follows from Theorem 5 that

$$\int_0^1 s\{\Theta_1(x)\}\,dx \leqq \int_0^1 s\left(\frac{1}{x}\int_0^x f^*(t)\,dt\right)dx;$$

and it is obvious from symmetry that the corresponding integral with Θ_2 has the same upper bound.[1] This proves the theorem.

The factor 2 is the best possible constant. For suppose that $a=0$, $b=1$, and that $f(x)$ is 1 between $\frac{1}{2}(1-\delta)$ and $\frac{1}{2}(1+\delta)$ and 0 elsewhere, so that $f^*(x)$ is 1 between 0 and δ. An elementary calculation shows that the two integrals of Theorem 7 are then

$$\delta + \delta\log\frac{1}{\delta},\quad \delta + 2\delta\log\frac{1+\delta}{2\delta}$$

respectively, and their ratio tends to 2 when $\delta \to 0$.

[1] Θ_2 depends on averages over intervals to the right from x, and the f^* which arises then is an increasing function: this does not affect the final result.

IV.

Inequalities deduced from the maximal theorems.

14. Suppose in particular that

(14. 1) $$s(x) = x^k \qquad (k > 1).$$

It is known[1] that

(14. 2) $$\sum_1^n \left(\frac{a_1 + a_2 + \cdots + a_\nu}{\nu}\right)^k \leqq \left(\frac{k}{k-1}\right)^k \sum_1^n a_\nu^k$$

and

(14. 3) $$\int_0^a \left(\frac{1}{x}\int_0^x f(t)\,dt\right)^k dx \leqq \left(\frac{k}{k-1}\right)^k \int_0^a f^k(x)\,dx,$$

for finite or infinite n and a. Since

$$A(\nu, 1, a^*) = \frac{a_1^* + a_2^* + \cdots + a_\nu^*}{n}$$

and $\Sigma a_\nu^{*k} = \Sigma a_\nu^k$, with analogous formulae for integrals, we obtain the following theorems.

Theorem 8. *If $A, \mu = \mu(\nu)$, and a_ν are defined as in Theorems 2 and 3, then*

$$\sum_1^n A^k(\nu, \mu, a) \leqq \left(\frac{k}{k-1}\right)^k \sum_1^n a_\nu^k$$

and

$$\sum_1^n \alpha_\nu^k \leqq \left(\frac{k}{k-1}\right)^k \sum_1^n a_\nu^k.$$

Here n may be finite or infinite.

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[1] See for example G. H. Hardy, 'Note on a theorem of Hilbert', *Math. Zeitschrift,* 6 (1919), 314—317, and 'Notes on some points in the integral calculus', *Messenger of Math.,* 54 (1925), 150—156; and E. B. Elliott, 'A simple exposition of some recently proved facts as to convergency', *Journal London Math. Soc.,* 1 (1926), 93—96. A considerable number of other proofs have been given by other writers in the *Journal of the London Mathematical Society.*

Theorem 9. *If A, $\xi=\xi(x)$, and Θ are defined as in Theorems 5 and 6, then*

$$\int_0^a A^k(x, \xi, f)\, dx \leqq \left(\frac{k}{k-1}\right)^k \int_0^a f^k(x)\, dx$$

and

$$\int_0^a \Theta^k(x)\, dx \leqq \left(\frac{k}{k-1}\right)^k \int_0^a f^k(x)\, dx.$$

Here a may be finite or infinite.

Theorem 10. *If Θ is defined as in Theorem 7, then*

$$\int_0^a \Theta^k(x)\, dx \leqq 2\left(\frac{k}{k-1}\right)^k \int_a^b f^k(x)\, dx.$$

Here a and b may be finite or infinite.

We do not assert that the 2 here is a best possible constant.

15. 1. All the theorems of the last section become false for $k=1$. Thus

$$\int_0^a f\, dx \qquad (0<a<1)$$

is convergent when $f=x^{-1}\left(\log \frac{1}{x}\right)^{-2}$, but

$$\int_0^a \left(\frac{1}{x}\int_0^x f\, dt\right) dx$$

is divergent. There is, however, an interesting theorem corresponding to this case.

We shall say that $f(x)$ belongs to Z in a finite interval (a, b) if

$$\int_a^b |f| \log^+ |f|\, dx$$

exists. Here, as usual, $\log^+|f|$ is $\log|f|$ if $|f| \geqq 1$ and zero otherwise. Since

$$|f| \leqq \mathrm{Max}\,(e, |f|\log^+|f|),$$

any function of Z is integrable.[1] The importance of the class Z in the theory of functions has been shown recently by Zygmund.[2]

The following theorem contains rather more than we shall actually require, but is of sufficient interest to be stated completely.

Theorem 11. *Suppose that a is positive and finite; that $B = B(a)$ denotes generally a number depending on a only; that $f(x)$ is positive; and that*

$$f_1(x) = \int_0^x f(t)\,dt, \tag{15.11}$$

$$J = \int_0^a f\log^+ f\,dx, \quad K = \int_0^a \frac{f_1}{x}\,dx. \tag{15.12}$$

(i) *If J is finite then K is also finite, and*

$$K < BJ + B. \tag{15.13}$$

(ii) *When f is a decreasing function the converse is also true: if K is finite then J is also finite and*

$$J < BK\log^+ K + B. \tag{15.14}$$

(iii) *A necessary and sufficient condition that f should belong to Z is that*

$$\int_0^a A(x, 0, f^*)\,dx, \tag{15.15}$$

the integral of Theorem 6, with $s(x) = x$, should be finite.

It is not necessary to state explicitly that f is integrable; in case (i) because we have seen that any function of Z is integrable, and in case (ii) because the integrability of f is implied in the existence of K.

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[1] This would not necessarily be true if the interval were infinite.

[2] A. Zygmund, 'Sur les fonctions conjuguées', *Fundamenta Math.*, 13 (1929), 284—303.

15.2. We begin by proving a lemma.

Lemma 7. *If $a > 0$ and f is positive and integrable, then*

(15.21) $$\int_0^a f \log \frac{1}{x}\, dx = \int_0^a \frac{f_1}{x}\, dx + f_1(a) \log \frac{1}{a}$$

whenever either integral is finite.

By partial integration

(15.22) $$\int_\varepsilon^a f \log \frac{1}{x}\, dx = \int_\varepsilon^a \frac{f_1}{x}\, dx + f_1(a) \log \frac{1}{a} - f_1(\varepsilon) \log \frac{1}{\varepsilon}$$

for $0 < \varepsilon < a$. The conclusion follows if only

(15.23) $$f_1(\varepsilon) \log \frac{1}{\varepsilon} \to 0$$

when $\varepsilon \to 0$. If the second integral in (15.21) is finite, and $\varepsilon < 1$, (15.22) gives

$$\int_\varepsilon^a f \log \frac{1}{x}\, dx \leqq \int_0^a \frac{f_1}{x}\, dx + f_1(a) \log \frac{1}{a},$$

so that the first integral is also finite. If, conversely, the first integral is finite, then

$$f_1(\varepsilon) \log \frac{1}{\varepsilon} = \log \frac{1}{\varepsilon} \int_0^\varepsilon f dx \leqq \int_0^\varepsilon f \log \frac{1}{x}\, dx$$

tends to 0 with ε, which proves (15.23) and therefore (15.21).

15.3. (i) Suppose now first that f belongs to Z. We have

(15.31) $$\int_0^a f dx \leqq \int_0^a f(\log^+ f + e)\, dx < J + B.$$

Next,

$$uv \leqq u \log u + e^{v-1}$$

for all positive u and real v.[1] Taking $u=f$, $v=\frac{1}{2}\log\frac{1}{x}$, we obtain

$$\frac{1}{2}f\log\frac{1}{x} \leqq f\log f + \frac{1}{e\sqrt{x}} \leqq f\log^+ f + \frac{1}{e\sqrt{x}},$$

and so

$$\int_0^a f\log\frac{1}{x}\,dx \leqq 2\int_0^a f\log^+ f\,dx + \frac{4\sqrt{a}}{e}.$$

Hence, by (15.21),

$$\int_0^a \frac{f_1}{x}\,dx \leqq 2\int_0^a f\log^+ f\,dx + \frac{4\sqrt{a}}{e} + \left|\log\frac{1}{a}\right|\int_0^a f\,dx;$$

and plainly this, with (15.31), gives (15.13).

(ii) Suppose now that f is a decreasing function, and that K is finite. Then

$$xf(x) \leqq \int_0^x f\,dt = f_1(x),$$

$$f_1(a) = \int_0^a f\,dx \leqq \int_0^a \frac{f_1}{x}\,dx = K,$$

$$f(x) \leqq \frac{1}{x}\int_0^x f\,dt \leqq \frac{K}{x},$$

$$\log^+ f \leqq \log^+\frac{1}{x} + \log^+ K,$$

$$J = \int_0^a f\log^+ f\,dx \leqq \int_0^a f\log^+\frac{1}{x}\,dx + \log^+ K\int_0^a f\,dx$$

$$\leqq \int_0^b f\log^+\frac{1}{x}\,dx + K\log^+ K,$$

[1] This very useful inequality is due to W. H. Young, 'On a certain series of Fourier', *Proc. London Math. Soc.* (2), 11 (1913), 357—366.

where $b = \mathrm{Min}(a, 1)$. We distinguish the cases $a \geqq 1$ and $a < 1$, using Lemma 7 in either case. If $a \geqq 1$, we have $b = 1$ and

$$J \leqq \int_0^1 f \log \frac{1}{x}\, dx + K \log^+ K$$

$$= \int_0^1 \frac{f_1}{x}\, dx + K \log^+ K \leqq K \log^+ K + K.$$

If $a < 1$, we have $b = a$ and

$$J \leqq \int_0^a f \log \frac{1}{x}\, dx + K \log^+ K$$

$$= \int_0^a \frac{f_1}{x}\, dx + f_1(a) \log \frac{1}{a} + K \log^+ K \leqq K \log^+ K + BK.$$

Since $K < BK \log^+ K + B$, we obtain (15. 14) in either case.

(iii) The last clause of the theorem is now obvious, since f^* is a decreasing function and belongs to Z if and only if f does so.

16. It is plain that we can now assert theorems corresponding to Theorems 9 and 10. That which corresponds to Theorem 10 is

Theorem 12. *If Θ is defined as in Theorems 7 and 10, then*

$$\int_a^b \Theta\, dx \leqq B \int_a^b f \log^+ f\, dx + B,$$

where $B = B(a, b)$ depends on a and b only.

V.

Applications to function-theory.

17. In what follows we are concerned with integrable and periodic functions. We take the period to be 2π.

We write

$$M(\theta) = M(\theta, f) = \operatorname*{Max}_{0<|t|\leq\pi} \left| \frac{1}{t} \int_0^t f(\theta + x)\, dx \right|, \tag{17.1}$$

$$\bar{M}(\theta) = \bar{M}(\theta, f) = \operatorname*{Max}_{0<|t|\leq\pi} \left(\frac{1}{t} \int_0^t |f(\theta + x)|\, dx \right), \tag{17.2}$$

$$N(\theta) = N(\theta, f) = \operatorname*{Max}_{|t|\leq\pi} \left(\frac{1}{t} \int_0^t |f(\theta + x)|\, dx \right). \tag{17.3}$$

It is to be understood that 'Max' is used here in the sense of upper bound, and that the mean value in (17.3) is to be interpreted as $|f(\theta)|$ when $t = 0$.

The three functions are all of the same type as the functions $\Theta(x, f)$ of Theorems 5 and 7. There are, however, slight differences; and it is convenient to use θ as the fundamental variable when we are considering periodic functions. We are therefore compelled to vary our notation to a certain extent, and it will probably be least confusing to change it completely.

The differences between the three functions are comparatively trivial. Thus

$$N(\theta) = \operatorname{Max}(\bar{M}(\theta), |f(\theta)|), \tag{17.4}$$

the value $t = 0$ being relevant to N but not to $\bar{M}$. Sometimes one function presents itself most naturally and sometimes another, and it is convenient to have all three at our disposal. It is obvious that

$$M \leqq \bar{M} \leqq N = \operatorname{Max}(\bar{M}, |f|). \tag{17.5}$$

18. We denote by $A(k)$ a number depending only on k (or any other parameters shown), by A a positive absolute constant, not always the same from one occurrence to another.[1]

Theorem 13. *If* $k > 1$ *and*

$$\frac{1}{2\pi} \int_{-\pi}^{\pi} |f|^k d\theta \leq C^k,$$

[1] A will not occur again in the sense of Section III. Constants B, C in future preserve their identity.

then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} F^k d\theta \leqq A(k) C^k,$$

where F is any one of M, $\bar{M}$, N.

We take $F = N$, which is, by (17. 5), the most unfavourable case. Now N is the upper bound of

$$\frac{1}{t}\int_{\theta}^{\theta+t} |f(u)|\, du,$$

an average of $|f|$ over a range included in $(-2\pi, 2\pi)$. Hence, by Theorem 10,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} N^k d\theta \leqq 2\left(\frac{k}{k-1}\right)^k \frac{1}{2\pi}\int_{-2\pi}^{2\pi} |f|^k d\theta,$$

which proves the theorem.

Similarly Theorem 12 gives

Theorem 14. *If*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f| \log^+ |f|\, d\theta \leqq C$$

then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} F d\theta < AC + A.$$

19. A number of important functions associated with $f(\theta)$ are expressible in terms of $f(\theta)$ by a formula of the type

(19. 1) $$h(\theta, p) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta + t)\chi(t, p)\, dt,$$

where p is a parameter, and χ, the 'kernel', satisfies

(19. 2) $$\frac{1}{2\pi}\int_{-\pi}^{\pi} \chi(t, p)\, dt = 1.$$

Examples are $s_n(\theta)$, the 'Fourier polynomial' of $f(\theta)$, $\sigma_n(\theta)$, the 'Fejér polynomial'[1], and $u(r, \theta)$, the 'Poisson integral', or the harmonic function having $f(\theta)$ as 'boundary function'. For $s_n(\theta)$,

$$p = n, \quad \chi = \frac{\sin\left(n + \frac{1}{2}\right)t}{\sin\frac{1}{2}t}; \tag{19.3}$$

for $\sigma_n(\theta)$

$$p = n, \quad \chi = \frac{\sin^2\frac{1}{2}nt}{n\sin^2\frac{1}{2}t}; \tag{19.4}$$

and for $u(r, \theta)$,

$$p = r, \quad \chi = \frac{1 - r^2}{1 - 2r\cos t + r^2}. \tag{19.5}$$

20. The applications of our maximal theorems depend upon the fact that a number of functions $h(\theta, p)$ satisfy inequalities

$$|h(\theta, p)| \leqq KN(\theta), \tag{20.1}$$

where K is independent of θ and p. These inequalities in their turn depend upon inequalities

$$\frac{1}{2\pi}\int_{-\pi}^{\pi}\left|t\frac{\partial\omega}{\partial t}\right|dt \leqq B \qquad (\omega = \chi \text{ or } X), \tag{20.2}$$

in which B is independent of p, and ω is either χ itself or some majorant X of χ. Thus when $h(\theta, p) = u(r, \theta)$, χ satisfies (20.2). When $h(\theta, p) = \sigma_n(\theta)$, χ does not itself satisfy (20.1), but

$$0 < \chi \leqq \frac{A}{1 + n^2t^2} = X; \tag{20.3}$$

and X satisfies (20.2). When $h(\theta, p) = s_n(\theta)$, there is no such majorant. It is familiar that the differences between the 'convergence theory' of Fourier series

[1] $s_n(\theta)$ is formed from the first $n + 1$ terms of the Fourier series of $f(\theta)$, $\sigma_n(\theta)$ from the first n.

and the 'summability theory' depend primarily on the fact that $\int |\chi| \, dt$ is bounded in one case and not in the other; here we push this distinction a little further.

21. We begin by considering the case in which χ itself satisfies (20. 2).

Lemma 8. *If χ is periodic and satisfies* (19. 2) *and* (20. 2), *then*

(21. 1) $$|\chi(\pi)| \leqq B + 1,$$

(21. 2) $$\frac{1}{2\pi}\int_{-\pi}^{\pi} |\chi| \, dt \leqq 2B + 1.$$

(i) We have

$$\chi(\pi) = \frac{1}{2\pi}[t\chi]_{-\pi}^{\pi} = \frac{1}{2\pi}\int_{-\pi}^{\pi} \chi \, dt + \frac{1}{2\pi}\int_{-\pi}^{\pi} t\frac{\partial \chi}{\partial t} \, dt,$$

from which (21. 1) follows immediately.

(ii) Also

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |\chi| \, dt = \frac{1}{2\pi}[t|\chi|]_{-\pi}^{\pi} - \frac{1}{2\pi}\int_{-\pi}^{\pi} t\frac{\partial|\chi|}{\partial t} \, dt$$

$$\leqq |\chi(\pi)| + \frac{1}{2\pi}\int_{-\pi}^{\pi} \left| t\frac{\partial \chi}{\partial t}\right| dt \leqq 2B + 1.$$

22. **Theorem 15.** *If χ is periodic and satisfies* (19. 2) *and* (20. 2), *then*

(22. 1) $$|h(\theta, p)| \leqq (AB + A)M(\theta).$$

Let

$$f_1(t) = f_1(t, \theta) = \int_0^t f(\theta + u) \, du.$$

Then

$$h = \frac{1}{2\pi}[f_1(t)\chi]_{-\pi}^{\pi} - \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f_1(t)}{t} \cdot t\frac{\partial \chi}{\partial t} \cdot dt = h_1 + h_2,$$

say. Here

$$|h_1| = |\chi(\pi)| \left| \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta + u)\, du \right|$$

$$\leqq (B + 1) \operatorname{Max} \left\{ \left| \frac{1}{\pi}\int_{0}^{\pi} f(\theta + u)\, du \right|, \left| \frac{1}{\pi}\int_{-\pi}^{0} f(\theta + u)\, du \right| \right\}$$

$$\leqq (B + 1) M(\theta);$$

and

$$|h_2| \leqq M(\theta) \frac{1}{2\pi}\int_{-\pi}^{\pi} \left| t\frac{\partial \chi}{\partial t} \right| dt \leqq B M(\theta).$$

Hence we obtain (22. 1).

23. We can now prove our principal results concerning the harmonic function $u(r, \theta)$.

Theorem 16. *If* $u(r, \theta)$ *is the harmonic function whose boundary function is* $f(\theta)$, *then*

$$|u(r, \theta)| \leqq A M(\theta) \tag{23. 1}$$

for $r < 1$.

Here

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \chi\, dt = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{1 - r^2}{1 - 2r\cos t + r^2}\, dt = 1,$$

$$t\frac{\partial \chi}{\partial t} = -\frac{2r(1 - r^2)\, t \sin t}{(1 - 2r\cos t + r^2)^2} \leqq 0,$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \left| t\frac{\partial \chi}{\partial t} \right| dt = -\frac{1}{2\pi}\int_{-\pi}^{\pi} t\frac{\partial \chi}{\partial t}\, dt = \frac{2r}{1 + r} < 1.$$

Hence χ satisfies the conditions of Theorem 15, with $B = 1$, and the conclusion follows.

Theorem 17. *If $k > 1$ and*

(23.2) $$U(\theta) = \max_{r<1} |u(r, \theta)|,$$

then

(23.3) $$\frac{1}{2\pi}\int_{-\pi}^{\pi} U^k(\theta)\,d\theta \leqq A(k)\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta)|^k\,d\theta.$$

The result is false for $k = 1$.

The positive assertion is an immediate corollary of Theorems 13 and 16. To prove the result false for $k = 1$, take

$$u = \frac{R^2 - r^2}{R^2 - 2Rr\cos\theta + r^2} \qquad (R > 1).$$

An elementary calculation shows that u is a maximum, for a given θ and $r \leqq 1$, when

$$r = R(\sec\theta - |\tan\theta|),$$

provided that this is positive and less than 1, that is to say provided

$$\sigma = \arcsin\frac{R^2 - 1}{R^2 + 1} < |\theta| < \frac{1}{2}\pi\text{[1]};$$

and that then $U = \operatorname{cosec}\theta$. Hence

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} U(\theta)\,d\theta > \frac{1}{\pi}\int_{\sigma}^{\frac{1}{2}\pi} \frac{d\theta}{\sin\theta},$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta)\,d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{R^2 - 1}{R^2 - 2R\cos\theta + 1}\,d\theta = 1.$$

Since the first integral tends to infinity when $R \to 1$, $\sigma \to 0$, we can falsify (23.3), for $k = 1$ and any A, by taking R sufficiently near to 1.

The theorem corresponding to the case $k = 1$ is

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[1] When $|\theta| < \sigma$ the maximum is given by $r = 1$, and when $|\theta| > \frac{1}{2}\pi$ by $r = 0$.

Theorem 18. *If $f(\theta)$ belongs to Z and*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta)| \log^+ |f(\theta)| \, d\theta \leqq C,$$

then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} U(\theta)\, d\theta < AC + A.$$

This is a corollary of Theorems 14 and 16.

24. Before going further with the theory of harmonic and analytic functions, we consider the case $h(\theta, p) = \sigma_n(\theta)$, which is typical of the second possibility mentioned in § 20. In this case χ does not satisfy (20. 2); but

$$0 \leqq \chi = \frac{\sin^2 \frac{1}{2} nt}{n \sin^2 \frac{1}{2} t} < \frac{An}{1 + n^2 t^2} = X^1, \tag{24. 1}$$

and

$$0 < X(\pi) = X(-\pi) \leqq B \tag{24. 2}$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \left| t \frac{\partial X}{\partial t} \right| dt = \frac{A}{\pi}\int_{-\pi}^{\pi} \frac{n^3 t^2 dt}{(1 + n^2 t^2)^2} < \frac{A}{\pi}\int_{-\infty}^{\infty} \frac{u^2 du}{(1 + u^2)^2},$$

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \left| t \frac{\partial X}{\partial t} \right| dt \leqq C, \tag{24. 3}$$

where B and C are independent of n.

Theorem 19. *If χ is periodic and has a majorant X which satisfies (24. 2) and (24. 3), then*

$$|h(\theta, p)| \leqq (B + C)\, M(\theta). \tag{24. 4}$$

[1] The usefulness of a kernel of the type of X was first pointed out by Fejér. See L. Fejér, 'Über die arithmetischen Mittel erster Ordnung der Fourierreihe', *Göttinger Nachrichten*, 1925, 13—17.

We have

$$(24.5) \qquad |h(\theta, p)| \leqq \frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta+t)|\, X\, dt = H(\theta, p),$$

say. If now

$$f_1(t) = f_1(t, \theta) = \int_0^t |f(\theta+u)|\, du,$$

(24.5) gives

$$H(\theta, p) = \frac{1}{2\pi}[f_1(t) X]_{-\pi}^{\pi} - \frac{1}{2\pi}\int_{-\pi}^{\pi} \frac{f_1(t)}{t} \cdot t \frac{\partial X}{\partial t} \cdot dt = H_1 + H_2,$$

say. Here

$$|H_1| = X(\pi)\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta+u)|\, du$$

$$\leqq B \operatorname{Max}\left(\frac{1}{\pi}\int_0^{\pi} |f(\theta+u)|\, du,\ \frac{1}{\pi}\int_{-\pi}^{0} |f(\theta+u)|\, du\right)$$

$$\leqq B\bar{M}(\theta),$$

and

$$|H_2| \leqq \bar{M}(\theta)\frac{1}{2\pi}\int_{-\pi}^{\pi} \left| t\frac{\partial X}{\partial t}\right| dt \leqq C\bar{M}(\theta).$$

Hence we obtain (24.4).

25. **Theorem 20.** *If $\sigma_n(\theta)$ is the Fejér polynomial formed from the first n terms of the Fourier series of $f(\theta)$, then*

$$(25.1) \qquad |\sigma_n(\theta)| \leqq A\bar{M}(\theta).$$

This is a corollary of Theorem 19, since we have already verified that the kernel of $\sigma_n(\theta)$ satisfies the conditions of Theorem 19, with $B=A$, $C=A$.

The theorems corresponding to Theorems 17 and 18 are

Theorem 21. *If $k > 1$ and*

$$\Sigma(\theta) = \operatorname*{Max}_{(n)} |\sigma_n(\theta)|, \tag{25.2}$$

then

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \Sigma^k(\theta)\,d\theta \leqq A(k)\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta)|^k d\theta. \tag{25.3}$$

The result is false for $k=1$.

Theorem 22. *If* $f(\theta)$ *satisfies the conditions of Theorem* 18, *then*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} \Sigma(\theta)\,d\theta < A\,C + A. \tag{25.4}$$

The positive assertions are corollaries of Theorems 13, 14 and 20. The negative one becomes obvious when we remember that a bound of $\sigma_n(\theta)$ for varying n is also a bound of $u(r,\theta)$ for varying r, so that $U(\theta) \leqq \Sigma(\theta)$. For the same reason (23.3) is a corollary of (25.3).

Theorem 23. *The results of Theorems* 20, 21 *and* 22 *remain true when* $\sigma_n(\theta)$ *denotes a Cesàro mean of any positive order* δ, *provided that* A *and* $A(k)$ *are replaced by* $A(\delta)$ *and* $A(k,\delta)$.

It is only necessary to verify that the χ now corresponding to $\sigma_n(\theta)$ has a majorant X which satisfies (24.2) and (24.3), with values of B and C of the type $A(\delta)$.

We may suppose $\delta < 1$ (an upper bound of a lower mean being *a fortiori* one of a higher mean). We have then[1]

$$\chi = \chi_1 + \chi_2 = \frac{\Gamma(\delta+1)\Gamma(n+1)}{\Gamma(n+\delta+1)} \frac{\sin\left\{\left(n+\frac{1}{2}\delta+\frac{1}{2}\right)t - \frac{1}{2}\delta\pi\right\}}{2^\delta\left(\sin\frac{1}{2}t\right)^{\delta+1}} + \chi_2,$$

$$|\chi| \leqq A(\delta)n, \quad |\chi_2| \leqq \frac{A(\delta)}{nt^2}.$$

[1] E. Kogbetliantz, 'Les séries trigonométriques et les séries sphériques', *Annales de l'Ecole Normale* (3), 40 (1923), 259—323.

Hence $|\chi| \leqq A(\delta)n$ if $|nt| \leqq 1$ and

$$|\chi| \leqq A(\delta)\,\mathrm{Max}\left(\frac{1}{n^{\delta}|t|^{\delta+1}}, \frac{1}{nt^2}\right) = \frac{A(\delta)}{n^{\delta}|t|^{\delta+1}}$$

if $|nt| > 1$. We may therefore take

$$X = \frac{A(\delta)n}{1 + n^{\delta}|t|^{\delta+1}},$$

and it may be verified at once that this X has the properties required.

26. We return now to the case $h = u(r, \theta)$. We suppose that $0 \leqq \alpha < \frac{1}{2}\pi$ and that $S_\alpha(\theta)$ is the kite-shaped area defined by drawing two lines through $e^{i\theta}$ at angles α with the radius vector and dropping perpendiculars upon them from the origin[1]; and we denote by $U(\theta, \alpha)$ the upper bound of $|u(r, \theta)|$ for $z = re^{i\theta}$ interior to $S_\alpha(\theta)$.

Theorem 24. *If $k > 1$ then*

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} U^k(\theta, \alpha)\,d\theta \leqq A(k, \alpha)\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(\theta)|^k\,d\theta. \tag{26. 1}$$

If $z_1 = r_1 e^{i\theta_1}$ is any point in $S_\alpha(\theta)$, then

$$u(z_1) = \frac{1}{2\pi}\int_{-\pi}^{\pi} f(\theta + t)\,\chi(t, z_1)\,dt,$$

where

$$\chi(t, z_1) = \frac{1 - r_1^2}{1 - 2r_1\cos(t - \theta + \theta_1) + r_1^2}.$$

It is easily verified that $\chi(t, z_1)$ satisfies (19. 2) and (20. 2), the B being of the form $A(k, \alpha)$. This proves the theorem.

27. An equivalent form of Theorem 24 is as follows.

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[1] There is of course no particular point in the precise shape of $S_\alpha(\theta)$; it is an area of fixed size and shape including all 'Stolz-paths' to $e^{i\theta}$ inside an angle 2α. The radius vector corresponds to $\alpha = 0$

Theorem 25. *If $k > 1$, $u(r, \theta)$ is harmonic for $r < 1$ and satisfies*

(27. 1) $$\frac{1}{2\pi}\int_{-\pi}^{\pi} |u(r, \theta)|^k d\theta \leqq C^k,$$

and $U(\theta, \alpha)$ is the upper bound of $|u|$ in $S_\alpha(\theta)$, then

(27. 2) $$\frac{1}{2\pi}\int_{-\pi}^{\pi} U^k(\theta, \alpha) d\theta \leqq A(k, \alpha) C^k.$$

Let $S_\alpha(r, \theta)$ be the region related to $re^{i\theta}$ as $S_\alpha(\theta)$ is to $e^{i\theta}$, and let $U(r, \theta, \alpha)$ be the upper bound of $|u|$ in $S_\alpha(r, \theta)$. By Theorem 24,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} U^k(r, \theta, \alpha)\, d\theta \leqq A(k, \alpha)\frac{1}{2\pi}\int_{-\pi}^{\pi} |u(r, \theta)|^k d\theta \leqq A(k, \alpha) C^k.$$

But $U(r, \theta, \alpha)$ tends by increasing values to $U(\theta, \alpha)$, and we may take limits under the integral sign. This proves (27. 2).

28. **Theorem 26.** *If $k > 1$, $w(r, \theta)$ is positive and subharmonic for $r < 1$,*

(28. 1) $$\frac{1}{2\pi}\int_{-\pi}^{\pi} w^k(r, \theta) d\theta \leqq C^k$$

for $r < 1$, and $W(\theta, \alpha)$ is the upper bound of w in $S_\alpha(\theta)$, then

(28. 2) $$\frac{1}{2\pi}\int_{-\pi}^{\pi} W^k(\theta, \alpha) d\theta \leqq A(k, \alpha) C^k.$$

There is a harmonic function $u(r, \theta)$ such that

$$w \leqq u, \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} u^k(r, \theta) d\theta \leqq C^k.$$ [1]

Hence Theorem 26 is a corollary of Theorem 25.

[1] J. E. Littlewood, 'On functions subharmonic in a circle', *Journal Lond. Math. Soc.*, 2 (1927), 192—196.

• We can if we please avoid any appeal to this theorem of Littlewood. Suppose for simplicity of writing that $\alpha = 0$ and that all integrations are over $(-\pi, \pi)$, and let

29. **Theorem 27.** *Suppose that $\lambda > 0$, that $f(z)$ is an analytic function regular for $r < 1$, that*

(29. 1)
$$\frac{1}{2\pi}\int_{-\pi}^{\pi} |f(z)|^{\lambda} d\theta \leqq C^{\lambda} \qquad (r < 1),$$

and that

(29. 2)
$$F = F(\theta, \alpha) = \underset{S_\alpha(\theta)}{\text{Max}}\, |f(z)|.$$

Then

(29. 3)
$$\frac{1}{2\pi}\int_{-\pi}^{\pi} F^{\lambda} d\theta \leqq A(\lambda, \alpha)\, C^{\lambda}.$$

The most important case is that in which $\alpha = 0$, $S_\alpha(\theta)$ is the radius vector, and $A(\lambda, \alpha) = A(\lambda)$. It is to be observed that λ, unlike the k of previous theorems, is not restricted to be greater than 1.

Theorem 27 is an immediate corollary of Theorem 26, since

$$w = |f|^{\frac{1}{2}\lambda}$$

is a positive subharmonic function satisfying

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} w^2 d\theta = \frac{1}{2\pi}\int_{-\pi}^{\pi} |f|^{\lambda} d\theta \leqq C^{\lambda}.$$

$$u_\varrho(r, \theta) = \frac{1}{2\pi}\int \frac{w(\varrho, \varphi)\,d\varphi}{\varrho^2 - 2r\varrho\cos(\varphi - \theta) + r^2} \qquad (0 < r < \varrho).$$

Then u_ϱ is harmonic and assumes the values w for $r = \varrho$, and, by F. Riesz's fundamental theorem on sub-harmonic functions, $w \leqq u_\varrho$. Hence, using capital letters to denote radial maxima, and observing that $\int u_\varrho^k(r, \theta)\,d\theta$ increases with r, we have

$$\int W^k(r, \theta)\,d\theta \leqq \int U_\varrho^k(r, \theta)\,d\theta \leqq A(k)\int u_\varrho^k(r, \theta)\,d\theta$$
$$\leqq A(k)\int u_\varrho^k(\varrho, \theta)\,d\theta = A(k)\int w^k(\varrho, \theta)\,d\theta \leqq A(k)C^k.$$

The theorem is, however, the most obviously interesting of all those which we have proved, and it may be desirable to give a proof independent of the theory of sub-harmonic functions.

(i) Suppose that $\lambda = 2, f = u + iv$. Then $|f|^2 = u^2 + v^2$, $F^2 \leqq U^2 + V^2$,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} u^2 d\theta \leqq C^2, \quad \frac{1}{2\pi}\int_{-\pi}^{\pi} v^2 d\theta \leqq C^2,$$

and by Theorem 25,

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} F^2 d\theta \leqq \frac{1}{2\pi}\int_{-\pi}^{\pi} U^2 d\theta + \frac{1}{2\pi}\int_{-\pi}^{\pi} V^2 d\theta \leqq A(\alpha)\, C^2.$$

This proves the theorem in the particular case $\lambda = 2$.

(ii) Suppose next that λ is any positive number and that f has no zeros for $r < 1$. Then

$$f^{\frac{1}{2}\lambda} = g$$

is regular for $r < 1$ and $|f|^\lambda = g^2$, $F^\lambda = G^2$, so that the theorem follows from (i).

(iii) Finally consider the general case. It was proved by F. Riesz[1] that an f satisfying (29. 1) can be expressed as a product gh, where $|h| \leqq 1$ for $r < 1$, while g has no zeros for $r < 1$ and satisfies (29. 1). Hence $F \leqq G$ and, from case (ii),

$$\frac{1}{2\pi}\int_{-\pi}^{\pi} F^\lambda d\theta \leqq \frac{1}{2\pi}\int_{-\pi}^{\pi} G^\lambda d\theta \leqq A(\lambda, \alpha)\, C^\lambda.$$

30. It is well known that a function $f(z)$ satisfying (29. 1) tends to a »boundary function» $f(e^{i\theta})$, for almost all θ, as z tends radially to $e^{i\theta}$. Theorem 27 carries with it as corollaries several well known theorems concerning the behaviour of f near the boundary, which can be read out of it with the help of a well-known principle. Thus, the functions

$$|f(z)|^\lambda, \quad |f(z) - f(e^{i\theta})|^\lambda$$

[1] F. Riesz, 'Über die Randwerte einer analytischen Funktion', *Math. Zeitschrift*, 18 (1923), 87—95.

are majorized by $A(\lambda) F^{\lambda}(\theta)$, an integrable function; their convergence to their limit functions $|f(e^{i\theta})|^{\lambda}$ and o is »dominated» convergence, and we may proceed to the limit under the sign of integration. The three rather subtle results

$$\lim_{r \to 1} \frac{1}{2\pi} \int |f(re^{i\theta})|^{\lambda} d\theta = \frac{1}{2\pi} \int |f(e^{i\theta})|^{\lambda} d\theta,$$

$$\lim_{r \to 1} \frac{1}{2\pi} \int |f(re^{i\theta}) - f(e^{i\theta})|^{\lambda} d\theta = 0,$$

and

$$\lim_{r \to 1} \frac{1}{2\pi} \int_{E(r)} |f(re^{i\theta})|^{\lambda} d\theta = 0$$

thus become immediate. In the last of them $E(r)$ denotes a set of θ, varying with r, whose measure tends to zero as $r \to 1$.

In the same way the convergence of

$$|\sigma_n(\theta)|^k, \quad |\sigma_n(\theta) - f(\theta)|^k$$

to $|f(\theta)|^k$ and o is »dominated» convergence when $k > 1$; and we may infer at once the well-known results

$$\lim_{n \to \infty} \frac{1}{2\pi} \int |\sigma_n(\theta)|^k d\theta = \frac{1}{2\pi} \int |f(\theta)|^k d\theta,$$

$$\lim_{n \to \infty} \frac{1}{2\pi} \int |\sigma_n(\theta) - f(\theta)|^k d\theta = 0.$$