CONCENTRATION OF MASS AND CENTRAL LIMIT PROPERTIES OF ISOTROPIC CONVEX BODIES

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Abstract. We discuss the following question: Do there exist an absolute constant $c > 0$ and a sequence $\phi(n)$ tending to infinity with n, such that for every isotropic convex body K in $\mathbb{R}^n$ and every $t \geq 1$ the inequality $\text{Prob}\big(\{x \in K : \|x\|_2 \geq c\sqrt{n}L_K t\}\big) \leq \exp\big(-\phi(n)t\big)$ holds true? Under the additional assumption that K is 1-unconditional, Bobkov and Nazarov have proved that this is true with $\phi(n) \simeq \sqrt{n}$. The question is related to the central limit properties of isotropic convex bodies. Consider the spherical average $f_K(t) = \int_{S^{n-1}} |K \cap (\theta^\perp + t\theta)|\sigma(d\theta)$. We prove that for every $\gamma \geq 1$ and every isotropic convex body K in $\mathbb{R}^n$, the statements (A) "for every $t \geq 1$, $\text{Prob}\big(\{x \in K : \|x\|_2 \geq \gamma\sqrt{n}L_K t\}\big) \leq \exp\big(-\phi(n)t\big)$" and (B) "for every $0 < t \leq c_1(\gamma)\sqrt{\phi(n)}L_K$, $f_K(t) \leq \frac{c_2}{L_K}\exp\big(-t^2/(c_3(\gamma)^2L_K^2)\big)$, where $c_i(\gamma) \simeq \gamma$", are equivalent.

1. Introduction

Let K be an isotropic convex body in $\mathbb{R}^n$. This means that K has volume equal to 1, its centre of mass is at the origin and its inertia matrix is a multiple of the identity. Equivalently, there exists a positive constant L_K such that

$$\int_K \langle x, \theta\rangle^2 dx = L_K^2 \tag{1.1}$$

for every $\theta \in S^{n-1}$. As a direct consequence of (1.1) we have

$$\int_K \|x\|_2^2 dx = nL_K^2, \tag{1.2}$$

where $\|\cdot\|_2$ denotes the Euclidean norm. Applying Markov's inequality we see that $|K \cap (3\sqrt{n}L_K)B_2^n| \geq 8/9$, and Borell's lemma (see [12], Appendix I) proves the following:

Fact 1: *If K is an isotropic convex body in $\mathbb{R}^n$, then*

$$\text{Prob}\big(\{x \in K : \|x\|_2 \geq 3\sqrt{n}L_K t\}\big) \leq \exp(-t) \tag{1.3}$$

for every $t \geq 1$.

Alesker [1] showed that if K is isotropic, then the Euclidean norm $f(x) = \|x\|_2$ satisfies the ψ_2-estimate

$$\|f\|_{\psi_2} \leq c\|f\|_1 \leq c\sqrt{n}L_K, \tag{1.4}$$

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where $c>0$ is an absolute constant and

$$\|f\|_{\psi_2} = \inf\left\{\lambda > 0 : \int_K \exp\left((|f(x)|/\lambda)^2\right) dx \leq 2\right\}. \tag{1.5}$$

In particular, we have the following improvement of the estimate in Fact 1:

FACT 2: *There exists an absolute constant $c>0$ such that if K is an isotropic convex body in $\mathbb{R}^n$, then*

$$\mathrm{Prob}\left(\{x \in K : \|x\|_2 \geq c\sqrt{n}L_K t\}\right) \leq 2\exp(-t^2) \tag{1.6}$$

for every $t \geq 1$.

Bobkov and Nazarov [7] have recently obtained a striking stronger result in the case of 1-unconditional isotropic convex bodies.

FACT 3: *There exists an absolute constant $c>0$ such that if K is a 1-unconditional isotropic convex body in $\mathbb{R}^n$, then*

$$\mathrm{Prob}\left(\{x \in K : \|x\|_2 \geq c\sqrt{n}t\}\right) \leq \exp\left(-\sqrt{n}t\right) \tag{1.7}$$

for every $t \geq 1$.

Note that $L_K \simeq 1$ in the case of 1-unconditional convex bodies (see [11]). Since the circumradius $R(K)$ of an isotropic convex body K in $\mathbb{R}^n$ is always bounded by $(n+1)L_K$ [10], the estimate in Fact 3 is stronger than the previous ones for all $t \geq 1$. A question which arises naturally and was actually stated in [7] is the following:

QUESTION: *Do there exist an absolute constant $c>0$ and a function $\phi : \mathbb{N} \to \mathbb{R}^+$ with $\phi(n) \to \infty$ as $n \to \infty$, such that for every isotropic convex body K in $\mathbb{R}^n$ the inequality*

$$\mathrm{Prob}\left(\{x \in K : \|x\|_2 \geq c\sqrt{n}L_K t\}\right) \leq \exp\left(-\phi(n)t\right) \tag{1.8}$$

holds true for every $t \geq 1$?

As we shall see, the question is related to the central limit properties of isotropic convex bodies. It has been conjectured that the $(n-1)$-dimensional volume $f_{K,\theta}(t)$ of the intersections $K \cap (\theta^\perp + t\theta)$ of an isotropic convex body K with hyperplanes perpendicular to a fixed direction $\theta \in S^{n-1}$, seen as a function of the distance $t \geq 0$ of the hyperplanes to the origin, is - with high probability - close to the centered Gaussian density of variance L_K^2. This conjecture can be stated precisely in several different ways (see [8], [2]) and has been verified only for some special classes of bodies. Bobkov and Koldobsky [6] (see also [8]) have considered the spherical average

$$f_K(t) = \int_{S^{n-1}} f_{K,\theta}(t)\sigma(d\theta), \tag{1.9}$$

and showed that if K is an isotropic convex body in $\mathbb{R}^n$, then

$$\left| f_K(t) - \frac{1}{\sqrt{2\pi}L_K}\exp(-t^2/(2L_K^2))\right| \leq C\left(\frac{\sigma_K L_K}{t^2\sqrt{n}} + \frac{1}{n}\right) \tag{1.10}$$

for all $0 < t \leq c\sqrt{n}$, where $c, C > 0$ are absolute constants and the parameter σ_K is defined by

$$\sigma_K^2 = \frac{\mathrm{Var}(\|x\|_2^2)}{nL_K^4}. \tag{1.11}$$

It is conjectured that σ_K is bounded by an absolute constant (this has been verified for all ℓ_p^n-balls by Ball and Perissinaki [3]).

The main result of this note shows that the original question is closely related to the behavior of the function f_K.

Theorem 1.1. *Let $1 \ll \phi(n) \ll n$ be a positive constant. For every isotropic convex body K in $\mathbb{R}^n$, the following statements are equivalent:*

(a) *For some $\gamma \geq 1$ and for every $t \geq 1$,*

$$\text{Prob}\big(\{x \in K : \|x\|_2 \geq \gamma\sqrt{n}L_K t\}\big) \leq \exp\big(-\phi(n)t\big). \tag{1.12}$$

(b) *For every $0 < t \leq c_1(\gamma)\sqrt{\phi(n)}L_K$,*

$$f_K(t) \leq \frac{c_2}{L_K}\exp\big(-t^2/(c_3(\gamma)^2L_K^2)\big), \tag{1.13}$$

where $c_i(\gamma) \simeq \gamma$.

(c) *For every $2 \leq q \leq c_4\phi(n)$,*

$$I_q(K) = \left(\int_K \|x\|_2^q dx\right)^{1/q} \leq c_5(\gamma)\sqrt{n}L_K, \tag{1.14}$$

where $c_5(\gamma) \simeq \gamma$.

In a few words, the volume of an isotropic convex body outside a ball of radius $\sqrt{n}L_K$ "suddenly" decreases if and only if f_K is subgaussian for a "long initial interval". Both conditions are in turn equivalent to the fact that the moments I_q of the Euclidean norm remain of the order of $\sqrt{n}L_K$ for large values of q. The dependence of the constants $c(\gamma)$ on γ is linear in each of the implications of the theorem; this will become clear in §3.

We do not know if the question has an affirmative answer. However, our result has the following consequence (which gives some positive evidence, combined with the conjectured bound for the parameter σ_K).

Theorem 1.2. *Let K be an isotropic convex body in $\mathbb{R}^n$. Then,*

$$\text{Prob}\big(\{x \in K : \|x\|_2 \geq C\sqrt{n}L_K t\}\big) \leq \exp\big(-\phi(K)t\big) \tag{1.15}$$

for all $t \geq 1$, where

$$\phi(K) = \min\Big\{\log\left(\frac{n^2}{\text{Var}(\|x\|_2^2)}\right), \log n\Big\} \tag{1.16}$$

and $C > 0$ is an absolute constant.

It is easy to check that if σ_K and L_K are uniformly bounded, then $\phi(K) \simeq \log n$, in which case

$$\text{Prob}\big(\{x \in K : \|x\|_2 \geq C_1\sqrt{n}t\}\big) \leq n^{-t}, \tag{1.17}$$

for all $t \geq 1$, where $C_1 > 0$ is an absolute constant.

Notation. We work in $\mathbb{R}^n$, which is equipped with a Euclidean structure $\langle\cdot,\cdot\rangle$. We denote by $\|\cdot\|_2$ the corresponding Euclidean norm, and write B_2^n for the Euclidean unit ball, and S^{n-1} for the unit sphere. Volume is denoted by $|\cdot|$. We write ω_n for the volume of B_2^n and σ for the rotationally invariant probability measure on S^{n-1}. The circumradius of K is the quantity $R(K) = \max\{\|x\|_2 : x \in K\}$.

Whenever we write $a \simeq b$, we mean that there exist absolute constants $c_1, c_2 > 0$ such that $c_1 a \leq b \leq c_2 a$ (also, $a \gg b$ means that a exceeds Cb for some (large) absolute constant $C > 1$). The letters c, c', C, c_1, c_2 etc. denote absolute positive constants which may change from line to line. We refer to [12], [14] for background information on convex bodies and finite dimensional normed spaces, and to [11] for more information on the isotropic position.

2. Preliminaries

Let $\mathcal{P}_{d,n}$ denote the space of polynomials $f : \mathbb{R}^n \to \mathbb{R}$ of degree less than or equal to d. Bourgain [5] (see also Bobkov [4]) proved that for every $1 \leq q, r \leq \infty$ there exists a constant $c_{q,r,d} > 0$ depending only on q, r and d, such that $\|f\|_{L^q(K)} \leq c_{q,r,d}\|f\|_{L^r(K)}$ for every $f \in \mathcal{P}_{d,n}$ and every convex body K of volume 1 in $\mathbb{R}^n$. Carbery and Wright [9] have recently established the best possible dependence of the constant $c_{q,r,d}$ on q, r and d. We will use some estimates which follow directly from their work.

Lemma 2.1. *There exists an absolute constant $\alpha > 0$ such that for every convex body K of volume 1 in $\mathbb{R}^n$ and for every $f \in \mathcal{P}_{d,n}$*

$$\|f^{\#}\|_q \leq \alpha \frac{q}{r} \|f^{\#}\|_r \tag{2.1}$$

whenever $1 \leq r \leq q < \infty$, and

$$\|f^{\#}\|_\infty \leq \alpha \|f^{\#}\|_n, \tag{2.2}$$

where $f^{\#}(x) = |f(x)|^{1/d}$. □

Using Lemma 2.1 one can obtain a variety of tail estimates for polynomials $f \in \mathcal{P}_{d,n}$.

Lemma 2.2. *Let K be a convex body of volume 1 in $\mathbb{R}^n$ and let $f \in \mathcal{P}_{d,n}$. Then,*

$$\mathrm{Prob}\left(\left\{x \in K : f^{\#}(x) \geq 3\alpha \|f^{\#}\|_q \cdot s\right\}\right) \leq e^{-qs} \tag{2.3}$$

for all $q \geq 1$ and $s \geq 1$, where α is the constant in Lemma 2.1.

Proof. Let $q \geq 1$. Lemma 2.1 implies that

$$\int_K f^{\#}(x)^{qp} dx \leq (\alpha p)^{qp} \|f^{\#}\|_q^{qp}. \tag{2.4}$$

for every $p \geq 1$. With a simple application of Markov's inequality we get

$$\mathrm{Prob}\left(\left\{x \in K : f^{\#}(x) \geq 3\alpha \|f^{\#}\|_q \cdot s\right\}\right) \leq \left(\frac{p}{3s}\right)^{qp}. \tag{2.5}$$

Then, the choice $p = 3s/e \geq 1$ gives the assertion of the lemma. □

For every $q > 0$ we consider the q-th moment of the Euclidean norm

$$I_q(K) = \left(\int_K \|x\|_2^q dx\right)^{1/q}. \tag{2.6}$$

Applying Lemma 2.2 for the linear functionals $x \mapsto \langle x, \theta\rangle$ and for the polynomial $f(x) = \|x\|_2^2$, we have the following immediate consequence.

Lemma 2.3. *Let K be a convex body of volume 1 in $\mathbb{R}^n$. If $q \geq 1$, then*

$$\mathrm{Prob}\left(\left\{x \in K : |\langle x, \theta\rangle| \geq 3\alpha \|\langle \cdot, \theta\rangle\|_q s\right) \leq e^{-qs} \tag{2.7}$$

for all $\theta \in S^{n-1}$ and $s \geq 1$, and

$$\mathrm{Prob}\left(\left\{x \in K : \|x\|_2 \geq 3\alpha I_q(K) s\right\}\right) \leq e^{-qs} \tag{2.8}$$

for all $s \geq 1$, where α is the constant in Lemma 2.1. □

Definition. Let K be an isotropic convex body in $\mathbb{R}^n$. For every $q > 0$ and $t > 0$ we define

$$Z(q) = \left(\int_{S^{n-1}} \int_K |\langle x, \theta \rangle|^q dx \; \sigma(d\theta) \right)^{1/q} \tag{2.9}$$

and

$$Z(q,t) = \left(\int_{S^{n-1}} \int_{B_{K,\theta}(t)} |\langle x, \theta \rangle|^q dx \; \sigma(d\theta) \right)^{1/q} \tag{2.10}$$

where

$$B_{K,\theta}(t) = \{ x \in K : |\langle x, \theta \rangle| \leq t \}. \tag{2.11}$$

Lemma 2.4. *Let K be a convex body of volume 1 in $\mathbb{R}^n$. For every $t > 0$ we have the identity*

$$Z^q(q,t) = 2 \int_0^t r^q f_K(r) dr. \tag{2.12}$$

Proof. It is an immediate consequence of Fubini's theorem:

$$\begin{aligned} Z^q(q,t) &= 2 \int_{S^{n-1}} \int_0^t r^q f_{K,\theta}(r) dr \sigma(d\theta) = 2 \int_0^t r^q \int_{S^{n-1}} f_{K,\theta}(r) \sigma(d\theta) dr \\ &= 2 \int_0^t r^q f_K(r) dr, \end{aligned}$$

by the definition of f_K. □

The quantities $Z(q)$ and $I_q(K)$ are related through the following simple lemma (for a proof see [13]).

Lemma 2.5. *Let K be a convex body of volume 1 in $\mathbb{R}^n$. Then,*

$$Z(q) \simeq \sqrt{\frac{q}{q+n}} I_q(K). \tag{2.13}$$

for every $q \geq 1$. □

For every $\theta \in S^{n-1}$ and $q \geq 1$ we write $H_q(\theta) = \| \langle \cdot, \theta \rangle \|_q$. The next lemma shows that integration of the function $|\langle \cdot, \theta \rangle|^q$ on the strip $B_{K,\theta}(3\alpha H_q(\theta) s)$ essentially captures the value of $H_q^q(\theta)$.

Lemma 2.6. *Let K be a convex body of volume 1 in $\mathbb{R}^n$. Then, for every $\theta \in S^{n-1}$ and every $q, s \geq 1$,*

$$\left(1 - e^{-qs/2} (2\alpha)^q \right) H_q^q(\theta) \leq \int_{B_{K,\theta}(3\alpha H_q(\theta) s)} |\langle x, \theta \rangle|^q dx, \tag{2.14}$$

where $\alpha > 0$ is the constant in Lemma 2.1.

Proof. Lemma 2.3 shows that

$$|K \setminus B_{K,\theta}(3\alpha H_q(\theta) s)| \leq \exp(-qs) \tag{2.15}$$

for all $q, s \geq 1$. We write

$$\begin{aligned} H_q^q(\theta) &= \int_{B_{K,\theta}(3\alpha H_q(\theta)s)} |\langle x,\theta\rangle|^q dx + \int_{K\setminus B_{K,\theta}(3\alpha H_q(\theta)s)} |\langle x,\theta\rangle|^q dx \\ &\leq \int_{B_{K,\theta}(3\alpha H_q(\theta)s)} |\langle x,\theta\rangle|^q dx + \exp(-qs/2)\left(\int_K |\langle x,\theta\rangle|^{2q} dx\right)^{1/2} \\ &\leq \int_{B_{K,\theta}(3\alpha H_q(\theta)s)} |\langle x,\theta\rangle|^q dx + \exp(-qs/2)(2\alpha)^q H_q^q(\theta), \end{aligned}$$

where we have used (2.15), Cauchy-Schwarz inequality and Lemma 2.1 (for the pair $q, 2q$). $\square$

Our main technical lemma is the next one: it shows that $Z(q) \simeq Z(q,t)$ when t becomes of the order of $Z(q)$.

Lemma 2.7. *There exists an absolute constant $\beta > 0$ with the following property: for every convex body K of volume 1 in $\mathbb{R}^n$ and every $q \geq 1$,*

$$Z^q(q) \leq 2Z^q(q, \beta Z(q)). \tag{2.16}$$

Proof. For every $t > 0$ we set $U_t = \{\theta \in S^{n-1} : H_q(\theta) \geq tZ(q)\}$. Markov's inequality shows that $\sigma(U_t) \leq t^{-q}$. Using Lemma 2.6, for every $s \geq 1$ we write

$$\begin{aligned} (1 - e^{-qs/2}(2\alpha)^q)Z^q(q) &\leq \int_{S^{n-1}\setminus U_t} \int_{B_{K,\theta}(3\alpha H_q(\theta)s)} |\langle x,\theta\rangle|^q dx\sigma(d\theta) \\ &\quad + \int_{U_t} \int_{B_{K,\theta}(3\alpha H_q(\theta)s)} |\langle x,\theta\rangle|^q dx\sigma(d\theta) \\ &\leq \int_{S^{n-1}} \int_{B_{K,\theta}(3\alpha tsZ(q))} |\langle x,\theta\rangle|^q dx\sigma(d\theta) \\ &\quad + \sigma(U_t)^{1/2}\left(\int_{S^{n-1}} \int_K |\langle x,\theta\rangle|^{2q} dx\sigma(d\theta)\right)^{1/2} \\ &\leq Z^q(q, 3\alpha tsZ(q)) + t^{-q/2}Z^q(2q) \\ &\leq Z^q(q, 3\alpha tsZ(q)) + (c\alpha)^q t^{-q/2} Z^q(q), \end{aligned}$$

because $Z(2q) \leq c\alpha Z(q)$, where $c > 0$ is an absolute constant (this follows from Lemma 2.5 and the fact that $I_{2q}(K) \leq 2\alpha I_q(K)$ by Lemma 2.1 applied to the polynomial $f(x) = \|x\|_2^2$).

We now choose s, t so that $\sqrt{t} = 4c\alpha$ $e^{s/2} = 8\alpha$. Then,

$$(1 - 4^{-q})Z^q(q) \leq Z^q(q, 3\alpha tsZ(q)) + 4^{-q}Z^q(q). \tag{2.17}$$

Inserting the values of t, s in (2.17) we compute the value of β. $\square$

Finally, we will use an integral formula for f_K (see [6], [8]).

Lemma 2.8. *Let K be a convex body of volume 1 in $\mathbb{R}^n$. Then, for every $t > 0$,*

$$f_K(t) = c_n \int_{U_K(t)} \frac{1}{\|x\|_2}\left(1 - \frac{t^2}{\|x\|_2^2}\right)^{\frac{n-3}{2}} dx, \tag{2.18}$$

where $c_n \simeq \sqrt{n}$ as $n \to \infty$ and $U_K(t) = \{x \in K : \|x\|_2 \geq t\}$. $\square$

Remark: From Lemma 2.8 we readily see that f_K is a decreasing function.

3. Proofs of Theorems 1.1 and 1.2

Theorem 1.1 is a direct consequence of the following three Propositions.

Proposition 3.1. *Let $\gamma \geq 1$ and let K be an isotropic convex body in $\mathbb{R}^n$. If $1 \ll \phi(n) \ll n$ and*

$$\mathrm{Prob}\big(\{x \in K : \|x\|_2 \geq \gamma\sqrt{n}L_K t\}\big) \leq \exp\big(-\phi(n)t\big) \tag{3.1}$$

for every $t \geq 1$, then

$$f_K(t) \leq \frac{c_1}{L_K}\exp\left(-c_2t^2/\gamma^2L_K^2\right) \tag{3.2}$$

for all $0 < t \leq c_3\gamma\sqrt{\phi(n)}L_K$.

Proof. We assume that $n > 3$. From Lemma 2.8 we have

$$f_K(t) = c_n\int_{U_K(t)} g_t(\|x\|_2)dx \tag{3.3}$$

for all $t > 0$, where g_t is defined by

$$g_t(s) = \frac{1}{s}\left(1 - \frac{t^2}{s^2}\right)^{\frac{n-3}{2}} \tag{3.4}$$

on $[t, \infty)$. Differentiating g_t we see that it is increasing on $[t, t\sqrt{n-2}]$ and then decreasing. Let $0 < t \leq c_3\gamma\sqrt{\phi(n)}L_K$, where the absolute constant $c_3 > 0$ is to be chosen. Assume first that $\gamma\sqrt{n}L_K \leq t\sqrt{n-2}$ (this is satisfied if $t \geq \sqrt{2}\gamma L_K$). Then, we write

$$\begin{aligned}
f_K(t) &= c_n\int_{K\cap\{t\leq\|x\|_2\leq\gamma\sqrt{n}L_K\}} g_t(\|x\|_2)dx + c_n\int_{U_K(\gamma\sqrt{n}L_K)} g_t(\|x\|_2)dx\\
&\leq c_ng_t(\gamma\sqrt{n}L_K) + \exp(-\phi(n))c_ng_t(t\sqrt{n-2})\\
&= \frac{c_n}{\gamma\sqrt{n}L_K}\left(1 - \frac{t^2}{\gamma^2nL_K^2}\right)^{\frac{n-3}{2}} + \exp(-\phi(n))\frac{c_n}{t\sqrt{n-2}}\left(1 - \frac{1}{n-2}\right)^{\frac{n-3}{2}}\\
&\leq \frac{c_1}{L_K}\exp(-c_2t^2/\gamma^2L_K^2) + \frac{c_1}{L_K}\exp(-\phi(n))\\
&\leq \frac{2c_1}{L_K}\exp(-c_2t^2/\gamma^2L_K^2),
\end{aligned}$$

because $\phi(n) \geq c_2t^2/\gamma^2L_K^2$ if we choose $c_3 = 1/\sqrt{c_2}$ (we have also used the fact that $c_n \simeq \sqrt{n}$).

If $0 < t \leq \min\{\sqrt{2}\gamma L_K, c_3\gamma\sqrt{\phi(n)}L_K\}$, then

$$f_K(t) \leq \frac{c_5}{L_K} \leq \frac{2c_5}{L_K}\exp(-c_7t^2/\gamma^2L_K^2), \tag{3.5}$$

because $f_{K,\theta}(t) \leq c_5/L_K$ for all $\theta \in S^{n-1}$ (see [11]) and $\exp(-c_7t^2/\gamma^2L_K^2) \geq \exp(-2c_7) \geq 1/2$ if $c_7 > 0$ is suitably chosen. It follows that (3.2) holds true for all $0 < t \leq c_3\gamma\sqrt{\phi(n)}L_K$. □

Proposition 3.2. *Let $\gamma \geq 1$ and let K be an isotropic convex body in $\mathbb{R}^n$. Assume that $\beta \leq \gamma\psi(n) < R(K)/L_K$, where $\beta > 0$ is the constant in Lemma 2.7, and*

$$f_K(t) \leq \frac{c_1}{L_K}\exp\left(-t^2/\gamma^2L_K^2\right) \tag{3.6}$$

for all $0 < t \leq \gamma\psi(n)L_K$*. Then, for every* $2 \leq q \leq c_2\psi^2(n)$ *we have*

$$I_q(K) \leq c_3\gamma\sqrt{n}L_K. \tag{3.7}$$

Proof. Note that $Z(2) = L_K$ and $\lim_{s\to\infty} Z(s) = R(K)$. Since $\beta \leq \gamma\psi(n) < R(K)/L_K$, there exists $s \geq 2$ such that $\beta Z(s) = \gamma\psi(n)L_K$. Then Lemmas 2.7 and 2.4 show that

$$\begin{aligned} Z^s(s) &\leq 2Z^s(s,\beta Z(s)) = 4\int_0^{\beta Z(s)} r^s f_K(r)dr \\ &\leq 4\int_0^{\gamma\psi(n)L_K} r^s f_K(r)dr \\ &\leq \frac{4c_1}{L_K}\int_0^{\gamma\psi(n)L_K} r^s \exp(-r^2/\gamma^2L_K^2)dr \\ &\leq \frac{4c_1}{L_K}\int_0^{\infty} r^s \exp(-r^2/\gamma^2L_K^2)dr \\ &\leq (c_1'\gamma\sqrt{s}L_K)^s. \end{aligned}$$

In other words,

$$Z(s) \leq c_1'\gamma\sqrt{s}L_K. \tag{3.8}$$

Lemma 2.5 implies that

$$I_s(K) \leq c_1''\sqrt{n/s}Z(s) \leq c_3\gamma\sqrt{n}L_K, \tag{3.9}$$

and Hölder's inequality gives

$$I_q(K) \leq I_s(K) \leq c_3\gamma\sqrt{n}L_K \tag{3.10}$$

for all $q \leq s$. On the other hand, by the definition of s nd (3.8),

$$s \geq \frac{Z^2(s)}{c_3^2\gamma^2L_K^2} = \frac{\psi^2(n)}{c_3^2\beta^2} =: c_2\psi^2(n), \tag{3.11}$$

which completes the proof. □

Remark: The range $\beta \leq \gamma\psi(n) \leq R(K)/L_K$ is the interesting range for the parameter $\psi(n)$. If $0 < \gamma\psi(n) < \beta$, then the conclusion of Proposition 3.1 is trivially true. If $\gamma\psi(n) \geq R(K)/L_K$, then we have (3.6) for every $t > 0$. Following the previous argument, we check that $I_n(K) \simeq Z(n) \leq c\gamma\sqrt{n}L_K$. But $I_n(K) \simeq R(K)$, and this implies (3.7) for every $q \geq 2$.

Proposition 3.3. *Let* $\gamma \geq 1$ *and let* K *be an isotropic convex body in* $\mathbb{R}^n$*. If*

$$I_q(K) \leq \gamma\sqrt{n}L_K \tag{3.12}$$

for all $2 \leq q \leq \phi(n)$*, then*

$$\mathrm{Prob}\left(\{x \in K : \|x\|_2 \geq c\gamma\sqrt{n}L_Kt\}\right) \leq \exp\left(-\phi(n)t\right) \tag{3.13}$$

for every $t \geq 1$*, where* $c > 0$ *is an absolute constant.*

Proof. From Lemma 2.3 we have

$$\mathrm{Prob}\left(\{x \in K : \|x\|_2 \geq 3\alpha I_q(K)t\}\right) \leq e^{-qt} \tag{3.14}$$

for every $t \geq 1$, where $\alpha > 0$ is the constant in Lemma 2.1. Setting $q = \phi(n)$ and using (3.12), we get

$$\mathrm{Prob}\left(\{x \in K : \|x\|_2 \geq 3\alpha\gamma\sqrt{n}L_Kt\}\right) \leq \exp(-\phi(n)t) \tag{3.15}$$

for every $t \geq 1$, and the result follows with $c := 3\alpha$. $\square$

Theorem 1.1 and the result of Boblov and Nazarov [7] show that f_K is subgaussian on $[0, c\sqrt[4]{n}]$ in the 1-unconditional case.

Corollary 3.4. *There exist absolute constants $c_i > 0$ such that if K is an isotropic 1-unconditional convex body in $\mathbb{R}^n$, then*

$$f_K(t) \leq c_1 \exp(-c_2 t^2) \tag{3.16}$$

for all $0 < t \leq c_3 \sqrt[4]{n}$. $\square$

Note: We can construct examples of isotropic 1-unconditional convex bodies in $\mathbb{R}^n$ for which the length of the interval of t's on which (3.16) holds cannot have order greater than $\sqrt[4]{n}$.

For our last remark, recall the estimate of Bobkov and Koldobsky [6]: if K is an isotropic convex body in $\mathbb{R}^n$ then, for every $0 < t \leq c\sqrt{n}$,

$$\left| f_K(t) - \frac{1}{\sqrt{2\pi} L_K} \exp(-t^2/(2L_K^2)) \right| \leq C \left(\frac{\sigma_K L_K}{t^2 \sqrt{n}} + \frac{1}{n} \right), \tag{3.17}$$

where $c, C > 0$ are absolute constants and $\sigma_K^2 = \mathrm{Var}(\|x\|_2^2)/(nL_K^4)$. Using Theorem 1.1 we get the following.

Theorem 3.5. *Let K be an isotropic convex body in $\mathbb{R}^n$. Then,*

$$\mathrm{Prob}\left(\{ x \in K : \|x\|_2 \geq C_1 \sqrt{n} L_K t \} \right) \leq \exp\big(-\phi(K) t \big) \tag{3.18}$$

for every $t \geq 1$, where

$$\phi(K) \simeq \min\{ \log(n^2/\mathrm{Var}(\|x\|_2^2)), \log n \}, \tag{3.19}$$

and $C_1 > 0$ is an absolute constant.

Proof. Let C be the constant in (3.17) and let $c > 0$ be an absolute constant to be chosen (small enough). From (3.19) and the definition of σ_K we have

$$\phi(K) \leq \log\left(\frac{n}{\sigma_K^2 L_K^4} \right) = \frac{1}{2} \log\left(\frac{\sqrt{n}}{\sigma_K L_K^2} \right). \tag{3.20}$$

If $\sqrt{C} \leq t \leq c\sqrt{\phi(K)} L_K$, then (3.20) shows that

$$\frac{\sigma_K L_K^2}{\sqrt{n}} \leq e^{-2t^2/c^2 L_K^2}, \tag{3.21}$$

Observe that $C/t^2 \leq 1$, and hence,

$$C \frac{\sigma_K L_K}{t^2 \sqrt{n}} \leq \frac{1}{L_K} e^{-2t^2/c^2 L_K^2}. \tag{3.22}$$

Also, if c is small enough and $n \gg 1$, we have $\exp(2t^2/L_K^2) \leq n^{2c^2} \leq n/(CL_K)$ since $c_1 \leq L_K \leq c_2\sqrt{n}$ (these are the simple bounds on L_K; see [11]). This implies

$$\frac{C}{n} \leq \frac{1}{L_K} \exp(-2t^2/L_K^2). \tag{3.23}$$

Therefore, (3.17) gives

$$\begin{aligned} f_K(t) &\leq \frac{1}{\sqrt{2\pi} L_K} \exp(-t^2/(2L_K^2)) + C \frac{\sigma_K L_K}{t^2 \sqrt{n}} + \frac{C}{n} \\ &\leq \frac{c'}{L_K} \exp(-c'' t^2 / L_K^2), \end{aligned}$$

for all $t \in [\sqrt{C}, c\sqrt{\phi(K)}L_K]$, where $c', c'' > 0$ are absolute constants. A similar bound is trivially true if $0 < t \leq \sqrt{C}$. We can now use the implication (b)⇒(a) of Theorem 1.1 to conclude the proof. □

Assuming that σ_K and L_K are uniformly bounded, we have $\phi(K) \simeq \log n$. Then, Theorem 1.2 would give a positive answer to our original question: for every isotropic convex body K in $\mathbb{R}^n$,

$$\mathrm{Prob}\left(\{x \in K : \|x\|_2 \geq C_2\sqrt{n}t\}\right) \leq n^{-t}, \tag{3.22}$$

for every $t \geq 1$, where $C_2 > 0$ is an absolute constant.

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