TROPICAL VARIETIES FOR EXPONENTIAL SUMS AND THEIR DISTANCE TO AMOEBAE

ALPEREN ERGÜR, GRIGORIS PAOURIS, AND J. MAURICE ROJAS

Abstract. Given any n-variate exponential sum, g, the real part of the complex zero set of g forms a sub-analytic variety $\Re(Z(g))$ generalizing the amoeba of a complex polynomial. We extend the notion of Archimedean tropical hypersurface to derive a piecewise linear approximation, $\mathrm{Trop}(g)$, of $\Re(Z(g))$, with explicit bounds — solely as a function of n, the number of terms, and the minimal distance between frequencies — for the Hausdorff distance $\Delta(\Re(Z(g)), \mathrm{Trop}(g))$. We also discuss the membership complexity of $\mathrm{Trop}(g)$ relative to the Blum-Shub-Smale computational model over $\mathbb{R}$. Along the way, we also estimate the number of roots of univariate exponential sums in axis-parallel rectangles, refining earlier work of Wilder and Voorhoeve.

1. Introduction

Since the late 20th century (see, e.g., [Voo77, Kaz81, Kho91]) it has been known that many of the quantitative results relating algebraic sets and polyhedral geometry can be extended to more general analytic functions, including exponential sums. Here, we show that the recent estimates on the distance between amoebae and Archimedean tropical varieties from [AKNR14] admit such an extension. Metric estimates for amoebae of polynomials are useful for coarse approximation of solution sets of polynomial systems, as a step toward finer approximation via, say, homotopy methods (see, e.g., [AGGR13, HL14]). Polynomial systems are ubiquitous in numerous applications, and via a logarithmic change of variables, are clearly equivalent to systems of exponential sums with integer frequencies. Exponential sums with real frequencies are important in Signal Processing, Model Theory, and 3-manifold invariants (see Remark 1.8 below).[1]

Definition 1.1. *We use the abbreviations* $[N] := \{1, \ldots, N\}$, $w := (w_1, \ldots, w_n)$, $z := (z_1, \ldots, z_n)$, $w \cdot z := w_1 z_1 + \cdots + w_n z_n$, *and* $\mathbb{C}^* := \mathbb{C} \setminus \{0\}$. *We also let* $\Re(z)$ *denote the vector whose* $i^{\underline{th}}$ *coordinate is the real part of* z_i, *and* $\Re(S) := \{\Re(z) \mid z \in S\}$ *for any subset* $S \subseteq \mathbb{C}^n$. *Henceforth, we let* $A := \{a_1, \ldots, a_t\} \subset \mathbb{R}^n$ *have cardinality* $t \geq 2$, $b_j \in \mathbb{C}$ *for all* $j \in [t]$, *and set* $g(z) := \sum_{j=1}^{t} e^{a_j \cdot z + b_j}$. *We call* g *an* n-variate exponential t-sum *and call* A *the* spectrum *of* g. *We also call the* a_j *the* frequencies *of* g *and define their* minimal spacing *to be* $\delta(g) := \min_{p \neq q} |a_p - a_q|$ *where* $|\cdot|$ *denotes the standard* L^2*-norm on* $\mathbb{C}^n$. *Finally, let* $Z(g)$ *denote the zero set of* g *in* $\mathbb{C}^n$, *and define the* (Archimedean) tropical variety of g *to be*

$$\mathrm{Trop}(g) := \Re\left(\left\{z \in \mathbb{C}^n \ : \ \max_j \left|e^{a_j \cdot z + b_j}\right| \ \textit{is attained for at least two distinct } j\right\}\right). \qquad \diamond$$

Note that while we restrict to real frequencies for our exponential sums, we allow complex coefficients. $\mathrm{Trop}(g)$ also admits an equivalent (and quite tractable) definition as the dual of a polyhedral subdivision of A depending on the real parts of the b_j (see Thm. 1.12 and Prop. 2.4 below).

Example 1.2. *When* $n = 1$ *and* $g(z) = e^{\sqrt{2} z_1} + e^{\log(3) + \pi\sqrt{-1}}$, *we see that* $Z(g)$ *is a countable, discrete, and unbounded subset of the vertical line* $\left\{z_1 \in \mathbb{C} \ \middle| \ \Re(z_1) = \frac{\log 3}{\sqrt{2}}\right\}$. *So* $\Re(Z(g)) = \left\{\frac{\log 3}{\sqrt{2}}\right\}$. $\diamond$

Date: March 31, 2015.

Key words and phrases. exponential sum, amoeba, Hausdorff distance, tropical variety, complexity, metric.

A.E. and J.M.R. were partially supported by NSF grant CCF-1409020. Rojas' work was also supported by LABEX MILYON (ANR-10-LABX-0070) of Université de Lyon, within the program "Investissements d'Avenir" (ANR-11-IDEX-0007) operated by the French National Research Agency (ANR). G.P. was partially supported by BSF grant 2010288 and NSF CAREER grant DMS-1151711.

[1]Lest there be any confusion, let us immediately clarify that we do *not* consider terms of the form $e^{p(x)}$ with p a polynomial of degree ≥ 2. The latter type of exponential sums are of great importance in analytic number theory and the study of zeta functions.

Example 1.3. *When* $g(z) := e^{a_1 z_1 + b_1} + e^{a_2 z_1 + b_2}$ *for some distinct* $a_1, a_2 \in \mathbb{R}$ *(and any* $b_1, b_2 \in \mathbb{C}$*) it is easily checked that* $\mathrm{Trop}(g) = \Re(Z(g)) = \left\{\frac{\Re(b_1 - b_2)}{a_2 - a_1}\right\}$*. More generally, for any* n*-variate exponential* 2*-sum* g*,* $\mathrm{Trop}(g)$ *and* $\Re(Z(g))$ *are the same affine hyperplane. However, the univariate exponential* 3*-sum* $g(z_1) := (e^{z_1} + 1)^2$ *gives us* $\mathrm{Trop}(g) = \{\pm \log 2\}$*, which is neither contained in, nor has the same number of points, as* $\Re(Z(g)) = \{0\}$*.* ⋄

When $A \subset \mathbb{Z}^n$, $\Re(Z(g))$ is the image of the complex zero set of the polynomial $\sum_{j=1}^{t} e^{b_j} x^{a_j}$ under the coordinate-wise log-absolute value map, i.e., an *amoeba* [GKZ94]. Piecewise linear approximations for amoebae date back to work of Viro [Vir01] and, in the univariate case, Ostrowski [Ost40]. More recently, Alessandrini has associated piecewise linear approximations to log-limit sets of semi-algebraic sets and definable sets in an o-minimal stucture [Ale13]. However, other than Definition 1.1 here, we are unaware of any earlier formulation of such approximations for real parts of complex zero sets of n-variate exponential sums.

Our first main results are simple and explicit bounds for how well $\mathrm{Trop}(g)$ approximates $\Re(Z(g))$, in arbitrary dimension.

Definition 1.4. *Given any subsets* $R, S \subseteq \mathbb{R}^n$*, their* Hausdorff distance *is*

$$\Delta(R, S) := \max\left\{\sup_{r \in R} \inf_{s \in S} |r - s|, \sup_{s \in S} \inf_{r \in R} |r - s|\right\}. \qquad \diamond$$

Theorem 1.5. *For any* n*-variate exponential* t*-sum* $g(z) := \sum_{j=1}^{t} e^{a_j \cdot z + b_j}$ *with* $a_j \in \mathbb{R}^n$ *and* $b_j \in \mathbb{C}$ *for all* j*, let* d *be the dimension of the smallest affine subspace containing* $a_1, \ldots, a_t$*, and set* $\delta(g) := \min_{p \neq q} |a_p - a_q|$*. Then* $t \geq d + 1$ *and*

(0) *If* $t = d + 1$ *then* $\mathrm{Trop}(g) \subseteq \Re(Z(g))$ *(and thus* $\sup_{w \in \mathrm{Trop}(g)} \inf_{r \in \Re(Z(g))} |r - w| = 0$*).*

(1) *For* $t \geq 2$ *we have:*

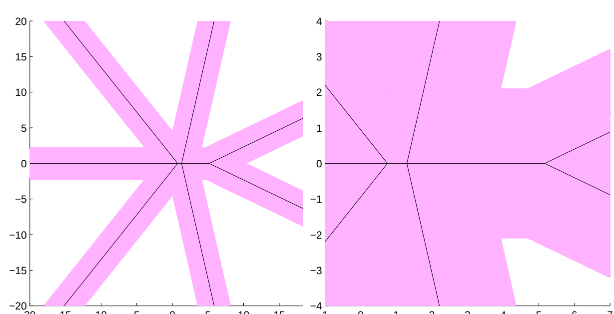
  (a) $\sup_{r \in \Re(Z(g))} \inf_{w \in \mathrm{Trop}(g)} |r - w| \leq \log(t-1)/\delta(g)$

  (b) $\Delta(\Re(Z(g)), \mathrm{Trop}(g)) \leq \frac{\sqrt{ed}\, t^2 (2t-3) \log 3}{\delta(g)}$.

(2) *Defining the* n*-variate exponential* t*-sum* $g_{t,n}(x) := (e^{\delta z_1} + 1)^{t-n} + e^{\delta z_2} + \cdots + e^{\delta z_n}$*, we have* $\Delta(\Re(Z(g_{t,n})), \mathrm{Trop}(g_{t,n})) \geq \log(t-n)/\delta$ *for* $t \geq n+1$ *and* $\delta > 0$*.*

We prove Theorem 1.5 in Section 4. Fundamental results on the geometric and topological structure of $\Re(Z(g))$ have been derived in recent decades by Favorov and Silipo [Fav01, Sil08]. However, we are unaware of any earlier explicit bounds for the distance between $\Re(Z(g))$ and $\mathrm{Trop}(g)$ when $A \not\subset \mathbb{Z}^n$.

Example 1.6. *When* g *is the* 2*-variate exponential* 7*-sum* $\sum_{j=0}^{6} \binom{7}{j} e^{\cos(2\pi j/7) z_1 + \sin(2\pi j/7) z_2}$*, Assertion (1) of Theorem 1.5 tells us that every point of* $\Re(Z(g))$ *lies within distance* $\log(6)/\sqrt{(1 - \cos(2\pi/7))^2 + \sin(2\pi/7)^2} < 2.065$ *of some point of* $\mathrm{Trop}(g)$*. To the right, we can see* $\mathrm{Trop}(g)$ *as the black piecewise linear curve drawn on the right, along with the stated neighborhood of* $\mathrm{Trop}(g)$ *containing* $\Re(Z(g))$*. The magnified view reveals that* $\mathrm{Trop}(g)$ *has exactly* 3 *vertices.* ⋄



The special case $A \subset \mathbb{Z}^n$ of Theorem 1.5 was known earlier, with a bound independent of n: Our $\mathrm{Trop}(g)$ agrees with the older definition of (Archimedean) tropical variety for the polynomial

$f(x) := \sum_{j=1}^{t} e^{b_j} x^{a_j}$, and the simpler bound $\Delta(\text{Amoeba}(f), \text{Trop}(f)) \leq (2t-3)\log(t-1)$ holds [AKNR14]. Earlier metric results for the special case $A \subset \mathbb{Z}$ date back to work of Ostrowski on Graeffe iteration [Ost40]. Viro and Mikhalkin touched upon the special case $A \subset \mathbb{Z}^2$ in [Vir01] and [Mik05, Lemma 8.5, pg. 360].

We derive our distance bounds by using a projection trick arising from the study of random convex sets (see [GPV12] and Section 3 below) to reduce to the $d=1$ case. The $d=1$ case then follows from specially tailored extensions of existing results for the polynomial case (see Section 2 below). This approach results in succinct proofs for our bounds. However, it is not yet clear if the dependence on d is actually necessary or just an artifact of our techniques.

A consequence of our approach is a refinement of an earlier estimate of Wilder (see [Wil17], [Voo77], and Section 2.2 below) on the number of roots of univariate exponential sums in infinite horizontal strips of $\mathbb{C}$: Theorem 2.11 (see Section 2.2) allows us to estimate the number of roots in certain axis-parallel rectangles in $\mathbb{C}$. A very special case of Theorem 2.11 is the fact that *all* the roots of g are confined to an explicit union of infinite *vertical* strips explicitly determined by $\text{Trop}(g)$. In what follows, the *open ε-neighborhood* of a subset $X \subseteq \mathbb{R}$ is simply $\{x' \in \mathbb{R} \ : \ |x - x'| < \varepsilon \text{ for some } x \in X\}$.

Corollary 1.7. *Suppose g is any univariate t-sum with real spectrum and W is the open $\frac{\log 3}{\delta(g)}$-neighborhood of* $\text{Trop}(g)$*. Then all the complex roots of g lie in $W \times \mathbb{R}$. In particular,* $\sup\limits_{r \in \Re(Z(g))} \inf\limits_{w \in \text{Trop}(g)} |r - w| \leq \frac{\log 3}{\delta(g)}$ *in the univariate case.* ■

Unlike the distribution of roots of g in horizontal strips, where there is a kind of equidistribution (see, e.g., [Voo77, AGS13] and Section 2 below), Corollary 1.7 tells us that the roots of g cluster only within certain deterministically predictable vertical strips.

Our next main results concern the complexity of deciding whether a given point lies in the real part of the complex zero set of a given exponential sum, and whether checking membership in a neighborhood of a tropical variety instead is more efficient.

1.1. On the Computational Complexity of $\Re(Z(g))$ and $\text{Trop}(g)$.

We have tried to balance generality and computational tractability in the family of functions at the heart of our paper. In particular, the use of arbitrary real inputs causes certain geometric and algorithmic subtleties. We will see below that these difficulties are ameliorated by replacing exact queries with approximate queries.

Remark 1.8. *"Polynomials" with real exponents — sometimes called* posinomials *— occur naturally in many applications. For example, the problem of finding the directions of a set of unknown signals, using a radar antenna built from a set of specially spaced sensors, can easily be converted to an instance of root-finding in the univariate case* [FH95, HAGY08]. *Approximating roots in the higher-dimensional case is the fundamental computational problem of Geometric Programming* [DPZ67, Chi05, BKVH07]. *Pathologies with the phases of complex roots can be avoided through a simple exponential change of variables, so this is one reason that exponential sums are more natural than posinomials. Among other applications, exponential sums occur in the calculation of* 3*-manifold invariants (see, e.g.,* [McM00, Appendix A] *and* [Had14]*), and have been studied from the point of view of Model Theory and Diophantine Geometry (see, e.g.,* [Wil96, Zil02, Zil11]*).* ⋄

To precisely compare the computational complexity of $\Re(Z(g))$ and $\text{Trop}(g)$ we will first need to fix a suitable model of computation: We will deal mainly with the *BSS model over* $\mathbb{R}$

[BCSS98]. This model naturally augments the classical *Turing machine* [Pap95, AB09, Sip12] by allowing field operations and comparisons over $\mathbb{R}$ in unit time. We are in fact forced to move beyond the Turing model since our exponential sums involve arbitrary real numbers, and the Turing model only allows finite bit strings as inputs. We refer the reader to [BCSS98] for further background.

We are also forced to move from *exact* equality and membership questions to questions allowing a margin of uncertainty. One reason is that exact arithmetic involving exponential sums still present difficulties, even for computational models allowing field operations and comparisons over $\mathbb{R}$.

Proposition 1.9. *The problem of determining, for an input $(z_1, z_2) \in \mathbb{R}^2$, whether $z_1 = e^{z_2}$, is undecidable*[2] *in the BSS model over $\mathbb{R}$, i.e., there is no algorithm terminating in finite time for all inputs.*

(We were unable to find a precise statement of Proposition 1.9 in the literature, so we provide a proof at the end of this section.) Note that when the input is restricted, deciding whether $z_1 = e^{z_2}$ can be tractable (and even trivially so). For instance, a famous result of Lindemann [Lin82] tells us that e^{z_2} is transcendental if $z_2 \in \mathbb{C}$ is nonzero and algebraic.

Proposition 1.9 may be surprising in light of there being efficient iterations for approximating the exponential function [BB88, Ahr99]. Determining which questions are tractable for expressions involving exponentials has in fact been an important impetus behind parts of Computational Algebra, Model Theory, and Diophantine Geometry in recent decades (see, e.g., [Ric83, Wil96, Zil02, HP14, SY14]). As for the complexity of $\Re(Z(g))$, deciding membership turns out to be provably hard, already for the simplest bivariate exponential 3-sums.

Theorem 1.10. *Determining, for arbitrary input $r_1, r_2 \in \mathbb{R}$ whether $(r_1, r_2) \in \Re(Z(1 - e^{z_1} - e^{z_2}))$ is undecidable in the BSS model over $\mathbb{R}$.*

(We prove Theorem 1.10 at the end of this section.) The intractability asserted in Theorem 1.10 can be thought of as an amplification of the **NP**-hardness of deciding amoeba membership when $A \subset \mathbb{Z}$ [AKNR14, Thm. 1.9]. (See also [Pla84] for an important precursor.) However, just as in Proposition 1.9, there are special cases of the membership problem from Theorem 1.10 that are perfectly tractable. For instance, when $e^{r_1}, e^{r_2} \in \mathbb{Q}$, deciding whether $(r_1, r_2) \in \Re(Z(1 - e^{z_1} - e^{z_2}))$ is in fact doable — even on a classical Turing machine — in polynomial-time (see, e.g., [The02, TdW13] and [AKNR14, Thm. 1.9]).

More to the point, Theorem 1.10 above is yet another motivation for approximating $\Re(Z(g))$, and our final main result shows that membership queries (and even distance queries) involving $\mathrm{Trop}(g)$ are quite tractable in the BSS model over $\mathbb{R}$. We refer the reader to [Grü03, Zie95, dLRS10] further background on polyhedral geometry and subdivisions.

Definition 1.11. *For any n-variate exponential t-sum g, let $\Sigma(\mathrm{Trop}(g))$ denote the polyhedral complex whose cells are exactly the (possibly improper) faces of the closures of the connected components of $\mathbb{R}^n \setminus \mathrm{Trop}(g)$.* $\diamond$

Theorem 1.12. *Suppose n is fixed. Then there is a polynomial-time algorithm that, for any input $w \in \mathbb{R}^n$ and n-variate exponential t-sum g, outputs the closure — described as an explicit intersection of $O(t^2)$ half-spaces — of the unique cell σ_w of $\Sigma(\mathrm{Trop}(g))$ containing w.*

[2][Poo14] provides an excellent survey on undecidability, in the classical Turing model, geared toward non-experts in complexity theory.

We prove Theorem 1.12 in Section 5. An analogue of Theorem 1.12, for the classical Turing model (assuming $A\subset\mathbb{Z}^n$ and $w\in\mathbb{Q}^n$) appears in [AGGR13, Thm. 1.5]. Extending to $A\subset\mathbb{R}^n$ and real coefficients, and using the BSS model over $\mathbb{R}$, in fact conceptually simplifies the underlying algorithm and helps us avoid certain Diophantine subtleties.

By applying the standard formula for point-hyperplane distance, and the well-known efficient algorithms for approximating square-roots (see, e.g., [BB88]), Theorem 1.12 implies that we can also efficiently check membership in any ε-neighborhood about $\mathrm{Trop}(g)$. This means, thanks to Theorem 1.5, that membership in a neighborhood of $\mathrm{Trop}(g)$ is a tractable and potentially useful relaxation of the problem of deciding membership in $\Re(Z(g))$.

For completeness, we now prove Proposition 1.9 and Theorem 1.10.

Proof of Proposition 1.9: The key is to consider the shape of the space of inputs $\mathcal{I}$ that lead to a "Yes" answer in a putative BSS machine deciding membership in the curve in $\mathbb{R}^2$ defined by $y=e^x$. In particular, [BCSS98, Thm. 1, Pg. 52] tells us that any set of inputs leading to a "Yes" answer in a BSS machine over $\mathbb{R}$ must be a countable union of semi-algebraic sets. So if $\mathcal{I}$ is indeed decidable relative to this model then $\mathcal{I}$ must contain a bounded connected neighborhood W of a real algebraic curve (since $\mathcal{I}$ has infinite length). Since $\mathcal{I}$ is the graph of e^x, W extends by analytic continuation to the graph of an entire algebraic function. But this impossible: One simple way to see this is that an entire algebraic function must have polynomial growth order. However, the function e^x clearly has non-polynomial growth order. ■

Proof of Theorem 1.10: Similar to our last argument, one can easily show that $\mathcal{I} := \Re(Z(1-e^{z_1}-e^{z_2}))$ being decidable by a BSS machine over $\mathbb{R}$ implies that a neighborhood W of the boundary of I must be real algebraic. (We may in fact assume that W is the part of the boundary that lies in the curve defined by $y=\log(1-e^x)$.) So, via analytic continuation to $U:=\mathbb{C}\setminus\{(2k+1)\sqrt{-1}\pi \mid k\in\mathbb{Z}\}$, it suffices to show that $\log(1-e^x)$ is not an algebraic function that is analytic on U. But this is easy since an algebraic function can only have finitely many branch points, whereas $\log(1-e^x)$ has infinitely many. (Moreover, each branch point of $\log(1-e^x)$ has infinite monodromy whereas algebraic functions can only have branch points with finite monodromy.) ■

2. Tropically Extending Classical Polynomials Root Bounds to Exponential Sums

2.1. Basics on Roots of Univariate Exponential Sums.

Let $\#S$ denote the cardinality of a set S. It is worth noting that although $\#\mathrm{Trop}(g)$ and our bounds for $\Delta(\Re(Z(g)),\mathrm{Trop}(g))$ are independent of the maximal distance between frequencies $D:=\max_{p,q}|a_p-a_q|$, the cardinality $\#\Re(Z(g))$ can certainly depend on D, and even be infinite for $n=1$.

Example 2.1. *For any integer $D\geq 2$, $g(z_1):=e^{Dz_1}+e^{z_1}+1$ satisfies $\#\mathrm{Trop}(g)=1$ but $\#\Re(Z(g))=\lceil D/2\rceil$. The latter cardinality is easily computed by observing that the non-real roots of the trinomial $f(x_1):=x_1^D+x_1+1$ occur in conjugate pairs, and at most 2 roots of f can have the same norm. (The latter fact is a very special case of* [TdW14, Prop. 4.3].*)* ⋄

Example 2.2. *Considering the decimal expansion of $\sqrt{2}$, and the local continuity of the roots of $e^{Dz_1}+e^{z_1}+1$ as a function of $D\in\mathbb{R}$, it is not hard to show that $X:=\Re\left(Z\left(e^{\sqrt{2}z_1}+e^{z_1}+1\right)\right)$ is in fact countably infinite, and Corollary 2.6 below tells us that X is also contained in the open interval $\left(-\frac{\log 2}{\sqrt{2}-1},\frac{\log 2}{\sqrt{2}-1}\right)$.* ⋄

To derive our main results we will need the following variant of the Newton polytope, specially suited for studying real parts of roots of exponential sums.

Definition 2.3. *Let* $\mathrm{Conv}(S)$ *denote the convex hull of a subset* $S \subseteq \mathbb{R}^n$*, i.e., the smallest convex set containing* S*. Given any* n*-variate exponential* t*-sum* $g(z) = \sum_{j=1}^{t} e^{a_j \cdot z + b_j}$ *with real frequencies* a_j*, we then define its* Archimedean Newton polytope *to be* $\mathrm{ArchNewt}(g) := \mathrm{Conv}\left(\{(a_j, -\Re(b_j))\}_{j\in[t]}\right)$*. We also call any face of a polytope* $P \subset \mathbb{R}^{n+1}$ *having an outer-normal vector with* negative *last coordinate a* lower face*.* $\diamond$

Proposition 2.4. *For any* n*-variate exponential* t*-sum* g *with real spectrum we have* $\mathrm{Trop}(g) = \{w \mid (w, -1)$ *is an outer normal of a positive-dimensional face of* $\mathrm{ArchNewt}(g)\}$*. Furthermore, when* $n=1$*,* $\mathrm{Trop}(g)$ *is also the set of slopes of the lower edges of* $\mathrm{ArchNewt}(g)$*.* ■

We refer the reader to [AKNR14] for further background on the polynomial cases of ArchNewt and Trop.

A key trick we will use is relating the points of $\mathrm{Trop}(g)$ to (vertical) half-planes of $\mathbb{C}$ where certain terms of the univariate exponential sum g dominate certain sub-summands of g.

Proposition 2.5. *Suppose* $g(z_1) := \sum_{j=1}^{t} e^{a_j z_1 + b_j}$ *satisfies* $a_1 < \cdots < a_t$ *and* $b_j \in \mathbb{C}$ *for all* j*. Suppose further that* $w \in \mathrm{Trop}(g)$*,* ℓ *is the unique index such that* $(a_\ell, \Re(b_\ell))$ *is the right-hand vertex of the lower edge of* $\mathrm{ArchNewt}(g)$ *of slope* w*, and let* $\delta_\ell := \min\limits_{p,q\in[\ell]\&p\neq q} |a_p - a_q|$*. Then for any* $N \in \mathbb{N}$ *and* $z_1 \in \left[w + \frac{\log(N+1)}{\delta_\ell}, \infty\right) \times \mathbb{R}$ *we have* $\left|\sum_{j=1}^{\ell-1} e^{a_j z_1 + b_j}\right| < \frac{1}{N}\left|e^{a_\ell z_1 + b_\ell}\right|$*.*

Proof: First note that $2 \leq \ell \leq t$ by construction. Let $\beta_j := \Re(b_j)$, $r := \Re(z_1)$, and note that

$$\left|\sum_{j=1}^{\ell-1} e^{a_j z_1 + b_j}\right| \leq \sum_{j=1}^{\ell-1} \left|e^{a_j z_1 + b_j}\right| = \sum_{j=1}^{\ell-1} e^{a_j r + \beta_j} = \sum_{j=1}^{\ell-1} e^{a_j(r-w) + a_j w + \beta_j}$$

Now, since $a_{j+1} - a_j \geq \delta_\ell$ for all $j \in \{1, \ldots, \ell-1\}$, we obtain $a_j \leq a_\ell - (\ell - j)\delta_\ell$. So for $r > w$ we have $\left|\sum_{j=1}^{\ell-1} e^{a_j z_1 + b_j}\right| \leq \sum_{j=1}^{\ell-1} e^{(a_\ell - (\ell-j)\delta_\ell)(r-w) + a_j w + \beta_j} \leq \sum_{j=1}^{\ell-1} e^{(a_\ell - (\ell-j)\delta_\ell)(r-w) + a_\ell w + \beta_\ell}$, where the last inequality follows from Definition 1.1. So then

$$\begin{aligned}
\left|\sum_{j=1}^{\ell-1} e^{a_j z_1 + b_j}\right| &\leq e^{(a_\ell - (\ell-1)\delta_\ell)(r-w) + a_\ell w + \beta_\ell} \sum_{j=1}^{\ell-1} e^{(j-1)\delta_\ell (r-w)} \\
&= e^{(a_\ell - (\ell-1)\delta_\ell)(r-w) + a_\ell w + \beta_\ell} \left(\frac{e^{(\ell-1)\delta_\ell(r-w)} - 1}{e^{\delta_\ell(r-w)} - 1}\right) \\
&< e^{(a_\ell - (\ell-1)\delta_\ell)(r-w) + a_\ell w + \beta_\ell} \left(\frac{e^{(\ell-1)\delta_\ell(r-w)}}{e^{\delta_\ell(r-w)} - 1}\right) = \frac{e^{a_\ell r + \beta_\ell}}{e^{\delta_\ell(r-w)} - 1}
\end{aligned}$$

So to prove our desired inequality, it clearly suffices to enforce $e^{\delta_\ell(r-w)} - 1 \geq N$. The last inequality clearly holds for all $r \geq w + \frac{\log(N+1)}{\delta_\ell}$, so we are done. ■

It is then easy to prove that the largest (resp. smallest) point of $\Re(Z(g))$ can't be too much larger (resp. smaller) than the largest (resp. smallest) point of $\mathrm{Trop}(g)$. Put another way, we can give an explicit vertical strip containing all the complex roots of g.

Corollary 2.6. *Suppose g is a univariate exponential t-sum with real spectrum and minimal spacing $\delta(g)$, and $w_{\min}$ (resp. $w_{\max}$) is* $\max \operatorname{Trop}(g)$ *(resp.* $\min \operatorname{Trop}(g)$*). Then $\Re(Z(g))$ is contained in the open interval* $\left(w_{\min} - \frac{\log 2}{\delta(g)}, w_{\max} + \frac{\log 2}{\delta(g)}\right)$.

The $\log 2$ in Corollary 2.6 can not be replaced by any smaller constant: For $g(z_1) = e^{(t-1)z_1} - e^{(t-2)z_1} - \cdots - e^{z_1} - 1$ we have $\delta(g) = 1$, $\operatorname{Trop}(g) = \{0\}$, and it is easily checked that $\Re(Z(g))$ contains points approaching $\log 2$ as $t \longrightarrow \infty$. While the polynomial analogue of Corollary 2.6 goes back to work of Cauchy, Birkhoff, and Fujiwara pre-dating 1916 (see [RS02, pp. 243–249, particularly bound 8.1.11 on pg. 247] and [Fuj16] for further background) we were unable to find an explicit bound for exponential sums like Corollary 2.6 in the literature. So we supply a proof below.

Proof of Corollary 2.6: Replacing z_1 by its negative, it clearly suffices to prove $\Re(Z(g)) \subset \left(-\infty, w_{\max} + \frac{\log 2}{\delta}\right)$. Writing $g(z_1) = \sum_{j=1}^{t} e^{a_j z_1 + b_j}$ with $a_1 < \cdots < a_t$, let ζ denote any root of g, $r := \Re(\zeta)$, and $\beta_j := \Re(b_j)$ for all j. Since we must have $\sum_{j=1}^{t-1} e^{a_j \zeta + b_j} = -e^{a_t \zeta + b_t}$, taking absolute values implies that $\left|\sum_{j=1}^{t-1} e^{a_j \zeta + b_j}\right| = \left|e^{a_t \zeta + b_t}\right|$. However, this equality is contradicted by Proposition 2.5 for $\Re(z_1) \geq w_{\max} + \frac{\log 2}{\delta}$. So we are done. ■

Another simple consequence of our term domination trick (Proposition 2.5 above) is that we can give explicit vertical strips in $\mathbb{C}$ free of roots of g.

Corollary 2.7. *Suppose $g(z_1) := \sum_{j=1}^{t} e^{a_j z_1 + b_j}$ satisfies $a_1 < \cdots < a_t$, $b_j \in \mathbb{C}$ for all j, and that w_1 and w_2 are* consecutive *points of* $\operatorname{Trop}(g)$ *satisfying $w_2 \geq w_1 + \frac{2\log 3}{\delta(g)}$. Let ℓ be the unique index such that $(a_\ell, \Re(b_\ell))$ is the vertex of* $\operatorname{ArchNewt}(g)$ *incident to lower edges of slopes w_1 and w_2. Then the vertical strip $\left[w_1 + \frac{\log 3}{\delta(g)}, w_2 - \frac{\log 3}{\delta(g)}\right] \times \mathbb{R}$ contains* no *roots of g.*

Proof: By Proposition 2.5, we have $\left|\sum_{j=1}^{\ell-1} e^{a_j z_1 + b_j}\right| < \frac{1}{2}\left|e^{a_\ell z_1 + b_\ell}\right|$ for all $z_1 \in \left[w_1 + \frac{\log 3}{\delta(g)}, \infty\right)$ and (employing the change of variables $z_1 \mapsto -z_1$) $\left|\sum_{j=\ell+1}^{t} e^{a_j z_1 + b_j}\right| < \frac{1}{2}\left|e^{a_\ell z_1 + b_\ell}\right|$ for all $z_1 \in \left(-\infty, w_2 - \frac{\log 3}{\delta(g)}\right]$. So we obtain $\left|\sum_{j \neq \ell} e^{a_j z_1 + b_j}\right| < \left|e^{a_\ell z_1 + b_\ell}\right|$ in the stated vertical strip, and this inequality clearly contradicts the existence of a root of g in $\left[w_1 + \frac{\log 3}{\delta(g)}, w_2 - \frac{\log 3}{\delta(g)}\right] \times \mathbb{R}$. ■

Remark 2.8. *Corollary 1.7 from the introduction follows immediately from Corollaries 2.6 and 2.7.* ◇

Let us now recall a result of Wilder [Wil17] (later significantly refined by Voorhoeve [Voo77]) that tightly estimates the number of roots of exponential sums in infinite horizontal strips of $\mathbb{C}$. Let $\Im(\alpha)$ denote the imaginary part of $\alpha \in \mathbb{C}$ and let $\langle x \rangle := \min_{u \in \mathbb{Z}} |x - u|$ be the distance of x to the nearest integer.

Wilder-Voorhoeve Theorem. [Voo77, Thm. 5.3] *For any univariate exponential t-sum g with real frequencies $a_1 < \cdots < a_t$ and $u \leq v$ let $H_{u,v}$ denote the number of roots of g, counting multiplicity, in the infinite horizontal strip $\{z_1 \in \mathbb{C} \mid \Im(z_1) \in [u, v]\}$. Then*

$$\left|H_{u,v} - \frac{v-u}{2\pi}(a_t - a_1)\right| \leq t - 1 - \sum_{j=2}^{t} \left\langle \frac{(v-u)(a_j - a_{j-1})}{2\pi} \right\rangle. \qquad ■$$

We will ultimately refine the Wilder-Voorhoeve Theorem into a *localized* deviation bound (Theorem 2.11 below) counting the roots of g in special axis parallel rectangles in $\mathbb{C}$. For this, we will need to look more closely at the variation of the argument of g on certain vertical and horizontal segments.

2.2. Winding Numbers and Density of Roots in Rectangles and Vertical Strips.

To count roots of exponential sums in rectangles, it will be useful to observe a basic fact on winding numbers for *non*-closed curves.

Proposition 2.9. *Suppose $I\subset\mathbb{C}$ is any compact line segment and g and h are functions analytic on a neighborhood of I with $|h(z)|<|g(z)|$ for all $z\in I$. Then* $\left|\Im\left(\int_I \frac{g'+h'}{g+h}dz - \int_I \frac{g'}{g}dz\right)\right| < \pi$.

Proof: The quantity $V_1:=\Im\left(\int_I \frac{g'}{g}dz\right)$ (resp. $V_2:=\Im\left(\int_I \frac{g'+h'}{g+h}dz\right)$) is nothing more than the variation of the argument of g (resp. $g+h$) along the segment I. Since I is compact, $|g|$ and $|g+h|$ are bounded away from 0 on I by construction. So we can lift the paths $g(I)$ and $(g+h)(I)$ (in $\mathbb{C}^*$) to the universal covering space induced by the extended logarithm function. Clearly then, V_1 (resp. V_2) is simply a difference of values of $\Im(\mathrm{Log}(g))$ (resp. $\Im(\mathrm{Log}(g+h))$), evaluated at the endpoints I, where different branches of Log may be used at each endpoint. In particular, at any fixed endpoint z, our assumptions on $|g|$ and $|h|$ clearly imply that $g(z)+h(z)$ and $g(z)$ both lie in the open half-plane normal (as a vector in $\mathbb{R}^2$) to $g(z)$. So $|\Im(\mathrm{Log}(g(z)+h(z)))-\Im(\mathrm{Log}(g(z)))|<\frac{\pi}{2}$ at the two endpoints of I, and thus $|V_1-V_2|<\frac{\pi}{2}+\frac{\pi}{2}=\pi$. ■

Re-examining Corollary 1.7 from the last section, one quickly sees that the vertical strips in $\mathbb{C}$ containing the roots of a univariate exponential sum g correspond exactly to clusters of "closely spaced" consecutive points of $\mathrm{Trop}(g)$. These clusters of points in $\mathrm{Trop}(g)$ in turn correspond to certain sub-summands of g. In particular, sets of consective "large" (resp. "small") points of $\mathrm{Trop}(g)$ correspond to sums of "high" (resp. "low") order terms of g. Our next step will then be to relate the roots of a high (or low) order summand of g to an *explicit portion* of the roots of g.

Lemma 2.10. *Let $g(z_1):=\sum_{j=1}^t e^{a_jz_1+b_j}$ with $a_1<\cdots<a_t$ and $b_j\in\mathbb{C}$ for all j, $u\leq v$, and let $w_{\min}$ (resp. $w_{\max}$) be* $\min\mathrm{Trop}(g)$ *(resp.* $\max\mathrm{Trop}(g)$*). Also let w_1 and w_2 be* consecutive *points of* $\mathrm{Trop}(g)$ *satisfying $w_{\min}<w_1<w_2<w_{\max}$ and let ℓ be the unique index such that $(a_\ell,\Re(b_\ell))$ is the vertex of* $\mathrm{ArchNewt}(g)$ *incident to lower edges of slopes w_1 and w_2 (so $2\leq\ell\leq t-1$). Finally, assume $w_2-w_1\geq\frac{2\log 3}{\delta(g)}$. and let $R^1_{u,v}$ and $R^2_{u,v}$ respectively denote the number of roots of g, counting multiplicity, in the rectangles $\left(w_{\min}-\frac{\log 2}{\delta(g)}, w_1+\frac{\log 3}{\delta(g)}\right)\times[u,v]$ and $\left(w_2-\frac{\log 3}{\delta(g)}, w_{\max}+\frac{\log 2}{\delta(g)}\right)\times[u,v]$. Then*

$$\left|R^1_{u,v}-\frac{v-u}{2\pi}(a_\ell-a_1)\right|\leq\varepsilon_1+1 \quad \textit{and} \quad \left|R^2_{u,v}-\frac{v-u}{2\pi}(a_t-a_\ell)\right|\leq\varepsilon_2+1,$$

where $\varepsilon_1,\varepsilon_2\geq 0$ and $\varepsilon_1+\varepsilon_2\leq t-1-\sum_{j=2}^t\left\langle\frac{(v-u)(a_j-a_{j-1})}{2\pi}\right\rangle$.

When $\mathrm{Trop}(g)$ has two adjacent points sufficiently far apart (as detailed above), Lemma 2.10 thus refines the Wilder-Voorhoeve Theorem. Lemma 2.10 also considerably generalizes an earlier root count for the polynomial case presented in [AKNR14, Lemma 2.8]: Rephrased in terms of the notation above, the older root count from [AKNR14, Lemma 2.8] becomes the equalities $R^1_{0,2\pi}=a_\ell-a_1$ and $R^2_{0,2\pi}=a_t-a_\ell$ for the special case $A\subset\mathbb{Z}$.

Proof of Lemma 2.10: By symmetry (with respect to replacing z_1 by $-z_1$) it clearly suffices to prove the estimate for $R^2_{u,v}$. Since g is analytic, the Argument Principle (see, e.g., [Ahl79]) tells us that

$$R^2_{u,v} = \frac{1}{2\pi\sqrt{-1}} \int_{I_-\cup I_+\cup J_-\cup J_+} \frac{g'}{g} dz$$

where I_- (resp. I_+, J_-, J_+) is the oriented line segment from

$$\left(w_2 - \tfrac{\log 3}{\delta(g)}, v\right) \text{ (resp. } \left(w_{\max} + \tfrac{\log 2}{\delta(g)}, u\right), \left(w_2 - \tfrac{\log 3}{\delta(g)}, u\right), \left(w_{\max} + \tfrac{\log 2}{\delta(g)}, v\right))$$

to

$$\left(w_2 - \tfrac{\log 3}{\delta(g)}, u\right) \text{ (resp. } \left(w_{\max} + \tfrac{\log 2}{\delta(g)}, v\right), \left(w_{\max} + \tfrac{\log 2}{\delta(g)}, u\right), \left(w_2 - \tfrac{\log 3}{\delta(g)}, v\right)),$$

assuming no root of g lies on $I_-\cup I_+\cup J_-\cup J_+$. By Corollaries 2.6 and 2.7, there can be no roots of g on $I_-\cup I_+$. So let assume temporarily that there are no roots of g on $J_-\cup J_+$.

Since $w_2 - \frac{\log 3}{\delta(g)} \geq w_1 + \frac{\log 3}{\delta(g)}$ by assumption, Proposition 2.5 tells us that

$$\tfrac{1}{2}\left|c_\ell e^{a_\ell\left(w_2 - \frac{\log 3}{\delta(g)} + \sqrt{-1}v\right)+b_\ell}\right| > \left|\sum_{j=1}^{\ell-1} e^{a_j\left(w_2-\frac{\log 3}{\delta(g)}+\sqrt{-1}v\right)+b_j}\right|$$

and, by symmetry and another application of Proposition 2.5,

$$\tfrac{1}{2}\left|c_\ell e^{a_\ell\left(w_2 - \frac{\log 3}{\delta(g)} + \sqrt{-1}v\right)+b_\ell}\right| > \left|\sum_{j=\ell+1}^{t} e^{a_j\left(w_2-\frac{\log 3}{\delta(g)}+\sqrt{-1}v\right)+b_j}\right|.$$

So we can apply Proposition 2.9 and deduce that $\left|\Im\left(\int_{I_-}\frac{g'}{g}dz - \int_{I_-}\frac{(e^{a_\ell z+b_\ell})'}{e^{a_\ell z+b_\ell}}dz\right)\right| < \pi$. So then, since the last integral has imaginary part easily evaluating to $a_\ell(u-v)\sqrt{-1}$, we clearly obtain $\left|\left(\frac{1}{2\pi\sqrt{-1}}\int_{I_-}\frac{g'}{g}dz\right) - a_\ell(u-v)\right| < \frac{1}{2}$. An almost identical argument (applying Propositions 2.5 and 2.9 again, but with the term $e^{a_t z+b_t}$ dominating instead) then also yields $\left|\left(\frac{1}{2\pi\sqrt{-1}}\int_{I_+}\frac{g'}{g}dz\right) - a_t(v-u)\right| < \frac{1}{2}$.

So now we need only prove sufficiently sharp estimates on $\frac{1}{2\pi\sqrt{-1}}\int_{J_\pm}\frac{g'}{g}dz$:

$$\begin{aligned}
\left|\int_{J_-\cup J_+}\Im\left(\frac{g'}{g}\right)dz\right| &= \left|\int_{w_2-\frac{\log 3}{\delta(g)}}^{w_{\max}+\frac{\log 2}{\delta(g)}} \Im\left(\frac{g'\left(z+u\sqrt{-1}\right)}{g\left(z+u\sqrt{-1}\right)} - \frac{g'\left(z+v\sqrt{-1}\right)}{g\left(z+v\sqrt{-1}\right)}\right)dz\right| \\
&\leq \int_{w_2-\frac{\log 3}{\delta(g)}}^{w_{\max}+\frac{\log 2}{\delta(g)}} \left|\Im\left(\frac{g'\left(z+u\sqrt{-1}\right)}{g\left(z+u\sqrt{-1}\right)} - \frac{g'\left(z+v\sqrt{-1}\right)}{g\left(z+v\sqrt{-1}\right)}\right)\right|dz \\
&=: K\left(w_2 - \frac{\log 3}{\delta(g)}, w_{\max} + \frac{\log 2}{\delta(g)}; u, v; g\right).
\end{aligned}$$

A quantity closely related to $K(x_1, x_2; u, v; g)$ was, quite fortunately, already studied in Voorhoeve's 1977 Ph.D. thesis: In our notation, the proof of [Voo77, Thm. 5.3] immediately yields $\lim\limits_{x\to\infty} K(-x, x; u, v; g) = t-1-\sum_{j=2}^{t}\left\langle\frac{(v-u)(a_j-a_{j-1})}{2\pi}\right\rangle$. In particular, by the additivity of integration, the nonnegativity of the underlying integrands, and taking $\varepsilon_1 := K\left(w_{\min} - \frac{\log 2}{\delta(g)}, w_1 + \frac{\log 3}{\delta(g)}; u, v; g\right)$ and $\varepsilon_2 := K\left(w_2 - \frac{\log 3}{\delta(g)}, w_{\max} + \frac{\log 2}{\delta(g)}; u, v; g\right)$, we obtain $\left|\int_{J_-\cup J_+}\Im\left(\frac{g'}{g}\right)dz\right| \leq \varepsilon_2$, with $\varepsilon_1, \varepsilon_2 \geq 0$ and $\varepsilon_1 + \varepsilon_2 \leq t-1-\sum_{j=2}^{t}\left\langle\frac{(v-u)(a_j-a_{j-1})}{2\pi}\right\rangle$.

Adding terms and errors, we then clearly obtain $\left|R^2_{u,v} - \frac{v-u}{2\pi}(a_t - a_\ell)\right| < \varepsilon_2 + 1$, in the special case where no roots of g lie on $J_- \cup J_+$. To address the case where a root of g lies on $J_- \cup J_+$, note that the analyticity of g implies that the roots of g are a discrete subset of $\mathbb{C}$. So we can find arbitrarily small $\eta > 0$ with the boundary of the slightly stretched rectangle $\left(w_2 - \frac{\log 3}{\delta(g)}, w_{\max} + \frac{\log 2}{\delta(g)}\right) \times [u-\eta, v+\eta]$ not intersecting any roots of g. So then, by the special case of our lemma already proved, $\left|R^2_{u-\eta,v+\eta} - \frac{v-u+2\eta}{2\pi}(a_t - a_\ell)\right| < \varepsilon'_2 + 1$, with $\varepsilon'_1, \varepsilon'_2 \geq 0$ and $\varepsilon'_1 + \varepsilon'_2 \leq t - 1 - \sum_{j=2}^t \left\langle \frac{(v-u+2\eta)(a_j - a_{j-1})}{2\pi} \right\rangle$. Clearly, $R^2_{u-\eta,v+\eta} = R^2_{u,v}$ for η sufficiently small, and the limit of the preceding estimate for $R^2_{u-\eta,v+\eta}$ tends to the estimate stated in our lemma. So we are done. ■

We at last arrive at our strongest refinement of the Wilder-Voorhoeve Theorem.

Theorem 2.11. *Suppose $g(z_1) = \sum_{j=1}^t e^{a_j z_1 + b_j}$, $a_1 < \cdots < a_t$, and C is any connected component of the open $\frac{\log 3}{\delta(g)}$-neighborhood of* $\mathrm{Trop}(g)$. *Also let $w_{\min}(C)$ (resp. $w_{\max}(C)$) be* $\min(\mathrm{Trop}(g) \cap C)$ *(resp.* $\max(\mathrm{Trop}(g) \cap C)$*) and let i (resp. j) be the unique index such that $(a_i, \Re(b_i))$ is the left-most (resp. right-most) vertex of the lower edge of* $\mathrm{ArchNewt}(g)$ *of slope $w_{\min}(C)$ (resp. $w_{\max}(C)$). Finally, let $R_{C,u,v}$ denote the number of roots of g, counting multiplicity, in the rectangle $C \times [u, v]$. Then*

$$\left|R_{C,u,v} - \frac{v-u}{2\pi}(a_j - a_i)\right| \leq \varepsilon_C + 1,$$

where $\varepsilon_C \geq 0$ and the sum of ε_C over all such connected components C is no greater than $t - 1 - \sum_{j=2}^t \left\langle \frac{(v-u)(a_j - a_{j-1})}{2\pi} \right\rangle$.

Note that Lemma 2.10 is essentially the special case of Theorem 2.11 where C is the leftmost or rightmost connected component specified above. Note also that a special case of Theorem 2.11 implies that the fraction of roots of g lying in $C \times \mathbb{R}$ (i.e., the ratio $\lim\limits_{y \to \infty} \frac{R_{C,u,v}}{H_{u,v}}$, using the notation from our statement of the Wilder-Voorhoeve Theorem) is exactly $\frac{a_j - a_i}{a_t - a_1}$. This density of roots localized to a vertical strip can also be interpreted as the average value of the function 1, evaluated at all root of g in $C \times \mathbb{R}$. Soprunova has studied the average value of general analytic functions h, evaluated at the roots (in a sufficiently large vertical strip) of an exponential sum [Sop03]. Theorem 2.11 thus refines the notion of the "average value of 1 over the roots of g in $\mathbb{C}$" in a different direction.

Proof of Theorem 2.11: The argument is almost identical to the proof of Lemma 2.10, save for the horizontal endpoints of the rectangle and the dominating terms in the application of Proposition 2.5 being slightly different. ■

A consequence of our development so far, particularly Corollary 1.7, is that every point of $\Re(Z(g))$ is close to some point of $\mathrm{Trop}(g)$. We now show that every point of $\mathrm{Trop}(g)$ is close to some point of $\Re(Z(g))$. The key trick is to break $\mathrm{Trop}(g)$ into clusters of closely spaced points, and use the fact that every connected component C (from Theorem 2.11) contains at least one real part of a complex root of g.

Theorem 2.12. *Suppose g is any univariate exponential t-sum with real spectrum and $t \geq 2$. Let s be the maximum cardinality of* $\mathrm{Trop}(g) \cap C$ *for any connected component C of the open*

$\frac{\log 3}{\delta(g)}$*-neighborhood of* Trop(g). *(So* $1 \leq s \leq t-1$ *in particular.) Then for any* $v \in \text{Trop}(g)$ *there is a root* $z \in \mathbb{C}$ *of* g *with* $|\Re(z) - v| \leq \frac{(2s-1)\log 3}{\delta(g)}$.

Proof: For convenience, for the next two paragraphs we will allow negative indices i for $\sigma_i \in \text{Trop}(g)$ (but we will continue to assume σ_i is increasing in i).

Let us define R to be the largest j with $v, \sigma_1, \ldots, \sigma_j$ being consecutive points of Trop(g) in increasing order, $\sigma_1 - v \leq \frac{2\log 3}{\delta(g)}$, and $\sigma_{i+1} - \sigma_i \leq \frac{2\log 3}{\delta(g)}$ for all $i \in [j-1]$. (We set $R=0$ should no such j exist.) Similarly, let us define L to be the largest j with $v, \sigma_{-1}, \ldots, \sigma_{-j} \in \text{Trop}(g)$ being consecutive points of Trop(g) in decreasing order, $v - \sigma_{-1} \leq \frac{2\log 3}{\delta(g)}$, and $\sigma_{-i} - \sigma_{-i-1} \leq \frac{2\log 3}{\delta(g)}$ for all $i \in [j-1]$. (We set $L=0$ should no such j exist.) Note that $L + R + 1 \leq s$.

By Theorem 2.11 there must then be at least one point of $\Re(Z(g))$ in the interval $\left[v - (2L+1)\frac{\log 3}{\delta(g)}, v + (2R+1)\frac{\log 3}{\delta(g)}\right]$. So there must be a point of $\Re(Z(g))$ within distance $(2\max\{L, R\} + 1)\frac{\log 3}{\delta(g)}$ of v. Since $2L+2, 2R+2 \leq 2s$, we are done. ■

At this point, we are almost ready to prove our main theorems. The remaining fact we need is a generalization of Corollary 1.7 to arbitrary dimension.

2.3. **A Quick Distance Bound in Arbitrary Dimension.** Having proved an upper bound for the largest point of $\Re(Z(g))$, one may wonder if there is a *lower bound* for the largest point of $\Re(Z(g))$. Montel proved (in different notation) the univariate polynomial analogue of the assertion that the largest points of $\Re(Z(g))$ and Trop(g) differ by no more than $\log(t-1)$ [Mon23]. One can in fact guarantee that *every* point of $\Re(Z(g))$ is close to some point of Trop(g), and in arbitrary dimension.

Lemma 2.13. *For any* n*-variate exponential* t*-sum* g *with real spectrum and* $t \geq 2$ *we have* $\sup\limits_{r \in \Re(Z(g))} \inf\limits_{w \in \text{Trop}(g)} |r - w| \leq \log(t-1)/\delta(g)$.

Let $z \in Z(g)$ and assume without loss of generality that

$$\left|e^{a_1 \cdot z + b_1}\right| \geq \left|e^{a_2 \cdot z + b_2}\right| \geq \cdots \geq \left|e^{a_t \cdot z + b_t}\right|.$$

Since $g(z) = 0$ implies that $\left|e^{a_1 \cdot z + b_1}\right| = \left|e^{a_2 \cdot z + b_2} + \cdots + e^{a_t \cdot z + b_t}\right|$, the Triangle Inequality immediately implies that $\left|e^{a_1 \cdot z + b_1}\right| \leq (t-1)\left|e^{a_2 \cdot z + b_2}\right|$. Taking logarithms, and letting $\rho := \Re(z)$ and $\beta_i := \Re(b_i)$ for all i, we then obtain

$$a_1 \cdot \rho + \beta_1 \geq \cdots \geq a_t \cdot \rho + \beta_t \quad \text{and} \tag{1}$$

$$a_1 \cdot \rho + \beta_1 \leq \log(t-1) + a_2 \cdot \rho + \beta_2 \tag{2}$$

For each $i \in \{2, \ldots, t\}$ let us then define η_i to be the shortest vector such that

$$a_1 \cdot (\rho + \eta_i) + \beta_1 = a_i \cdot (\rho + \eta_i) + \beta_i.$$

Note that $\eta_i = \lambda_i(a_i - a_1)$ for some nonnegative λ_i since we are trying to affect the dot-product $\eta_i \cdot (a_1 - a_i)$. In particular, $\lambda_i = \frac{(a_1 - a_i)\cdot\rho + \beta_1 - \beta_i}{|a_1 - a_i|^2}$ so that $|\eta_i| = \frac{(a_1 - a_i)\cdot\rho + \beta_1 - \beta_i}{|a_1 - a_i|}$. (Indeed, Inequality (1) implies that $(a_1 - a_i) \cdot \rho + \beta_1 - \beta_i \geq 0$.)

Inequality (2) implies that $(a_1 - a_2) \cdot \rho + \beta_1 - \beta_2 \leq \log(t-1)$. We thus obtain $|\eta_2| \leq \frac{\log(t-1)}{|a_1 - a_2|} \leq \frac{\log(t-1)}{\delta(g)}$. So let $i_0 \in \{2, \ldots, t\}$ be any i minimizing $|\eta_i|$. We of course have $|\eta_{i_0}| \leq \log(t-1)/\delta(g)$, and by the definition of η_{i_0} we have

$$a_1 \cdot (\rho + \eta_{i_0}) + \beta_1 = a_{i_0} \cdot (\rho + \eta_{i_0}) + \beta_{i_0}.$$

Moreover, the fact that η_{i_0} is the shortest among the η_i implies that

$$a_1 \cdot (\rho + \eta_{i_0}) + \beta_1 \geq a_i \cdot (\rho + \eta_{i_0}) + \beta_i$$

for all i. Otherwise, we would have $a_1 \cdot (\rho + \eta_{i_0}) + \beta_1 < a_i \cdot (\rho + \eta_{i_0}) + \beta_i$ and $a_1 \cdot \rho + \beta_1 \geq a_i \cdot \rho + \beta_i$ (the latter following from Inequality (1)). Taking a convex linear combination of the last two inequalities, it is then clear that there must be a $\mu \in [0, 1)$ such that

$$a_1 \cdot (\rho + \mu\eta_{i_0}) + \beta_1 = a_i \cdot (\rho + \mu\eta_{i_0}) + \beta_i.$$

Thus, by the definition of η_i, we would obtain $|\eta_i| \leq \mu|\eta_{i_0}| < |\eta_{i_0}|$ — a contradiction.

We thus have the following: (i) $a_1 \cdot (\rho + \eta_{i_0}) - (-\beta_1) = a_{i_0} \cdot (\rho + \eta_{i_0}) - (-\beta_{i_0})$, (ii) $a_1 \cdot (\rho + \eta_{i_0}) - (-\beta_1) \geq a_i \cdot (\rho + \eta_{i_0}) - (-\beta_i)$ for all i, and (iii) $|\eta_{i_0}| \leq \log(t-1)/\delta(g)$. Together, these inequalities imply that $\rho + \eta_{i_0} \in \mathrm{Trop}(g)$. In other words, we've found a point in $\mathrm{Trop}(g)$ sufficiently near ρ to prove our desired upper bound. ■

3. Small Ball Probability

Let $G_{n,k}$ be the Grassmanian of k-dimensional subspaces of $\mathbb{R}^n$, equipped with its unique rotation-invariant Haar probability measure $\mu_{n,k}$. The following "small ball probability" estimate holds.

Lemma 3.1. [GPV12, Lemma 3.2] *Let $1 \leq k \leq n-1$, $x \in \mathbb{R}^n$, and $\varepsilon \leq \frac{1}{\sqrt{e}}$. Then*

$$\mu_{n,k}\left(\left\{F \in G_{n,k} \;\middle|\; |P_F(x)| \leq \varepsilon\sqrt{\frac{k}{n}}|x|\right\}\right) \leq \left(\sqrt{e}\varepsilon\right)^k,$$

where P_F is the surjective orthogonal projection mapping $\mathbb{R}^n$ onto F. ■

An important precursor, in the context of bounding distortion under more general Euclidean embeddings, appears in [Mat90].

A simple consequence of the preceding metric result is the following fact on the existence of projections mapping a high-dimensional point set onto a lower-dimensional subspace in a way that preserves the minimal spacing as much as possible.

Proposition 3.2. *Let $\gamma > 0$ and $x_1, \ldots, x_N \in \mathbb{R}^n$ be such that $|x_i - x_j| \geq \gamma$ for all distinct i, j. Then, following the notation of Lemma 3.1, there exist $F \in G_{n,k}$ such that $|P_F(x_i) - P_F(x_j)| \geq \sqrt{\frac{k}{en}}\frac{\gamma}{N^{2/k}}$ for all distinct i, j.*

Proof: Let $z_{\{i,j\}} := x_i - x_j$. Then our assumption becomes $z_{\{i,j\}} \geq \gamma$ for all distinct i, j and there are no more than $N(N-1)/2$ such pairs $\{i, j\}$. By Lemma 3.1 we have, for any fixed $\{i, j\}$, that $|P_F(z_{\{i,j\}})| \geq \varepsilon\sqrt{\frac{k}{n}}|z_{\{i,j\}}|$ with probability at least $1 - \left(\sqrt{e}\varepsilon\right)^k$. So if $\varepsilon < \frac{1}{\sqrt{e}}\left(\frac{2}{N(N-1)}\right)^{1/k}$, the union bound for probabilities implies that, for all distinct i, j, we have $|P_F(z_{\{i,j\}})| \geq \varepsilon\sqrt{\frac{k}{n}}|z_{\{i,j\}}| \geq \varepsilon\gamma\sqrt{\frac{k}{n}}$ (and thus $|P_F(x_i) - P_F(x_j)| \geq \varepsilon\gamma\sqrt{\frac{k}{n}}$) with probability at least $1 - \frac{N(N-1)}{2}(\sqrt{e}\varepsilon)^k$. Since this lower bound is positive by construction, we can conclude by choosing $\varepsilon := \frac{1}{\sqrt{e}N^{2/k}}$. ■

4. Proof of Theorem 1.5

The assertion that $t \geq d+1$ is easy since any d-dimensional polytope always has at least $d+1$ vertices. So we now focus on the rest of the theorem. We prove Assertion 1(b) last.

In what follows, for any real $n \times n$ matrix M and $z \in \mathbb{R}^n$, we assume that z is a column vector when we write Mz. Also, for any subset $S \subseteq \mathbb{R}^n$, the notation $MS := \{Mz \mid z \in S\}$ is understood. The following simple functoriality properties of $\mathrm{Trop}(g)$ and $\Re(Z(g))$ will prove useful.

Proposition 4.1. *Suppose g_1 and g_2 are n-variate exponential t-sums, $\alpha \in \mathbb{C}^*$, $a \in \mathbb{R}^n$, $\beta := (\beta_1, \ldots, \beta_n) \in \mathbb{C}^n$, and g_2 satisfies the identity $g_2(z) = \alpha e^{a \cdot z} g_1(z_1 + \beta_1, \ldots, z_n + \beta_n)$. Then $\Re(Z(g_2)) = \Re(Z(g_1)) - \Re(\beta)$ and $\mathrm{Trop}(g_2) = \mathrm{Trop}(g_1) - \Re(\beta)$. Also, if $M \in \mathbb{R}^{n \times n}$ and we instead have the identity $g_2(z) = g_1(Mz)$, then $M\Re(Z(g_2)) = \Re(Z(g_1))$ and $M\mathrm{Trop}(g_2) = \mathrm{Trop}(g_1)$.* ■

4.1. **Proof of Assertion (0).** First note that, thanks to Proposition 4.1, an invertible linear change of variables allows us to reduce to the special case $A = \{\mathbf{O}, e_1, \ldots, e_n\}$, where $\mathbf{O}$ and $\{e_1, \ldots, e_n\}$ are respectively the origin and standard basis vectors of $\mathbb{R}^n$. But this special case is well known: One can either prove it directly, or avail to earlier work of Rullgård on the spines of amoebae (see, e.g., the remark following Theorem 8 on Page 33, and Theorem 12 on Page 36, of [Rul03]). In fact, observing that our change of variables can in fact be turned into an isotopy (by the connectivity of $\mathbb{GL}_n^+(\mathbb{R})$), we can further assert that $\mathrm{Trop}(g)$ is a deformation retract of $\Re(Z(g))$ in this case. ■

4.2. **Proof of Assertion 1(a).** This is simply Lemma 2.13, which was proved in Section 2. ■

4.3. **Proof of Assertion (2).** The special case $\delta = 1$ follows immediately from Assertion (2) of Theorem 1.5 of [AKNR14] (after setting $x_i = e^{z_i}$ in the notation there). Proposition 4.1 tells us that scaling the spectrum of g by a factor of δ scales $\Re(Z(g))$ and $\mathrm{Trop}(g)$ each by a factor of $1/\delta$. So we are done. ■

4.4. **Proof of Assertion 1(b).** First note that the Hausdorff distance in question is invariant under rotation in $\mathbb{R}^n$. So we may in fact assume that g involves just the variables $z_1, \ldots, z_d$ and thus assume $d = n$.

By the $k = 1$ case of Proposition 3.2 we deduce that there exists a unit vector $\theta \in \mathbb{R}^n$ such that

$$\min_{i \neq j} |a_i \cdot \theta - a_j \cdot \theta| \geq \frac{\delta(g)}{\sqrt{ent^2}} \tag{3}$$

Now let $v \in \mathrm{Trop}(g)$ and write $v = v_\theta \theta + v_\theta^\perp$ for some $v_\theta^\perp$ perpendicular to θ. Also let $u_\theta \in \mathbb{C}$ and $u \in \mathbb{C}^n$ satisfy $u = u_\theta \theta + v_\theta^\perp$. For $z_1 \in \mathbb{C}$ define the univariate exponential t-sum $\tilde{g}(z_1) = \sum_{j=1}^{t} e^{(a_j \cdot (z_1 \theta + v_\theta^\perp)) + b_i}$. By Inequality (3) we see that $\delta(\tilde{g}) \geq \frac{\delta(g)}{\sqrt{ent^2}}$. We also see that $\tilde{g}(u_\theta) = g(u)$ and $\tilde{g}(v_\theta) = g(v)$. By Theorem 2.12 there exists a value for u_θ such that $0 = \tilde{f}(u_\theta) = f(u)$ and $|\Re(u_\theta) - v_\theta| \leq \frac{(2t-3)\log 3}{\delta(\tilde{g})} \leq \frac{\sqrt{ent^2}(2t-3)\log 3}{\delta(g)}$.

So $|\Re(u) - v| = |(\Re(u_\theta) - v_\theta)\theta| \leq \frac{\sqrt{ent^2}(2t-3)\log 3}{\delta(g)} = \frac{\sqrt{edt^2}(2t-3)\log 3}{\delta(g)}$ since we've already reduced to the case $d = n$. ■

5. Proving Theorem 1.12

We will need some supporting results on linear programming before starting our proof.

Definition 5.1. *Given any matrix $M \in \mathbb{R}^{N \times n}$ with $i^{\underline{th}}$ row m_i, and $c := (c_1, \ldots, c_N) \in \mathbb{R}^N$, the notation $Mx \leq c$ means that $m_1 \cdot x \leq c_1, \ldots, m_N \cdot x \leq c_N$ all hold. These inequalities are called* constraints, *and the set of all $x \in \mathbb{R}^N$ satisfying $Mx \leq c$ is called the* feasible region *of $Mx \leq c$. We also call a constraint* active *if and only if it holds with equality. Finally, we call* *a constraint* redundant *if and only if the corresponding row of M and corresponding entry of c can be deleted without affecting the feasible region of $Mx \leq c$.* $\diamond$

Lemma 5.2. *Suppose n is fixed. Then, given any $c \in \mathbb{R}^N$ and $M \in \mathbb{R}^{N \times n}$, we can, in time polynomial in N, find a submatrix M' of M, and a subvector c' of c, such that the feasible regions of $Mx \leq c$ and $M'x \leq c'$ are equal, and $M'x \leq c'$ has no redundant constraints. Furthermore, in time polynomial in N, we can also enumerate all maximal sets of active constraints defining vertices of the feasible region of $Mx \leq c$.* ■

Note that we are using the BSS model over $\mathbb{R}$ in the preceding lemma. In particular, we are only counting field operations and comparisons over $\mathbb{R}$ (and these are the only operations needed). We refer the reader to the excellent texts [Sch86, GLS93, Gri13] for further background and a more leisurely exposition on linear programming.

Proof of Theorem 1.12: Let $w \in \mathbb{R}^n$ be our input query point. Using $O(t \log t)$ comparisons, we can isolate all indices such that $\max_j |e^{a_j \cdot z + b_j}|$ is attained, so let j_0 be any such index. Taking logarithms, we then obtain, say, J equations of the form $a_j \cdot w + \Re(b_j) = a_{j_0} \cdot w + \Re(b_{j_0})$ and K inequalities of the form $a_j \cdot w + \Re(b_j) > a_{j_0} \cdot w + \Re(b_{j_0})$ or $a_j \cdot w + \Re(b_j) < a_{j_0} \cdot w + \Re(b_{j_0})$.

Thanks to Lemma 5.2, we can determine the exact cell of $\mathrm{Trop}(f)$ containing w if $J \geq 2$. Otherwise, we obtain the unique cell of $\mathbb{R}^n \setminus \mathrm{Trop}(f)$ with relative interior containing w. Note also that an $(n-1)$-dimensional face of either kind of cell must be the dual of an edge of $\mathrm{ArchNewt}(g)$. Since every edge has exactly 2 vertices, there are at most $t(t-1)/2$ such $(n-1)$-dimensional faces, and thus σ_w is the intersection of at most $t(t-1)/2$ half-spaces. So we are done. ■

Acknowledgements

We are grateful to Timo de Wolff for pointing out Silipo's work [Sil08]. We also thank Pascal Koiran, Gregorio Malajovich, Jiří Matoušek, and Klaus Meer for useful discussions.

References

[Ahl79] Ahlfors, Lars, *Complex Analysis,* McGraw-Hill Science/Engineering/Math; 3rd edition, 1979.

[Ahr99] Ahrendt, Timm, *"Fast computations of the exponential function,"* in proceedings of STACS '99 (16th annual conference on Theoretical aspects of computer science), pp. 302–312, Springer-Verlag Berlin, 1999.

[AGS13] D'Andrea, Carlos; Galligo, André; and Sombra, Martin, *"Quantitative Equidistribution for the Solution of a System of Sparse Polynomial Equations,"* Advances in Mathematics, to appear.

[AGGR13] Anthony, Eleanor; Grant, Sheridan; Gritzmann, Peter; and Rojas, J. Maurice, *"Polynomial-time Amoeba Neighborhood Membership and Faster Localized Solving,"* Chapter 15 of: Topological and Statistical Methods for Complex Data – Tackling Large-Scale, High-Dimensional, and Multivariate Data Sets, (Bennett, Janine; Vivodtzev, Fabien; Pascucci, Valerio (Eds.)), series on Mathematics and Visualization, Springer-Verlag, 2014.

[AB09] Arora, Sanjeev and Barak, Boaz, *Computational complexity. A modern approach.* Cambridge University Press, Cambridge, 2009.

[Ale13] Alessandrini, Daniele, *"Logarithmic limit sets of real semi-algebraic sets,"* Advances in Geometry 13, no. 1 (2013), pp. 155–190.

[AKNR14] Avendaño, Martín; Kogan, Roman; Nisse, Mounir; and Rojas, J. Maurice, *"Metric Estimates and Membership Complexity for Archimedean Amoebae and Tropical Hypersurfaces,"* submitted for publication, also available as Math ArXiV preprint `1307.3681`

[BCSS98] Blum, Lenore; Cucker, Felipe; Shub, Mike; and Smale, Steve, *Complexity and Real Computation,* Springer-Verlag, 1998.

[BB88] Borwein, John M. and Borwein, Peter B., *"On the Complexity of Familiar Functions and Numbers,"* SIAM Review, Vol. 30, No. 4, (Dec., 1988), pp. 589–601.

[BKVH07] Boyd, S.; Kim, S.-J.; Vandenberghe, L.; and Hassibi, A., *"A Tutorial on Geometric Programming,"* Optimization and Engineering, 8(1):67-127, 2007.

[Chi05] Chiang, Mung, *Geometric Programming for Communication Systems,* now Publishers Inc., Massachusetts, 2005.

[dLRS10] De Loera, Jesús A.; Rambau, Jörg; Santos, Francisco, *Triangulations,* Structures for algorithms and applications, Algorithms and Computation in Mathematics, 25, Springer-Verlag, Berlin, 2010.

[DPZ67] Duffin, Richard J.; Peterson, Elmor L.; Zener, Clarence, *Geometric Programming,* John Wiley and Sons, 1967.

[Fav01] Favorov, Sergei Yu., *"Holomorphic almost periodic functions in tube domains and their amoebas,"* Computational Methods and Function Theory, v. 1 (2001), No. 2, pp. 403–415.

[FH95] Forsythe, Keith and Hatke, Gary, *"A Polynomial Rooting Algorithm for Direction Finding,"* preprint, MIT Lincoln Laboratories, 1995.

[Fuj16] Fujiwara, M., *"Über die obere Schranke des absoluten Betrags der Wurzeln einer algebraischen Gleichung,"* Tôhoku Mathematical Journal, **10**, pp. 167–171.

[GKZ94] Gel'fand, Israel Moseyevitch; Kapranov, Misha M.; and Zelevinsky, Andrei V.; *Discriminants, Resultants and Multidimensional Determinants,* Birkhäuser, Boston, 1994.

[GPV12] Giannopoulos, Apostolos; Paouris, Grigoris; and Valettas, Petros, *"On the distribution of the ψ_2-norm of linear functionals on isotropic convex bodies,"* GAFA Seminar Volume (2050), 2012.

[Gri13] Gritzmann, Peter, *Grundlagen der Mathematischen Optimierung: Diskrete Strukturen, Komplexitätstheorie, Konvexitätstheorie, Lineare Optimierung, Simplex-Algorithmus, Dualität,* Springer Vieweg, 2013.

[GLS93] Grötschel, Martin; Lovász, Laszlo; and Schrijver, Alexander, *Geometric Algorithms and Combinatorial Optimization,* Springer-Verlag, New York, 1993.

[Grü03] Grünbaum, Branko, *Convex Polytopes,* Wiley-Interscience, London, 1967; 2nd ed. (edited by Ziegler, G.), Graduate Texts in Mathematics, vol. 221, Springer-Verlag, 2003.

[HP14] Habegger, Philipp and Pila, Jonathan, *"o-minimality and certain atypical intersections,"* Math ArXiV preprint `1409.0771` .

[Had14] Hadari, Asaf, *"The Spectra of Polynomial Equations with Varying Exponents,"* Math ArXiV preprint `1410.0064` .

[HL14] Hauenstein, Jonathan D. and Levandovskyy, Viktor, *"Certifying solutions to square systems of polynomial-exponential equations,"* Math ArXiV preprint `1109.4547` .

[HAGY08] Hwang, H. K.; Aliyazicioglu, Zekeriya; Grice, Marshall; and Yakovlev, Anatoly, *"Direction of Arrival Estimation using a Root-MUSIC Algorithm,"* Proceedings of the International MultiConference of Engineers and Computer Scientists 2008, Vol. II, IMECS 2008, 19–21 March, 2008, Hong Kong.

[Kaz81] Kazarnovskiĭ, B. Ja., *"On Zeros of Exponential Sums,"* Soviet Math. Doklady, **23** (1981), no. 2, pp. 347–351.

[Kho91] Khovanskiĭ, Askold G., *Fewnomials,* AMS Press, Providence, Rhode Island, 1991.

[Lin82] von Lindemann, Ferdinand, *"Über die Ludolph'sche Zahl,"* Sitzungber. Knigl. Preuss. Akad. Wissensch. zu Berlin 2 (1882), pp. 679–682.

[Len99] Lenstra (Jr.), Hendrik W., *"Finding Small Degree Factors of Lacunary Polynomials,"* Number Theory in Progress, Vol. 1 (Zakopane-Kóscielisko, 1997), pp. 267–276, de Gruyter, Berlin, 1999.

[Mat90] Matoušek, Jiří, *"Bi-Lipschitz embeddings into low-dimensional Euclidean spaces,"* Commentationes Mathematicae Universitatis Carolinae, vol. 31 (1990), issue 3, pp. 589–600.

[McM00] McMullen, Curtis T., *"Polynomial invariants for fibered* 3*-manifolds and Teichmuller geodesics for foliations,"* Ann. Sci. Ecole Norm. Super, 33(4): 519–560, 2000.

[Mik05] Mikhalkin, Grigory, *"Enumerative Tropical Algebraic Geometry in $\mathbb{R}^2$,"* Journal of the American Mathematical Society, Vol. 18, No. 2, pp. 313–377, 2005.

[Mon23] Montel, Paul, *"Sur les modules des zéros des polnômes,"* Annales Scientifiques de l'École Normale Suérieure (3), **40**, pp. 1–34, 1923.

[Ost40] Ostrowski, Alexandre, *"Recherches sur la méthode de Graeffe et les zéros des polynomes et des séries de Laurent,"* Acta Math. **72**, (1940), pp. 99–155.

[Pap95] Papadimitriou, Christos H., *Computational Complexity,* Addison-Wesley, 1995.

[Pla84] Plaisted, David A., *"New NP-Hard and NP-Complete Polynomial and Integer Divisibility Problems,"* Theoret. Comput. Sci. 31 (1984), no. 1–2, 125–138.

[Poo14] Poonen, Bjorn, *"Undecidable problems: a sampler,"* Interpreting Gödel: Critical essays (J. Kennedy ed.), Cambridge Univ. Press, pp. 211–241, 2014.

[RS02] Rahman, Q. I. and Schmeisser, G., *Analytic Theory of Polynomials,* London Mathematical Society Monographs 26, Oxford Science Publications, 2002.

[Ric83] Richardson, Daniel, *"Roots of real exponential functions,"* J. London Math. Soc. (2), 28 (1983), pp. 46–56.

[Rul03] Rullgård, Hans, *Topics in geometry, analysis and inverse problems,* doctoral dissertation, Department of Mathematics, Stockholm University, Sweden, 2003. Downloadable from `http://su.diva-portal.org/smash/record.jsf?pid=diva2:190169` .

[SY14] Scanlon, Thomas and Yasufuku, Yu, *"Exponential-Polynomial Equations and Dynamical Return Sets,"* International Mathematics Research Notices, doi:10.1093/imrn/rnt081 .

[Sch86] Schrijver, Alexander, *Theory of Linear and Integer Programming,* John Wiley & Sons, 1986.

[Sil08] Silipo, James, *"Amibes de sommes d'exponentielles,"* The Canadian Journal of Mathematics, Vol. 60, No. 1 (2008), pp. 222–240.

[Sip12] Sipser, Michael, *Introduction to the Theory of Computation,* 3rd edition, Cengage Learning, 2012.

[Sop03] Soprunova, Evgenia, *"Exponential Gelfond-Khovanskii Formula in Dimension One,"* Math ArXiV preprint `0312433` .

[The02] Theobald, Thorsten, *"Computing Amoebas,"* Experiment. Math. Volume 11, Issue 4 (2002), pp. 513–526.

[TdW13] Theobald, Thorsten and de Wolff, Timo, *"Approximating Amoebas and Coamoebas by Sums of Squares,"* Math. Comp., to appear. Also downloadable as Math ArXiV preprint `1101.4114` .

[TdW14] Theobald, Thorsten and de Wolff, Timo, *"Norms of Roots of Trinomials,"* Math ArXiV preprint `1411.6552` .

[Vir01] Viro, Oleg Ya., *"Dequantization of Real Algebraic Geometry on a Logarithmic Paper,"* Proceedings of the 3rd European Congress of Mathematicians, Birkhäuser, Progress in Math, 201, (2001), pp. 135–146.

[Voo77] Voorhoeve, Marc, *"Zeros of Exponential Polynomials,"* Ph.D. thesis, University of Leiden, 1977.

[Wil17] Wilder, Charles E., *"Expansion problems of ordinary linear differential equations,"* Trans. Amer. Math. Soc., 18 (1917), pp. 415–442.

[Wil96] Wilkie, A. J., *"Model completeness results for expansions of the ordered field of real numbers by restricted pfaffian functions and the exponential functions,"* J. Amer. Math. Soc. 9 (1996), pp. 1051–1094.

[Zie95] Ziegler, Günter M., *Lectures on Polytopes*, Graduate Texts in Mathematics, Springer Verlag, 1995.

[Zil02] Zilber, Boris, *"Exponential sum equations and the Schanuel conjecture,"* J. London Math. Soc. (2) **65** (2002), pp. 27–44.

[Zil11] Zilber, Boris, *"Exponentially Closed Fields and the Conjecture on Intersections with Tori,"* Math ArXiV preprint `1108.1075v2` .

Department of Mathematics, Texas A&M University TAMU 3368, College Station, Texas 77843-3368, USA.

E-mail address: `alperen@math.tamu.edu`

Department of Mathematics, Texas A&M University TAMU 3368, College Station, Texas 77843-3368, USA.

E-mail address: `grigoris@math.tamu.edu`

Department of Mathematics, Texas A&M University TAMU 3368, College Station, Texas 77843-3368, USA.

E-mail address: `rojas@math.tamu.edu`