# ON THE CLUSTERING OF PADÉ ZEROS AND POLES OF RANDOM POWER SERIES 

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#### Abstract

We estimate non-asymptotically the probability of uniform clustering around the unit circle of the zeros of the $[m, n]$-Padé approximant of a random power series $f(z)=\sum_{j=0}^{\infty} a_{j} z^{j}$ for $a_{j}$ independent, with finite first moment, and Lévy function satisfying $\mathcal{L}\left(a_{j}, \varepsilon\right) \leq K \varepsilon$. Under the same assumptions we show that almost surely $f$ has infinitely many zeros in the unit disc, with the unit circle serving as a natural boundary for $f$. For $R_{m}$ the radius of the largest disc containing at most $m$ zeros of $f$, a deterministic result of Edrei implies that in our setting the poles of the [ $m, n$ ]-Padé approximant almost surely cluster uniformly at the circle of radius $R_{m}$ as $n \rightarrow \infty$ and $m$ stays fixed, and we provide almost sure rates of converge of these $R_{m}$ 's to 1 . We also show that our results on the clustering of the zeros hold for log-concave vectors $\left(a_{j}\right)$ with not necessarily independent coordinates.


## 1. Introduction

The $[m, n]$-Padé approximant of a power series $f(z)=\sum_{j=0}^{\infty} a_{j} z^{j}$ is the rational function

$$
\begin{equation*}
\mathfrak{P}_{m n}(z)=\frac{P_{m n}(z)}{Q_{m n}(z)}=\frac{p_{0}+p_{1} z+\ldots+p_{m} z^{m}}{q_{0}+q_{1} z+\ldots+q_{n} z^{n}} \tag{1.1}
\end{equation*}
$$

with Taylor series at 0 that matches $f$ as much as possible, as introduced by Frobenius in [Fro81]. The aim of this article is to examine the behavior of Pade approximants when the coefficients $a_{j}$, and therefore the $p$ 's and $q$ 's, are random variables. Our interest is in the behavior, with high probability, of the random zeros and poles of $\mathfrak{P}_{m n}$ and how these cluster around specific geometric loci, cf. [Fro69], [GP97a], [GP97b]. Under our assumptions here on the randomness of the $a_{j}$ 's, and for the range of $m$ and $n$ we examine, this locus turns out to be the unit circle.

In particular, as the $[m, n]$-Padé approximant of the power series $\sum_{j=0}^{\infty} a_{j} z^{j}$ is the same as the Padé approximant of the polynomial $\sum_{j=0}^{N} a_{j} z^{j}$ for $N \geq m+n$, see section 2.1 , we show as Theorem 3.2 that: Whenever $\left(a_{j}\right)_{j=0}^{N}$ for $N \geq m+n$ is an isotropic random vector with independent coordinates and with a certain anti-concentration property, see (3.9), then the zeros of the numerator $P_{m n}$ of the $[m, n]$-Padé approximant of the random polynomial $\sum_{j=0}^{N} a_{j} z^{j}$ cluster uniformly around the unit circle for $m$ and $n$ in a certain range, see (3.8). We understand uniform clustering as in Szegö [Sze22], Rosenbloom [Ros55], and Erdős-Turán, [ET50]. We measure how much clustering is achieved either via the Erdős-Turán ratio (2.22) or via the distance of the empirical measure of the zeros of the $P_{m n}$ from the uniform measure on the unit circle in the bounded Lipschitz metric (2.26). The result is non-asymptotic in that we find the range of $m$ and $n$ for which the desired degree of clustering happens. Theorem 3.2 follows from the more general Theorem 3.1, whose proof relies on the deterministic Erdős-Turán result [ET50], a standard application of Jensen's formula on zeros of holomorphic functions [Jen99], the calculation of the coefficients of the Padé numerator using Toeplitz matrices (2.4), (2.5), and our estimate on the probability of invertibility of Toeplitz matrices, Proposition 2.3.

Theorem 4.2 provides another way of understanding why the zeros of Padé numerators cluster uniformly around the unit circle. One of the conclusions of that Theorem is that the unit circle is almost surely a natural boundary for the power series $f(z)=\sum_{j=0}^{\infty} a_{j} z^{j}$ when the $a_{j}$ 's are independent, have moment bounds, and anti-concentrate. Our proof of this relies on the Ryll-Nardzewski theorem [Kah85, p. 41] and complements the classical result on symmetric random $a_{j}$ 's [Kah85, p. 39] and the results of Breuer and

[^0]Simon [BS11] on bounded $a_{j}$ 's. Now a deterministic result of Edrei [Edr78], recasting Hadamard [Had92] and generalizing Szegö [Sze22], shows that for meromorphic $f$, as $n$ stays constant and $m \rightarrow \infty$, the zeros of the Padé numerator $P_{m n}$ cluster, up to subsequence, around the boundary of the largest disc that contains not more than $n$ poles of $f$. In the presence of a natural boundary all such discs are the unit disc, therefore clustering takes place around the unit circle. Note that Edrei's deterministic result is asymptotic and holds for some $m$-subsequence whereas our probabilistic Theorem 3.2 holds for all $m$ 's and $n$ 's that satisfy our non-asymptotic condition (3.8). Note also that part of Edrei's proof is devoted to showing that the Toeplitz determinants that calculate the Padé numerators and denominators are not singular for his subsequences. In our work, this is reflected by our estimate on the probability of the invertibility of random Toeplitz matrices, Proposition 2.3.

Regarding the poles of the Padé approximant $\mathfrak{P}_{m n}$, we rely in Section 4 on the fact the the $[m, n]$ denominator for $f$ is the $[n, m]$-numerator of $1 / f$ and appeal directly to Edrei's result for $1 / f$. For this, we need information on the position of the poles of $1 / f$, equivalently of the zeros of $f$. This we garner in Theorem 4.12 that shows that almost surely $f$ has infinitely many zeros in the unit disc and estimates how fast the radius $R_{m}$ of the largest disc that contains at most $m$ zeros converges to 1 . The assumptions here are the same as for Theorem 4.2. The proof uses Jensen's formula [Jen99]. The application of Edrei's result for almost any realization of the random series takes place in Theorem 4.16 showing that for fixed $m$ the zeros of $Q_{m n}$ cluster uniformly as $n \rightarrow \infty$ around the circle of radius $R_{m}$, with $1-R_{m}=O(\log m / m)$.

For $n=0$ the Padé approximants are, of course, nothing but the Taylor polynomials that approximate the power series. For random coefficients, the study of these fits in the extensive literature on random polynomials, see for example [HN08, IZ13, vv62].

Some background on Padé approximants and preliminary probabilistic results are included in Section 2. The final Section 5 provides the estimate on the probability of the invertibility of random Toeplitz matrices when the $a_{j}$ 's are not independent but come from a log-concave random vector. An Appendix shows how the Erdős-Turán ratio controls the bounded Lipschitz distance of the empirical measure of the zeros of a polynomial from the uniform measure on the unit circle.

Whereas we made some effort to give precise values to the various universal constants that appear in the estimates we do not claim that they are the best possible. When we do not specify the constants we reserve the right to change them freely using the same notation.

## 2. Padé, Toeplitz, and Erdős -Turán

Padé approximants are inextricably linked with Toeplitz matrices, and a result of Erdős and Turán is a standard way to show clustering of roots of polynomials. In this section we present these connections along with preliminary results, some probabilistic, that we use later.

### 2.1. Padé approximants and formulas. Recall that given a power series

$$
\begin{equation*}
f(z)=\sum_{j=0}^{\infty} a_{j} z^{j} \tag{2.1}
\end{equation*}
$$

the $[m, n]$-Pade approximant of $f$ is the (unique) rational function

$$
\begin{equation*}
\mathfrak{P}_{m n}=P_{m n} / Q_{m n} \tag{2.2}
\end{equation*}
$$

with $P_{m n}$ and $Q_{m n}$ polynomials on $\mathbb{C}$ of degree at most $m$ and $n$, respectively, and $Q_{m n}(z) \not \equiv 0$, such that

$$
\begin{equation*}
f(z) Q_{m n}(z)-P_{m n}(z)=\sum_{j \geq m+n+1} c_{j} z^{j} \tag{2.3}
\end{equation*}
$$

for some $c_{j}$ 's, [Fro81], [Gra72]. (For the definition in [Bak75] and its place in the probabilistic approach, see Remark 2.1 below.)

To determine the coefficients $\mathbf{p}=\left(p_{0}, \ldots, p_{m}\right)$ of $P_{m n}$ and $\mathbf{q}=\left(q_{0}, \ldots, q_{n}\right)$ of $Q_{m n}$, the Frobenius formulation (2.3) leads to the following linear systems:

$$
\left[\begin{array}{ccccc}
a_{0} & 0 & 0 & \ldots & 0  \tag{2.4}\\
a_{1} & a_{0} & 0 & \ldots & 0 \\
\vdots & & & & \\
a_{m} & a_{m-1} & \ldots & \ldots & a_{m-n}
\end{array}\right] \mathbf{q} \equiv C_{m}^{(n)} \mathbf{q}=\mathbf{p}
$$

and

$$
\left[\begin{array}{ccccc}
a_{m+1} & a_{m} & a_{m-1} & \ldots & a_{m-(n-1)}  \tag{2.5}\\
a_{m+2} & a_{m+1} & a_{m} & \ldots & a_{m-(n-2)} \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
a_{m+n} & a_{m+(n-1)} & a_{m+(n-2)} & \cdots & a_{m}
\end{array}\right] \mathbf{q} \equiv T_{m}^{(n)} \mathbf{q}=\mathbf{0}
$$

Note that $C_{m}^{(n)}$ is an $(m+1) \times(n+1)$ matrix, whereas $T_{m}^{(n)}$ is an $n \times(n+1)$ matrix. It is a standard convention that $a_{l}=0$ when $l$ is negative.

Prominent in what follows will be the $n \times n$ square matrices $A_{m}^{(n)}$ (following the notation in [Edr78]), a submatrix of $T_{m}^{(n)}$,

$$
A_{m}^{(n)}=\left[\begin{array}{cccc}
a_{m} & a_{m-1} & \ldots & a_{m-(n-1)}  \tag{2.6}\\
a_{m+1} & a_{m} & \ldots & a_{m-(n-2)} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m+(n-1)} & a_{m+(n-2)} & \cdots & a_{m}
\end{array}\right] .
$$

These are Toeplitz matrices, i.e. matrices with constant entries on each diagonal.
Remark 2.1. It is important to consider the case

$$
\begin{equation*}
\operatorname{det} A_{m}^{(n)} \neq 0, \quad \text { for all } m, n \tag{2.7}
\end{equation*}
$$

Then the Padé polynomials can be written as

$$
\begin{align*}
& P_{m n}(z)=a_{0}+\ldots+\frac{\operatorname{det} A_{m}^{(n+1)}}{\operatorname{det} A_{m}^{(n)}} z^{m}, \\
& Q_{m n}(z)=1-\frac{\operatorname{det}\left(T_{m}^{(n)}[2]\right)}{\operatorname{det} A_{m}^{(n)}}+\ldots+(-1)^{n} \frac{\operatorname{det} A_{m+1}^{(n)}}{\operatorname{det} A_{m}^{(n)}} z^{n}, \tag{2.8}
\end{align*}
$$

with $T_{m}^{(n)}[k]$ the square matrix that results from $T_{m}^{(n)}$ from (2.5) after the $k$-th column is omitted, see [Gra72, Corollary 1, p. 18].

Note that (2.7) then implies that the top order coefficients of (2.8) do not vanish. This in turn implies that any pair of polynomials satisfying (2.3) are constant multiples of those in (2.8). In this case the Baker condition $Q_{m n}(0) \neq 0[\operatorname{Bak} 75$, p. 6] is satisfied and we also have

$$
\begin{equation*}
f(z)-\frac{P_{m n}(z)}{Q_{m n}(z)}=\sum_{j \geq m+n+1} \hat{c}_{j} z^{j} . \tag{2.9}
\end{equation*}
$$

As in our power series $\sum_{j \geq 0} a_{j} z^{j}$ will have random coefficients, we examine next conditions for (2.7) to be satisfied almost always, or at least with high probability. In particular, pathological examples like $f(z)=$ $1+\sum_{j \geq 2} z^{j}$ when $P_{11}(z)=z$ and $Q_{11}(z)=z$ satisfy (2.3) but not (2.9), either almost surely or with high probability will not happen.
2.2. Random Toeplitz determinants. We now quantify the singularity of random Toeplitz matrices. Recall first the Lévy concentration function for a random variable $\xi$ for $\delta>0$ :

$$
\begin{equation*}
\mathcal{L}(\xi, \delta)=\sup _{t \in \mathbb{R}} \mathbb{P}(|\xi-t|<\delta) \tag{2.10}
\end{equation*}
$$

This measure of dispersion was introduced in the works of Doeblin \& Lévy [DL36], Kolmogorov [Kol58], and Esseen [Ess66] concerning the spread of sums of independent random variables. In recent years it has
become an indispensable tool in the study of quantitative invertibility of (unstructured) random matrices, see [RV10] for a detailed exposition. In the present work it occupies a central role, too, since it quantifies the invertibility of random patterned matrices (Toeplitz), and it is related to the existence of a naturally boundary for a random power series.
Lemma 2.2. For any (deterministic) monic polynomial $P$ with $\operatorname{deg} P=d$, for any random variable $\xi$, and for any $\varepsilon>0$, we have

$$
\begin{equation*}
\mathbb{P}\left(|P(\xi)|<\varepsilon^{d}\right) \leq d \mathcal{L}(\xi, \varepsilon) \tag{2.11}
\end{equation*}
$$

Proof. Write $P(z)=\prod_{j=1}^{d}\left(z-r_{j}\right)$, for $r_{j}$ the (complex) roots of $P$. Then

$$
\begin{equation*}
\mathbb{P}\left(|P(\xi)|<\varepsilon^{d}\right) \leq \sum_{j=1}^{d} \mathbb{P}\left(\left|\xi-r_{j}\right|<\varepsilon\right) \leq \sum_{j=1}^{d} \mathbb{P}\left(\left|\xi-\operatorname{Re}\left(r_{j}\right)\right|<\varepsilon\right) \leq d \mathcal{L}(\xi, \varepsilon) \tag{2.12}
\end{equation*}
$$

using the union bound.
Proposition 2.3 (quantitative invertibility). For $\mathbf{a}=\left(a_{k}\right)_{0 \leq k \leq 2 n-2}$ a random vector with independent coordinates, and for A the Toeplitz matrix

$$
A=\left[\begin{array}{cccc}
a_{n-1} & a_{n-2} & \ldots & a_{0}  \tag{2.13}\\
a_{n} & a_{n-1} & \ldots & a_{1} \\
\vdots & \vdots & \ddots & \vdots \\
a_{2 n-2} & a_{2 n-3} & \ldots & a_{n-1}
\end{array}\right]
$$

and for any $\varepsilon>0$,

$$
\begin{equation*}
\mathbb{P}\left(|\operatorname{det} A|^{1 / n}<\varepsilon\right) \leq n \mathcal{L}\left(a_{n-1}, \varepsilon\right) \tag{2.14}
\end{equation*}
$$

Proof. Write

$$
\begin{equation*}
A=a_{n-1} I+\sum_{k \neq n-1} a_{k} B_{k} \tag{2.15}
\end{equation*}
$$

where each $B_{k}$ has 1's on a single, not the main, diagonal and 0's everywhere else. The random matrices $a_{n-1} I$ and $B:=\sum_{k \neq n-1} a_{k} B_{k}$ are independent. Conditioning on $a_{k}$ for all $k \neq n-1$,

$$
\begin{align*}
\mathbb{P}\left(|\operatorname{det} A|^{1 / n}<\varepsilon\right) & =\mathbb{E}\left[\mathbb{P}\left(\left|\operatorname{det}\left(a_{n-1} I+B\right)\right|<\varepsilon^{n} \mid\left(a_{k}\right)_{k \neq n-1}\right)\right]  \tag{2.16}\\
& =\mathbb{E}\left[\mathbb{P}\left(\left|P_{-B}\left(a_{n-1}\right)\right|<\varepsilon^{n} \mid\left(a_{k}\right)_{k \neq n-1}\right)\right]
\end{align*}
$$

for $P_{-B}(\cdot)$ the characteristic polynomial of a matrix $-B$. As $P_{-B}$ is monic, apply Lemma 2.2 conditionally to conclude.
2.3. Clustering of zeros. Let $P(z)=\alpha_{0}+\alpha_{1} z+\ldots+\alpha_{N} z^{N}, \alpha_{N} \neq 0$ be a polynomial with roots $z_{1}, \ldots, z_{N}$, and let

$$
\begin{equation*}
\nu_{P}=\frac{1}{N} \sum_{j=1}^{N} \delta_{z_{j}} \tag{2.17}
\end{equation*}
$$

the normalized zero-counting measure associated with $P$.
For a sequence of polynomials $P_{m}$, each of degree $m$, we write $\nu_{m}$ for $\nu_{P_{m}}$.
The following makes precise the meaning of clustering of zeros around a circle, following Szegö [Sze22], Rosenbloom [Ros55], Erdős-Turán [ET50], and Edrei [Edr78].
Definition 2.4 (clustering). The zeros of a sequence of polynomials $P_{m}$, each of degree $m$, cluster uniformly around the circle of radius $r$ if both of the following are satisfied:
(a) Radial clustering. For any $\rho>0$, for the annulus

$$
\begin{equation*}
R(r, \rho)=\{z \in \mathbb{C}:(1-\rho) r<|z|<(1+\rho) r\} \tag{2.18}
\end{equation*}
$$

we have

$$
\begin{equation*}
\nu_{m}(R(r, \rho)) \rightarrow 1, \quad m \rightarrow \infty \tag{2.19}
\end{equation*}
$$

(b) Angular clustering. For $0 \leq \theta<\phi<2 \pi$, for the angular sector sector

$$
\begin{equation*}
S(\theta, \phi)=\{z \in \mathbb{C}: \theta<\operatorname{Arg} z \leq \phi\} \tag{2.20}
\end{equation*}
$$

we have

$$
\begin{equation*}
\nu_{m}(S(\theta, \phi)) \rightarrow \frac{\phi-\theta}{2 \pi}, \quad m \rightarrow \infty \tag{2.21}
\end{equation*}
$$

Our goal is to establish such clustering for the polynomial numerators and denominators of Padé approximants of random Taylor series, and to estimate how much clustering takes place at large but finite degree.

As is well known, both radial and angular clustering of polynomials around the unit circle are controlled by comparing the end coefficients to all of the coefficients: For any polynomial $P(z)=\alpha_{0}+\alpha_{1} z+\ldots+\alpha_{N} z^{N}$ with $\alpha_{0} \alpha_{N} \neq 0$, set

$$
\begin{equation*}
L(P):=\frac{\left|\alpha_{0}\right|+\left|\alpha_{1}\right|+\ldots+\left|\alpha_{N}\right|}{\sqrt{\left|\alpha_{0}\right| \cdot\left|\alpha_{N}\right|}} \tag{2.22}
\end{equation*}
$$

Then for radial clustering around the unit circle we have:
Proposition 2.5. For $P(z)=\alpha_{0}+\alpha_{1} z+\ldots+\alpha_{N} z^{N}$ with $\alpha_{0} \alpha_{N} \neq 0$ and for $0<\rho \leq 1$,

$$
\begin{equation*}
1-\nu_{P}(R(1, \rho)) \leq \frac{\log L(P)}{\rho N} \tag{2.23}
\end{equation*}
$$

A proof of this is part of Theorem 1 in [Ros55, p. 268]. Edrei reproves it [Edr78, p. 264]. See also [HN08] for a more recent take connected to probabilistic considerations independent of Padé approximants. All these use the classical formula of Jensen [Jen99] which, for a polynomial $P$ links the measure $\nu_{P}$ and the average of $P$ on a circle. For a useful presentation of Jensen's formula, including the case of zeros on the circle, see [AN07, §4.8]. We state the formula here in a form that will be useful later.
Proposition 2.6 (Jensen Formula). For $f$ holomorphic on the closed disc of radius $r$ and with $f(0) \neq 0$,

$$
\begin{equation*}
\frac{1}{2 \pi} \int_{0}^{2 \pi} \log \left|f\left(r e^{i \theta}\right)\right| d \theta+\sum_{j=1}^{n} \log \frac{\left|z_{j}\right|}{r}=\log |f(0)| \tag{2.24}
\end{equation*}
$$

where $z_{1}, \ldots, z_{n}$ are the zeros of $f$ in the interior of the disc of radius $r$, repeated according to multiplicity.
As for angular clustering, we have the celebrated Erdős-Turán result [ET50], see also [Ros55]. For a more recent treatment see [AB02].
Proposition 2.7 (Erdős-Turán). For $P(z)=\alpha_{0}+\alpha_{1} z+\ldots+\alpha_{N} z^{N}$ with $\alpha_{0} \alpha_{N} \neq 0$ and for $0 \leq \theta<\phi<2 \pi$,

$$
\begin{equation*}
\left|\frac{\phi-\theta}{2 \pi}-\nu_{P}(S(\theta, \phi))\right| \leq 16 \sqrt{\frac{\log L(P)}{N}} \tag{2.25}
\end{equation*}
$$

For $\mu$ the uniform measure on the unit circle $\mathbb{T}=\{z \in \mathbb{C}:|z|=1\}$, Proposition 2.5 and Proposition 2.7 show that $\nu_{P}$ is close to $\mu$ in certain sense. This proximity can be measured in terms of the bounded Lipschitz metric

$$
\begin{equation*}
d_{\mathrm{BL}}\left(\mu, \nu_{P}\right)=\sup \left\{\left|\int f d \mu-\int f d \nu\right|: f \in B_{1}\right\} \tag{2.26}
\end{equation*}
$$

for $B_{1}:=\left\{f: \mathbb{C} \rightarrow \mathbb{R} \mid\|f\|_{\text {BL }}:=\|f\|_{\text {Lip }}+\|f\|_{\infty} \leq 1\right\}$. It is well known that this metric induces weak convergence in the space of probability distributions, c.f. [Bog07, §8.3]. We then have
Proposition 2.8. For $P(z)=\alpha_{0}+\alpha_{1} z+\ldots+\alpha_{N} z^{N}$ with $\alpha_{0} \alpha_{N} \neq 0$,

$$
\begin{equation*}
d_{\mathrm{BL}}\left(\nu_{P}, \mu\right) \leq 32\left(\frac{\log L(P)}{N}\right)^{1 / 4} \tag{2.27}
\end{equation*}
$$

For completeness, we include a proof of this proposition in the Appendix A.
2.4. Erdős-Turán for Padé numerators and denominators. We now estimate the Erdős-Turán ratio (2.22) in the case of Padé numerators of $f(z)=\sum_{j=0}^{\infty} a_{j} z^{j}$ in terms of the coefficients $a_{j}$. Toeplitz matrices inevitably appear and we assume here that their determinants do not vanish. As mentioned in Remark 2.1, this assumption will hold in what follows either almost always or with high probability.

From (2.8) the coefficient vectors of the numerator and denominator of the $[m, n]$-Pade of $f$ will be $\mathbf{p}=\left(a_{0}, p_{1}, \ldots, p_{m}\right), \mathbf{q}=\left(1, q_{1}, \ldots, q_{n}\right)$. From (2.4) we have

$$
\begin{equation*}
\|\mathbf{p}\|_{1}=\left\|C_{m}^{(n)} \mathbf{q}\right\|_{1} \leq\|\mathbf{q}\|_{1} \cdot \max _{j \leq n}\left\|C_{m}^{(n)} e_{j}\right\|_{1}=\|\mathbf{q}\|_{1} \cdot \sum_{j=0}^{m}\left|a_{j}\right| . \tag{2.28}
\end{equation*}
$$

Along with the expression for $q_{n}$ from (2.8), this estimates the Erdős-Turán ratio of $P_{m n}$ in terms of the ratio of $Q_{m n}$ :

$$
\begin{align*}
L\left(P_{m n}\right) & \leq \frac{\|\mathbf{q}\|_{1}}{\sqrt{\left|a_{0}\right|\left|q_{n}\right|}} \cdot\left(\frac{\left|\operatorname{det} A_{m+1}^{(n)}\right|}{\left|\operatorname{det} A_{m}^{(n+1)}\right|}\right)^{1 / 2} \cdot \sum_{j=0}^{m}\left|a_{j}\right| \\
& =\frac{\sum_{j=0}^{m}\left|a_{j}\right|}{\sqrt{\left|a_{0}\right|}} \cdot\left(\frac{\left|\operatorname{det} A_{m+1}^{(n)}\right|}{\left|\operatorname{det} A_{m}^{(n+1)}\right|}\right)^{1 / 2} \cdot L\left(Q_{m n}\right) . \tag{2.29}
\end{align*}
$$

On the other hand, the expression for $q_{k}$ from (2.8) gives

$$
\begin{equation*}
L\left(Q_{m n}\right)=\|\mathbf{q}\|_{1}\left(\frac{\left|\operatorname{det} A_{m}^{(n)}\right|}{\left|\operatorname{det} A_{m+1}^{(n)}\right|}\right)^{1 / 2}=\left(1+\sum_{k=1}^{n} \frac{\operatorname{det}\left(T_{m}^{(n)}[k+1]\right)}{\left|\operatorname{det} A_{m}^{(n)}\right|}\right) \cdot\left(\frac{\left|\operatorname{det} A_{m}^{(n)}\right|}{\left|\operatorname{det} A_{m+1}^{(n)}\right|}\right)^{1 / 2} \tag{2.30}
\end{equation*}
$$

Then (2.29) becomes

$$
\begin{align*}
L\left(P_{m n}\right) & \leq \frac{\sum_{j=0}^{m}\left|a_{j}\right|}{\sqrt{\left|a_{0}\right|}} \cdot\left(1+\sum_{k=1}^{n} \frac{\operatorname{det}\left(T_{m}^{(n)}[k+1]\right)}{\left|\operatorname{det} A_{n}^{(m)}\right|}\right) \cdot\left(\frac{\left|\operatorname{det} A_{m}^{(n)}\right|}{\left|\operatorname{det} A_{m}^{(n+1)}\right|}\right)^{1 / 2}  \tag{2.31}\\
& \leq \frac{\sum_{j=0}^{m}\left|a_{j}\right|}{\sqrt{\left|a_{0}\right|}} \frac{\sqrt{n \operatorname{det}\left(T_{m}^{(n)} T_{m}^{(n)^{*}}\right)}}{\left(\left|\operatorname{det} A_{m}^{(n)}\right| \cdot\left|\operatorname{det} A_{m}^{(n+1)}\right|\right)^{1 / 2}} \tag{2.32}
\end{align*}
$$

after using the Cauchy-Schwarz inequality to go to the sum of the squares of the sub-determinants and the Cauchy-Binet formula for $\operatorname{det}\left(T_{m}^{(n)} T_{m}^{(n)^{*}}\right)$ :

$$
\begin{equation*}
\operatorname{det}\left(T_{m}^{(n)} T_{m}^{(n)^{*}}\right)=\sum_{k=0}^{n} \operatorname{det}\left(T_{m}^{(n)}[k+1]\right)^{2} \tag{2.33}
\end{equation*}
$$

Finally, the arithmetic-geometric mean inequality (applied on the singular values of $T_{m}^{(n)}$ ) gives

$$
\begin{equation*}
L\left(P_{m n}\right) \leq \frac{\sum_{j=0}^{m}\left|a_{j}\right|}{\sqrt{\left|a_{0}\right|}} \cdot \frac{n^{1 / 2}\left(n^{-1 / 2}\left|\left\|T_{m}^{(n)} \mid\right\|_{1}\right)^{n}\right.}{\left(\left|\operatorname{det} A_{m}^{(n)}\right| \cdot\left|\operatorname{det} A_{m}^{(n+1)}\right|\right)^{1 / 2}} \tag{2.34}
\end{equation*}
$$

where $\left\|\mid T_{m}^{(n)}\right\|\left\|_{1}:=\sum_{j=1}^{n+1}\right\| T_{m}^{(n)}\left(e_{j}\right) \|_{1}$.

## 3. NON-ASYMPTOTIC CLUSTERING OF ROOTS

Given a random power series of the form

$$
\begin{equation*}
f(z)=\sum_{j=0}^{\infty} a_{j} z^{j}, \quad a_{j}:(\Omega, \mathbb{P}) \rightarrow \mathbb{C} \text { random variables }, \tag{3.1}
\end{equation*}
$$

it is clear from (2.4) and (2.5) that calculating the [ $m, n$ ]-Pade approximant of $f$ is the same as caclulating the $[m, n]$-Padé approximant of $f_{N}=\sum_{j=0}^{N} a_{j} z^{j}$ for any $N \geq m+n$. As our focus in this section is on
non-asymptotic results, we work here with $f_{N}$. We deal with full power series in the next section. The following is our main probabilistic, non-asymptotic estimate for the Erdős-Turán ratio of the numerator of the Padé approximants when the $a_{j}$ 's are independent random variables. It is applied when $\mathcal{L}\left(a_{j}, \varepsilon\right) \leq K \varepsilon$ in Theorem 3.2 and we show its use for discrete distributions in Example 3.3.

Theorem 3.1 (Bound on the Erdős-Turán ratio). Let $m, n \in \mathbb{N}, 1 \leq \gamma<\infty$, and $\varepsilon$ and $b$ in $(0,1)$.
Then for any $N \geq m+n$ and $\left(a_{j}\right)_{j=0}^{N}$ random vector with independent coordinates, $\mathbb{E}\left|a_{j}\right| \leq \gamma$ and $\mathcal{L}\left(a_{j}, \varepsilon\right) \leq b$ for all $j$, the numerator $P_{m n}$ of the $[m, n]$-Padé approximant of $f_{N}(z)=\sum_{k=0}^{N} a_{j} z^{j}$ satisfies

$$
\begin{equation*}
\mathbb{P}\left(\left\{\log L\left(P_{m n}\right)>\log m+2 n \log \left(\frac{4 \gamma}{b \varepsilon}\right)\right\}\right) \leq 5 n b \tag{3.2}
\end{equation*}
$$

Proof. Take $5 n b<1$, as otherwise there is nothing to prove. In view of the estimate (2.34), consider the following events:

$$
\begin{align*}
& \mathcal{E}_{0}:=\left\{\left|a_{0}\right|<\varepsilon\right\},  \tag{3.3}\\
& \mathcal{E}_{1}:=\left\{\sum_{j=0}^{m}\left|a_{j}\right|>t(m+1) \gamma\right\}, \quad \mathcal{E}_{2}:=\left\{\left|\left\|T_{m}^{(n)} \mid\right\|_{1}>\operatorname{tn}(n+1) \gamma\right\}\right.  \tag{3.4}\\
& \mathcal{E}_{3}:=\left\{\left|\operatorname{det} A_{m}^{(n)}\right| \cdot\left|\operatorname{det} A_{m}^{(n+1)}\right|<\varepsilon^{2 n}\right\}, \tag{3.5}
\end{align*}
$$

where $t>0$ will be suitably chosen below. Note that

$$
P\left(\mathcal{E}_{0}\right) \leq b, \quad \mathbb{P}\left(\mathcal{E}_{3}\right) \leq n \mathcal{L}\left(a_{m}, \varepsilon\right)+(n+1) \mathcal{L}\left(a_{m}, \varepsilon\right) \leq 3 n b
$$

by the assumption on $\mathcal{L}$ and Proposition 2.3, and that

$$
\mathbb{P}\left(\mathcal{E}_{1} \cup \mathcal{E}_{2}\right) \leq \frac{2}{t}
$$

by Markov's inequality. Therefore in the complement of $\bigcup_{i=0}^{3} \mathcal{E}_{i}$ we have

$$
L\left(P_{m n}\right) \leq \frac{2 t m \gamma n^{1 / 2}}{\varepsilon} \cdot\left(\frac{2 t \gamma n^{3 / 2}}{\varepsilon}\right)^{n}
$$

with probability greater than $1-b-3 n b-2 / t$. For $t \geq \frac{2}{n b}$ this gives

$$
\begin{equation*}
L\left(P_{m n}\right) \leq m\left(\frac{2 n t \gamma}{\varepsilon}\right)^{2 n} \tag{3.6}
\end{equation*}
$$

with probability greater than $1-5 n b$. In particular, the choice $t=2 /(n b)$ yields

$$
\begin{equation*}
L\left(P_{m n}\right) \leq m\left(\frac{4 \gamma}{b \varepsilon}\right)^{2 n} \tag{3.7}
\end{equation*}
$$

with probability greater than $1-5 n b$, as claimed.
The following theorem holds with the same integrability conditions as in Theorem 3.1. We present it here for isotropic random vectors, a standard class of random vectors in high-dimensional probability. Recall that a random vector is called isotropic if it is centered and its covariance matrix is the identity, see e.g., [BGVV14, Definition 2.3.7]. Applying Theorem 3.1 to this case we have

Theorem 3.2. Let $m, n \in \mathbb{N}, K \geq 1$ and $\delta \in(0,1)$. If $m, n$ satisfy

$$
\begin{equation*}
m \geq C \delta^{-4} n \log (e K n / \delta) \tag{3.8}
\end{equation*}
$$

then for any $N \geq m+n$, and for any isotropic random vector $\mathbf{a}=\left(a_{j}\right)_{j=0}^{N} \in \mathbb{R}^{N+1}$ with independent coordinates, and

$$
\begin{equation*}
\mathcal{L}\left(a_{j}, \varepsilon\right) \leq K \varepsilon, \quad \text { for all } j \text { and } \varepsilon>0 \tag{3.9}
\end{equation*}
$$

the numerator $P_{m n}$ of the $[m, n]$-Padé approximant of $f_{N}(z)=\sum_{j=0}^{N} a_{j} z^{j}$ satisfies

$$
\begin{equation*}
\mathbb{P}\left(\frac{\log L\left(P_{m n}\right)}{m}>\delta^{4}\right)<\delta \tag{3.10}
\end{equation*}
$$

In particular, for $\mu$ the uniform measure on the unit circle we have

$$
\begin{equation*}
\mathbb{P}\left(d_{\mathrm{BL}}\left(\nu_{P_{m n}}, \mu\right)>40 \delta\right)<\delta \tag{3.11}
\end{equation*}
$$

Proof. Isotropicity implies that in applying Theorem 3.1 we can take $\gamma=1$, as $\mathbb{E}\left|a_{j}\right| \leq\left(\mathbb{E}\left|a_{j}\right|^{2}\right)^{1 / 2}=1$. If we take $\varepsilon=\delta /(5 K n)$ we see that $b=\delta /(5 n)$ in Theorem 3.1 will do. Then (3.2) becomes

$$
\begin{equation*}
\mathbb{P}\left(\frac{\log L\left(P_{m n}\right)}{m}>\frac{\log m}{m}+\frac{2 n \log \left(100 K n^{2} / \delta^{2}\right)}{m}\right)<\delta \tag{3.12}
\end{equation*}
$$

Finally, observe that there is a universal ${ }^{1}$ constant $C>0$, such that for $m \geq C \delta^{-4} n \log (e K n / \delta)$ we have

$$
\begin{equation*}
\delta^{4}>\frac{\log m}{m}+\frac{2 n \log \left(100 K n^{2} / \delta^{2}\right)}{m} \tag{3.13}
\end{equation*}
$$

so that for $m$ and $n$ in the same range

$$
\begin{equation*}
\mathbb{P}\left(\frac{\log L\left(P_{m n}\right)}{m}>\delta^{4}\right)<\delta \tag{3.14}
\end{equation*}
$$

For the last statement, combine (3.14) with Proposition 2.8.
The (static) estimate on the probability of failure in Theorem 3.1 is meaningful even for discrete distributions, provided that they have sufficient anti-concentration bounds, as the following example shows:
Example 3.3. Fix $M$ a large positive integer and take the random variables $a_{j}$ to have common distribution supported uniformly on $\{0, \pm 1, \ldots, \pm M\}$, so that $\mathcal{L}\left(a_{j}, 1 / 2\right) \asymp 1 / M$ and $\mathbb{E}\left|a_{j}\right| \asymp M$. Then estimate (3.2) yields

$$
\log L\left(P_{m n}\right) \leq \log m+C_{1} n \log \left(C_{2} n M\right)
$$

and the clustering of the Padé numerator at $|z|=1$ is measured by

$$
\frac{\log m}{m}+\frac{1}{m} C_{1} n \log \left(C_{2} n M\right)
$$

with probability greater than $1-c n / M$. For $n$ small compared to $M$, e.g. for $n \sim \sqrt{M}$, clustering will be manifested whenever $m$ is sufficiently large, and in particular much larger than $n \log M$.

## 4. Asymptotic clustering of poles

We now turn our attention to the poles of Padé approximants of random power series. For random $f_{\omega}(z)=\sum_{j=0}^{\infty} a_{j}(\omega) z^{j}$ we shall apply on $1 / f_{\omega}$, and separately for almost any $\omega$, Edrei's deterministic result for the clustering of Padé numerators of meromorphic functions on circles defined by the distance of the poles from the orgin, [Edr78]. This will determine the clustering of the denominator $Q_{m n}$ via the elementary property

$$
\begin{equation*}
P_{n m}^{(1 / f)}=Q_{m n}^{(f)} \tag{4.1}
\end{equation*}
$$

where we use superscripts to indicate the power series that the Padé approximates, see [Gra72, p. 216, Theorem 22(i)] or [Gil78, p. 216]. For this we show first as Theorem 4.2 that under our assumptions the power series has almost always radius of convergence 1 and that the unit circle is in fact a natural boundary. Then, after a few preliminary facts, we establish in Theorem 4.12 that the random power series $f$ has zeros in the open unit disc $\mathbb{D}$ with high probability and gather information for the location of these zeros. We conclude by applying Edrei's result in the last subsection.

[^1]4.1. Natural boundary. Recall first the following:

Definition 4.1 (Natural Boundary). Let $r_{f}<\infty$ be the radius of convergence of the power series $f(z)=$ $\sum_{j=0}^{\infty} a_{j} z^{j}$. Then $|z|=r_{f}$ is the natural boundary of $f$ if $f$ cannot be extended to a holomorphic function through any arc of this circle.
Theorem 4.2. Let $\left(a_{j}\right)_{j=0}^{\infty}$ be independent random variables with $\mathbb{E}\left|a_{j}\right| \leq \gamma<\infty$, and $\liminf _{j} \mathcal{L}\left(a_{j}, \varepsilon\right)<1$ for some $\varepsilon>0$. Then, the random power series $f(z)=\sum_{j=0}^{\infty} a_{j} z^{j}$ almost surely has radius of convergence 1 and the unit circle is almost surely a natural boundary for $f$.
Proof. Let $\tau=\liminf _{j} \mathcal{L}\left(a_{j}, \varepsilon\right)$. In particular, we have

$$
\limsup _{j} \mathbb{P}\left(\left|a_{j}\right| \geq \varepsilon\right) \geq 1-\tau>0 \quad \Longrightarrow \quad \sum_{j=0}^{\infty} \mathbb{P}\left(\left|a_{j}\right| \geq \varepsilon\right)=\infty
$$

Then, the second Borel-Cantelli lemma yields that $\left\{\left|a_{j}\right|^{1 / j} \geq 1\right.$ i.o. $\}$ is a sure event. That is, the radius of convergence $r_{f}$ of $f$ satisfies $r_{f} \leq 1$ a.s. For the converse inequality we just observe that, by Markov's inequality, we have

$$
\sum_{j=0}^{\infty} \mathbb{P}\left(\left|a_{j}\right|>(1+\delta)^{j}\right) \leq \gamma \sum_{j=0}^{\infty}(1+\delta)^{-j}<\infty
$$

and the result follows from the first Borel-Cantelli lemma.
To show that the unit circle is a natural boundary, recall that the Ryll-Nardzewksi theorem [RN53] guarantees that ${ }^{2}$ if that were not the case there would exist deterministic power series $g(z)=\sum_{j=0}^{\infty} b_{j} z^{j}$ such that $f-g$ would have radius of convergence $r_{f-g}>1$ and the circle of this radius would be a natural boundary of $f-g$. In particular, almost surely for all $z \in \mathbb{C}$ with $|z|=1$ we would have

$$
\begin{equation*}
\sum_{j=0}^{\infty}\left(a_{j}(\omega)-b_{j}\right) z^{j}<\infty \tag{4.2}
\end{equation*}
$$

so that

$$
\begin{equation*}
a_{j}(\omega)-b_{j} \rightarrow 0 \tag{4.3}
\end{equation*}
$$

with probability 1 . As almost sure convergence implies convergence in probability, this contradicts the fact

$$
\limsup _{j} \mathbb{P}\left(\omega:\left|a_{j}(\omega)-b_{j}\right| \geq \varepsilon\right) \geq 1-\tau>0
$$

The proof is complete.
Remark 4.3. The almost sure existence of a natural boundary appears often in the study of random power series. For symmetric random variables see [RN53]. Our conditions in Theorem 4.2 are more relaxed than those in [BS11] where the $a_{j}$ 's are independent, $\sup _{j}\left|a_{j}(\omega)\right|<M<\infty$ almost surely, and limsup $\left.\operatorname{sar}_{j} \operatorname{Var} a_{j}\right]>$ 0 , see [BS11, Theorem 6.1]. This follows from:
Fact 4.4. Let $\xi$ be a random variable with $\operatorname{Var}[\xi] \geq \theta>0$ and let $\mathbb{E}|\xi|^{4}=\sigma_{4}<\infty$. Then

$$
\mathcal{L}(\xi, \sqrt{\theta / 2})<1-c \frac{\theta^{2}}{\sigma_{4}}
$$

where $c>0$ is a universal constant.
We leave the details to the reader. However, we stress that the authors in [BS11] can show that in their case there is strong natural boundary, see [BS11, p. 4905].
Remark 4.5. Note that the existence of natural boundary at $|z|=1$, in light of Hadamard's theorem [Had92], yields almost surely that for each $n$ we have

$$
\begin{equation*}
\underset{m}{\limsup }\left|\operatorname{det} A_{m}^{(n)}\right|^{1 / m}=1 \tag{4.4}
\end{equation*}
$$

However, under the assumptions of Theorem 4.2 one can show that the limits exist almost surely:

[^2]Fact 4.6. For $\left(a_{j}\right)_{j=0}^{\infty}$ independent random variables with $\mathbb{E}\left|a_{j}\right| \leq \gamma<\infty$, and $\mathcal{L}\left(a_{j}, \varepsilon\right) \leq K \varepsilon$ for all $\varepsilon>0$ and for some $K \geq 1$, it holds that almost surely, for each $n$,

$$
\begin{equation*}
\lim _{m}\left|\operatorname{det} A_{m}^{(n)}\right|^{1 / m}=1 \tag{4.5}
\end{equation*}
$$

Indeed: We fix $n \geq 1$. From the arithmetic-geometric mean inequality

$$
\begin{equation*}
\mathbb{E}\left|\operatorname{det} A_{m}^{(n)}\right|^{1 / n} \leq \frac{\mathbb{E}\left\|A_{m}^{(n)}\right\|_{\mathrm{HS}}}{\sqrt{n}} \leq \gamma n^{3 / 2} \tag{4.6}
\end{equation*}
$$

From Markov's inequality

$$
\begin{equation*}
\mathbb{P}\left(\left|\operatorname{det} A_{m}^{(n)}\right|^{1 / n}>(1+\varepsilon)^{m / n}\right) \leq \frac{\mathbb{E}\left|\operatorname{det} A_{m}^{(n)}\right|^{1 / n}}{(1+\varepsilon)^{m / n}} \leq \frac{n}{(1+\varepsilon)^{m / n}} \tag{4.7}
\end{equation*}
$$

and the first (direct) part of the Borel-Cantelli lemma gives that almost surely

$$
\begin{equation*}
\underset{m}{\limsup }\left|\operatorname{det} A_{m}^{(n)}\right|^{1 / m} \leq 1+\varepsilon \tag{4.8}
\end{equation*}
$$

On the other hand, again by Markov,

$$
\begin{equation*}
\mathbb{P}\left(\left|\operatorname{det} A_{m}^{(n)}\right|^{-1 / 2 n}>(1+\varepsilon)^{m / 2 n}\right) \leq \frac{\mathbb{E}\left|\operatorname{det} A_{m}^{(n)}\right|^{-1 / 2 n}}{(1+\varepsilon)^{m / 2 n}} \leq \frac{2 K n}{(1+\varepsilon)^{m / n}} \tag{4.9}
\end{equation*}
$$

where we have used Proposition 2.3 and that for any random variable $\xi$

$$
\begin{equation*}
\mathbb{P}(\xi<\varepsilon) \leq \gamma \varepsilon \Rightarrow \mathbb{E}\left[\xi^{-1 / 2}\right] \leq 1+\gamma \tag{4.10}
\end{equation*}
$$

Therefore, almost surely for any $\varepsilon>0$

$$
\begin{equation*}
\limsup _{m}\left|\operatorname{det} A_{m}^{(n)}\right|^{-1 / m} \leq 1+\varepsilon \Rightarrow \liminf _{m}\left|\operatorname{det} A_{m}^{(n)}\right|^{1 / m} \geq \frac{1}{1+\varepsilon} \tag{4.11}
\end{equation*}
$$

4.2. Random zeros in the unit disc. The following lemma is standard. We shall use it to estimate averages of random polynomials on the unit circle.

Lemma 4.7. Let $(\Sigma, \mu)$ be a probability space and let $\left(Y_{\sigma}\right)_{\sigma \in \Sigma}$ be a family of random variables with $Y:=$ $\int Y_{\sigma} \mu(d \sigma)$ well defined. If

$$
\sup _{\sigma} \mathbb{P}\left(Y_{\sigma}>t\right) \leq A e^{-a t}
$$

for some $A \geq 1, a>0$, and for all $t>0$, then

$$
\begin{equation*}
\mathbb{E}\left[e^{\lambda Y}\right] \leq 3 A, \quad \text { for all } 0<\lambda \leq \frac{a}{2} \tag{4.12}
\end{equation*}
$$

In particular,

$$
\begin{equation*}
\mathbb{P}(Y>t) \leq 3 A e^{-a t / 2}, \quad \text { for all } t>0 \tag{4.13}
\end{equation*}
$$

Proof. Let $\lambda>0$. Using Jensen's inequality with respect to $\sigma$ and Tonelli's theorem,

$$
\mathbb{E}\left[e^{\lambda Y}\right] \leq \mathbb{E}\left[\int e^{\lambda Y_{\sigma}} \mu(d \sigma)\right]=\int \mathbb{E}\left[e^{\lambda Y_{\sigma}}\right] \mu(d \sigma)
$$

For fixed $\sigma \in \Sigma$ we have

$$
\mathbb{E}\left[e^{\lambda Y_{\sigma}}\right] \leq 1+\mathbb{E}\left[e^{\lambda Y_{\sigma}} \mathbf{1}_{\left\{Y_{\sigma}>0\right\}}\right] \leq 1+\mathbb{E}\left[e^{\lambda Y_{\sigma}^{+}}\right]
$$

Finally, we have

$$
\mathbb{E}\left[e^{\lambda Y_{\sigma}^{+}}\right]=1+\lambda \int_{0}^{\infty} e^{\lambda t} \mathbb{P}\left(Y_{\sigma}>t\right) d t \leq 1+\lambda A \int_{0}^{\infty} e^{(\lambda-a) t} d t \leq 1+A
$$

since $\lambda-a \leq-\lambda$. Integrating this with respect to $\sigma$ gives (4.12), since $A>1$. Markov's inequality for $\lambda=a / 2$ gives (4.13).

We next examine random polynomials $f_{N}(z)=\sum_{j=0}^{N} a_{j} z^{j}$ associated to isotropic vectors $\mathbf{a}=\left(a_{j}\right)_{j=0}^{N}$ in $\mathbb{R}^{N+1}$ for which there is $K \geq 1$ such that for all $\varepsilon>0$

$$
\begin{equation*}
\mathbb{P}(|\langle\mathbf{a}, u\rangle|<\varepsilon) \leq K \varepsilon, \quad \text { for all } u \text { with }\|u\|_{2}=1 \tag{4.14}
\end{equation*}
$$

and we set

$$
\begin{equation*}
X_{z}:=\left|f_{N}(z)\right|^{2} \tag{4.15}
\end{equation*}
$$

Note that $X_{z}$ is also given by

$$
X_{z}=\left\|V_{z}(\mathbf{a})\right\|_{2}^{2}
$$

for $V_{z}: \mathbb{R}^{N+1} \rightarrow \mathbb{R}^{2}$ the linear map with

$$
V_{z}\left(e_{j}\right)=\left(|z|^{j} \cos (j \theta),|z|^{j} \sin (j \theta)\right), \quad \theta=\operatorname{Arg}(z)
$$

for $j=0,1, \ldots, N$ and $\left(e_{j}\right)_{j=0}^{N}$ denotes the standard basis of $\mathbb{R}^{N+1}$. The isotropicity of a implies ${ }^{3}$

$$
\begin{equation*}
\mathbb{E}\left[X_{z}\right]=\left\|V_{z}\right\|_{\mathrm{HS}}^{2}=\sum_{k=0}^{n}|z|^{2 k} \tag{4.16}
\end{equation*}
$$

As we intend to apply Lemma 4.7 to $\log \left(X_{z} / \mathbb{E}\left[X_{z}\right]\right)$, we first need the following:
Lemma 4.8. For all $z \in \mathbb{C}$ and all $t>0$ it holds that

$$
\begin{equation*}
\mathbb{P}\left(\left|\log X_{z}-\log \mathbb{E}\left[X_{z}\right]\right|>t\right) \leq 3 K e^{-t / 2} \tag{4.17}
\end{equation*}
$$

Proof. Fix $z \in \mathbb{C}$. By (4.16) we have

$$
\mathbb{P}\left(\left|\log X_{z}-\log \mathbb{E}\left[X_{z}\right]\right|>t\right)=\mathbb{P}\left(X_{z}>e^{t} \mathbb{E}\left[X_{z}\right]\right)+\mathbb{P}\left(X_{z} \leq e^{-t}\left\|V_{z}\right\|_{\mathrm{HS}}^{2}\right)
$$

The first probability is bounded by $e^{-t}$ using Markov's inequality.
For the second, notice that either $\left\|V_{z}^{*}\left(e_{1}\right)\right\|_{2}^{2} \geq\left\|V_{z}\right\|_{\mathrm{HS}}^{2} / 2$ or $\left\|V_{z}^{*}\left(e_{2}\right)\right\|_{2}^{2} \geq\left\|V_{z}\right\|_{\mathrm{HS}}^{2} / 2$. Without loss of generality assume the former. By Hölder's inequality and for the unit vector $u_{z}:=V_{z}^{*}\left(e_{1}\right) /\left\|V_{z}^{*}\left(e_{1}\right)\right\|_{2}$ we have

$$
\begin{aligned}
\mathbb{P}\left(X_{z} \leq e^{-t}\left\|V_{z}\right\|_{\mathrm{HS}}^{2}\right) & \leq \mathbb{P}\left(\left|\left\langle\mathbf{a}, V_{z}^{*}\left(e_{1}\right)\right\rangle\right|^{2} \leq e^{-t}\left\|V_{z}\right\|_{\mathrm{HS}}^{2}\right) \\
& \leq \mathbb{P}\left(\left|\left\langle\mathbf{a}, u_{z}\right\rangle\right| \leq e^{-t / 2} \frac{\left\|V_{z}\right\|_{\mathrm{HS}}}{\left\|\left(V_{z}\right)^{*}\left(e_{1}\right)\right\|_{2}}\right) \\
& \leq \mathbb{P}\left(\left|\left\langle\mathbf{a}, u_{z}\right\rangle\right| \leq e^{-t / 2} \sqrt{2}\right) \\
& \leq e^{-t / 2} K \sqrt{2} .
\end{aligned}
$$

Proposition 4.9. Let $\mathbf{a}=\left(a_{j}\right)_{j=0}^{N}$ be an isotropic random vector in $\mathbb{R}^{N+1}$ satisfying (4.14). Then for all $r, t>0$ the random polynomial $f_{N}(z)=\sum_{j=0}^{N} a_{j} z^{j}$ satisfies

$$
\begin{equation*}
\mathbb{P}\left(\left|\frac{1}{2 \pi} \int_{0}^{2 \pi} \log \right| f_{N}\left(r e^{i \theta}\right)\left|d \theta-\frac{1}{2} \log \rho_{N}(r)-\log \right| a_{0}| |>t\right) \leq C K e^{-c t} \tag{4.18}
\end{equation*}
$$

where $\rho_{N}(r)=\sum_{k=0}^{N} r^{2 k}$. In particular,

$$
\begin{equation*}
\mathbb{P}\left(\nu_{f_{N}}(\mathbb{D})>0\right) \geq 1-C K N^{-c} \tag{4.19}
\end{equation*}
$$

Proof. For any fixed $r>0$ we set

$$
\begin{equation*}
X_{\theta}:=X_{r e^{i \theta}}, \quad Y_{\theta}:=\log \left(\frac{X_{\theta}}{\mathbb{E}\left[X_{\theta}\right]}\right), \quad Y:=\frac{1}{2 \pi} \int_{0}^{2 \pi} Y_{\theta} d \theta \tag{4.20}
\end{equation*}
$$

In view of Lemma 4.8, apply Lemma 4.7 for $\left( \pm Y_{\theta}\right)_{\theta \in[0,2 \pi]}$ to have

$$
\begin{equation*}
\mathbb{P}(|Y|>t) \leq 18 K e^{-t / 4}, \quad t>0 \tag{4.21}
\end{equation*}
$$

[^3]Since

$$
\begin{equation*}
\mathbb{P}\left(|\log | a_{0} \|>s\right) \leq e^{-2 s}+\mathbb{P}\left(\left|a_{0}\right|<e^{-s}\right) \leq 2 K e^{-s}, \quad s>0, \tag{4.22}
\end{equation*}
$$

we get

$$
\begin{equation*}
\mathbb{P}\left(|Y+2 \log | a_{0}| |>2 t\right) \leq 18 K e^{-t / 4}+2 K e^{-t / 2} \leq 20 K e^{-t / 4}, \tag{4.23}
\end{equation*}
$$

which proves (4.18).
For (4.19) we use Jensen's formula from Proposition 2.6:

$$
\frac{1}{2 \pi} \int_{0}^{2 \pi} \log \left|f_{N}\left(e^{i \theta}\right)\right| d \theta-\log \left|a_{0}\right|=N \nu_{f_{N}}(\mathbb{D})
$$

Applying (4.18) for $r=1$ and $t=\frac{1}{4} \log (N+1)$ we find that

$$
\mathbb{P}\left(\nu_{f_{N}}(\mathbb{D}) \geq \frac{\log (N+1)}{4 N}\right)>1-20 K e^{-\frac{1}{16} \log (N+1)},
$$

as asserted.
Notation 4.10. $\mathfrak{N}_{f}(r)$ will denote the number of zeros of $f$, counted with multiplicity, in the open disc of radius $r$ centered at the origin.

For $s=0,1,2, \ldots$ let $R_{s}(f)$ denote the largest radius $r>0$ for which the function $f$ has no more than $s$ zeros in the open disc of radius $r$ centered at the origin.

Recall here that, by a straightforward calculation, if we arrange the zeros of a holomorphic function in increasing distance from the origin $\left|z_{1}\right| \leq\left|z_{2}\right| \leq \ldots$, then

$$
\begin{equation*}
\int_{0}^{r} \frac{\mathfrak{N}_{f}(t)}{t} d t=\sum_{j=1}^{\mathfrak{N}_{f}(r)} \log \frac{r}{\left|z_{j}\right|} . \tag{4.24}
\end{equation*}
$$

We shall need the following:
Lemma 4.11. Let $f: \mathbb{D} \rightarrow \mathbb{C}$ be a holomorphic function with $f(0) \neq 0$. If $f_{n}: \mathbb{D} \rightarrow \mathbb{C}$ is a sequence of holomorphic functions which converges uniformly on compacts to $f$, then for any $0<r<1$ we have

$$
\lim _{n \rightarrow \infty} \int_{0}^{2 \pi} \log \left|f_{n}\left(r e^{i \theta}\right)\right| d \theta=\int_{0}^{2 \pi} \log \left|f\left(r e^{i \theta}\right)\right| d \theta
$$

Proof. As each $f_{N}$ is holomorphic on a domain that includes $|z| \leq r$ then, regardless of zeros on the circle $|z|=r$ [AN07, Theorem 4.8.3], Jensen's formula yields

$$
\begin{equation*}
\frac{1}{2 \pi} \int_{0}^{2 \pi} \log \left|f_{n}\left(r e^{i \theta}\right)\right| d \theta=\log |f(0)|+\sum_{j=1}^{\mathfrak{N}_{f_{n}}(r)} \log \frac{\left|z_{j}^{(n)}\right|}{r} \tag{4.25}
\end{equation*}
$$

where $\mathfrak{N}_{f_{n}}(r)$ is the number of zeros of $f_{n}$ in $|z|<r$ and $z_{j}^{(n)}$ denote the zeros of $f_{n}$. Also,

$$
\begin{equation*}
\frac{1}{2 \pi} \int_{0}^{2 \pi} \log \left|f\left(r e^{i \theta}\right)\right| d \theta=\log \left|f_{n}(0)\right|+\sum_{j=1}^{\mathfrak{N}_{f}(r)} \log \frac{\left|z_{j}\right|}{r} . \tag{4.26}
\end{equation*}
$$

Given any $\varepsilon>0$, by the theorem of Hurwitz [AN07, Theorem 5.1.3], after a certain $n$ we have $\mathfrak{N}_{f}(r)=\mathfrak{N}_{f_{n}}(r)$, and for any $z_{j}$ there is unique $z_{l}^{(n)}$, call it $z^{(n)}(j)$, with $\left|z_{j}-z^{(n)}(j)\right|<\varepsilon$, so that

$$
\begin{equation*}
\left|\sum_{j=1}^{\mathfrak{N}_{f}(r)} \log \frac{\left|z_{j}\right|}{r}-\sum_{j=1}^{\mathfrak{N}_{f_{n}}(r)} \log \frac{\left|z_{j}^{(n)}\right|}{r}\right| \leq \sum_{j=1}^{\mathfrak{N}_{f}(r)}|\log | z_{j}|-\log | z^{(n)}(j)| |, \tag{4.27}
\end{equation*}
$$

which can be made arbitrarily small.
We now prove the main result of this section, Theorem 4.12, which should be compared to [Kah85, p. 180, Theorem 1] that considers only Gaussian $a_{j}$ 's and also uses Jensen's formula.

Theorem 4.12. Let $\left(a_{j}\right)_{j=0}^{\infty}$ be a sequence of independent random variables on a probability space $(\Omega, \Sigma, \mathbb{P})$ with $\mathbb{E}\left[a_{j}\right]=0, \mathbb{E}\left[a_{j}^{2}\right]=1$, and $\mathcal{L}\left(a_{j}, \varepsilon\right) \leq K \varepsilon$ for all $\varepsilon>0$ and all $j$. Then, for the random power series $f_{\omega}(z)=\sum_{j=0}^{\infty} a_{j}(\omega) z^{j}$ the following hold true:
(1) We have

$$
\begin{equation*}
\underset{r \uparrow 1}{\liminf }\left|\frac{1}{\log \left(1-r^{2}\right)} \int_{0}^{r} \frac{\mathfrak{N}_{f}(t)}{t} d t+\frac{1}{2}\right|=0, \quad \text { a.s. } \tag{4.28}
\end{equation*}
$$

(2) The zero-set of $f$ in $\mathbb{D}=\{z \in \mathbb{C}:|z|<1\}$ is infinite a.s., $0<R_{s}(f)<1$ for $s=0,1, \ldots$, and $R_{s}(f) \uparrow 1$.

Moreover, for almost every $\omega$, we have

$$
\begin{equation*}
1-R_{s}\left(f_{\omega}\right) \geq R_{0}\left(f_{\omega}\right)^{c s} \tag{4.29}
\end{equation*}
$$

for all sufficiently large s, and

$$
\begin{equation*}
1-R_{s}\left(f_{\omega}\right)=O\left(\frac{\log s}{s}\right), \quad s \rightarrow \infty \tag{4.30}
\end{equation*}
$$

Proof. (1) Let $f_{N}(\omega, z)=\sum_{j=0}^{N} a_{j}(\omega) z^{j}$ be the $N$-th partial sum of $f_{\omega}$. Then, the random vector $\mathbf{a}_{N}=$ $\left(a_{0}, \ldots, a_{N}\right)$ is isotropic and has independent coordinates with $\mathcal{L}\left(a_{j}, \varepsilon\right) \leq K \varepsilon$ for all $\varepsilon>0$. Therefore, from [RV15, Corollary 1.4], we conclude that for all $\|u\|_{2}=1$ the anti-concentration estimate

$$
\begin{equation*}
\mathbb{P}\left(\left|\left\langle\mathbf{a}_{N}, u\right\rangle\right| \leq \varepsilon\right) \leq C K \varepsilon \tag{4.31}
\end{equation*}
$$

holds for all $\varepsilon>0$. Then if $t_{N, r, \delta}=\delta \log \rho_{N}(r), 0<r<1, \delta>0$, and

$$
\begin{equation*}
\mathcal{E}_{N}(r, \delta):=\left\{\left|\frac{1}{2 \pi} \int_{0}^{2 \pi} \log \right| f_{N}\left(\omega, r e^{i \theta}\right)\left|d \theta-\frac{1}{2} \log \rho_{N}(r)-\log \right| a_{0}| |>t_{N, r, \delta}\right\} \tag{4.32}
\end{equation*}
$$

Proposition 4.9, applied for $t=t_{N, r, \delta}$, shows that

$$
\begin{equation*}
\mathbb{P}\left(\mathcal{E}_{N}(r, \delta)\right) \leq C K e^{-t_{N, r, \delta} / 4}=C K e^{-c \delta \log \rho_{N}(r)} \tag{4.33}
\end{equation*}
$$

Now if

$$
\mathcal{E}_{\infty}(r, \delta):=\left\{\left|\frac{1}{2 \pi} \int_{0}^{2 \pi} \log \right| f_{\omega}\left(r e^{i \theta}\right)|d \theta-\log | a_{0}\left|+\frac{1}{2} \log \left(1-r^{2}\right)\right|>-\delta \log \left(1-r^{2}\right)\right\}
$$

Lemma 4.11 implies that $\mathcal{E}_{\infty}(r, \delta) \subset \liminf _{N} \mathcal{E}_{N}(r, \delta)$, and Fatou's lemma implies that

$$
\mathbb{P}\left(\liminf _{N} \mathcal{E}_{N}(r, \delta)\right) \leq \liminf _{N} \mathbb{P}\left(\mathcal{E}_{N}(r, \delta)\right) \stackrel{(4.33)}{\leq} C K\left(1-r^{2}\right)^{c \delta}
$$

Having derived

$$
\begin{equation*}
\mathbb{P}\left(\mathcal{E}_{\infty}(r, \delta)\right) \leq C K\left(1-r^{2}\right)^{c \delta}, \quad 0<r<1, \delta>0 \tag{4.34}
\end{equation*}
$$

Jensen's formula for $f$ shows that

$$
\begin{equation*}
\mathcal{E}_{\infty}(r, \delta)=\left\{\left|\frac{1}{2}+\frac{1}{\log \left(1-r^{2}\right)} \int_{0}^{r} \frac{\mathfrak{N}_{f}(t)}{t} d t\right|>\delta\right\} \tag{4.35}
\end{equation*}
$$

Now the choice $r=r_{m, \delta}:=\sqrt{1-m^{-2 /(c \delta)}}, m=1,2, \ldots$ yields that

$$
\sum_{m} \mathbb{P}\left(\mathcal{E}_{\infty}\left(r_{m, \delta}, \delta\right)\right) \leq C K \sum_{m}\left(1-r_{m, \delta}^{2}\right)^{c \delta} \leq C K \sum_{m} \frac{1}{m^{2}}<\infty
$$

Hence, the first Borel-Cantelli lemma in turn shows that the set

$$
\begin{equation*}
\mathcal{N}_{\delta}:=\limsup _{m} \mathcal{E}_{\infty}\left(r_{m, \delta}, \delta\right) \tag{4.36}
\end{equation*}
$$

is null for each $\delta>0$. Further, let $\mathcal{N}:=\bigcup_{q \in \mathbb{Q}^{+}} \mathcal{N}_{q}$. Clearly, $\mathcal{N}$ is a null set and for every $\omega \notin \mathcal{N}$ it holds that: for any $\varepsilon>0$ there is subsequence of $r$ 's (of the form $r_{m, 1 / n}$ for $1 / n<\varepsilon$ ) such that for that subsequence we eventually have

$$
\begin{equation*}
\left|\frac{1}{2}+\frac{1}{\log \left(1-r^{2}\right)} \int_{0}^{r} \frac{\mathfrak{N}_{f}(t)}{t} d t\right|<\varepsilon \tag{4.37}
\end{equation*}
$$

for any $\varepsilon>0$. In particular, $\liminf _{r}\left|\frac{1}{2}+\frac{1}{\log \left(1-r^{2}\right)} \int_{0}^{r} \frac{\mathfrak{N}_{f}(t)}{t} d t\right|<\varepsilon$ for all $\varepsilon>0$.
(2) Part(1) implies that for a.e. $\omega \in \Omega$ the zero-set of $f_{\omega}$ in $\mathbb{D}$ is infinite, therefore we have strict inequality $R_{s}\left(f_{\omega}\right)<1$ for all $s$. Since $a_{0} \neq 0$ a.s. we have that $R_{0}(f)>0$ a.s. By definition, $\left(R_{s}\right)_{s=0}^{\infty}$ is a non-decreasing sequence of numbers. Then as

$$
\begin{equation*}
\sum_{j=1}^{\mathfrak{N}_{f}(r)} \log \frac{r}{\left|z_{j}\right|} \leq \mathfrak{N}_{f}(r) \log \frac{1}{R_{0}} \tag{4.38}
\end{equation*}
$$

and $R_{\mathfrak{N}_{f}(r)} \geq r,(4.24)$ and (4.28) show that $R_{s} \rightarrow 1$ as $s \rightarrow \infty$.
A more careful analysis of the argument leading up to (4.28) reveals the announced rates of convergence. Indeed; for (4.29), consider $\mathcal{N}_{1 / 3}$ from (4.36). Then, for $\omega \notin \mathcal{N}_{1 / 3}$ there exists $m_{0}(\omega) \in \mathbb{N}$ such that for all $m \geq m_{0}$ we have

$$
\begin{equation*}
\frac{1}{6} \log \left[\left(1-r_{m}^{2}\right)^{-1}\right] \leq \int_{0}^{r_{m}} \frac{\mathfrak{N}_{f}(t)}{t} d t \leq \frac{5}{6} \log \left[\left(1-r_{m}^{2}\right)^{-1}\right] \tag{4.39}
\end{equation*}
$$

where $r_{m} \equiv r_{m, 1 / 3}=\sqrt{1-m^{-6 / c}}$. It follows from the first inequality in (4.39) that

$$
\mathfrak{N}_{f}\left(r_{m}\right) \log \left(1 / R_{0}\right) \geq \int_{R_{0}}^{r_{m}} \frac{\mathfrak{N}_{f}(t)}{t} d t=\int_{0}^{r_{m}} \frac{\mathfrak{N}_{f}(t)}{t} d t \geq c \log m
$$

for all $m \geq m_{0}$. Then, for any $s>\frac{c \log m_{0}}{\log \left(1 / R_{0}\right)}$, and for the smallest $m$ so that $s \leq \frac{c \log m}{\log \left(1 / R_{0}\right)}$ we have $s \leq \mathfrak{N}_{f}\left(r_{m}\right)$. By the definition of $R_{s}$, we also have $R_{s} \leq r_{m}$. Finally use $r_{m} \leq 1-m^{-c^{\prime}}$ to conclude (4.29).

For (4.30), first note that $\mathfrak{N}_{f}(t) \geq s$ for all $t>R_{s}$. Therefore, using the second inequality in (4.39),

$$
\begin{equation*}
\frac{s}{2} \log \frac{1}{R_{s}} \leq \int_{R_{s}}^{\sqrt{R_{s}}} \frac{\mathfrak{N}_{f}(t)}{t} d t \leq \int_{0}^{r_{q_{s}+1}} \frac{\mathfrak{N}_{f}(t)}{t} d t \leq C \log q_{s} \tag{4.40}
\end{equation*}
$$

where $q_{s}:=\min \left\{m \mid r_{m+1}^{2} \geq R_{s}\right\}$. By the definition of $q_{s}$ we have

$$
\begin{equation*}
R_{s}>r_{q_{s}}^{2}=1-q_{s}^{-c} \quad \Longrightarrow \quad c \log q_{s}<\log \frac{1}{1-R_{s}} \tag{4.41}
\end{equation*}
$$

Combining (4.40) with (4.41) we obtain

$$
c s \log \frac{1}{R_{s}} \leq \log \frac{1}{1-R_{s}}
$$

Since $1 / 2<R_{s}<1$ for all sufficiently large $s$, we conclude ${ }^{4}$ that $1-R_{s} \leq C \log s / s$, as claimed.
4.3. Clustering of poles. Recall first the following:

Definition 4.13 (Radius of meromorphicity). The radius of meromorphicity of a power series $g(z)=$ $\sum_{j=0}^{\infty} b_{j} z^{j}$ is the largest radius $r>0$ with the property that there is polynomial $p$ of finite degree with $p(z) g(z)$ holomorphic on $|z|<r$.

In Theorem 4.2 we showed that, for $a_{j}$ satisfying the assumptions there, $f(z)=\sum_{j=0}^{\infty} a_{j} z^{j}$ has natural boundary. The following translates this into $1 / f$ :

Lemma 4.14. If $f$ has natural boundary at $|z|=1$ then $1 / f$ has radius of meromorphicity 1 .

[^4]Proof. As $f$ is holomorphic on $|z|<1,1 / f$ is meromorphic there, cf. [AN07, Remarks 4.2.5], therefore the radius of meromorphicity of $1 / f$ is at least 1 . If it were more than 1 there would exist polynomial $P(z)$ such that $P(z) / f(z)$ would be holomorphic with radius of convergence $r>1$ and $f(z) / P(z)$ would be meromorphic on $|z|<r$. In particular, there would exist neighborhoods of points on the unit circle where $f / P$, and therefore $f$, would be holomorphic. Then $|z|=1$ would not be the natural boundary of $f$.

Now recall Edrei's deterministic result [Edr78, Theorem 1] that we shall apply for $1 / f$ :
Theorem 4.15 (Edrei). Let $g(z)=\sum_{j=0}^{\infty} b_{j} z^{j}$ be the power series expansion at 0 of a function with radius of meromorphicity $0<\tau<\infty$. Then for any $n \geq 0$ there exists $m$-subsequence of Padé numerators $P_{m n}^{(g)}$ of $g$ whose zeros cluster uniformly around the circle of radius $\sigma_{n}$, defined as the largest radius of a disc centered at the origin containing no more than $n$ poles of $f$.

In the notation of Definition 2.4, this means that there is subsequence $m_{l}$ such that

$$
\begin{equation*}
\nu_{P_{m_{l} n}}\left(R\left(\sigma_{n}, \rho\right)\right) \rightarrow 1, \quad \nu_{P_{m_{l} n}}(S(\theta, \phi)) \rightarrow \frac{\phi-\theta}{2 \pi}, \quad l \rightarrow \infty \tag{4.42}
\end{equation*}
$$

for all $\rho>0$, and all $0 \leq \theta<\phi<2 \pi$.
Theorem 4.16 (Clustering of poles for Padé approximant). Let $f$ be as in Theorem 4.12 and $R_{m}\left(f_{\omega}\right)<1$ the radius of the largest disc that contains no more than $m$ zeros of $f_{\omega}(z)$, also defined in Theorem 4.12. Then for almost all $\omega$ the zeros of the Padé denominator $Q_{m n}^{\left(f_{\omega}\right)}$ cluster uniformly, up to subsequence as $n \rightarrow \infty$, at $|z|=R_{m}\left(f_{\omega}\right)<1$, with $1-R_{m}\left(f_{\omega}\right)=O(\log m / m)$.

Proof. By Lemma 4.14, the radius of meromorphicity of $1 / f$ is almost surely 1 . And the largest radius that contains no more than $m$ poles of $1 / f$ is $R_{m}$. Therefore, by Theorem 4.15, the Padé numerators $P_{n s}^{(1 / f)}$ of $1 / f$ cluster uniformly, as $n \rightarrow \infty$ at $|z|=R_{m}$. It remains to notice that the Padé numerator $P_{n m}^{(1 / f)}$ of $1 / f$ is the same as the Padé denominator $Q_{m n}^{(f)}$ of $f$, see [Gra72, p. 216, Theorem 22(i)]. Finally use Theorem 4.12.

## 5. LOGARITHMICALLY CONCAVE DATA

In this section we return to the clustering of zeros of the Padé approximant initiated in Section 3 and examine the case when the $a_{j}$ 's in $\sum_{j=1}^{N} a_{j} z^{j}$ are not necessarily independent. Our focus is on the fairly large class of (isotropic) log-concave vectors.

Recall that a random vector $X$ on $\mathbb{R}^{n}$ is log-concave if for any two compact sets $K, L$, and for any $0<\lambda<1$, we have

$$
\begin{equation*}
\mathbb{P}(X \in(1-\lambda) K+\lambda L) \geq[\mathbb{P}(X \in K)]^{1-\lambda}\left[\left.\mathbb{P}(X \in L)\right|^{\lambda}\right. \tag{5.1}
\end{equation*}
$$

It is well known, see e.g., [Bor75], [BGVV14, Theorem 2.1.2], that if $X$ is a log-concave random vector on $\mathbb{R}^{n}$, non-degenerate in the sense that $\mathbb{P}(X \in H)<1$ for every hyperplane $H$, then the distribution of $X$ has density $f_{X}$, log-concave on its support, i.e.

$$
\begin{equation*}
f_{X}((1-\lambda) x+\lambda y) \geq f_{X}(x)^{1-\lambda} f_{X}(y)^{\lambda}, \quad \text { for all } x, y, \quad 0<\lambda<1 \tag{5.2}
\end{equation*}
$$

See [BGVV14] for background material on log-concave distributions and their geometric properties.
From Section 3, notice that in establishing non-asymptotic clustering of roots the independence of the entries of the data vector $\mathbf{a}=\left(a_{0}, \ldots, a_{N}\right)$ is barely used. The properties required to establish the counterpart of Theorem 3.2 for log-concave vectors $\mathbf{a} \in \mathbb{R}^{N+1}$ can be summarized as follows:

P 1 : Upper bound for $\mathcal{L}\left(a_{j}, \varepsilon\right), \varepsilon>0$.
P2: Upper bound for $\mathbb{P}\left(\left\|P_{\sigma}(\mathbf{a})\right\|_{1}>t\right), t>0$, where $\sigma \subset\{0,1, \ldots, N\}$ and $P_{\sigma}$ stands for the coordinate projection onto $\mathbb{R}^{\sigma}=\operatorname{span}\left\{e_{i}: i \in \sigma\right\}$.

P3: Upper bound for $\mathbb{P}\left(\left|\operatorname{det} A_{m}^{(n)}\right|^{1 / n}<\varepsilon\right), \varepsilon>0$, for $A_{m}^{(n)}$ the $n \times n$ Toeplitz matrix associated to a as in (2.6):

$$
A_{m}^{(n)}=\left[\begin{array}{cccc}
a_{m} & a_{m-1} & \ldots & a_{m-n+1}  \tag{5.3}\\
a_{m+1} & a_{m} & \ldots & a_{m-n+2} \\
\vdots & \vdots & \ddots & \vdots \\
a_{m+n-1} & a_{m+n-2} & \ldots & a_{m}
\end{array}\right]
$$

Properties P1 and P2 will follow easily is subsections 5.1 and 5.2 , respectively. We shall rely on the following fact which can be directly verified from the definitions:
Fact 5.1. Let $X$ be a log-concave vector on $\mathbb{R}^{n}$ and let $T: \mathbb{R}^{n} \rightarrow \mathbb{R}^{k}$ be a linear mapping. Then, the vector $T X$ is also log-concave. In particular, if $E$ is a $k$-dimensional subspace of $\mathbb{R}^{n}$, the marginal $P_{E} X$ is $\log$-concave, where $P_{E}$ denotes the orthogonal projection onto $E$. Furthermore, if $X$ is isotropic, so is $P_{E} X$.

The more involved property P3 is established in subsection 5.3.
5.1. Anti-concentration for coordinates of log-concave vectors. We recall first that 1-dimensional marginals of an isotropic, log-concave vector $X$ satisfy anti-concentration bounds of the following form:
Lemma 5.2. Let $X$ be an isotropic, log-concave random vector on $\mathbb{R}^{n}$. Then, for any $\theta \in \mathbb{R}^{n}$ with $\|\theta\|_{2}=1$ we have

$$
\begin{equation*}
\mathcal{L}(\langle X, \theta\rangle, \varepsilon) \leq C \varepsilon \tag{5.4}
\end{equation*}
$$

for all $\varepsilon>0$, where $C>0$ is a universal constant.
Proof. Fix $\theta \in \mathbb{R}^{n}$ with $\|\theta\|_{2}=1$. Then, in view of Fact 5.1, the random variable $P_{\theta} X:=\langle X, \theta\rangle$ has mean zero, variance 1 , and is log-concave. Let $f_{\theta}$ be its density function. It is well known, see e.g. [Fra97], [BGVV14, Theorem 2.2.2], that $\left\|f_{\theta}\right\|_{\infty} \leq e f_{\theta}(0)$. Also, [BGVV14, Theorem 2.3.3] implies that $f_{\theta}(0) \leq C$. Thus, for any $a \in \mathbb{R}$ we may write

$$
\begin{equation*}
\mathbb{P}(|\langle X, \theta\rangle-a|<\varepsilon)=\int_{a-\varepsilon}^{a+\varepsilon} f_{\theta}(t) d t \leq 2 \varepsilon\left\|f_{\theta}\right\|_{\infty} \tag{5.5}
\end{equation*}
$$

and the assertion follows.
5.2. Large deviation estimates for projections of log-concave vectors. As noticed, the projections of isotropic (log-concave) vectors are also isotropic (and log-concave). Therefore, we may simply bound

$$
\begin{equation*}
\mathbb{P}\left(\left\|P_{\sigma} X\right\|_{1}>t\right) \leq \frac{|\sigma|}{t}, \quad \forall t>0 \tag{5.6}
\end{equation*}
$$

by using Markov's inequality and the basic fact that $\mathbb{E}\left|X_{i}\right| \leq 1$.
Note 5.3. For isotropic log-concave vectors stronger large deviation estimates are available: The celebrated deviation inequality of Paouris [Pao06] implies

$$
\mathbb{P}\left(\left\|P_{\sigma} X\right\|_{2} \geq C t \sqrt{|\sigma|}\right) \leq C \exp (-t \sqrt{|\sigma|}), \quad \forall t \geq 1
$$

where $C>0$ is a universal constant. (See also [LS16] for kindred estimates with respect to other $\ell_{p}$-norms.) Nonetheless, since this event will later be unified with a less rare event, the weaker probability of the latter will persist. Hence, we have argued with the trivial bound (5.6) which is adequate for our purposes.
5.3. Determinant of Toeplitz matrices with log-concave symbol. The rest of the section is devoted to proving the following estimate:

Theorem 5.4. Let $\mathbf{a}=\left(a_{0}, \ldots, a_{2 n-2}\right)$ be an isotropic, log-concave random vector in $\mathbb{R}^{N}, N:=2 n-1$, and let $T(\mathbf{a})$ be the $n \times n$ Toeplitz matrix

$$
T(\mathbf{a})=\left[\begin{array}{cccc}
a_{n-1} & a_{n-2} & \ldots & a_{0}  \tag{5.7}\\
a_{n} & a_{n-1} & \ldots & a_{1} \\
\vdots & \vdots & \ddots & \vdots \\
a_{2 n-2} & a_{2 n-3} & \ldots & a_{n-1}
\end{array}\right]
$$

Then the following anti-concetration bound holds:

$$
\begin{equation*}
\mathbb{P}\left(|\operatorname{det} T(\mathbf{a})|^{1 / n} \leq \varepsilon\right) \leq C n^{c} \varepsilon \tag{5.8}
\end{equation*}
$$

for all $\varepsilon>0$, where $C, c>0$ are universal constants.
This result will follow from [CW01, Theorem 8] once we have established a lower bound for $\mathbb{E}|\operatorname{det} T(\mathbf{a})|$. To this end, we have the following:

Proposition 5.5. Let a be a vector as above and let $g \sim N\left(0, I_{N}\right)$. The following estimate holds:

$$
\begin{equation*}
(\mathbb{E}|\operatorname{det} T(\mathbf{a})|)^{1 / n} \geq c n^{-3 / 2}(\mathbb{E}|\operatorname{det} T(g)|)^{1 / n} \tag{5.9}
\end{equation*}
$$

where $c>0$ is a universal constant.
Proof. Integration in polar coordinates along with the fact that $x \mapsto|\operatorname{det} T(x)|$ is $n$-homogeneous yields

$$
\begin{align*}
\mathbb{E}|\operatorname{det} T(\mathbf{a})| & =N \omega_{N} \int_{0}^{\infty} \int_{S^{N-1}} r^{N+n-1}|\operatorname{det} T(\theta)| f_{\mathbf{a}}(r \theta) d \sigma(\theta) d r \\
& =\frac{N \omega_{N}}{N+n} f(0) \int_{S^{N-1}}|\operatorname{det} T(\theta)| \rho_{K_{N+n}\left(f_{\mathbf{a}}\right)}^{N+n}(\theta) d \sigma(\theta) \tag{5.10}
\end{align*}
$$

where $\rho_{K_{p}\left(f_{\mathbf{a}}\right)}$ is the radial function of K. Ball's body associated with $f_{\mathbf{a}}$, see [Pao12, Section 3], [BGVV14, Section 2.5] for the definition. Applying the same formula for a replaced by $g$ we find

$$
\begin{equation*}
\mathbb{E}|\operatorname{det} T(g)|=\frac{N \omega_{N}}{N+n} \frac{2^{n / 2} \Gamma\left(\frac{N+n}{2}+1\right)}{\pi^{N / 2}} \int_{S^{N-1}}|\operatorname{det} T(\theta)| d \sigma(\theta) \tag{5.11}
\end{equation*}
$$

In view of [BGVV14, Proposition 2.5.7], [Pao12, Eqt. (3.11)] we have that

$$
\begin{equation*}
\rho_{K_{N+n}}(\theta) \geq e^{\frac{N}{N+1}-\frac{N}{N+n}} \rho_{K_{N+1}}(\theta) \tag{5.12}
\end{equation*}
$$

for all $\theta \in S^{N-1}$. Hence, we may lower bound (5.10) by

$$
\begin{equation*}
\mathbb{E}|\operatorname{det} T(\mathbf{a})| \geq \frac{N \omega_{N}}{N+n} f(0) e^{c n}\left|K_{N+1}\left(f_{\mathbf{a}}\right)\right|^{\frac{N+n}{N}} \int_{S^{N-1}}|\operatorname{det} T(\theta)| \rho \frac{N+n}{K_{N+1}\left(f_{\mathbf{a}}\right)}(\theta) d \sigma(\theta) \tag{5.13}
\end{equation*}
$$

where $\bar{A}$ denotes the homothetic image of $A$ of volume 1, i.e., $\bar{A}:=|A|^{-1 / N} A$.
Next, since $f_{\mathbf{a}}$ is isotropic it follows that $\overline{K_{N+1}\left(f_{\mathbf{a}}\right)}$ is almost isotropic, hence $\rho_{\overline{K_{N+1}}}(\theta) \geq c_{1} f_{\mathbf{a}}(0)^{1 / N}$ for all $\theta \in S^{N-1}$. Taking into account this, (5.11), and (5.13) we may write

$$
\begin{equation*}
\mathbb{E}|\operatorname{det} T(\mathbf{a})| \geq \frac{c_{2}^{n}}{\Gamma\left(\frac{N+n}{2}+1\right)} f_{\mathbf{a}}(0)^{2+\frac{n}{N}}\left|K_{N+1}\left(f_{\mathbf{a}}\right)\right|^{\frac{N+n}{N}} \mathbb{E}|\operatorname{det} T(g)| \tag{5.14}
\end{equation*}
$$

From [BGVV14, Proposition 2.5.8] we have that

$$
\begin{equation*}
\left|K_{N+1}\left(f_{\mathbf{a}}\right)\right| \geq c_{3} / f_{\mathbf{a}}(0) \tag{5.15}
\end{equation*}
$$

Plugging this estimate into the previous inequality we derive the following:

$$
\begin{equation*}
\mathbb{E}|\operatorname{det} T(\mathbf{a})| \geq \frac{c_{4}^{n}}{\left(c_{5} n\right)^{3 n / 2}} f_{\mathbf{a}}(0) \mathbb{E}|\operatorname{det} T(g)| \tag{5.16}
\end{equation*}
$$

Finally, since $f_{\mathbf{a}}$ is isotropic one has $f_{\mathbf{a}}(0)^{1 / N} \geq c_{6}>0$, see [BGVV14, Proposition 2.3.12], thus the result follows.

Proof of Theorem 5.4. Viewing $\operatorname{det} T(\mathbf{a})$ as a (homogeneous) polynomial on a of degree $n$, and employing the anti-concentration estimate due to Carbery and Wright [CW01, Theorem 8] we find

$$
\begin{equation*}
(\mathbb{E}|\operatorname{det} T(\mathbf{a})|)^{1 / n} \cdot \mathbb{P}\left(|\operatorname{det} T(\mathbf{a})|^{1 / n} \leq \varepsilon\right) \leq C n \varepsilon, \quad \varepsilon>0 \tag{5.17}
\end{equation*}
$$

On the other hand Proposition 5.5, in conjunction with Proposition 2.3, yields

$$
(\mathbb{E}|\operatorname{det} T(\mathbf{a})|)^{1 / n} \geq c n^{-3 / 2}(\mathbb{E}|\operatorname{det} T(g)|)^{1 / n} \geq c^{\prime} n^{-5 / 2}
$$

Combining all the above we get the desired result.
Following the line of argument of Theorem 3.2, with appropriate adjustments provided by the estimates proved in this section, we conclude that Theorem 3.2 holds for isotropic, log-concave vectors:

Corollary 5.6. Let $m, n \in \mathbb{N}$ and $\delta \in(0,1)$ which satisfy $m \geq C \delta^{-4} n \log (e n / \delta)$. Then, for any $N \geq m+n$, for any isotropic, log-concave random vector $\mathbf{a}=\left(a_{j}\right)_{j=0}^{N} \in \mathbb{R}^{N+1}$, the numerator $P_{m n}$ of the $[m, n]$-Padé approximant of $f_{N}(z)=\sum_{j=0}^{N} a_{j} z^{j}$ satisfies

$$
\mathbb{P}\left(\frac{\log L\left(P_{m n}\right)}{m}>\delta^{4}\right)<\delta
$$

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## Appendix A. Discrepancy of measures

We provide a proof for Proposition 2.8. To this end, we prove the following estimate:
Proposition A. 1 (Discrepancy). Let $P(z)=\sum_{j=0}^{N} a_{j} z^{j}$ with $a_{0} a_{N} \neq 0$, let $\nu_{P}$ be the normalized zerocounting measure associated with $P$, and let $\mu$ be the uniform probability measure on $\mathbb{T}=\{z \in \mathbb{C}:|z|=1\}$. Then, for any function $f: \mathbb{C} \rightarrow \mathbb{R}$ which is Lipschitz and bounded, we have

$$
\begin{equation*}
\left|\int f d \nu_{P}-\int f d \mu\right| \leq 31\|f\|_{\mathrm{BL}}(\log L(P) / N)^{1 / 4} . \tag{A.1}
\end{equation*}
$$

Proof. Without loss of generality let $\|f\|_{\text {BL }}:=\|f\|_{\infty}+\|f\|_{\text {Lip }}=1$ (otherwise we work with $f_{1}:=f /\|f\|_{\text {BL }}$ ). To ease the notation let us set $L=\log L(P)$. We assume, as we may, that $L \leq N$, otherwise the result holds trivially. Let $0<\rho \leq 1, m \geq 1$ (both to be chosen appropriately later), and $Z_{P}:=\{z \in \mathbb{C}: P(z)=0\}$. Note that

$$
\begin{equation*}
\int f d \nu_{P}=\frac{1}{N}\left[\sum_{z \in Z_{P} \cap R(\rho)} f(z)+\sum_{z \in Z_{P} \backslash R(\rho)} f(z)\right] . \tag{A.2}
\end{equation*}
$$

First, Jensen's result (Proposition 2.5) yields

$$
\begin{equation*}
\left|\sum_{z \in Z_{P} \backslash R(\rho)} f(z)\right| \leq\|f\|_{\infty}\left|Z_{P} \backslash R(\rho)\right| \leq \frac{L}{\rho} . \tag{A.3}
\end{equation*}
$$

If we decompose $R(\rho)$ into $m$ equi-angular polar rectangles, i.e.,

$$
R(\rho)=\bigcup_{j=1}^{m} R(\rho) \cap\left\{z: \frac{2(j-1) \pi}{m} \leq \arg z<\frac{2 j \pi}{m}\right\} \equiv \bigcup_{j=1}^{m} R_{j},
$$

then for each $j=1, \ldots, m$ and for any $w \in Z_{P} \cap R_{j}$ we get that

$$
\left|f(w)-\frac{m}{2 \pi} \int_{2 \pi(j-1) / m}^{2 \pi j / m} f\left(e^{i t}\right) d t\right| \leq\|f\|_{\text {Lip }} \operatorname{diam}\left(Z_{P} \cap R_{j}\right) \leq 2 \pi\left(\frac{1}{m}+\rho\right) \cdot{ }^{5}
$$

Hence, the choice $\rho=1 / m$ yields for each $j=1, \ldots, m$ that

$$
\begin{equation*}
\left|\bar{f}_{j}-\frac{1}{\left|Z_{P} \cap R_{j}\right|} \sum_{z \in Z_{P} \cap R_{j}} f(z)\right| \leq \frac{4 \pi}{m}, \quad \bar{f}_{j}:=\frac{m}{2 \pi} \int_{2 \pi(j-1) / m}^{2 \pi j / m} f\left(e^{i t}\right) d t \tag{A.4}
\end{equation*}
$$

[^5]Now we may write

$$
\begin{aligned}
\left|\int f d \nu_{P}-\int f d \mu\right| & \leq\left|\frac{1}{N} \sum_{z \in Z_{P} \cap R(\rho)} f(z)-\frac{1}{2 \pi} \int_{0}^{2 \pi} f\left(e^{i t}\right) d t\right|+\frac{m L}{N} \\
& \leq \sum_{j=1}^{m}\left|\frac{1}{N} \sum_{z \in Z_{P} \cap R_{j}} f(z)-\frac{1}{2 \pi} \int_{2 \pi(j-1) / m}^{2 \pi j / m} f\left(e^{i t}\right) d t\right|+\frac{m L}{N} \\
& =\sum_{j=1}^{m}\left|\frac{\left|Z_{P} \cap R_{j}\right|}{N}\left(\frac{1}{\left|Z_{P} \cap R_{j}\right|} \sum_{z \in Z_{P} \cap R_{j}} f(z)-\bar{f}_{j}\right)+\left(\frac{\left|Z_{P} \cap R_{j}\right|}{N}-\frac{1}{m}\right) \bar{f}_{j}\right| \\
& +\frac{m L}{N} \\
& \left(\begin{array}{l}
\text { A.4) } \\
\end{array}\right) \frac{4 \pi L}{m} \sum_{j=1}^{m} \frac{\left|Z_{P} \cap R_{j}\right|}{N}+m \max _{j \leq m}\left|\nu_{P}\left(R_{j}\right)-\frac{1}{m}\right| \int|f| d \mu+\frac{m L}{N} \\
& \leq \frac{4 \pi}{m}+m \max _{j \leq m}\left|\nu_{P}\left(R_{j}\right)-\frac{1}{m}\right|+\frac{m L}{N} .
\end{aligned}
$$

It remains to notice that

$$
\nu_{P}\left(R_{j}\right) \leq \nu_{P}\left(z: \frac{2 \pi(j-1)}{m} \leq \arg z<\frac{2 \pi j}{m}\right) \leq \frac{1}{m}+16 \sqrt{\frac{L}{N}}
$$

by the Erdős-Turán estimate (Proposition 2.7), and

$$
\begin{aligned}
\nu_{P}\left(R_{j}\right) & =\nu_{P}\left(z: \frac{2 \pi(j-1)}{m} \leq \arg z<\frac{2 \pi j}{m}\right)-\nu_{P}\left(\left\{z: \frac{2 \pi(j-1)}{m} \leq \arg z<\frac{2 \pi j}{m}\right\} \backslash R(\rho)\right) \\
& \geq \frac{1}{m}-16 \sqrt{\frac{L}{N}}-\frac{m L}{N}
\end{aligned}
$$

Plugging these estimates into inequality (A.5) we obtain

$$
\left|\int f d \nu_{P}-\int f d \mu\right| \leq \frac{4 \pi}{m}+16 m \sqrt{\frac{L}{N}}+\frac{2 m^{2} L}{N}
$$

and "optimizing" over $m$, by choosing $m=\left(\frac{N}{L}\right)^{1 / 4} \geq 1$, we conclude the assertion.
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[^1]:    ${ }^{1}$ For example, $C=e^{\theta}$ for $\theta \geq 5$ satisfying $e^{\theta}>\theta+20$ will do.

[^2]:    ${ }^{2}$ The existence of the natural boundary also follows from [Ahm59]. We choose to argue via the Ryll-Nardzewksi theorem as it is available, with a very direct proof, in [Kah85, p. 41].

[^3]:    ${ }^{3}$ Recall that $\|A\|_{\text {HS }}$ stands for the Hilbert-Schmidt norm of the matrix $A$, i.e., $\|A\|_{\mathrm{HS}}^{2}=\sum_{i, j}\left|a_{i j}\right|^{2}$.

[^4]:    ${ }^{4}$ Set $y:=\left(1-R_{s}\right)^{-1} \in(2, \infty)$. Then,

    $$
    \log y \geq c s \log \left(1+\frac{1}{y-1}\right) \geq c s \frac{1}{y} \quad \Longrightarrow \quad y \log y \geq c s
    $$

    It is easy to check that for $h(y)=y \log y, y>2$ we have $h^{-1}(u) \asymp \frac{u}{\log u}$ from which the (4.30) follows.

[^5]:    ${ }^{5}$ Let $0<a<b, 0<\theta<\phi<2 \pi$, then for $R:=\left\{r e^{i t} \mid a<r<b, \theta<t<\phi\right\}$ we have $\operatorname{diam}(R) \leq|a-b|+(|a| \wedge|b|)|\theta-\phi|$.

