SHARP INEQUALITIES FOR SYMMETRIC POLYNOMIALS, HUNTER'S CONJECTURE, AND MOMENTS OF EXPONENTIAL RANDOM VARIABLES

SILOUANOS BRAZITIKOS AND CHRISTOS PANDIS

Abstract. We prove Hunter's conjecture on complete homogeneous symmetric polynomials. For even n and every integer $k \geq 1$, we show that under the constraint $\sum_{i=1}^n a_i^2 = 1$ the global minimum of the even-degree polynomial $h_{2k}(a_1, \ldots, a_n)$ is attained precisely at the half-plus/half-minus vector and we compute the optimal value in closed form. The proof combines algebraic properties of h_{2k} with the probabilistic representation $k!\, h_k(a) = \mathbb{E}(\sum_{i=1}^n a_i X_i)^k$, where $X_1, \ldots, X_n$ are i.i.d. standard exponential random variables with density $e^{-x}\mathbf{1}_{x>0}$ and a combinatorial identity. This viewpoint further yields sharp upper and lower bounds for $\mathbb{E}|\sum_{i=1}^n a_i X_i|^q$ under natural constraints on the coefficients, including the spherical constraint $\sum a_i^2 = 1$ combined with the non-negative regime $a_i \geq 0$, or the centred regime $\sum a_i = 0$. Moreover, we determine the exact minimum of h_{2k} on the ℓ_∞-sphere $S_\infty = \{a \in \mathbb{R}^n : \|a\|_\infty = 1\}$, which yields sharp norm comparison inequalities between the matrix norms induced by complete homogeneous symmetric polynomials and the classical operator and Schatten norms.

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1. Introduction

Complete homogeneous symmetric (CHS) polynomials play a fundamental role in algebraic combinatorics, representation theory, and the study of moment inequalities in probability. Let h_k

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denote the degree-k CHS polynomial in n real variables $a_1, \ldots, a_n$, defined by

$$h_k(a_1, \ldots, a_n) = \sum_{1 \le i_1 \le \cdots \le i_k \le n} a_{i_1} \cdots a_{i_k}.$$

We adopt the convention that $h_0(a) = 1$. The interplay between the algebraic structure of h_k and the analytic properties of polynomials on $\mathbb{R}^n$ is particularly transparent in low degrees. For instance,

$$h_1(a) = \sum_{i=1}^{n} a_i, \qquad h_2(a) = \sum_{i=1}^{n} a_i^2 + \sum_{1 \le i < j \le n} a_i a_j.$$

Crucially, the quadratic polynomial admits the sum-of-squares decomposition

$$h_2(a) = \frac{1}{2} \sum_{i=1}^{n} a_i^2 + \frac{1}{2} \Big(\sum_{i=1}^{n} a_i \Big)^2.$$

Consequently, h_2 is manifestly positive definite. It is natural to inquire whether higher even-degree CHS polynomials enjoy similar positivity properties. While positivity fails trivially for odd degrees (since $h_{2k+1}(-a) = -h_{2k+1}(a)$), Hunter initiated the systematic study of this phenomenon for even degrees in [17]. In [18], he established that for any integer $k \ge 1$,

$$h_{2k}(a) \ge \frac{1}{k! \cdot 2^k} \left(\sum_{i=1}^{n} a_i^2 \right)^k. \tag{1.1}$$

Equality holds in (1.1) if and only if $k = 1$ and $\sum_{i=1}^{n} a_i = 0$. Hunter's positivity theorem has since been revisited and re-proved by several authors using various techniques (see, e.g., [21, 4]). More recently, Bouthat, Chávez and Garcia [4] developed a systematic probabilistic and operator-theoretic framework around Hunter's theorem, interpreting even-degree CHS polynomials as building blocks of "random vector norms" on spaces of matrices and surveying many of the existing proofs and generalizations.

In the same work [18], Hunter also *conjectured* a substantially stronger statement. Specifically, he conjectured that when n is even, under the normalization $\sum a_i^2 = 1$, the global minimum of h_{2k} is attained at the "half-plus/half-minus" vector

$$\tilde{a} = \Big(\underbrace{\tfrac{1}{\sqrt{n}}, \ldots, \tfrac{1}{\sqrt{n}}}_{n/2}, \underbrace{-\tfrac{1}{\sqrt{n}}, \ldots, -\tfrac{1}{\sqrt{n}}}_{n/2} \Big).$$

Progress on this conjecture has been incremental. Baston [2] sharpened Hunter's original bound by adding a correction term depending on $(\sum a_i)^{2k}$. More recently, Tao [21] demonstrated that each h_{2k} is Schur-convex on $\mathbb{R}^n$, which implies that if one fixes $\sum a_i = 1$, then the minimum is attained at the flat vector $(1/n, \ldots, 1/n)$. However, the global minimization problem on the sphere $\mathbb{S}^{n-1}$ has remained open.

A different line of work was recently initiated by Garcia and Volčič [14], who introduced noncommutative complete homogeneous symmetric (NCHS) polynomials and proved a noncommutative Hunter-type theorem. They obtained optimal operator-valued lower bounds

$$H_{2k}(X_1, \ldots, X_n) \succeq \mu_{n,k} \, (X_1^{2k} + \cdots + X_n^{2k})$$

for Hermitian operators X_i, together with explicit sum-of-hermitian-squares representations. In the commutative scalar case this yields new inequalities of the form

$$h_{2k}(a_1, \ldots, a_n) \geq \frac{\mu_{n,k}}{n^{k-1}} \|a\|_2^{2k},$$

which improve Hunter's original constant whenever d is sufficiently large compared to n. Nevertheless, the exact best lower bound for scalar CHS polynomials—that is, the optimal constant and the extremizing configurations on the Euclidean sphere—remained unknown in [14].

1.1. **Probabilistic and geometric perspectives.** Beyond their algebraic utility, CHS polynomials possess an elegant probabilistic representation, which will be central to our approach (see, e.g., [21, 4]). If $X_1, \ldots, X_n$ are independent and identically distributed (i.i.d.) standard exponential random variables (with density $e^{-x}\mathbf{1}_{x>0}$), then for every integer $k \geq 0$,

$$k!\, h_k(a) = \mathbb{E}\Big(\sum_{j=1}^{n} a_j X_j\Big)^k. \tag{1.2}$$

This identity bridges algebraic combinatorics with the study of optimal moment inequalities: the problem of minimizing h_{2k} on the sphere becomes the problem of minimizing the even moments of linear combinations of independent exponential random variables under a fixed variance constraint.

This probabilistic framework also interfaces naturally with the geometry of convex bodies. Moments of such sums are closely related to the volume of hyperplane sections of the regular simplex Δ_n. From the geometric viewpoint (see, for example, [22]), the constraint $\sum a_i = 0$ is precisely the condition that the hyperplane $a^\perp$ passes through the centroid of the simplex, and questions about extremal moments in the zero-sum regime are inherently linked to central sections of Δ_n.

There are thus two particularly compelling reasons to single out the zero-sum hyperplane

$$\sum_{i=1}^{n} a_i = 0.$$

Geometrically, as just noted, it encodes central sections of the simplex. From the probabilistic viewpoint, if $X_1, \ldots, X_n$ are i.i.d. standard exponentials with $\mathbb{E}X_i = 1$, then

$$\mathbb{E}\Big(\sum_{i=1}^{n} a_i X_i\Big) = \Big(\sum_{i=1}^{n} a_i\Big)\mathbb{E}X_1,$$

so the linear form $\sum a_i X_i$ is centred if and only if $\sum a_i = 0$. Thus the zero-sum regime is simultaneously the natural geometric setting for simplex slicing and the natural probabilistic setting for sharp moment inequalities of centred exponential distributions.

1.2. **Main contributions.** In this paper, we leverage the probabilistic perspective to provide a complete description of the extremal behaviour of complete homogeneous symmetric polynomials under natural constraints. Our analysis is organized into four regimes, according to the structure of the coefficient vector $a = (a_1, \ldots, a_n)$.

1. The unconditional regime (Hunter's conjecture). Our first main result is an affirmative resolution of Hunter's conjecture for all even degrees.

Theorem (Informal version of Theorem 3.2)**.** Let n be an even integer and $k \geq 1$. For any $a \in \mathbb{R}^n$ with $\sum a_i^2 = 1$,

$$h_{2k}(a) \geq h_{2k}(\tilde{a}),$$

where $\tilde{a}$ is the half-plus/half-minus vector. The explicit value of this minimum is given by

$$h_{2k}(\tilde{a}) = \frac{(n/2 + k - 1)!}{k!\,(n/2 - 1)!\,n^k},$$

and equality holds if and only if a is a permutation of $\tilde{a}$.

On the upper side, it is clear that the maximizers of h_{2k} under the constraint $\sum a_i^2 = 1$ coincide with those in the non-negative regime, which we describe in detail below.

2. The non-negative regime. We next analyze the behaviour of moments and CHS polynomials when the coefficients are constrained to be non-negative, $a_i \geq 0$, with $\sum a_i^2 = 1$. For small degrees, we use Schur-type arguments to show that for $k \leq 4$ the map

$$(x_1, \dots, x_n) \mapsto \mathbb{E}\Big(\sum_{j=1}^{n} \sqrt{x_j}\, X_j\Big)^k$$

is Schur-concave on $\mathbb{R}_+^n$, yielding sharp two-sided bounds for h_k in terms of extremal vectors with either one nonzero coordinate or all coordinates equal. Schur-concavity (or convexity) breaks for $k > 4$, thus for higher degrees we turn to an explicit interpolation formula for $\mathbb{E}\big(\sum a_i X_i\big)^k$ in terms of the coefficients a_i and Gamma functions. A detailed analysis of this formula shows that, for each fixed integer k and under the constraint $\sum a_i^2 = 1$, every minimizer among non-negative vectors has a very rigid structure: it is supported on a subset of coordinates on which all entries are equal, and all remaining coordinates are zero. In other words, all minimizers are of the form

$$(a_1, \dots, a_n) = (\underbrace{t, \dots, t}_{m \text{ times}}, 0, \dots, 0),$$

for some $m \in \{1, \dots, n\}$ and $t > 0$ determined by the normalization. The optimal support size m is characterized via a one-dimensional function. Dually, we prove that all maximizers in the non-negative regime are vectors with exactly $n - 1$ equal coordinates, that is, of the form

$$(a_1, \dots, a_n) = (s, \underbrace{t, \dots, t}_{n-1 \text{ times}}),$$

with $t \leq s$ and (s, t) explicitly determined by k and n as a root of an explicit polynomial. These maximizers are, of course, maximizers for the unconditional regime.

3. The centred (geometric) regime. Motivated by the simplex slicing problem and the probabilistic setting of centred random variables, we analyze in detail the case where

$$\sum_{i=1}^{n} a_i = 0, \qquad \sum_{i=1}^{n} a_i^2 = 1.$$

We prove that for even n the lower bound for h_{2k} under the zero-sum constraint coincides with the unconditional Hunter bound, and is again attained at the half-plus/half-minus vector. We also determine the exact maximizers under the same constraint, which turn out to be vectors with $n - 1$ equal coordinates and one opposite coordinate.

When dealing with the sum of three exponential random variables, we combine these ideas with Fourier-analytic formulas for moments to obtain sharp upper and lower bounds for nearly all exponents $q \in (-1, \infty)$. This leads to a complete description of the extremizers for $\mathbb{E}|\sum a_i X_i|^q$ under the zero-sum constraint.

4. Matrix-norm inequalities. A further motivation for our work comes from unitarily invariant norms on matrices induced by complete homogeneous symmetric polynomials. Following Aguilar, Chávez, Garcia and Volčič [1], given $A \in M_n(\mathbb{C})$ with singular values $s_1(A) \geq \cdots \geq s_n(A) \geq 0$ and an even integer $d = 2k$, one can define the CHS–norm

$$\|A\|_{H_d} := h_d\big(s_1(A), \ldots, s_n(A)\big)^{1/d}.$$

These norms interpolate between classical Schatten norms, and the authors proved two-sided comparisons with the operator norm $\|\cdot\|_{\text{op}}$; see in particular [1, Theorem 38]. The dependence of their constants on d and n is not optimal for the lower bound, and the authors explicitly asked for the sharp form of such inequalities. Our results in Section 6 answer this question, see Theorem 6.1, and lead to the optimal order of the best constant in the comparison between $\|\cdot\|_{H_d}$ and $\|\cdot\|_{\text{op}}$.

Organization of the paper. The rest of the paper is organized as follows: In Section 2, we collect some preliminaries and further develop the necessary background, focusing on Schur-convexity and majorization, which are central to the properties of the complete homogeneous symmetric polynomials and the Fourier-analytic formulas for moments. In Section 3, we provide an affirmative answer to Hunter's conjecture. Section 4 is devoted to the case of non-negative coefficients. Section 5 addresses the centred case and in Section 6 we study the minimisation of complete homogeneous symmetric polynomials under the constraint $\|a\|_\infty = 1$.

2. Preliminaries and Background

2.1. **Schur-convexity and majorization.** Schur-convexity-type arguments have recently appeared in probabilistic settings (see, for example, [10, 9]), leading to sharp results ranging from moment comparison inequalities to entropy inequalities. For a concise exposition on majorization and Schur-convexity, we refer to Chapter II of [3]. We recall here the basic notions that will be used throughout the paper.

Definition 2.1 (Decreasing rearrangement)**.** Given $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$, we denote by $x^* = (x_1^*, \ldots, x_n^*)$ its decreasing rearrangement, i.e.

$$x_1^* \geq x_2^* \geq \cdots \geq x_n^*.$$

Definition 2.2 (Majorization)**.** For any two vectors $x, y \in \mathbb{R}^n$, we say that x is majorized by y, and write $x \prec y$, if

$$\sum_{i=1}^n x_i = \sum_{i=1}^n y_i \quad \text{and} \quad \sum_{i=1}^k x_i^* \leq \sum_{i=1}^k y_i^* \quad \text{for every } k = 1, 2, \ldots, n.$$

As a direct consequence, for every vector $a = (a_1, \dots, a_n) \in \mathbb{R}^n_+$ such that $\sum_{i=1}^n a_i = 1$, we have

$$\left(\frac{1}{n}, \dots, \frac{1}{n}\right) \prec (a_1, \dots, a_n) \prec (1, 0, \dots, 0).$$

More specifically, if $\sum_{i=1}^n a_i^2 = 1$, then

$$\left(\frac{1}{n}, \dots, \frac{1}{n}\right) \prec (a_1^2, \dots, a_n^2) \prec (1, 0, \dots, 0). \tag{2.1}$$

Definition 2.3 (Schur-convexity/concavity)**.** A function $f : \mathbb{R}^n \to \mathbb{R}$ is said to be Schur-convex (resp. Schur-concave) if $x \prec y$ implies $f(x) \le f(y)$ (resp. $f(x) \ge f(y)$).

A central criterion for establishing the Schur-convexity or Schur-concavity of a function is due to Schur and Ostrowski.

Theorem 2.4 (Schur–Ostrowski)**.** *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a symmetric function with continuous partial derivatives. Then f is Schur-convex (resp. Schur-concave) if and only if*

$$(x_i - x_j)\left(\frac{\partial f}{\partial x_i} - \frac{\partial f}{\partial x_j}\right) \ge 0 \quad (\textit{resp.} \ \le 0)$$

for all $x \in \mathbb{R}^n$ and for all $1 \le i, j \le n$.

2.2. **Complete homogeneous symmetric polynomials.** In the Introduction we defined the complete homogeneous symmetric polynomial $h_k(a_1, \dots, a_n)$ by

$$h_k(a_1, a_2, \dots, a_n) := \sum_{1 \le i_1 \le i_2 \le \dots \le i_k \le n} a_{i_1} a_{i_2} \dots a_{i_k}.$$

One can also define all the complete homogeneous symmetric polynomials of n variables simultaneously by means of the generating function:

$$\sum_{k=0}^{\infty} h_k(a_1, a_2, \dots, a_n) t^k = \frac{1}{(1 - ta_1)(1 - ta_2) \dots (1 - ta_n)}. \tag{2.2}$$

As a direct consequence of the generating function representation, we obtain the following two important properties.

Lemma 2.5 (Lemmas 1 and 2 in [18])**.** *If $a \neq b$, then*

$$h_{k-1}(x, a) - h_{k-1}(x, b) = (a - b) h_{k-2}(x, a, b), \tag{2.3}$$

and

$$\frac{\partial}{\partial x_i} h_k(x) = h_{k-1}(x, x_i) \tag{2.4}$$

for every $k \ge 1$.

Another well-known formula for CHS polynomials is the Lagrange interpolation formula

$$h_k(x_1, \dots, x_n) = \sum_{i=1}^{n} \frac{x_i^{n+k-1}}{\prod_{j \neq i}(x_i - x_j)}. \tag{2.5}$$

We also recall the probabilistic representation already used in the introduction. Let $X_1, \dots, X_n$ be i.i.d. standard exponential random variables. Then for any $k \in \mathbb{N}$ we have

$$\begin{aligned}\mathbb{E}\Big(a_1 X_1 + \cdots + a_n X_n\Big)^k &= \mathbb{E}\left(\sum_{m_1+\cdots+m_n=k} \frac{k!}{m_1!\cdots m_n!} a_1^{m_1}\cdots a_n^{m_n} X_1^{m_1}\cdots X_n^{m_n}\right)\\ &= k! \sum_{m_1+\cdots+m_n=k} \frac{\mathbb{E}(X_1^{m_1})\cdots\mathbb{E}(X_n^{m_n})}{m_1!\cdots m_n!} a_1^{m_1}\cdots a_n^{m_n}\\ &= k!\cdot h_k(a_1,\dots,a_n),\end{aligned}$$

where in the last step we used the definition of h_k and the moment identity $\mathbb{E}(X_i^{m_i}) = m_i!$. This is exactly the representation (1.2).

In [18], Hunter was the first to show that even-degree CHS polynomials are positive definite.

Theorem 2.6 (Hunter)**.** *Let n, k be non-negative integers. Then $h_{2k}(x_1, \dots, x_n)$ is a positive definite function on $\mathbb{R}^n$, i.e. $h_{2k}(x_1, \dots, x_n) > 0$ for all $x \neq 0$.*

Tao established the positive definiteness and Schur-convexity of the CHS polynomials in [21].

Theorem 2.7 (Tao)**.** *Let n, k be non-negative integers. Then, for any $x \in \mathbb{R}^n$, the following hold.*

(i) ***Positive definiteness:*** *$h_{2k}(x) \geq 0$, with equality if and only if $x = 0$.*

(ii) ***Schur-convexity:*** *$h_{2k}(x) \leq h_{2k}(y)$ whenever $x \prec y$. Moreover, equality holds if and only if x is a permutation of y.*

(iii) ***Schur–Ostrowski criterion:*** *For every $1 \leq i < j \leq n$,*

$$(x_i - x_j)\Big(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial x_j}\Big) h_{2k}(x) \geq 0,$$

with strict inequality unless $x_i = x_j$.

Hunter's positivity theorem for CHS polynomials has been rediscovered and proved many times; for additional proofs and extensions we refer to [4] and the references therein.

2.3. **Power-sum symmetric polynomials.** The power-sum symmetric polynomial of degree m in the variables $x_1, \dots, x_n$ is defined, for $m \in \mathbb{N}$, by

$$p_m(x_1, \dots, x_n) = x_1^m + x_2^m + \cdots + x_n^m,$$

often written $p_m(\mathbf{x})$ or simply p_m when the variables are clear from context. The CHS polynomials and the power-sum polynomials are connected by the following well-known identity (see, for instance, [19, 20]):

$$h_k(\mathbf{x}) = \sum_{\substack{m_1+2m_2+\cdots+km_k=k\\ m_1\geq 0,\dots,m_k\geq 0}} \prod_{i=1}^{k} \frac{p_i(\mathbf{x})^{m_i}}{m_i!\, i^{m_i}}. \tag{2.6}$$

All coefficients in this expansion are nonnegative. This combinatorial identity will play a crucial role in our proof of Hunter's conjecture, as it allows us to compare h_{2k} on different vectors by comparing only the corresponding power sums.

2.4. **Fourier-analytic formulas.** Fourier-analytic formulas for moments and negative moments of random vectors have played a crucial role in the study of various slicing problems in convex geometry; see, for example, [7, 8, 10]. We recall here the classical formulas that we shall need.

Lemma 2.8 (Lemma 3 in [15]). *Let X be a random vector in $\mathbb{R}^d$ and let $p \in (0, d)$. Then*

$$\mathbb{E}\|X\|^{-p} = b_{p,d} \int_{\mathbb{R}^d} \phi_X(t) \|t\|^{p-d} \, dt,$$

provided that the right-hand side integral exists, where $\phi_X(t) = \mathbb{E}e^{i\langle t, X\rangle}$ is the characteristic function of X, $\|\cdot\|$ is the Euclidean norm on $\mathbb{R}^d$, and

$$b_{p,d} = 2^{-p}\pi^{-d/2} \frac{\Gamma\big((d-p)/2\big)}{\Gamma(p/2)}.$$

There are also Fourier-type formulas for positive moments.

Lemma 2.9 (Lemmas 2.3 and 2.4 in [16]). *Let*

$$C_p = \frac{2}{\pi}\Gamma(1+p)\sin\left(\frac{p\pi}{2}\right).$$

For a real-valued random variable X with characteristic function $\phi_X(t) = \mathbb{E}(e^{itX})$, we have, for $p \in (0, 2)$,

$$\mathbb{E}|X|^p = C_p \int_0^\infty \frac{1 - \Re(\phi_X(t))}{t^{p+1}} \, dt. \tag{2.7}$$

For $p \in (2, 4)$, assuming $\mathbb{E}(X^4) < \infty$, it holds that

$$\mathbb{E}|X|^p = -C_p \int_0^\infty \left(\Re(\phi_X(t)) - 1 + \frac{1}{2}\mathbb{E}(X^2)t^2\right) t^{-(p+1)} \, dt. \tag{2.8}$$

Using the method introduced in [16] to prove the above lemmas we can actually prove the following:

Lemma 2.10. *Let X be a real-valued random variable that satisfies $\mathbb{E}X^6 < \infty$. For $p \in (4, 6)$ we have*

$$\mathbb{E}|X|^p = C_p \int_0^\infty (-\Re(\phi_X(t)) + 1 - \frac{1}{2}\mathbb{E}(X^2)t^2 + \frac{1}{4!}\mathbb{E}(X^4)t^4)t^{-(p+1)}dt,$$

where C_p is the previous constant.

Proof. Let $x \in \mathbb{R}$, we will compute

$$M := \int_0^\infty (-\cos(xt) + 1 - \frac{1}{2!}x^2t^2 + \frac{1}{4!}x^4t^4)t^{-p-1} \, dx.$$

Notice that $-\cos(u) + 1 - \frac{1}{2}u^2 + \frac{1}{4!}u^4 > 0$ for $u > 0$. Since, $\cos(t) = 1 - \frac{1}{2}t^2 + \frac{1}{4!}t^4 + O(t^6)$ for $t \to 0$ and $\cos(t) = 1 - \frac{1}{2}t^2 + \frac{1}{4!}t^4 + O(t^4)$ for $t \to \infty$ we see that M is finite. Using the substitution $u = |x|t$ and integrating by parts, we get

$$\begin{aligned} M &= |x|^p \int_0^\infty (-\cos u + 1 - \tfrac{1}{2!}u^2 + \tfrac{1}{4!}u^4)u^{-p-1} \, du \\ &= \frac{|x|^p}{p(p-1)(p-2)(p-3)} \int_0^\infty (1 - \cos u)u^{-p+3} \, du \\ &= \frac{|x|^p}{p(p-1)(p-2)(p-3)} \frac{1}{C_{p-4}} = |x|^p \frac{1}{C_p}. \end{aligned}$$

In the last steps we used the facts that $p(p-1)C_{p-2} = -C_p$, $0 < p-4 < 2$ and

$$\int_0^\infty (1-\cos u)u^{-q-1} = \frac{1}{C_q},$$

for $0 < q < 2$ (see [16]). Thus,

$$|x|^p = C_p \int_0^\infty (-\cos(xt) + 1 - \frac{1}{2}x^2t^2 + \frac{1}{4!}x^4t^4)t^{-p-1}$$

The result follows from the fact $\Re(\phi_X(t)) = \Re(\mathbb{E}(e^{itx})) = \mathbb{E}(\cos(tX))$ combined with Fubini's Theorem. □

2.5. **Weighted sums of exponential random variables.** We next recall an explicit representation of the density of weighted sums of independent exponentials. It is a folklore result (see, e.g., [6]) that the density of the linear combination $a_1X_1 + \cdots + a_nX_n$, denoted by G, where $X_1, \ldots, X_n$ are i.i.d. standard exponential random variables, is given by

$$\begin{aligned} G(t) &= \sum_{\substack{j=1 \\ a_j>0}}^n \frac{1}{a_j} \prod_{\substack{k=1 \\ k\neq j}}^n \frac{a_j}{a_j - a_k} e^{-t/a_j}\, \mathbb{1}_{[0,\infty)}(t) \\ &= -\sum_{\substack{j=1 \\ a_j<0}}^n \frac{1}{a_j} \prod_{\substack{k=1 \\ k\neq j}}^n \frac{a_j}{a_j - a_k} e^{-t/a_j}\, \mathbb{1}_{(-\infty,0]}(t), \end{aligned}$$

that is, for $t \neq 0$,

$$G(t) = \sum_{\substack{j=1 \\ a_j>0}}^n \frac{1}{a_j} \prod_{\substack{k=1 \\ k\neq j}}^n \frac{a_j}{a_j - a_k} e^{-t/a_j} \mathbb{1}_{[0,\infty)}(t) - \sum_{\substack{j=1 \\ a_j<0}}^n \frac{1}{a_j} \prod_{\substack{k=1 \\ k\neq j}}^n \frac{a_j}{a_j - a_k} e^{-t/a_j} \mathbb{1}_{(-\infty,0]}(t), \tag{2.9}$$

and

$$G(0) = \frac{1}{2}\left(\sum_{\substack{j=1 \\ a_j>0}}^n \frac{1}{a_j} \prod_{\substack{k=1 \\ k\neq j}}^n \frac{a_j}{a_j - a_k} - \sum_{\substack{j=1 \\ a_j<0}}^n \frac{1}{a_j} \prod_{\substack{k=1 \\ k\neq j}}^n \frac{a_j}{a_j - a_k} \right). \tag{2.10}$$

This, in turn, implies the following interpolation formula:

$$\mathbb{E}\left|\sum_{j=1}^n a_jX_j\right|^q = \Gamma(1+q) \cdot \left(\sum_{j=1}^n |a_j|^q \prod_{i\neq j} \frac{a_j}{a_j - a_i} \right), \tag{2.11}$$

which remains valid for all $q+1 > 0$, and also for $q+1 < 0$ provided that $q+1$ is not an integer. Here Γ denotes the Euler gamma function, defined by

$$\Gamma(a) = \int_0^\infty t^{a-1}e^{-t}\,dt \quad \text{for } \Re(a) > 0,$$

and extended to all $a < 0$ except at its poles $\{0, -1, -2, \ldots\}$ by the recurrence $\Gamma(a) = \Gamma(a+1)/a$.

2.6. **Palindromic and anti-palindromic polynomials.** Finally, we record a simple algebraic notion that will be used in some auxiliary arguments.

Definition 2.11. Given a polynomial $P(x) = a_0 + a_1 x + \ldots + a_n x^n$, we say that it is *palindromic* if $a_i = a_{n-i}$ for all $i = 0, 1, \ldots, n$, i.e. if its coefficients, when the polynomial is written in the order of ascending or descending powers, form a palindrome.

Similarly, a polynomial P of degree n is called *anti-palindromic* if $a_i = -a_{n-i}$ for all $i = 0, 1, \ldots, n$.

An immediate property of an anti-palindromic polynomial $P(x)$ is that $x = 1$ is always a root.

3. A Proof of Hunter's Conjecture

In this section we prove our first main result, which gives a complete solution to Hunter's conjecture in the scalar case. We begin with the precise description of the extremizers for h_4, and then proceed to all even degrees.

Proposition 3.1. *Let $a_1, \ldots, a_n$ be real numbers such that $\sum_{i=1}^n a_i^2 = 1$. Then h_4 attains its maximum at the vector*

$$\bar{a} = \left(\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}} \right),$$

while it attains its minimum at the "half-plus/half-minus" vector

$$\tilde{a} = \left(\underbrace{\tfrac{1}{\sqrt{n}}, \ldots, \tfrac{1}{\sqrt{n}}}_{n/2}, \underbrace{-\tfrac{1}{\sqrt{n}}, \ldots, -\tfrac{1}{\sqrt{n}}}_{n/2} \right)$$

when n is even, and at a vector of the form

$$\left(\underbrace{a, \ldots, a}_{\frac{n-1}{2}}, \underbrace{b, \ldots, b}_{\frac{n+1}{2}} \right),$$

when n is odd. Here a appears $\frac{n-1}{2}$ times, b appears $\frac{n+1}{2}$ times, and $\frac{a}{b}$ minimizes the function

$$\frac{x^2 + 1 + \left(\frac{n+1}{2}x + \frac{n+3}{2}\right)^2}{\frac{n-1}{2}x^2 + \frac{n+1}{2}}.$$

Our main theorem settles Hunter's conjecture for all even degrees.

Theorem 3.2. *Let n be an even integer, and let $a_1, \ldots, a_n$ be real numbers such that $\sum_{i=1}^n a_i^2 = 1$. Then, for every integer $r \geq 1$,*

$$h_{2r}(a_1, \ldots, a_n) \geq \frac{\left(\frac{n}{2} + r - 1\right)!}{r! \cdot \left(\frac{n}{2} - 1\right)! \cdot n^r},$$

and this inequality is sharp, with equality achieved if and only if $(a_1, \ldots, a_n)$ is a permutation of the half-plus/half-minus vector $\tilde{a}$.

One may now ask about the maximum of h_{2r} under the normalization condition $\sum_i a_i^2 = 1$. To this end, we observe that a straightforward application of the triangle inequality in identity (1.2) reduces the problem to the case where $a_i \geq 0$ and $\sum_i a_i^2 = 1$. We will elaborate on this reduction later when treating the case involving positive coefficients.

Before proceeding to the proof of the conjecture, we present a useful Proposition suggesting that the extrema of h_{2k} are attained under specific structural conditions.

Proposition 3.3. *Let $n \geqslant 1$ and $d > 3$ be a non-negative even integer. Then the extrema of h_d on the unit sphere $\mathbb{S}^{n-1}$ are of the form*

$$\boldsymbol{x} = \left(\underbrace{a, \ldots, a}_{\gamma_1}, \underbrace{b, \ldots, b}_{\gamma_2}, \underbrace{c, \ldots, c}_{\gamma_3} \right).$$

Here, a appears γ_1 times, b appears γ_2 times, and c appears γ_3 times, subject to the constraints $\gamma_1 a^2 + \gamma_2 b^2 + \gamma_3 c^2 = 1$ and $\gamma_1 + \gamma_2 + \gamma_3 = n$.

Proof. The unit sphere in $\mathbb{R}^n$ is compact. Therefore, h_d must attain its extrema $\boldsymbol{x}$ on the unit sphere in $\mathbb{R}^n$. The method of Lagrange multipliers ensures that if $\boldsymbol{x}$ is a extrema, there exists λ such that

$$\frac{\partial h_d(\boldsymbol{x})}{\partial x_i} + 2\lambda x_i = 0 \tag{3.1}$$

for each $i = 1, 2, \ldots, n$. We multiply by x_i and sum over all i to obtain

$$\sum_{i=1}^{n} x_i \frac{\partial h_d(\boldsymbol{x})}{\partial x_i} + 2\lambda = 0. \tag{3.2}$$

From Euler's homogeneous function theorem, equation (3.2) becomes

$$d h_d(\boldsymbol{x}) + 2\lambda = 0.$$

Substituting and using the differentiating property of the CHS polynomials 2.4, we obtain

$$h_{d-1}(\boldsymbol{x}, x_i) = d x_i h_d(\boldsymbol{x}) \tag{3.3}$$

for each $i = 1, 2, \ldots, n$.

The vector with all coordinates equal satisfies equation (3.3). Thus, we may assume that there exist coordinates $x_i \neq x_j$. Applying equation (3.3) to x_i and x_j, and subtracting the results, combined with the difference property, suggests that

$$(x_i - x_j) h_{d-2}(\boldsymbol{x}, x_i, x_j) = h_{d-1}(\boldsymbol{x}, x_i) - h_{d-1}(\boldsymbol{x}, x_j) = d(x_i - x_j) h_d(\boldsymbol{x}).$$

Then,

$$h_d(\boldsymbol{x}) = \frac{h_{d-2}(\boldsymbol{x}, x_i, x_j)}{d}. \tag{3.4}$$

Assume that there exists a third distinct coordinate $x_k \neq x_i, x_j$. By applying relation (3.4) once again for x_i and x_k, and subtracting as before, we obtain

$$h_{d-3}(\boldsymbol{x}, x_i, x_j, x_k) = 0. \tag{3.5}$$

If we further consider, $x_l \neq x_i, x_j, x_k$ in the same manner we obtain

$$h_{d-4}(\boldsymbol{x}, x_i, x_j, x_k, x_l) = 0.$$

For $d \geqslant 4$, since $d - 4$ is even, the positivity of the even degree CHS polynomials leads to a contradiction. □

We proceed by deriving sharp bounds for h_4 through appropriate estimates of its extrema. This will play a crucial role as the inductive step.

Proof of Proposition 3.1. For the maximum notice that

$$h_4(a_1,\dots,a_n) = \frac{1}{4!}\mathbb{E}\left(\sum_{i=1}^n a_i X_i\right)^4 \leqslant \frac{1}{4!}\mathbb{E}\left(\sum_{i=1}^n |a_i| X_i\right)^4 \leqslant h_4\left(\frac{1}{\sqrt{n}},\dots,\frac{1}{\sqrt{n}}\right),$$

where we used Corollary 4.2 for $k=4$.

For the minimum, we shall use the method of Lagrange multipliers as in proof of Proposition 3.3 to bound the extrema of $h_4(a_1,\dots,a_n)$ which exist since the domain is compact. We are searching for all $\boldsymbol{x} = (x_1,\dots,x_n)$ and the real number λ. By following the preceding argument verbatim, we find that Relation (3.5) asserts that

$$h_1(\boldsymbol{x}, x_i, x_j, x_k) = 0$$

or equivalently, if we set $S := \sum_{i=1}^n x_i$,

$$S + x_i + x_j + x_k = 0. \tag{3.6}$$

We also obtained the following identity (3.4):

$$4\cdot h_4(\boldsymbol{x}) = h_2(\boldsymbol{x}, x_i, x_j) = \frac{1}{2}\left\{\sum_{m=1}^n x_m^2 + x_i^2 + x_j^2 + (S + x_i + x_j)^2\right\} = \frac{1}{2}\left(1 + x_i^2 + x_j^2 + x_k^2\right).$$

As established in Proposition 3.3, the extrema are attained under specific structural conditions, that is

$$\boldsymbol{x} = \left(\underbrace{a,\dots,a}_{\gamma_1}, \underbrace{b,\dots,b}_{\gamma_2}, \underbrace{c,\dots,c}_{\gamma_3}\right),$$

where $\gamma_1 a^2 + \gamma_2 b^2 + \gamma_3 c^2 = 1$. For the moment, we assume that the parameters a, b, and c are all distinct.

Thus, relation (3.6) suggests than it suffices to lower bound for $a^2+b^2+c^2$ under the conditions

$$\begin{cases} (\gamma_1+1)a + (\gamma_2+1)b + (\gamma_3+1)c = 0 \\ \gamma_1 a^2 + \gamma_2 b^2 + \gamma_3 c^2 = 1 \end{cases}$$

with $\gamma_1, \gamma_2, \gamma_3 \geqslant 1$ and $\gamma_1+\gamma_2+\gamma_3 = n$.

A direct computation shows that $h_4(\tilde{a}) = \frac{1}{8} + \frac{1}{4n}$. Thus, it remains to prove the inequality

$$a^2+b^2+c^2 \geqslant \frac{2}{n-1}$$

Without loss of generality, we may assume that $c \neq 0$. We write this as

$$a^2+b^2+c^2 = \frac{(a/c)^2 + (b/c)^2 + 1}{\gamma_1 (a/c)^2 + \gamma_2 (b/c)^2 + \gamma_3}.$$

By setting $x := \frac{a}{c}$ and $y := \frac{b}{c}$, and using the fact that $(\gamma_1+1)x + (\gamma_2+1)y + (\gamma_3+1) = 0$, we reduce the bound to the following quadratic inequality:

$$x^2\left[(\gamma_2+1)^2\left(1-\frac{2\gamma_1}{n-1}\right)+(\gamma_1+1)^2\left(1-\frac{2\gamma_2}{n-1}\right)\right]+2x(\gamma_1+1)(\gamma_3+1)\left(1-\frac{2\gamma_2}{n-1}\right)$$
$$+(\gamma_3+1)^2\left(1-\frac{2\gamma_2}{n-1}\right)+(\gamma_2+1)^2\left(1-\frac{2\gamma_3}{n-1}\right)\geqslant 0$$

Setting now $d_i := \gamma_i - 1 \geq 0$, we notice that the coefficient in front of x^2 is non-negative, since

$$(d_2+2)^2(d_2+d_3-d_1)+(d_1+2)^2(d_1+d_3-d_2) = (d_1-d_2)^2(d_1+d_2+4)+d_3\left[(d_2+2)^2+(d_1+2)^2\right]\geqslant 0.$$

The discriminant Δ, equals

$$-4(\gamma_2+1)^2\cdot\Big[(\gamma_3+1)^2\left(1-\frac{2\gamma_2}{n-1}\right)\left(1-\frac{2\gamma_1}{n-1}\right)+(\gamma_2+1)^2\left(1-\frac{2\gamma_1}{n-1}\right)\left(1-\frac{2\gamma_3}{n-1}\right)$$
$$+(\gamma_1+1)^2\left(1-\frac{2\gamma_2}{n-1}\right)\left(1-\frac{2\gamma_3}{n-1}\right)\Big].$$

We will prove that Δ is non-positive. Due to symmetry, we may assume that $\gamma_1 \geq \gamma_2 \geq \gamma_3$. Then substituting $n = \gamma_1+\gamma_2+\gamma_3$, it suffices to prove that

$$\sum (d_1+2)^2(d_1+d_2-d_3)(d_1+d_3-d_2)\geq 0.$$

If $d_3 = 0$, then the inequality can be rewritten in the form

$$(d_1-d_2)^2(d_1^2+2d_1d_2+d_2^2+4d_1+4d_2-4)\geq 0,$$

which is true.
If $d_3 = 1$, then the inequality is equivalent to

$$(d_1^2-d_2^2)^2+(d_1+d_2)(d_1-d_2)^2+(2d_1^3-10d_1^2+4d_1+1+17d_1d_2)+(2d_2^3-10d_2^2+4d_2+1+17d_1d_2)\geq 0,$$

which holds.
Finally, if $d_1, d_2, d_3 \geq 2$, we rewrite the inequality as

$$\sum (d_1+2)^2(d_1-d_3)(d_1-d_2)+\sum (d_1+2)^2 d_3(d_1-d_3)+\sum (d_1+2)^2 d_2d_3\geq 0.$$

The last sum is clearly non-negative. For the first sum notice that it can be expressed as

$$(d_1-d_2)\left[(d_1+2)^2(d_1-d_2)-(d_2+2)^2(d_2-d_3)\right]+(d_3+2)^2(d_3-d_2)(d_3-d_1)\geqslant 0$$

since $d_1 \geqslant d_2 \geqslant d_3$. For the second one, after collecting the same terms, equals to

$$\sum (d_1-d_3)^2(d_1d_3-4)\geq 0,$$

which is again true, since $d_1, d_2, d_3 \geq 2$.

In the case where $\boldsymbol{x}$ has exactly two distinct coordinates, equation (3.5) does not hold. Without loss of generality assume $b \neq 0$. In this case

$$\boldsymbol{x} = (\underbrace{a,\dots,a}_{\gamma_1 \text{ times}},\underbrace{b,\dots,b}_{\gamma_2 \text{ times}})$$

and $\gamma_1 a^2 + \gamma_2 b^2 = 1$ holds. In this case, from relation (3.4), we need to lower bound

$$a^2+b^2+\left((\gamma_1+1)a+(\gamma_2+1)b\right)^2.$$

Then we will find the best constant $\frac{2}{n-1} \geq c \geq \frac{2}{n}$, such that the inequality

$$a^2 + b^2 + [(\gamma_1+1)a + (\gamma_2+1)b]^2 \geqslant c(\gamma_1 a^2 + \gamma_2 b^2) \tag{3.7}$$

holds for all a, b. This can be equivalently expressed as

$$x^2\left[1 + (\gamma_1+1)^2 - c\gamma_1\right] + 2x(\gamma_1+1)(\gamma_2+1) + (\gamma_2+1)^2 + 1 - c\gamma_2 \geqslant 0.$$

Notice that since $c \leq 1$ we have

$$1 + (\gamma_1+1)^2 - c\gamma_1 \geqslant 0$$

and that

$$\frac{\Delta}{4} = (\gamma_1+1)^2(\gamma_2+1)^2 - \left(1 + (\gamma_1+1)^2 - c\gamma_1\right)\left(1 + (\gamma_2+1)^2 - c\gamma_2\right).$$

We use the fact that $\gamma_1 + \gamma_2 = n$, to write the last one as a function of γ_1. The derivative of this function with respect to γ_1 equals to

$$(-c^2 + cn + 4c + 2)(n - 2\gamma_1).$$

The first parenthesis is of course non-negative, therefore the function is increasing for $\gamma_1 \leq n/2$ and decreasing for $\gamma_1 \geq n/2$. If n is even, then it takes its maximum for $\gamma_1 = n/2$. The maximum equals to

$$\frac{1}{4}(-2 + cn)(6 + 4n - cn + n^2),$$

which is non-positive for $c \leq \frac{2}{n}$. Therefore, for even n we have that

$$a^2 + b^2 + [(\gamma_1+1)a + (\gamma_2+1)b]^2 \geqslant \frac{2}{n}(\gamma_1 a^2 + \gamma_2 b^2) \tag{3.8}$$

and the equality holds when $\gamma_1 = \gamma_2 = n/2$ and $a = -b$.

In the case where n is odd, γ_1 cannot be equal to $n/2$, therefore the function takes its maximum for $\gamma_1 = \frac{n-1}{2}$ (or $\gamma_1 = \frac{n+1}{2}$). In the first case, the discriminant is equal to

$$\frac{1}{4}\left(c^2(1-n^2) + c(-4 + 7n + 4n^2 + n^3) - 2(7 + 4n + n^2)\right).$$

The last one is non-positive if and only if $c \leq \rho_1(n)$ or $c \geq \rho_2(n)$. However, $\rho_2(n) \geq \frac{2}{n-1}$, therefore, the largest value that c can take is

$$c = \rho_1(n) = \frac{-4 + 7n + 4n^2 + n^3 - (n+3)\sqrt{8 - 8n + n^2 + 2n^3 + n^4}}{2n^2 - 2}.$$

Note that

$$\rho_1(n) \sim \frac{2}{n}$$

as $n \to +\infty$. We conclude that for n odd the inequality

$$a^2 + b^2 + [(\gamma_1+1)a + (\gamma_2+1)b]^2 \geqslant \rho_1(n)(\gamma_1 a^2 + \gamma_2 b^2) \tag{3.9}$$

holds, and we have equality when $\gamma_1 = (n-1)/2$, $\gamma_2 = (n+1)/2$ and $\frac{a}{b} = x$, where x is the minimum value of the function

$$\frac{x^2 + 1 + \left(\frac{n+1}{2}x + \frac{n+3}{2}\right)^2}{\frac{n-1}{2}x^2 + \frac{n+1}{2}}.$$

□

We now proceed with the proof of Hunter's conjecture.

Proof of Theorem 3.2. We will prove, by induction on k, that every extremum $\boldsymbol{x}$ of h_{2k} on the sphere $\mathbb{S}^{n-1}$, when n is even, satisfies

$$h_{2k}(\boldsymbol{x}) \geqslant h_{2k}(\tilde{a}) = \frac{(n/2+k-1)!}{k!\cdot(n/2-1)!\cdot n^k}$$

We have already established the cases $k=1$ and $k=2$. Now, assume the statement holds for $k-1$ and that the extrema are of the form

$$\boldsymbol{x} = \Big(\underbrace{a,\dots,a}_{\gamma_1},\underbrace{b,\dots,b}_{\gamma_2},\underbrace{c,\dots,c}_{\gamma_3}\Big),$$

where $\gamma_1 a^2+\gamma_2 b^2+\gamma_3 c^2 = 1$.

Due to the symmetry, we can assume that $a > |b| > |c|$.

If $\gamma_1 \geqslant \gamma_2+\gamma_3$ then $h_{2k}(\boldsymbol{x}) \geqslant h_{2k}(\tilde{a})$. Indeed, in this case we have that

$$p_{2m+1}(\boldsymbol{x}) = \gamma_1 a^{2m+1}+\gamma_2 b^{2m+1}+\gamma_3 c^{2m+1} \geqslant 0 = p_{2m+1}(\tilde{a})$$

and using the Power-Mean inequality

$$\left(\frac{\gamma_1 a^{2m}+\gamma_2 b^{2m}+\gamma_3 c^{2m}}{\gamma_1+\gamma_2+\gamma_3}\right)^{1/m} \geqslant \frac{\gamma_1 a^2+\gamma_2 b^2+\gamma_3 c^2}{\gamma_1+\gamma_2+\gamma_3} = \frac{1}{n},$$

which can be written as

$$p_{2m}(\boldsymbol{x}) \geqslant p_{2m}(\tilde{a}).$$

Then, from identity (2.6), which expresses the CHS polynomial solely in terms of the power-sum polynomials, we conclude the desired inequality.

If $\gamma_1 \leqslant \gamma_2+\gamma_3$, we proceed using the already established relation (3.4)

$$\begin{aligned} h_{2k}(\boldsymbol{x}) &= \frac{h_{2k-2}(a[\gamma_1+1], b[\gamma_2+1])}{2k} \\ &= \frac{(1+a^2+b^2)^{k-1}}{2k} h_{2(k-1)}\left(\frac{\boldsymbol{x}}{\sqrt{1+a^2+b^2}}, \frac{a}{\sqrt{1+a^2+b^2}}, \frac{b}{\sqrt{1+a^2+b^2}}\right). \end{aligned}$$

In this case, we obtain that

$$a^2+b^2 \geqslant \frac{2}{n},$$

which helps us to complete the induction. Assume now that the extrema is of the form

$$\boldsymbol{x} = (\underbrace{a,\dots,a}_{\gamma_1 \text{ times}},\underbrace{b,\dots,b}_{\gamma_2 \text{ times}})$$

where a, b are distinct and appear γ_1 and γ_2 times respectively and thus also $\gamma_1 a^2+\gamma_2 b^2 = 1$. Setting $b = c$ in the argument above which helps us to complete the induction.

□

4. The Non-Negative Coefficients Case

In this section we study the extremal behaviour of moments and complete homogeneous symmetric polynomials when the coefficients are constrained to be non-negative. Throughout we assume $a_i \geq 0$ and $\sum_{i=1}^n a_i^2 = 1$.

For positive integer moments up to order four we have the following Schur-concavity result.

Theorem 4.1. *Let $X_1, X_2, \ldots$ be independent and identically distributed standard exponential random variables. For any positive integer $k \leq 4$ and $n \in \mathbb{N}$, the function*

$$(x_1, \ldots, x_n) \mapsto \mathbb{E}\left(\sum_{j=1}^{n} \sqrt{x_j} X_j\right)^k$$

is Schur-concave on $\mathbb{R}_+^n$.

Note that for $k > 4$ Schur-concavity or Schur-convexity breaks.

As an immediate corollary, we obtain two-sided moment bounds in terms of the extreme non-negative configurations.

Corollary 4.2. *For $X_1, X_2, \ldots$ i.i.d standard exponential random variables. For any positive integer $k \leq 4$ and $n \in \mathbb{N}$,*

$$\mathbb{E}X_1^k \leq \mathbb{E}\left(\sum_{j=1}^{n} a_j X_j\right)^k \leq \mathbb{E}\left(\frac{X_1 + \cdots + X_n}{\sqrt{n}}\right)^k.$$

Remark 4.3. By a similar argument, Theorem 4.1 remains valid when the standard exponential random variables are replaced with Gamma(γ) random variables, for any positive integer $k \leq 2\gamma + 2$.

Fot our next results we need the following definition.

Definition 4.4. Let $X_1, X_2, \ldots, X_n$ be i.i.d. standard exponential random variables. For a real number q, we define

$$\rho(1, q) := \mathbb{E}[X_1^q], \quad \rho(2, q) := \mathbb{E}\left(\frac{X_1 + X_2}{\sqrt{2}}\right)^q,$$

and for general $n \in \mathbb{N}$,

$$\rho(n, q) := \mathbb{E}\left(\frac{X_1 + \cdots + X_n}{\sqrt{n}}\right)^q.$$

Theorem 4.5. *Let k be a non-negative integer, and let $a_1, \ldots, a_n$ be non-negative real numbers such that $\sum_{i=1}^n a_i^2 = 1$. If $X_1, \ldots, X_n$ are i.i.d. standard exponential random variables, then*

$$\mathbb{E}\left(a_1 X_1 + \cdots + a_n X_n\right)^k \geq \min\{\rho(1, k), \ldots, \rho(n, k)\},$$

while the maximum will occur at a unit vector with nonzero coordinates, $(n-1)$ of which are equal, that is, of the form

$$(a_1, \ldots, a_n) = (s, \underbrace{t, \ldots, t}_{n-1 \text{ times}}),$$

with $t \le s$ and (s,t) explicitly determined by k and n as a root of an explicit polynomial, see the Remark below for details.

In the following remarks we explain in details the behavior of the minimizer and the maximizer respectively.

Remark 4.6. We fix some k. In order to find the minimum of $\rho(s,k)$, consider the function $g:(0,\infty)\to(0,\infty)$,

$$g(n) = \frac{\Gamma(n+k)}{n^{k/2}\,\Gamma(n)} = \frac{\prod_{j=0}^{k-1}(n+j)}{n^{k/2}}. \tag{4.1}$$

Differentiating the logarithm with respect to n gives

$$\frac{g'(n)}{g(n)} = \frac{d}{dn}\big(\ln g(n)\big) = \sum_{j=0}^{k-1}\frac{1}{n+j} - \frac{k}{2}\frac{1}{n} := h(n).$$

Since $g(n) > 0$ for all $n > 0$, the sign of $g'(n)$ coincides with the sign of $h(n)$, and the critical points of g in $(0,\infty)$ are exactly the zeros of h.

To analyse h, first multiply by $n > 0$:

$$n\,h(n) = \sum_{j=0}^{k-1}\frac{n}{n+j} - \frac{k}{2} = \sum_{j=0}^{k-1}\frac{1}{1+\frac{j}{n}} - \frac{k}{2}.$$

Introduce the new variable $x = \frac{1}{n} > 0$ and define

$$F(x) = \sum_{j=0}^{k-1}\frac{1}{1+jx} - \frac{k}{2}.$$

Then

$$h(n) = 0 \quad\Longleftrightarrow\quad F\left(\frac{1}{n}\right) = 0.$$

The derivative of F is

$$F'(x) = \sum_{j=0}^{k-1}\frac{d}{dx}\left(\frac{1}{1+jx}\right) = -\sum_{j=1}^{k-1}\frac{j}{(1+jx)^2}.$$

For $k \ge 2$ and $x > 0$ every term in the last sum is negative, hence

$$F'(x) < 0 \qquad \text{for all } x > 0 \text{ and } k \ge 2.$$

Thus F is strictly decreasing on $(0,\infty)$ whenever $k \ge 2$.

The limits of F at 0^+ and $+\infty$ are easily computed. One has

$$\lim_{x\to 0^+} F(x) = \sum_{j=0}^{k-1} 1 - \frac{k}{2} = k - \frac{k}{2} = \frac{k}{2},$$

and, since for $j \ge 1$ we have $1 + jx \to \infty$ as $x \to \infty$,

$$\lim_{x\to\infty} F(x) = 1 - \frac{k}{2}.$$

Combining these observations with the monotonicity of F leads to the following conclusions about the equation $F(x) = 0$ (equivalently $h(n) = 0$). For $k = 2$ one has

$$F(0^+) = 1, \qquad \lim_{x\to\infty} F(x) = 0,$$

and F is strictly decreasing. Therefore $F(x) > 0$ for all $x > 0$, again implying that $h(n)$ has no zero in $(0, \infty)$.

For $k \geq 3$ one has

$$F(0^+) = \frac{k}{2} > 0, \qquad \lim_{x\to\infty} F(x) = 1 - \frac{k}{2} < 0,$$

and F is strictly decreasing on $(0, \infty)$. Hence, by the intermediate value theorem, there exists a unique $x_k > 0$ such that $F(x_k) = 0$. This implies that there is a unique $n_k > 0$ with

$$\frac{1}{n_k} = x_k, \qquad h(n_k) = 0.$$

Therefore, for every integer $k \geq 3$, the function g has exactly one critical point n_k in $(0, \infty)$.

For $k \geq 3$ the expression (4.1) can be rewritten as

$$g(n) = \frac{n(n+1)\cdots(n+k-1)}{n^{k/2}}.$$

As $n \to 0^+$, the factors $n+1, \ldots, n+k-1$ tend to positive constants, so the behaviour is dominated by

$$g(n) \sim C_k\, n^{1-k/2}$$

for some constant $C_k > 0$. Since $1 - k/2 < 0$ for $k \geq 3$, one has $g(n) \to \infty$ as $n \to 0^+$. As $n \to \infty$, the product $n(n+1)\cdots(n+k-1)$ behaves like n^k, hence

$$g(n) \sim n^{k-k/2} = n^{k/2} \to \infty$$

as $n \to \infty$. Together with the fact that g' changes sign only once (because h has exactly one zero), this shows that for $k \geq 3$ the function g is strictly decreasing on $(0, n_k)$, strictly increasing on (n_k, ∞), and attains a unique global minimum at $n = n_k$.

The preceding discussion also ensures that, for fixed k and n, the maximum of $\rho(s, k)$, for $s = 1, \ldots, n$, is attained either at $\rho(1, k)$ or at $\rho(n, k)$.

It is also useful to obtain an asymptotic approximation for the location of this minimum when k is moderately large. The defining equation for n_k is $h(n_k) = 0$, that is

$$\sum_{j=0}^{k-1} \frac{1}{n_k + j} = \frac{k}{2n_k}.$$

Approximating the sum by an integral gives

$$\sum_{j=0}^{k-1} \frac{1}{n+j} \approx \int_0^k \frac{dx}{n+x} = \ln \frac{n+k}{n} = \ln\Big(1 + \frac{k}{n}\Big).$$

Thus, for $n = n_k$, one expects approximately

$$\ln\Big(1 + \frac{k}{n_k}\Big) \approx \frac{k}{2n_k}.$$

Introducing the ratio

$$u = \frac{k}{n_k}$$

this becomes the transcendental equation

$$\ln(1+u) = \frac{u}{2},$$

which no longer involves k. This equation has a unique positive solution u_0, and a simple numerical computation shows that

$$u_0 \approx 2.51.$$

Consequently,

$$n_k \approx \frac{k}{u_0} \approx 0.40\, k$$

for large k. In other words, the location of the continuous minimizer grows asymptotically linearly in k with slope slightly below 0.4.

For concrete values of k, one can solve the equation $h(n) = 0$ numerically. The following table lists, for $k = 5, \dots, 15$, an approximation of the unique minimizer n_k in $(0, \infty)$ together with its integer part $\lfloor n_k \rfloor$.

k	n_k (approx.)	$\lfloor n_k \rfloor$
5	1.2900	1
6	1.6958	1
7	2.0989	2
8	2.5006	2
9	2.9014	2
10	3.3015	3
11	3.7012	3
12	4.1005	4
13	4.4997	4
14	4.8986	4
15	5.2974	5

TABLE 1. Approximate continuous minimizer n_k of $g(n)$ and its integer part for various values of k.

Remark 4.7. Let $x = s/t \leqslant 1$. The maximizing configuration occurs as a root of the polynomial (see the proof below)

$$g(x) = \binom{k}{1}\Gamma(n)\Gamma(k) + \sum_{j=0}^{k-2} x^{j+1}\left[\binom{k}{j}\Gamma(j+n-1)\Gamma(k-j+1)(n-1)(j-k) + (j+2)\binom{k}{j+2}\Gamma(j+n+1)\Gamma(k-j-1)\right] - \binom{k}{k-1}\Gamma(k+n-2)2(n-1)x^k.$$

Moreover, let $f(x)$ be the logarithm of

$$\mathbb{E}\left(s\Gamma(n-1)+t\Gamma(1)\right)^k = \frac{\mathbb{E}\left(x\Gamma(n-1)+\Gamma(1)\right)^k}{((n-1)x^2+1)^{k/2}}.$$

There are inherent limitations in giving an exact description of the maximizer. In some cases it occurs at $x = 1$ (see Figure 1), whereas in other cases it may occur at one of the two additional roots of g (see Figures 2 and 3).

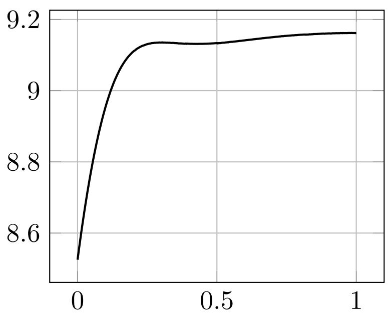

FIGURE 1. Plot of $f(x)$ for $n = k = 7$.

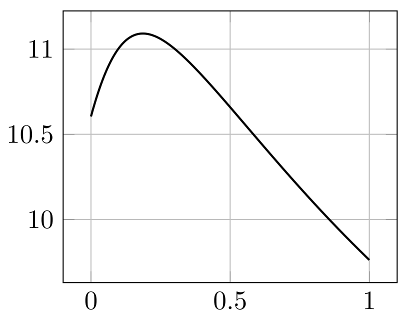

FIGURE 2. Plot of $f(x)$ for $n = 7$, $k = 8$.

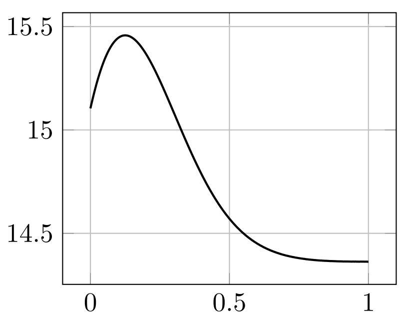

FIGURE 3. Plot of $f(x)$ for $n = 7$ and $k = 10$

We also conjecture that this behavior holds for every real $q > 0$, as expressed in the following conjecture:

Conjecture 4.8. Let $q > 0$ be a real number, and let $a_1, \ldots, a_n$ be non-negative real numbers satisfying $\sum_{i=1}^{n} a_i^2 = 1$. If $X_1, \ldots, X_n$ are i.i.d. standard exponential random variables, then

$$\mathbb{E}\big(a_1X_1 + \cdots + a_nX_n\big)^q \geqslant \min\{\rho(1,q), \ldots, \rho(n,q)\}.$$

4.1. **Characterization of Extrema.** We first introduce a Lemma that will be useful for the characterization of the global extrema.

Lemma 4.9. *Let x, y, z and a, b, c be non-negative real numbers such that $x + y + z = a + b + c$, $x^2 + y^2 + z^2 = a^2 + b^2 + c^2$, and $xyz \leqslant abc$. Then, for any integer $k \geqslant 1$, we have*

$$x^k + y^k + z^k \leq a^k + b^k + c^k.$$

Respectively, if $xyz \geqslant abc$, then

$$x^k + y^k + z^k \geqslant a^k + b^k + c^k.$$

Proof of Lemma 4.9. Let a, b and c be pairwise distinct numbers and set $a+b+c=u$, $ab+ac+bc=v$ and $abc=w$.

Since $a^k+b^k+c^k$ is a symmetric polynomial, it can be expressed as

$$a^k+b^k+c^k := f(u,v,w),$$

and we must show that f increases as a function of w. For this, it is enough to show that $\frac{\partial f}{\partial w}\geq 0$.

Computing the partial derivatives we get

$$\begin{aligned} 1 &= \frac{\partial(a+b+c)}{\partial u} = \frac{\partial a}{\partial u}+\frac{\partial b}{\partial u}+\frac{\partial c}{\partial u}, \\ 0 = \frac{\partial(ab+ac+bc)}{\partial u} &= \frac{\partial a}{\partial u}b+\frac{\partial b}{\partial u}a+\frac{\partial a}{\partial u}c+\frac{\partial c}{\partial u}a+\frac{\partial b}{\partial u}c+\frac{\partial c}{\partial u}b \\ &= (b+c)\frac{\partial a}{\partial u}+(a+c)\frac{\partial b}{\partial u}+(a+b)\frac{\partial c}{\partial u}. \end{aligned}$$

Moreover,

$$0=\frac{\partial(abc)}{\partial u}=bc\frac{\partial a}{\partial u}+ac\frac{\partial b}{\partial u}+ab\frac{\partial c}{\partial u}.$$

The determinant of this system equals

$$\Delta=\left|\begin{pmatrix} 1 & 1 & 1 \\ b+c & a+c & a+b \\ bc & ac & ab \end{pmatrix}\right| = \sum_{cyc}(ab(a+c)-bc(a+c)) = (a-b)(a-c)(b-c),$$

which gives

$$\frac{\partial a}{\partial w}=\frac{1}{(a-b)(a-c)}.$$

Similarly,

$$\frac{\partial b}{\partial w}=\frac{1}{(b-a)(b-c)}$$

and

$$\frac{\partial c}{\partial w}=\frac{1}{(c-a)(c-b)}.$$

To this end, note that

$$\frac{\partial f}{\partial w}=\sum_{cyc}\frac{\partial f}{\partial a}\frac{\partial a}{\partial w}=\sum_{cyc}\frac{ka^{k-1}}{(a-b)(a-c)}=k\sum_{cyc}\frac{a^{k-1}}{(a-b)(a-c)}=kh_{k-3}(a,b,c)\geq 0,$$

for all $k\geq 3$, where in the last equality we used (2.5).

Since the above proof is valid for $a\to b^+$ and for $b\to c^+$ and since f is a continuous function, we obtain that f increases for any non-negative a, b and c. □

Proposition 4.10. *Let $n\geqslant 1$ and $k>2$ be a non-negative integer. Then the extrema of h_k in the the space $\mathbb{S}_+^{n-1}$ are of the form*

$$\boldsymbol{x}=\Bigg(\underbrace{a,\ldots,a}_{\gamma_1},\underbrace{b,\ldots,b}_{\gamma_2}\Bigg).$$

Here, a appears γ_1 times, b appears γ_2 times, subject to the constraints $\gamma_1a^2+\gamma_2b^2=1$ and $\gamma_1+\gamma_2=n$.

Moreover, if $a \geqslant b > 0$, the global minimum is attained at a vector of the form

$$\boldsymbol{x} = \left(\underbrace{a, \ldots, a}_{n-1}, b\right),$$

where $(n-1)a^2 + b^2 = 1$, while the global maximum is attained at a vector of the form

$$\boldsymbol{x} = \left(\underbrace{b, \ldots, b}_{n-1}, a\right),$$

where $(n-1)b^2 + a^2 = 1$, or where the extrema are attained at a vector with some components equal to zero and the remainder equal; specifically,

$$\left(\underbrace{\frac{1}{\sqrt{\gamma}}, \ldots, \frac{1}{\sqrt{\gamma}}}_{\gamma}, \underbrace{0, \ldots, 0}_{n-\gamma}\right),$$

for $\gamma = 1, \ldots, n$.

Proof. We follow verbatim the proof of Proposition 3.3 for the case of extrema lying in the interior; unlike before, we now have a boundary. Since all coefficients are positive, the argument terminates at relation (3.5), where for distinct x_i, x_j, x_k, we have $h_{d-3}(\boldsymbol{x}, x_i, x_j, x_k) = 0$, which is clearly a contradiction. The extrema on the boundary will occur when at least one coordinate is zero, then our argument shows that the remaining non-zero coordinates must be equal completing the first part of the proof.

For the second part, without loss of generality, we may assume that $a > b > 0$. (If $b = 0$ we get exactly the third described form.) Suppose, for the sake of contradiction, that the vector attaining the global minimum is not of the desired form, and assume that there exist at least two occurrences of b. A similar argument provides the maximum.

Define the set

$$S := \left\{(x, y, z) \in \mathbb{R}^3_+ : x + y + z = 2b + a \quad \text{and} \quad x^2 + y^2 + z^2 = 2b^2 + a^2\right\}.$$

It is clear that S is non-empty and compact. Thus, we can choose x, y, z in S such that the product xyz is minimized.

Observe that $xyz < b^2 a$ and the inequality is strict since Corollary 1.8 in [11] suggests that the minimum cannot be attained at (a, b, b).

Applying Lemma 4.9 we obtain that for every non-negative integer $m > 2$

$$p_m(a, \ldots, b, b) > p_m(x, a, \ldots, a, y, z),$$

where the right-hand side is obtained from the vector $(a, \ldots, b, b)$ by replacing one occurrence of a with x, and by replacing two occurrences of b with y and z, respectively.

Therefore, by the representation (2.6), we arrive at $h_k(a, \ldots, a, b, b) > h_k(x, a, \ldots, a, y, z)$, which is clearly a contradiction. ☐

4.2. **Proofs.**

Proof of Theorem 4.1. We view the moment generating function of $\sum_{j=1}^n \sqrt{x_j}X_j$ in two ways, using on the one hand independence and on the other the Taylor expansion of the exponential function, namely

$$\begin{aligned}\mathbb{E}e^{t\sum_{j=1}^n \sqrt{x_j}X_j} &= \sum_{k=1}^\infty \frac{t^k}{k!}\mathbb{E}\left(\sum_{j=1}^n \sqrt{x_j}X_j\right)^k \\ &= \prod_{j=1}^n (1-t\sqrt{x_j})^{-1},\end{aligned}$$

for all t such that the above is well defined. If we let $F_t(x_1,\dots,x_n) = \prod_{j=1}^n(1-t\sqrt{x_j})^{-1}$, differentiating we get

$$\frac{\partial F}{\partial x_i} = F\cdot \frac{t}{2\sqrt{x_i}(1-t\sqrt{x_i})}$$

for every $i=1,\dots,n$, which leads to

$$\begin{aligned}\left(\frac{\partial}{\partial x_j}-\frac{\partial}{\partial x_i}\right)F &= F\cdot\frac{\sqrt{x_j}-\sqrt{x_i}}{2\sqrt{x_ix_j}}\frac{t^2(\sqrt{x_j}+\sqrt{x_i})-t}{(1-\sqrt{x_i}t)(1-\sqrt{x_j}t)} \\ &= \prod_{k\neq i,j}(1-t\sqrt{x_k})\cdot\frac{\sqrt{x_j}-\sqrt{x_i}}{2\sqrt{x_ix_j}}\frac{t^2(\sqrt{x_j}+\sqrt{x_i})-t}{(1-\sqrt{x_i}t)^2(1-\sqrt{x_j}t)^2},\end{aligned}$$

for every $i\neq j$. Taylor expanding around 0, we see that

$$\frac{(a+b)t^2-t}{(1-at)^2(1-bt)^2} = -t-(a+b)t^2-(a^2+b^2)t^3-(a-b)^2(a+b)t^4+O(t^5).$$

In particular, the coefficient of t^k is non-positive for any $a,b>0$ when $k\leqslant 4$. The wanted statement follows then by the Schur-Ostrowski criterion. □

Remark 4.11. It is worth to note that the above argument works only for $k\leqslant 4$. For example the Taylor coefficient of t^5 in the expansion of $\frac{(a+b)t^2-t}{(1-at)^2(1-bt)^2}$ is $-a^4+2a^3b+3a^2b^2+2ab^3-b^4$, which does not preserve sign, e.g. is positive when $a=b$.

Proof of Corollary 4.2. This is a direct application of the function to the majorization sequence (2.1). □

Proof of Theorem 4.5. The importance of Proposition 4.10 lies in the fact that it reduces the problem to one involving only two variables, namely the study of $\mathbb{E}(aG_1+bG_2)^k$ when $\gamma_1a^2+\gamma_2b^2 = 1$. We can further reduce this to a one-variable problem by dividing by $(\gamma_1a^2+\gamma_2b^2)^{k/2}$ and factoring out a common factor of b. Thus, we are ultimately led to study the following:

For any $a,b>0$, let $x:=a/b$, and define the function $f:(0,\infty)\to\mathbb{R}$ by

$$f(x) = \log\left(\sum_{j=0}^k\binom{k}{j}\Gamma(j+\gamma_1)\Gamma(k-j+\gamma_2)x^j\right)-\log\left(\Gamma(\gamma_1)\Gamma(\gamma_2)\right)-\frac{k}{2}\log(\gamma_1x^2+\gamma_2).$$

its derivative can be computed as:

$$f'(x) = \frac{g(x)}{(\gamma_1 x^2 + \gamma_2)\left(\sum_{j=0}^{k} \binom{k}{j} \Gamma(j+\gamma_1)\Gamma(k-j+\gamma_2)x^j\right)},$$

where

$$g(x) = \gamma_2 \binom{k}{1} \Gamma(1+\gamma_1)\Gamma(k-1+\gamma_2) + \sum_{j=0}^{k-2} x^{j+1} \left[\binom{k}{j} \Gamma(j+\gamma_1)\Gamma(k-j+\gamma_2)\gamma_1(j-k) \right.$$
$$\left. + (j+2)\gamma_2 \binom{k}{j+2} \Gamma(j+2+\gamma_1)\Gamma(k-j-2+\gamma_2) \right] - \binom{k}{k-1} \Gamma(k-1+\gamma_1)\Gamma(1+\gamma_2)\gamma_1 x^k.$$

Observe that $g(0) > 0$ and $\lim_{x\to+\infty} g(x) = -\infty$.

We aim to better understand the coefficient in front of x^{j+1}, say z_j, which can be simplified as

$$z_j := -\frac{1}{k-j-1}(k-j-1+\gamma_2)(k-j-2+\gamma_2)\gamma_1 + \frac{1}{j+1}\gamma_2(j+1+\gamma_1)(j+\gamma_1),$$

by factoring out

$$\frac{k!\,(j+\gamma_1-1)!\,(k-j-3+\gamma_2)!}{j!\,(k-j-2)!}.$$

This can be rewritten as

$$\frac{1}{(j+1)(k-j+1)} \Big[-j^3(\gamma_1+\gamma_2) + j^2\left(k(2\gamma_1+\gamma_2) - 4\gamma_1 - 2\gamma_2\right) - j\big(k^2\gamma_1 + k(-5\gamma_1-\gamma_2) + \gamma_1^2\gamma_2$$
$$+ \gamma_1(\gamma_2^2 - 2\gamma_2 + 5) + \gamma_2\big) - \gamma_1\big(k^2 + k(-\gamma_1\gamma_2 + \gamma_2 - 3) + \gamma_1\gamma_2 + \gamma_2^2 - 2\gamma_2 + 2\big)\Big].$$

According to Descartes' Rule of Signs, the number of positive real roots of a polynomial is at most equal, or is less than it by an even number, to the number of sign changes in the sequence of its coefficients. Since the numerator of z_j is a cubic polynomial, it can exhibit up to three sign changes, and therefore, in the worst-case scenario g can have three sign changes in the sequence of its coefficients.

Combining all the above, we conclude that g has at most three positive roots.

We now claim that $f'(1) = 0$, and thus the obvious root of the polynomial g is $x = 1$. To verify this, we compute the derivatives of $\mathbb{E}[(xG_1 + G_2)^k]$. By differentiting under the integral sign we obtain

$$\frac{d}{dx}\mathbb{E}\left[(xG_1+G_2)^k\right] = k\mathbb{E}\left[(xG_1+G_2)^{k-1}G_1\right].$$

Moreover

$$\mathbb{E}[(xG_1+G_2)^{k-1}G_1] = \mathbb{E}\left(\sum_{j=0}^{k-1}\binom{k-1}{j}(xG_1)^{j+1}G_2^{k-1-j}\right) = \sum_{j=0}^{k-1}\binom{k-1}{j}x^{j+1}\mathbb{E}(G_1^{j+1})\,\mathbb{E}(G_2^{k-1-j}).$$

The moment formula for Gamma distributions gives:

$$\mathbb{E}(G_1^{j+1}) = \frac{\Gamma(\gamma_1+j+1)}{\Gamma(\gamma_1)} \quad \text{and} \quad \mathbb{E}(G_2^{k-1-j}) = \frac{\Gamma(\gamma_2+k-1-j)}{\Gamma(\gamma_2)}.$$

Therefore,

$$\mathbb{E}[(xG_1+G_2)^{k-1}G_1] = \sum_{j=0}^{k-1}\binom{k-1}{j}x^{j+1}\frac{\Gamma(\gamma_1+j+1)}{\Gamma(\gamma_1)}\cdot\frac{\Gamma(\gamma_2+k-1-j)}{\Gamma(\gamma_2)}.$$

This sum simplifies to

$$\gamma_1 \cdot \mathbb{E}[(xG_3 + G_4)^{k-1}]$$

where $G_3 \sim \Gamma(\gamma_1 + 1)$, $G_4 \sim \Gamma(\gamma_2)$ where $G_3 \sim \Gamma(\gamma_1 + 1)$ and is independent of G_1, G_2. In particular, for $x = 1$, either expression above can be evaluated directly. Similarly, we obtain the second derivative to be

$$\frac{d^2}{dx^2}\mathbb{E}[(xG_1 + G_2)^k] = k(k-1)\gamma_1(\gamma_1 - 1)\mathbb{E}[(xG_4 + G_2)^{k-1}],$$

where $G_4 \sim \Gamma(\gamma_1 + 2)$ and is independed from the others. We observe also that

$$\mathbb{E}[(G_1 + G_2)^k] = \frac{\Gamma(n+k)}{\Gamma(n)}.$$

Thus, putting everything together:

$$f'(1) = \gamma_1 \frac{k}{n} - \frac{k\gamma_1}{n} = 0.$$

A similar argument as above yields the second derivative at $x = 1$:

$$f''(1) = \frac{k\gamma_1}{n} \frac{\gamma_2(k-n-2)}{n(n+1)}.$$

Therefore, by the second-order derivative test, we conclude the following:

- If $k > n + 2$, then $f''(1) > 0$, so f has a local minimum at $x = 1$.
- If $k < n + 2$, then $f''(1) < 0$, so f has a local maximum at $x = 1$.

Since $g(0) > 0$ and $g(+\infty) < 0$, we conclude that for $k > n+2$, the other two roots $s < t$ must be positioned as $0 < s < 1 < t$ and thus the function f may attain its minimum at one of the points $x = 0$, $x = 1$, or $x = +\infty$. That is, the possible candidates for the minimum of f are $f(0)$, $f(1)$, and $f(+\infty)$. The maximum will be attained at one of $f(s), f(t)$.

In the case where $k = n + 2$, f has a root of multiplicity two at $x = 1$. Therefore, the possible minima occur at $f(0)$ and $f(+\infty)$.

In the case where $k < n+2$, there are two possible configurations. In the first scenario, the global maximum is attained at $x = 1$, and the potential minima are at $f(0)$ and $f(+\infty)$. In the second scenario, the polynomial g admits two distinct roots s and t such that $0 < s < t < 1$ (or $1 < s < t$). In this case, the maximum of f occurs at $x = s$ or $x = 1$ and the minimum could potentially occur at $f(t)$, in addition to $f(0)$ and $f(+\infty)$. Both scenarios are possible so this argument cannot work.

For $k \leq n + 1$, we employ a different approach. Recall (see Proposition 4.10) that the global minimum will be attained at a vector of the form

$$\boldsymbol{x} = \left(\underbrace{a, \ldots, a}_{n-1}, b \right),$$

for $a \geqslant b > 0$ or at a vector of the form

$$\left(\underbrace{\frac{1}{\sqrt{\gamma}}, \dots, \frac{1}{\sqrt{\gamma}}}_{\gamma}, \underbrace{0, \dots, 0}_{n-\gamma}\right)$$

where $\gamma = 1, \dots, n$. We proceed by induction on k to show that for every natural number n and $x \geq 1$, the following inequality holds:

$$\frac{\mathbb{E}\,(xG_1 + G_2)^k}{((n-1)x^2+1)^{k/2}} \geq \frac{\Gamma(n-1+k)}{\Gamma(n-1)\,(n-1)^{k/2}},$$

where G_1, G_2 are independent $\Gamma(n-1)$ and $\Gamma(1)$ random variables respectively. This would complete the argument. We observe that the cases $k = 1, 2$ hold, so we assume the statement holds for $k-1$.

Consider the function

$$h(x) = \frac{\mathbb{E}\,(xG_1 + G_2)^k}{((n-1)x^2+1)^{k/2}}.$$

As before, we compute its derivative:

$$h'(x) = \frac{k(n-1)\,\mathbb{E}\,(xG_3 + G_2)^{k-1}\left((n-1)x^2+1\right)^{k/2} - k(n-1)x\,\mathbb{E}\,(xG_1+G_2)^k\left((n-1)x^2+1\right)^{k/2-1}}{\left(x^2(n-1)+1\right)^k},$$

where G_1, G_2 and G_3 are independent Gamma random variables with distributions $G_1 \sim \Gamma(n-1)$, $G_2 \sim \Gamma(1)$ and $G_3 \sim \Gamma(n)$, respectively.

Therefore, at any root y of $h'(x)$ we have

$$\begin{aligned} h(y) &= \frac{\mathbb{E}\,(yG_1 + G_2)^k}{((n-1)y^2+1)^{k/2}} \\ &= \frac{\mathbb{E}\,(yG_3 + G_2)^{k-1}}{(ny^2+1)^{(k-1)/2}} \cdot \frac{\left(ny^2+1\right)^{(k-1)/2}}{y\left((n-1)y^2+1\right)^{k/2-1}}. \end{aligned}$$

Notice that the function

$$g(y) = \frac{\left(ny^2+1\right)^{(k-1)/2}}{y\left((n-1)y^2+1\right)^{k/2-1}},$$

has derivative

$$g'(y) = \frac{(y^2(k-n-1)-1)(ny^2+1)^{\frac{k-3}{2}}}{y^2((n-1)y^2+1)^{k/2}},$$

and thus is decreasing for $k \leq n+1$. Therefore,

$$g(y) \geqslant \frac{n^{(k-1)/2}}{(n-1)^{k/2-1}}.$$

Combining this estimate with the inductive hypothesis completes the proof. □

To support Conjecture 4.8 concerning real exponents, we provide a proof for the two-dimensional case.

Proposition 4.12. *Let q be a non-negative real number and $a, b \geqslant 0$ such that $a^2 + b^2 = 1$. Then*

$$\mathbb{E}(aX + bY)^q \geq \min\left\{\mathbb{E}\Big(\frac{X+Y}{\sqrt{2}}\Big)^q, \mathbb{E}\big(X_1^q\big)\right\}.$$

For the regime $q \leqslant 4$ we will need the following log-concavity Lemma (see for example [5]).

Lemma 4.13. *If $f : (0, +\infty) \to (0, +\infty)$ is log-concave then*

$$G(q) = \frac{1}{\Gamma(1+q)} \int_0^\infty t^q f(t)\, dt$$

is also log-concave on $(-1, +\infty)$.

Proof of Proposition 4.12. We will first work on the regime $q \leq 4$. We apply Lemma 4.13 for the density of $aX + bY$, given by

$$d(x) = \frac{1}{a-b}\left(e^{-x/a} - e^{-x/b}\right)\mathbf{1}_{(0,+\infty)}.$$

By log-concavity,

$$\frac{1}{\mathbb{E}X_1^q}\mathbb{E}\left(aX + bY\right)^q = G(q) \geqslant G(0)^{1-\frac{q}{4}} G(4)^{\frac{q}{4}} = \left(\frac{\mathbb{E}\left(aX+bY\right)^4}{4!}\right)^{\frac{q}{4}} \geqslant 1,$$

since, by Corollary 4.2, $\mathbb{E}\left(aX + bY\right)^4 \geqslant \mathbb{E}X_1^4 = 4!\,$.

For $q \geqslant 4$, the desired quantity becomes

$$\frac{a^{q+1} - b^{q+1}}{(a-b)(a^2+b^2)^{q/2}}$$

for all $a, b \geqslant 0$. Without loss of generality let $b \neq 0$, $x := a/b$ and define $f : (0, \infty) \to \mathbb{R}$ by

$$f(x) = \log\left(\frac{x^{q+1}-1}{x-1}\right) - \frac{q}{2}\log(x^2+1).$$

To this end, we examine the monotonicity of f.

$$\begin{aligned} f'(x) &= \frac{qx^{q+1} - (q+1)x^q + 1}{(x-1)(x^{q+1}-1)} - \frac{qx}{x^2+1} \\ &= \frac{-x^{q+2} + qx^{q+1} - x^q(q+1) + x^2(q+1) - qx + 1}{(x-1)(x^2+1)(x^{q+1}-1)} \end{aligned}$$

Note that the numerator, say g, has five sign changes in its coefficients, therefore by the extension of Descartes' rule of signs [13] it has at most five positive roots, and thus the same will hold for f'. It is easy to check that $g(x)$ has a root of multiplicity three at $x = 1$.

We observe that $g'''(1) = q(q+1)(q-4) > 0$ for $q > 4$ and thus from the higher order derivative test, $x = 1$ is a saddle point and a strictly increasing point of inflection. □

5. The Centred Case: Proofs

5.1. Characterization of Extrema.

For the **centred** case, where $\sum_{i=1}^n a_i = 0$ and $\sum_{i=1}^n a_i^2 = 1$, we have the following theorem.

When n is even, the lower bound for every even degree coincides with the one stated in Theorem 3.2, but in this case we are able to determine the exact upper bound.

Theorem 5.1. *Let n be an even integer and $k \geq 1$ be an integer, and let $a_1, \ldots, a_n$ be real numbers such that $\sum_{i=1}^n a_i^2 = 1$ and $\sum_{i=1}^n a_i = 0$. Then*

$$h_{2k}(a_1, \ldots, a_n) \geq \frac{\left(\frac{n}{2} + k - 1\right)!}{k! \cdot \left(\frac{n}{2} - 1\right)! \cdot n^k},$$

and equality is attained at the vector $\tilde{a}$.

The maximum, for both odd and even non-negative integers n, is attained at the vector where all of a_i except of one are equal, that is

$$\left(\underbrace{-\frac{1}{\sqrt{n(n-1)}}, \ldots, -\frac{1}{\sqrt{n(n-1)}}}_{n-1}, \sqrt{\frac{n-1}{n}} \right).$$

We also provide a proof of the lower bound for h_4 for both odd and even integers n.

Proposition 5.2. *Let $a_1, \ldots, a_n \in \mathbb{R}$ satisfy $\sum_{i=1}^n a_i = 0$ and $\sum_{i=1}^n a_i^2 = 1$. Then the quantity $h_4(a_1, \ldots, a_n)$ attains the minimum at*

$$\left(\underbrace{\frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}}}_{\frac{n}{2}}, \underbrace{-\frac{1}{\sqrt{n}}, \ldots, -\frac{1}{\sqrt{n}}}_{\frac{n}{2}} \right)$$

if n is even, or

$$\left(\underbrace{\sqrt{\frac{n+1}{n(n-1)}}, \ldots, \sqrt{\frac{n+1}{n(n-1)}}}_{\frac{n-1}{2}}, \underbrace{-\sqrt{\frac{n-1}{n(n+1)}}, \ldots, -\sqrt{\frac{n-1}{n(n+1)}}}_{\frac{n+1}{2}} \right),$$

if n is odd.

In the case $n = 3$, we are able to obtain sharp upper and lower bounds for nearly all moments.

Proposition 5.3. *Let*

$$x_1 = \left(\frac{1}{\sqrt{6}}, \frac{1}{\sqrt{6}}, -\frac{2}{\sqrt{6}} \right), \quad x_2 = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0 \right),$$

and define

$$f(a_1, a_2, a_3) := \mathbb{E} \left| a_1 X_1 + a_2 X_2 + a_3 X_3 \right|^q,$$

subject to the constraints $a_1 + a_2 + a_3 = 0$ and $a_1^2 + a_2^2 + a_3^2 = 1$.

- *(i) For $q \in (-1, 0) \cup (2, 4)$, the function f attains its minimum at x_1 and its maximum at x_2.*
- *(ii) For $q \in (0, 2) \cup (4, 6)$, the function f is minimized at x_2 and maximized at x_1.*
- *(iii) For $q = k > 3$, where k is a non-negative integer, the function f is minimized at x_2 and maximized at x_1.*

(iv) Moreover, for $q = 2$ and $q = 4$, the function f is constant.

Before proceeding with the proofs, we first characterize the extrema in the geometric case, that is $\sum_{i=1}^n a_i = 0$ and $\sum_{i=1}^n a_i^2 = 1$. Following almost verbatim the proof of Proposition 3.3 we obtain the following:

Proposition 5.4. *Let $n \geqslant 1$ and $d > 3$ be a even integer. Then the extrema of h_d on the space $\mathcal{S} := \{(x_1, \ldots, x_n) \in \mathbb{R}^n : \sum_{i=1}^n x_i = 0, \sum_{i=1}^n x_i^2 = 1\}$ are of the form*

$$\boldsymbol{x} = \left(\underbrace{a, \ldots, a}_{\gamma_1}, \underbrace{b, \ldots, b}_{\gamma_2}, \underbrace{c, \ldots, c}_{\gamma_3} \right).$$

Here, a appears γ_1 times, b appears γ_2 times, and c appears γ_3 times, subject to the constraints constraints $\gamma_1 a^2 + \gamma_2 b^2 + \gamma_3 c^2 = 1$ and $\gamma_1 + \gamma_2 + \gamma_3 = n$.

5.2. **Sharp Bounds for p_m.**

Proposition 5.5. *Let $n \geq 2$, and let a, b, c be real numbers such that*

$$\gamma_1 a + \gamma_2 b + \gamma_3 c = 0 \quad \textit{and} \quad \gamma_1 a^2 + \gamma_2 b^2 + \gamma_3 c^2 = 1,$$

and set

$$\boldsymbol{x} = \left(\underbrace{a, \ldots, a}_{\gamma_1}, \underbrace{b, \ldots, b}_{\gamma_2}, \underbrace{c, \ldots, c}_{\gamma_3} \right).$$

Then, for every positive integer m, $p_m(\boldsymbol{x}) = \gamma_1 a^m + \gamma_2 b^m + \gamma_3 c^m$ attains its maximum at

$$\left(\underbrace{-\frac{1}{\sqrt{n(n-1)}}, \ldots, -\frac{1}{\sqrt{n(n-1)}}}_{n-1}, \sqrt{\frac{n-1}{n}} \right),$$

that is, the vector in which all but one coordinate are equal.

Proof. Since $n = 2$ is trivial, assume that $n \geqslant 3$, $\gamma_1, \gamma_2, \gamma_3 \geqslant 1$ and $m \geqslant 3$ since $p_1(\boldsymbol{x}) = 0$, $p_2(\boldsymbol{x}) = 1$. We will distinguish two cases. If m is odd, say $m = 2k+1$, then using the Lagrange multiplier method, we find that an extreme point (a, b, c) satisfies

$$(2k+1)x^{2k} - 2\lambda x - \mu = 0,$$

for $x = a, b, c$. Since x^{2k} is convex, it has at most two points of intersection with a affine function, therefore, two of a, b, c are equal, say $a = b$. Then, if we set $\gamma_1 + \gamma_2 = \gamma$ we obtain

$$p_m = \gamma a^m + \gamma_3 c^m$$

under the conditions $\gamma a + \gamma_3 c = 0$ and $\gamma a^2 + \gamma_3 c^2 = 1$. Solving for a, c we and using $\gamma_3 = n - \gamma$, we write

$$p_m = \frac{1}{n^{m/2}} \left(\frac{(n-\gamma)^{m/2}}{\gamma^{m/2-1}} - \frac{\gamma^{m/2}}{(n-\gamma)^{m/2-1}} \right),$$

which is clearly maximized for $\gamma = 1$.

If m is even, say $m = 2k$, then using the Lagrange multiplier method we find that the vector (a, b, c) which attains the maximum satisfies

$$2k\, x^{2k-1} - 2\lambda x - \mu = 0, \tag{5.1}$$

for $x = a, b, c$. We can multiply this relation by $\gamma_1, \gamma_2, \gamma_3$, respectively, and add them to obtain $kp_{2k} = \lambda$. We multiply this relation by $\gamma_1 a, \gamma_2 b, \gamma_3 c$, respectively, and add them to get $2kp_{2k-1} = n\mu$. If substitute this into (5.1), and divide by $2m$,

$$x^{2k-1} = x\, p_{2k} + \frac{p_{2k-1}}{n}.$$

Due to the even power, suppose, without loss of generality, that a is positive and $|c| > a > |b|$. Under this assumption, we first prove that

$$\frac{1}{\sqrt{n(n-1)}} \leq a \leq \frac{1}{\sqrt{2}}.$$

For the upper bound, note that

$$2a^2 \leq (\gamma_1 + \gamma_3)a^2 \leq \gamma_1 a^2 + \gamma_2 b^2 + \gamma_3 c^2 = 1.$$

For the lower bound we write

$$(\gamma_1 + \gamma_2)a^2 + \frac{a^2(\gamma_1 + \gamma_2)^2}{\gamma_3} \geq \gamma_1 a^2 + \gamma_2 b^2 + \frac{(\gamma_1 a + \gamma_2 b)^2}{\gamma_3} = 1,$$

or

$$a^2 \geq \frac{\gamma_3}{n(n - \gamma_3)} \geq \frac{1}{n(n-1)}.$$

From the odd case, we know that

$$p_{2k-1} \geq -A_{n,2k-1} := -\left[\left(\frac{n-1}{n}\right)^{(2k-1)/2} + (-1)^{2k-1}(n-1)\left(\frac{1}{(n-1)n}\right)^{(2k-1)/2}\right].$$

On the other hand, since (a, b, c) is the point that achieves the maximum we have that

$$p_{2k} \geq A_{n,2k}.$$

Therefore, from the main relation for a satisfies the inequality

$$a^{2k-1} \geq a\, A_{n,2k} - \frac{A_{n,2k-1}}{n}.$$

We consider the following function:

$$f_k(x) = x^{2k-1} - x\, A_{n,2k} + \frac{A_{n,2k-1}}{n}.$$

and the constants:

$$s = \sqrt{\frac{n-1}{n}}, \quad t = \frac{1}{\sqrt{n(n-1)}}, \quad w = \frac{1}{\sqrt{2}}.$$

Notice that from Descartes's rule or signs, f_k has at most 2 positive roots.

We aim to prove that $f_k(w) < 0$ for every $n \geqslant 3$ and $k \geqslant 2$. The conclusion is that f_k has two positive roots, one of them is t and the other is larger than w, since the limit of f_k as x grows to $+\infty$ is $+\infty$. This means that in the open interval (t, w) f_k is negative. Therefore, $a = t$. In all the above inequalities, equality must hold, therefore, $\gamma_3 = 1$ and $a = b$.

Let us begin with $k \geq 2$ and $n \geq 6$.

We express $f_k(w)$ as:

$$f_k(w) = (w-t)\left[w^{2k-2} + w^{2k-3}t + \cdots + t^{2k-2} - A_{n,2k}\right]$$
$$:= (w-t)\, Q_k(w).$$

Since the prefactor $w - t > 0$, for $n \geq 3$ the sign of $Q_k(w)$ is the same as that of $f_k(w)$. If we set

$$q := \frac{t}{w} < 1$$

then

$$S(w) := \sum_{i=0}^{2k-2} t^i\, w^{2k-2-i} = w^{2k-2} \sum_{i=0}^{2k-2} q^i < \frac{w^{2k-2}}{1-q},$$

which gives

$$\frac{S(w)}{s^{2k}} < \frac{1}{1-q}\,\frac{w^{2k-2}}{s^{2k}} = \frac{w^{-2}}{1-q}\left(\frac{w^2}{s^2}\right)^k \leq \frac{w^{-2}}{1-q}\left(\frac{w^2}{s^2}\right)^2 = \frac{w^2}{s^4(1-q)} < 1,$$

since $s^4 \geq (5/6)^2$ and $q \leqslant \frac{1}{\sqrt{15}}$, for $n \geq 6$. This means

$$S(w) - \left(s^{2k} + (n-1)t^{2k}\right) < 0.$$

Finally, we can easily check that the above argument implies that $f_k(w) < 0$ also holds for $n \geq 4$ and $k \geq 3$. Indeed, For $n = 5$ we have $s^2 = 4/5$ and $q = t/w = \sqrt{1/10} < 1/3$, which implies

$$\frac{w^2}{s^4(1-q)} \leq \frac{1/2}{(16/25)\cdot(2/3)} = \frac{75}{64} < 1,$$

and the same conclusion follows. For $n = 4$ we retain the factor $(w^2/s^2)^k$ and use that

$$\frac{S(w)}{s^{2k}} < \frac{w^{-2}}{1-q}\left(\frac{w^2}{s^2}\right)^k = \frac{2}{1-q}\left(\frac{2}{3}\right)^k$$

with $q = 1/\sqrt{6}$. It is then easy to check that the right-hand side is < 1 for all $k \geq 4$, hence $f_k(w) < 0$ in this range. Finally, in the remaining case $(n,k) = (4,3)$ we compute explicitly

$$f_3(w) = \frac{20\sqrt{3} - 25\sqrt{2}}{288} < 0.$$

In particular, we have $f_k(w) < 0$ for all $n \geq 4$ and $k \geq 3$.

Therefore, it remains to check only the cases $n = 3$ and any $k \geqslant 2$, $(n,k) = (5,2)$ and $(n,k) = (4,2)$. The cases for $k = 2$ have been already settled, as it can be seen in Proposition 2.1 of [12]. For $n = 3$ and $k \geqslant 2$ it suffices to consider the quantity

$$\frac{a^{2k} + b^{2k} + c^{2k}}{(a^2 + b^2 + c^2)^k}.$$

Since $a + b + c = 0$ and $a^2 + b^2 + c^2 = 1$, we can without loss of generality assume that a, c are non-negative, $c \neq 0$ and set $x = a/c \geqslant 0$ and $y = b/c$ to obtain a function of one variable $f : (0, \infty) \to \mathbb{R}$ as follows:

$$f(x) = \frac{x^{2k} + (x+1)^{2k} + 1}{2^k(x^2 + x + 1)^k}.$$

We will prove that f attains its maximum at $x = 1$. Let g be the numerator of f', then a direct computation shows that

$$\frac{g(x)}{kx} = x^{2k+1} + 3x^{2k} + 2x^{2k-1} + (1-x)(x+1)^{2k} - 2x^2 - 3x - 1$$

Since k is a integer we can expand the binomial and by setting $a_j := \binom{2k}{j}$ we obtain

$$\begin{aligned}\frac{g(x)}{kx^2} =& -x^{2k-1}(-3 - a_{2k} + a_{2k-1}) - x^{2k-2}(-2 - a_{2k-1} + a_{2k-2}) \\ &+ \sum_{j=2}^{2k-3}(a_{j+1} - a_j)x^j + x(-2 + a_2 - a_1) + (-3 + a_1 - a_0).\end{aligned}$$

Since $a_j = a_{2k-j}$, it follows immediately that $\frac{g(x)}{kx^2}$ is an anti-palindromic polynomial. In particular, $x = 1$ is a root. The uniqueness of the positive root then follows from the sign of the sequence $a_{j+1} - a_j = a_j \frac{2k-2j-1}{j+1}$, together with Descartes' Rule of Signs.

□

Proof of Theorem 5.1. For the lower bound, applying Theorem 3.2 we conclude:

$$h_{2r}(\boldsymbol{x}) \geq \frac{(n/2 + r - 1)!}{r! \cdot (n/2 - 1)! \cdot n^r}.$$

For the maximum, notice that relation (2.6) suggest

$$h_k(\boldsymbol{x}) \leqslant \sum_{\substack{m_1 + 2m_2 + \cdots + km_k = k \\ m_1 \geq 0, \ldots, m_k \geq 0}} \prod_{i=1}^{k} \frac{|p_i|^{m_i}(\boldsymbol{x})}{m_i!\, i^{m_i}}.$$

Notice also that $p_{2k} \geq 0$. Moreover, if $\boldsymbol{w}$ is a maximizer of p_{2k-1}, then since p_{2k-1} is an odd function, it follows that $-\boldsymbol{x}$ is a minimizer. Thus,

$$|p_{2k-1}| \leq p_{2k-1}(\boldsymbol{w}).$$

The characterization of the extrema Proposition 5.4 and Proposition 5.5 now finishes the proof. □

Proof of Proposition 5.2. We will find upper and lower bounds for

$$\mathbb{E}\left|\sum_{i=1}^{n} a_i X_i\right|^4 = 3 + 6\sum_{i=1}^{n} a_i^4,$$

under the conditions $\sum_{i=1}^{n} a_i = 0$ and $\sum_{i=1}^{n} a_i^2 = 1$. We will use the method of Lagrange multipliers to find the maximum and the minimum which exist since the domain is compact. We are searching for all $\boldsymbol{a} = (a_1, \ldots, a_n)$ and the real numbers λ, μ such that for all indices $i = 1, \ldots, n$ we have

$$4a_i^3 - 2\lambda a_i + \mu = 0.$$

Multiplying by a_i and summing for all i we get

$$4\sum_{i=1}^{n} a_i^4 - 2\lambda = 0. \tag{5.2}$$

If for the indices $i \neq j$ it is true that $a_i \neq a_j$, then subtracting the two relations we get

$$4(a_i^2 + a_i a_j + a_j^2) - 2\lambda = 0. \tag{5.3}$$

Therefore if there exists one more index k, such that $a_k \neq a_i$ and $a_k \neq a_j$, then $a_i + a_j + a_k = 0$. Therefore the vector $\boldsymbol{a}$ has the form

$$(x, x, \ldots, x, y, y, \ldots, y, z, z, \ldots, z),$$

where $x + y + z = 0$. Suppose that x appears s times, y appears t times and z appears w times and due to the symmetry we assume that $s \geq t \geq w$. From (5.3) and (5.2), it suffices to find upper bounds for $x^2 + y^2 + z^2$, under the conditions

$$\begin{cases} x + y + z = 0 \\ sx + ty + wz = 0 \\ sx^2 + ty^2 + wz^2 = 1 \end{cases}$$

If $s = t = w$, then $x^2 + y^2 + z^2 = \frac{3}{n}$. Otherwise, we can solve the above system with respect to x, y, z and find that

$$x^2 + y^2 + z^2 = \frac{(s-t)^2 + (t-w)^2 + (s-w)^2}{w(s-t)^2 + s(t-w)^2 + t(s-w)^2}.$$

We will prove the following double inequality.

$$\frac{2}{n-1} \leq \frac{(s-t)^2 + (t-w)^2 + (s-w)^2}{w(s-t)^2 + s(t-w)^2 + t(s-w)^2}.$$

For the left-hand side inequality, note that $n = w + s + t$ and after some algebraic manipulations we end up proving that

$$(w-1)(w-s)(w-t) + (t-1)(t-s)(t-w) + (s-1)(s-w)(s-t) \geq 0.$$

To this end, note that the last one is equivalent to

$$(s-t)[(s-1)(s-w) - (t-1)(t-w)] + (w-1)(s-w)(t-w) \geq 0,$$

which is true, since $w, s, t \geq 1$ and $s \geq t \geq w$.

We also need to consider the case where exactly one group is zero, say z. In this case, since $x + y = 0$ and also $sx + ty = 0$, together with the condition $sx^2 + ty^2 = 1$, we are led to $s = t$, and a vector of the form

$$\left(\underbrace{\frac{1}{\sqrt{2s}}, \ldots, \frac{1}{\sqrt{2s}}}_{s\text{-times}}, \underbrace{-\frac{1}{\sqrt{2s}}, \ldots, -\frac{1}{\sqrt{2s}}}_{s\text{-times}}, \underbrace{0, \ldots, 0}_{(n-2s)\text{-times}} \right).$$

Note also that $2s = n - w$. Since we only need to bound $x^2 = \frac{1}{2s}$, which is decreasing in s, it suffices to consider the case $2s = n - 1$ for the minimum and $s = 1$ for the maximum.

It remains to check the case where the vector $\boldsymbol{a}$ has the form

$$(x, x, \ldots, x, y, y, \ldots, y),$$

where x appears s times and y appears t times. Without loss of generality, assume that $x > 0$ and $y < 0$. Then, from (5.3) and (5.2), it suffices to find upper the maximum and the minimum for x^2+xy+y^2, under the conditions

$$\begin{cases} sx+ty=0 \\ sx^2+ty^2=1 \end{cases}$$

Solving the system we get that

$$x^2+y^2+xy=\frac{1}{n}\left(\frac{t}{s}+\frac{s}{t}-1\right)=\frac{1}{n}\left(\frac{n-s}{s}+\frac{s}{n-s}-1\right).$$

The function in the parenthesis takes its maximum value for $s=1$ (or $s=n-1$) and its minimum value for $s=\lfloor\frac{n}{2}\rfloor$. It is easy now to compare the extrema. □

Proof of Proposition 5.3. For (iv), we observe that since $a_1+a_2+a_3=0$ and $a_1^2+a_2^2+a_3^2=1$, it follows that

$$\mathbb{E}\left|a_1X_1+a_2X_2+a_3X_3\right|^2=1,$$

and

$$\mathbb{E}\left|a_1X_1+a_2X_2+a_3X_3\right|^4=3+6(a_1^4+a_2^4+a_3^4)=6.$$

For (i), (ii) and (iii), a simple point-wise bound for $\Re\left(\phi_{\sum a_iX_i}\right)$, combined with the Fourier formulas (2.7), (2.8) and (2.10) suffices.

We observe that

$$\phi(t)=\frac{1}{(1+ia_1t)(1+ia_2t)(1+ia_3t)}=\frac{1-it(a_1+a_2+a_3)-t^2(a_1a_2+a_2a_3+a_1a_3)-i^3t^3a_1a_2a_3}{(1+a_1^2t^2)(1+a_2^2t^2)(1+a_3^2t^2)}.$$

Since $a_1+a_2+a_3=0$ and $a_1^2+a_2^2+a_3^2=1$, this simplifies to

$$\Re(\phi(t))=\frac{1+t^2/2}{(1+a_1^2t^2)(1+a_2^2t^2)(1+a_3^2t^2)}.$$

We now establish a bound for $\Re(\phi(t))$:

$$(1+t^2/2)^2\leq(1+a_1^2t^2)(1+a_2^2t^2)(1+a_3^2t^2)\leq(1+t^2/6)^2(1+2t^2/3),$$

where the upper bound is attained at x_2 and the lower bound at x_1.

This can be proven by expanding the product:

$$(1+a_1^2t^2)(1+a_2^2t^2)(1+a_3^2t^2)=1+t^2(a_1^2+a_2^2+a_3^2)+t^4(a_1^2a_2^2+a_2^2a_3^2+a_1^2a_3^2)+t^6a_1^2a_2^2a_3^2.$$

The lower bound is immediate. For the upper bound, we may assume without loss of generality that two of the variables share the same sign; let $a_1,a_2\geqslant 0$.

Applying the Cauchy–Schwarz inequality yields

$$1=a_1^2+a_2^2+a_3^2\geq 2\left(\frac{a_1+a_2}{2}\right)^2+a_3^2,$$

which implies $a_3^2\leq\frac{2}{3}$.

Using the AM–GM inequality, we obtain

$$a_1^2 a_2^2 a_3^2 \le \left(\frac{a_1+a_2}{2}\right)^4 a_3^2 = \frac{a_3^6}{2^4} \le \frac{(2/3)^3}{16} = \frac{1}{54}.$$

For integers $k > 3$, a different approach is required. We define the function

$$f(a_1,a_2,a_3) = \frac{\mathbb{E}|a_1X_1 + a_2X_2 + a_3X_3|^q}{(a_1^2+a_2^2+a_3^2)^{q/2}}.$$

Let $a_1, a_2 \geqslant 0$, $a_2 \neq 0$, and $a_3 \leqslant 0$, and write $-a_3 \leqslant 0$, so that the expression becomes $a_1X_1 + a_2X_2 - a_3X_3$. Let $x = a_1/a_2$ and $y = a_3/a_2$, then it equals

$$\frac{\mathbb{E}|xX_1 + X_2 - yX_3|^q}{(x^2+y^2+1)^{q/2}}.$$

Since $a_1 + a_2 - a_3 = 0$ and $a_1^2 + a_2^2 + a_3^2 = 1$, it follows that

$$f(x) = \frac{\mathbb{E}|xX_1 + X_2 - (1+x)X_3|^q}{2^{q/2}(x^2+x+1)^{q/2}}.$$

We now utilize formula (2.11) and drop the $\frac{\Gamma(1+q)}{2^{q/2+1}}$ constant for simplicity:

$$f(x) = \frac{-1-2x+2x^{2+q}+x^{3+q}+(x-1)(1+x)^{2+q}}{(x-1)(x+2)(2x+1)(x^2+x+1)^{q/2}}.$$

As a motivating example, when $q = 7$, we compute:

$$f'(x) = \frac{-3x(10x^9+80x^8+249x^7+363x^6+183x^5-183x^4-363x^3-249x^2-80x-10)}{2(x^2+(1+x))^{9/2}(1+2x)^2(2+x)^2}.$$

The function f' has a unique root at $x = 1$ by Descartes' rule of signs.

Claim. Let $q = k$ be a non-negative integer. The numerator of f', denoted g, has a root at $x = 1$ and exhibits exactly one sign change in its coefficients. More generally, g is an anti-palindromic polynomial with a single sign change in its coefficients.

Proof of Claim 5.2. From expanding $(x+1)^{k+2}$, we get

$$f(x) = \frac{2x^{k+2} + 2 + \sum_{j=1}^{k+1} x^j \left(\binom{k+2}{j} + 3\right)}{(x^2+x+1)^{k/2}(2x+1)(x+2)}.$$

Direct computation shows:

$$g(x) = (2x^4+7x^3+9x^2+7x+2)\left[2(k+2)x^{k+1} + \sum_{j=1}^{k+1} jx^{j-1}\left(\binom{k+2}{j}+3\right)\right]$$
$$-\frac{1}{2}(x^3(4k+8)+x^2(12k+18)+x(9k+18)+2k+10)\left(2x^{k+2}+2+\sum_{j=1}^{k+1}x^j\left(\binom{k+2}{j}+3\right)\right).$$

Define $a_j = \binom{k+2}{j} + 3$, we notice that the constant term, x^{k+5}, and x^{k+4} vanish, and thus a direct computation shows:

$$\begin{aligned}\frac{g(x)}{x} &= a_1(2-k) + 4a_2 - (9k+18) \\ &\quad + x\left(-\tfrac{9}{2}k\, a_1 + (9-k)a_2 + 6a_3 - (12k+18)\right) \\ &\quad + x^2\left[-(4k+8) - (6k+2)a_1 + \left(9 - \tfrac{9}{2}k\right)a_2 + (16-k)a_3 + 8a_4\right] \\ &\quad + \sum_{j=1}^{k-3} x^{j+2}\Big[(2j-2k-4)a_j + (7j-6k-2)a_{j+1} + \left(9j - \tfrac{9}{2}k + 9\right)a_{j+2} + (7j-k+16)a_{j+3} + (2j+8)a_{j+4}\Big] \\ &\quad + x^k\Big[(4k+8) + (6k+2)a_{k+1} - \left(9 - \tfrac{9}{2}k\right)a_k - (16-k)a_{k-1} - 8a_{k-2}\Big] \\ &\quad + x^{k+1}\left[\tfrac{9}{2}k\, a_{k+1} - (9-k)a_k - 6a_{k-1} + (12k+18)\right] \\ &\quad + x^{k+2}\left[-(2-k)a_{k+1} - 4a_k + (9k+18)\right]\end{aligned}$$

Note that $a_j = a_{k+2-j}$, and all coefficients of $g(x)$ are symmetric except for a sign flip. So we write:

$$g(x) = x\sum_{j=0}^{k+2} \beta_j x^j,$$

and its easy to see that $\beta_j = -\beta_{k+2-j}$, i.e., $g(x)/x$ is anti-palindromic. Let $z := z_{j,k}$ be the quantity inside the sum for j and $w := w_{j,k}$ for $k-j-2$. Then,

$$z = a_j(2j-2k-4) + a_{j+1}(7j-6k-2) + a_{j+2}(9j-9/2k+9) + a_{j+3}(7j-k+16) + a_{j+4}(2j+8)$$

and

$$w = a_{k-j-2}(-2j-8) + a_{k-j-1}(k-7j-16) + a_{k-j}(9/2k-9j-9) + a_{k-j+1}(6k-7j+2) + a_{k-j+2}(2k-2j+4)$$

We now notice that $z = -w$, proving anti-palindromicity and thus confirming $x = 1$ is a root.

To ensure uniqueness of the root, we show the first $\lfloor (k+2)/2 \rfloor$ coefficients have the same sign. Since we can easily deal we the terms outside of the sum, we want to prove that $z_{j,k} \geqslant 0$ for $j = 1, 2, \ldots, \lfloor (k-2)/2 \rfloor$. Direct computations show that

$$\begin{aligned} z_{j,k} &= \frac{(k+2)!(2j-k+2)(j^2+j(2-k)-2k^2-7k+3)}{2j!(k-j-2)!(j+1)(j+2)(j+3)(k-j+1)(k-j)(k-j-1)} + \frac{81}{2}(2j-k+2) \\ &= \frac{(k-2j-2)}{2}\left(\frac{(k+2)!(2k^2+7k-j^2+j(k-2)-3)}{(j+3)!(k-j+1)!} - 81\right). \end{aligned}$$

Since $k-2j-2 \geqslant 0$, it suffices to show that the second parenthesis is non-negative. Indeed, we have that for $k \geq 6$

$$2k^2 + 7k - j^2 + j(k-2) - 3 \geq \frac{7k^2}{4} + 8k - 4 > 81$$

and

$$\frac{(k+2)!}{(j+3)!(k-j+1)!} = \binom{k+2}{j+1}\frac{1}{(j+2)(j+3)} \geq \binom{k+2}{2}\frac{4}{(k+2)(k+4)} > 1.$$

This inequality holds for $k \geq 6$, and for $3 < k \leq 5$ it can be checked directly.

Finally, we verify positivity of the remaining coefficients:

$$\begin{aligned}\beta_0 &= (k-2)(k-4) \geq 0,\\ \beta_1 &= \frac{1}{2}(k-2)(k-4)(k+9) \geq 0,\\ \beta_2 &= \frac{1}{12}(k-4)(2k^3+15k^2+49k-270) \geq 0.\end{aligned}$$

□

By Descartes' rule of signs, f' has a unique root with sign pattern $+,-$ so $x=0$ is a minimum and $x=1$ a maximum. □

6. Minimum of h_{2k} under the $\|x\|_\infty$ constraint

In our setting, the sharp lower comparison between $\|A\|_{H_d}$ and the operator norm $\|A\|_{\text{op}}$ is controlled by the minimum of h_d on the ℓ_∞–sphere, since

$$\|A\|_{H_d}^d = h_d\big(s_1(A),\dots,s_n(A)\big) = \|A\|_{\text{op}}^d\, h_d\left(\frac{s_1(A)}{\|A\|_{\text{op}}},\dots,\frac{s_n(A)}{\|A\|_{\text{op}}}\right),$$

so the best constant $C_{n,d}$ in an inequality of the form

$$C_{n,d}\,\|A\|_{\text{op}} \leq \|A\|_{H_d}$$

is exactly the d-th root of $\min\{h_d(x) : \|x\|_\infty = 1\}$. In Theorem 6.1 below we show that every non-vertex local minimiser of h_{2k} on the ℓ_∞–sphere has the very rigid form $(t,\dots,t,1)$, with t determined by a one-variable polynomial equation. This reduces the optimal comparison between $\|A\|_{H_d}$ and $\|A\|_{\text{op}}$ to a one-dimensional optimisation problem and characterises the extremal matrices as those whose normalised singular-value vector has this $(t,\dots,t,1)$ structure; see Corollary 6.4.

Theorem 6.1. *Let $n \geq 2$ and $k \geq 1$ be integers, and let $S_\infty^{n-1} := \{x \in \mathbb{R}^n : \|x\|_\infty = 1\}$. The global minimum of h_{2k} on S_∞^{n-1} is always attained at a point of the form $(t,\dots,t,1)$ where t is the unique root of*

$$\varphi_{k,n}(t) := h_{2k-1}(\underbrace{t,\dots,t}_{n-1},1)$$

in $(-1,0)$. Moreover, the value of h_{2k} at this vector equals

$$\binom{n+2k-1}{2k} t^{2k}.$$

Proof. Since h_{2k} is continuous and S_∞^{n-1} is compact, h_{2k} attains a global minimum on S_∞^{n-1}. Let $x^* = (x_1^*,\dots,x_n^*) \in S_\infty^{n-1}$ be a global minimiser which is not a vertex of $\{\pm1\}^n$. Since h_{2k} is invariant under permutations of the coordinates and under the global sign change $x \mapsto -x$, we may assume without loss of generality that $x_n^* = 1$. We will first show that

$$|x_i^*| < 1,$$

for $i = 1, \ldots, n-1$. If this is not the case, then there exists some j, k, such that $|x_j^*| = 1$ and $|x_k^*| = s < 1$. Using Theorem 2.7

$$h_{2k}(x^*) > h_{2k}(y^*),$$

where y^* has exactly the same coordinates as x^*, except x_j^*, x_k^*, where y^* has their arithmetic mean. Since $y^* \in S_\infty^{n-1}$, we arrive at a contradiction. Thus x^* lies in the relative interior of the $(n-1)$–dimensional face

$$F := \{x \in [-1,1]^n : x_n = 1\}.$$

On the relative interior of F the coordinates $x_1, \ldots, x_{n-1}$ are unconstrained, so the restriction of h_{2k} to F has vanishing gradient at x^*, i.e.

$$\frac{\partial}{\partial x_i} h_{2k}(x^*) = 0,$$

for all $i = 1, \ldots, n-1$. By Lemma 2.5 we have, for each i,

$$\frac{\partial}{\partial x_i} h_{2k}(x) = h_{2k-1}(x, x_i),$$

Hence

$$h_{2k-1}(x^*, x_i^*) = 0$$

for all $i = 1, \ldots, n-1$. Fix distinct indices $i, j \in \{1, \ldots, n-1\}$. Using the difference identity (2.3) with

$$x = (x_1^*, \ldots, x_n^*), \quad a = x_i^*, \quad b = x_j^*,$$

we obtain

$$h_{2k-1}(x^*, x_i^*) - h_{2k-1}(x^*, x_j^*) = (x_i^* - x_j^*)\, h_{2k-2}(x^*, x_i^*, x_j^*).$$

The left-hand side vanishes, so

$$(x_i^* - x_j^*)\, h_{2k-2}(x^*, x_i^*, x_j^*) = 0.$$

By Hunter's positivity theorem (Theorem 2.6), the even-degree polynomial h_{2k-2} is strictly positive at every non-zero vector. The vector

$$(x^*, x_i^*, x_j^*) = (x_1^*, \ldots, x_n^*, x_i^*, x_j^*)$$

is non-zero because $x_n^* = 1$, hence $h_{2k-2}(x^*, x_i^*, x_j^*) > 0$. Therefore we must have $x_i^* = x_j^*$ for all $1 \leq i, j \leq n-1$. This shows that every non-vertex local minimiser has the form

$$x^* = (t, \ldots, t, 1)$$

for some $t \in (-1, 1)$.

To find the global minimiser we need to check the value of h_{2k} at every vertex of the cube $\{\pm 1\}^n$. However, if $(\epsilon_1, \ldots, \epsilon_{n-1}) \in \{\pm 1\}^{n-1}$, using Theorem 2.7, we obtain

$$h_{2k}(\epsilon_1, \ldots, \epsilon_{n-1}, 1) \geq h_{2k}(A, \ldots, A, 1),$$

where

$$A = \frac{\sum_{i=1}^{n-1} \epsilon_i}{n-1} \in [-1, 1],$$

therefore the minimum is not attained at a vertex, except possibly the vertices $(1, \ldots, 1)$ and $(-1, \ldots, -1, 1)$. We know that the global maximum is attained in the first one. For the latter, observe that Theorem 2.7 reassures that h_{2k} achieves a smaller value at $(1, 1, \ldots, 1, 0, 0)$. This observation completes the proof. The uniqueness of t and the exact form of the minimum will follow from the lemmata below.

Lemma 6.2. *Fix integers $n \geq 2$ and $k \geq 1$. For $t \in \mathbb{R}$ and $m \in \mathbb{N}$ set*

$$H_m^{(n)}(t) := h_m(\underbrace{t, \dots, t}_{n}, 1).$$

Then

$$H_m^{(n)}(t) = \sum_{j=0}^{m} \binom{n+j-1}{j} t^j. \tag{6.1}$$

In particular, there exists a unique $t_{n,k} \in (-1, 0)$ such that

$$H_{2k-1}^{(n)}(t_{n,k}) = h_{2k-1}(t_n[n], 1) = 0,$$

and the corresponding interior critical point for the minimization of h_{2k} on the ℓ_∞–sphere $\{a \in \mathbb{R}^n : \|a\|_\infty = 1\}$ is $a^* = (\underbrace{t_{n,k}, \dots, t_{n,k}}_{n-1}, 1)$. At this point we have the closed form

$$h_{2k}(a^*) = h_{2k}(\underbrace{t_{n,k}, \dots, t_{n,k}}_{n-1}, 1) = \binom{n+2k-1}{2k} t_{n,k}^{2k}. \tag{6.2}$$

Proof. The generating function representation

$$\sum_{m=0}^{\infty} h_m(a_1, \dots, a_{n+1}) z^m = \frac{1}{(1-a_1 z) \cdots (1-a_{n+1} z)}$$

with $a_1 = \cdots = a_n = t$ and $a_{n+1} = 1$ gives

$$\sum_{m=0}^{\infty} H_m^{(n)}(t)\, z^m = \frac{1}{(1-tz)^n (1-z)}.$$

Expanding

$$\frac{1}{(1-tz)^n} = \sum_{j=0}^{\infty} \binom{n+j-1}{j} t^j z^j, \qquad \frac{1}{1-z} = \sum_{r=0}^{\infty} z^r$$

and collecting the coefficient of z^m yields (6.1).

We next relate h_{2k} on $(t, \dots, t, 1)$ and on $(t, \dots, t, 1, 0)$. Using the difference identity $h_{m-1}(x, a) - h_{m-1}(x, b) = (a-b)\, h_{m-2}(x, a, b)$ from Lemma 2.5 with $a = t$ and $b = 0$ we obtain, for any $m \geq 1$,

$$h_m(t[n-1], 1) = h_m(t[n], 1, 0) - t\, h_{m-1}(t[n-1], 1, t, 0).$$

By symmetry of h_m this becomes

$$h_m(\underbrace{t, \dots, t}_{n-1}, 1) = h_m(\underbrace{t, \dots, t}_{n}, 1) - t\, h_{m-1}(\underbrace{t, \dots, t}_{n}, 1). \tag{6.3}$$

Let $t_{n,k} \in (-1, 0)$ be a solution of the stationarity equation

$$h_{2k-1}(t_{n,k}[n], 1) = H_{2k-1}^{(n)}(t_{n,k}) = 0.$$

(Existence and uniqueness of such a solution in $(-1,0)$ is stated in Lemma 6.3 below.) Plugging $m=2k$ and $t=t_{n,k}$ into (6.3) and using $h_{2k-1}(t_{n,k}[n],1)=0$ gives

$$h_{2k}(t_{n,k}[n-1],1) = h_{2k}(t_{n,k}[n],1) = H_{2k}^{(n)}(t_{n,k}). \tag{6.4}$$

Finally, applying (6.1) with $m=2k$ we have

$$H_{2k}^{(n)}(t) = \sum_{j=0}^{2k} \binom{n+j-1}{j} t^j = H_{2k-1}^{(n)}(t) + \binom{n+2k-1}{2k} t^{2k}.$$

Evaluating at $t=t_{n,k}$ and using $H_{2k-1}^{(n)}(t_{n,k})=0$ yields

$$H_{2k}^{(n)}(t_{n,k}) = \binom{n+2k-1}{2k} t_{n,k}^{2k}.$$

Combining this with (6.4) gives (6.2). □

Lemma 6.3. *For each $n \geq 1$ and $k \geq 1$ there exists a unique $t_{n,k} \in (-1,0)$ such that*

$$h_{2k-1}(\underbrace{t_{n,k},\dots,t_{n,k}}_{n},1) = 0.$$

The sequence $\{t_{n,k}\}_{k\geq 1}$ satisfies

$$\lim_{k\to\infty} t_{n,k} = -1.$$

Proof. Let $X_1,\dots,X_{n+1}$ be i.i.d. standard exponential random variables. By the moment representation of complete homogeneous symmetric polynomials,

$$\Phi(t) = h_{2k-1}(\underbrace{t,\dots,t}_{n},1) = \frac{1}{(2k-1)!}\,\mathbb{E}\big(t(X_1+\cdots+X_n)+X_{n+1}\big)^{2k-1}.$$

Set

$$S := X_1+\cdots+X_n, \qquad Y_t := tS + X_{n+1},$$

so that

$$\Phi(t) = \frac{1}{(2k-1)!}\,\mathbb{E}\big(Y_t^{2k-1}\big).$$

Since all moments of Y_t are finite and $t \mapsto Y_t$ is affine, we may differentiate under the expectation to obtain, for every $t \in \mathbb{R}$,

$$\Phi'(t) = \frac{1}{(2k-1)!}\,\mathbb{E}\big[(2k-1)S\,Y_t^{2k-2}\big] = \frac{1}{(2k-2)!}\,\mathbb{E}\big[S\,Y_t^{2k-2}\big].$$

Now fix $t \in (-1,0)$. Then $S>0$ almost surely, and $2k-2$ is even, so $Y_t^{2k-2} \geq 0$ almost surely. Moreover, the joint law of (S, X_{n+1}) is absolutely continuous, hence

$$\mathbb{P}\big(S=0\big) = \mathbb{P}\big(Y_t=0\big) = 0,$$

which implies

$$S\,Y_t^{2k-2} > 0 \quad \text{almost surely.}$$

Therefore

$$\Phi'(t) = \frac{1}{(2k-2)!}\,\mathbb{E}\big[S\,Y_t^{2k-2}\big] > 0 \qquad \text{for all } t \in (-1,0),$$

so Φ is strictly increasing on $(-1,0)$.

Consequently Φ can have at most one zero in $(-1,0)$. On the other hand,

$$\Phi(0) = h_{2k-1}(0,\dots,0,1) = 1 > 0,$$

and a direct computation from the explicit formula for $h_{2k-1}(t,\dots,t,1)$ shows that $\Phi(-1) < 0$. By the intermediate value theorem there exists $t_* \in (-1,0)$ with $\Phi(t_*) = 0$, and by strict monotonicity this zero is unique. Since $\Phi'(t_*) > 0$, the root is simple.

To understand the asymptotic behaviour, note that for $|\rho| < 1$

$$(1+\rho)^{-n} = \sum_{j=0}^{\infty} (-1)^j \binom{n+j-1}{j} \rho^j.$$

Thus, for $0 < \rho < 1$ we may write

$$(1+\rho)^{-n} = P_{n,k}(\rho) + R_{n,k}(\rho), \qquad R_{n,k}(\rho) := \sum_{j=2k}^{\infty} (-1)^j \binom{n+j-1}{j} \rho^j.$$

At the root $\rho = \rho_{n,k} = -t_{n,k}$ we have $P_{n,k}(\rho_{n,k}) = 0$, and hence

$$(1+\rho_{n,k})^{-n} = R_{n,k}(\rho_{n,k}). \tag{6.5}$$

Fix $\varepsilon \in (0,1)$ and set $q := 1-\varepsilon$. For $0 < \rho \le q$ and all $j \ge 2k$ we use the crude bound $\binom{n+j-1}{j} \le C_n j^{n-1}$ (for some constant C_n depending only on n) to obtain

$$|R_{n,k}(\rho)| \le \sum_{j=2k}^{\infty} \binom{n+j-1}{j} \rho^j \le C_n \sum_{j=2k}^{\infty} j^{n-1} q^j \le C'_n (2k)^{n-1} q^{2k},$$

for a suitable constant C'_n. The right-hand side tends to 0 as $k \to \infty$, uniformly in $0 < \rho \le q$. On the other hand $(1+\rho)^{-n} \ge (1+q)^{-n} > 0$ for all such ρ. Therefore, for k sufficiently large the identity (6.5) cannot hold with $\rho_{n,k} \le q$. Since $q < 1$ was arbitrary, it follows that $\rho_{n,k} \to 1$ as $k \to \infty$, and hence $t_{n,k} = -\rho_{n,k} \to -1$. □

□

Corollary 6.4. *Let $d = 2k$ be an even integer, and let $A \in M_n(\mathbb{C})$ have singular values $s_1(A) \ge \cdots \ge s_n(A) \ge 0$. Define the norm induced by the complete homogeneous symmetric polynomial h_d by*

$$\|A\|_{H_d} := h_d\big(s_1(A),\dots,s_n(A)\big)^{1/d}.$$

Then,

$$\binom{n+2k-1}{2k}^{1/2k} |t_{n,k}|\, \|A\|_{\mathrm{op}} \;\le\; \|A\|_{H_{2k}} \;\le\; \binom{n+2k-1}{2k}^{1/2k} \|A\|_{\mathrm{op}},$$

where $t_{n,k}$ is the unique number in $(-1,0)$ such that

$$h_{2k-1}(\underbrace{t_{n,k},\dots,t_{n,k}}_{n},1) = 0,$$

and the bound is optimal. Equality in the lower bound can occur only if the normalised singular-value vector $s(A)/\|A\|_{\mathrm{op}}$ is a minimiser of h_d on S_∞^{n-1}.

Proof. Let $A \in M_n(\mathbb{C})$ and put $x = s(A)/\|A\|_{\mathrm{op}}$. Then $\|x\|_\infty = 1$ and, by definition,

$$\|A\|_{H_d}^d = h_d\big(s_1(A), \ldots, s_n(A)\big) = \|A\|_{\mathrm{op}}^d \, h_d(x).$$

Taking the minimum and maximum of $h_d(x)$ over $\{x \in \mathbb{R}^n : \|x\|_\infty = 1\}$ yields the stated inequality. Optimality is immediate by evaluating at matrices whose normalised singular-value vector is a minimiser of h_d on S_∞^{n-1}; the structural description of such minimisers is given by Theorem 6.1. □

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Department of Mathematics & Applied Mathematics, University of Crete, Voutes Campus, 70013 Heraklion, Greece

Email address: silouanb@uoc.gr, chrpandis@gmail.com