

Continuous version of the approximate geometric Brascamp-Lieb inequalities

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Abstract

Given $\gamma > 1$ we say that a Borel measure ν on S^{n-1} is a γ -approximation of an isotropic measure if

$$I_n \preceq T_\nu = \int_{S^{n-1}} u \otimes u \, d\nu(u) \preceq \gamma I_n,$$

where I_n is the identity matrix. We provide a generalization of Barthe's continuous version of the Brascamp-Lieb inequalities to the context of these approximate isotropic measures, and we apply these inequalities to obtain stability results for some classical positions of convex bodies.

1 Introduction

This article is a continuation of the recent work [11] of the first named author. Our purpose is to extend Barthe's continuous version of the Brascamp-Lieb inequalities to the setting of approximately isotropic Borel measures on the sphere. Recall that a Borel measure ν on S^{n-1} is called isotropic if

$$(1.1) \quad I_n = \int_{S^{n-1}} u \otimes u \, d\nu(u),$$

where I_n is the identity matrix and, more generally, $(u \otimes v)(y) = \langle v, y \rangle u$. Barthe's theorem is a pair of inequalities for a family of functions (f_u) , $u \in S^{n-1}$, that satisfy some mild continuity conditions; namely,

- There exist a continuous function $F : S^{n-1} \times \mathbb{R} \rightarrow (0, +\infty)$ and two functions a, b on S^{n-1} with $a < b$ (a, b are either real-valued continuous or constant with value $\pm\infty$) such that for all $(u, t) \in S^{n-1} \times \mathbb{R}$

$$f_u(t) = \mathbf{1}_{a(u) \leq t \leq b(u)} F(u, t).$$

- There exists a function $U \in L_1(\mathbb{R}) \cap L_\infty(\mathbb{R})$ such that $0 \leq f_u \leq U$ for all $u \in S^{n-1}$.

Then, we say that (f_u) satisfies condition (H).

Theorem 1.1 (Barthe). *Let ν be an isotropic Borel measure on S^{n-1} and let (f_u) , $u \in S^{n-1}$ be a family of functions $f_u : \mathbb{R} \rightarrow [0, +\infty)$ that satisfy condition (H). Then,*

$$(1.2) \quad \int_{\mathbb{R}^n} \exp \left(\int_{S^{n-1}} \log f_u(\langle x, u \rangle) d\nu(u) \right) dx \leq \exp \left(\int_{S^{n-1}} \log \left(\int_{\mathbb{R}} f_u \right) d\nu(u) \right).$$

Also, if h is a measurable function such that

$$(1.3) \quad h \left(\int_{S^{n-1}} \theta(u) u \, d\nu(u) \right) \geq \exp \left(\int_{S^{n-1}} \log f_u(\theta(u)) \, d\nu(u) \right)$$

for every integrable function θ , then

$$(1.4) \quad \int_{\mathbb{R}^n} h(x) \, dx \geq \exp \left(\int_{S^{n-1}} \log \left(\int_{\mathbb{R}} f_u \right) d\nu(u) \right).$$

The discrete analogues of the two statements in Theorem 1.1, Ball's version of the Brascamp-Lieb inequality and Barthe's reverse Brascamp-Lieb inequality, have been the key to a number of sharp geometric inequalities and play an important role in convex geometric analysis. Recall that if $m \geq n$, $u_1, \dots, u_m \in \mathbb{R}^n$ and $c_1, \dots, c_m > 0$ with $c_1 + \dots + c_m = n$, then the Brascamp-Lieb inequality [10], as reformulated by Ball (see e.g. [2]), states that

$$(1.5) \quad G(f_1, \dots, f_m) := \int_{\mathbb{R}^n} \prod_{j=1}^m f_j^{c_j}(\langle x, u_j \rangle) dx \leq \frac{1}{\sqrt{D}} \prod_{j=1}^m \left(\int_{\mathbb{R}} f_j \right)^{c_j}$$

for all integrable functions $f_j : \mathbb{R} \rightarrow [0, \infty)$, where

$$(1.6) \quad D = \inf \left\{ \frac{\det \left(\sum_{j=1}^m c_j \lambda_j u_j \otimes u_j \right)}{\prod_{j=1}^m \lambda_j^{c_j}} : \lambda_j > 0 \right\}.$$

A reverse form of (1.5) was proved by Barthe in [6] (see also [7] for a multidimensional extension). Under the same assumptions on the data $\{u_j, c_j\}_{j \leq m}$, we have

$$(1.7) \quad K(h_1, \dots, h_m) := \int_{\mathbb{R}^n}^* \sup \left\{ \prod_{j=1}^m h_j^{c_j}(\theta_j) : \theta_j \in \mathbb{R}, x = \sum_{j=1}^m \theta_j c_j u_j \right\} dx \geq \sqrt{D} \prod_{j=1}^m \left(\int_{\mathbb{R}} h_j \right)^{c_j}$$

for all integrable functions $h_1, \dots, h_m : \mathbb{R} \rightarrow [0, \infty)$, where \int^* denotes the outer integral.

In order to apply (1.5) and (1.7) for a given set of vectors u_j and weights c_j , one has to compute the constant $D = D(\{u_j, c_j\}_{1 \leq j \leq m})$; this is not a simple problem. Ball's crucial observation is that if $u_1, \dots, u_m \in S^{n-1}$ and $c_1, \dots, c_m > 0$ satisfy the condition

$$(1.8) \quad I_n = \sum_{j=1}^m c_j u_j \otimes u_j,$$

then

$$(1.9) \quad D = D(\{u_j, c_j\}_{1 \leq j \leq m}) = 1.$$

Ball first used this fact in [2] to obtain estimates on the volume of sections and projections of the unit cube. A well-known decomposition of the form (1.8) appears in John's theorem regarding a convex body whose maximal volume ellipsoid is the Euclidean unit ball (see Section 2). The first and well-known application of (1.5) in this context is Ball's sharp reverse isoperimetric inequality in [3]. Subsequently, both (1.5) and (1.7) have been systematically used for the proof of several other sharp inequalities in convex geometric analysis (see [4] for references and the history of these ideas).

Note that $u_j \in S^{n-1}$ and $c_j > 0$ satisfy (1.8) if and only if the measure ν with $\text{supp}(\nu) = \{u_1, \dots, u_m\}$ and $\nu(\{u_j\}) = c_j$, $j = 1, \dots, m$ is an isotropic measure on S^{n-1} . It was later understood (and this point of view was put forward in [12]) that various classical positions of convex bodies are also characterized by the fact that an appropriate Borel measure on the sphere is isotropic (see Section 2 for background information). For example:

- A convex body K has minimal surface area among all its affine images of the same volume if and only if the surface area measure σ_K of K is an isotropic measure on S^{n-1} .
- A convex body K has minimal mean width among all its affine images of the same volume if and only if the measure ν_K with density h_K with respect to σ is an isotropic measure on S^{n-1} .

Barthe's theorem (Theorem 1.1) provides a continuous version (and extension) of (1.5) and (1.7) which potentially allows one to obtain geometric applications in situations where a non-discrete isotropic measure appears. This is certainly the case in the two examples above (see also [16] and [17] for a sample of applications).

We present a variant of Theorem 1.1 for a Borel measure ν on S^{n-1} which is approximately isotropic. For any non-negative and finite Borel measure ν on S^{n-1} we define the symmetric positive semi-definite $n \times n$ matrix

$$(1.10) \quad T_\nu = \int_{S^{n-1}} u \otimes u \, d\nu(u).$$

We say that ν is a γ -approximation of an isotropic measure (for some $\gamma > 1$) if

$$(1.11) \quad I_n \preceq T_\nu = \int_{S^{n-1}} u \otimes u \, d\nu(u) \preceq \gamma I_n.$$

First, we obtain a generalization of the next result of Lutwak, Yang and Zhang [15] which is the continuous version of Ball's observation (1.9): if ν is an isotropic measure on S^{n-1} and $t : \text{supp}(\nu) \rightarrow (0, \infty)$ is continuous, then

$$(1.12) \quad \det \left(\int_{S^{n-1}} t(u) u \otimes u \, d\nu(u) \right) \geq \exp \left[\int_{S^{n-1}} \log t(u) \, d\nu(u) \right].$$

This fact is a basic ingredient in Barthe's proof of Theorem 1.1. In the case of an approximate isotropic measure it takes the following form:

Theorem 1.2. *Let ν be a γ -approximation of an isotropic measure on S^{n-1} . For every continuous function $t : \text{supp}(\nu) \rightarrow (0, \infty)$ one has*

$$(1.13) \quad \begin{aligned} \gamma^n \det \left(\int_{S^{n-1}} t(u) \langle T_\nu^{-1} u, u \rangle u \otimes u \, d\nu(u) \right) &\geq \det \left(\int_{S^{n-1}} t(u) u \otimes u \, d\nu(u) \right) \\ &\geq \exp \left[\int_{S^{n-1}} \log t(u) \langle T_\nu^{-1} u, u \rangle \, d\nu(u) \right]. \end{aligned}$$

Then, following Barthe's argument from [8] we obtain a continuous Brascamp-Lieb inequality and its reverse form for a γ -approximation of an isotropic measure.

Theorem 1.3. *Let ν be a γ -approximation of an isotropic Borel measure in \mathbb{R}^n and let (f_u) , $u \in S^{n-1}$ be a family of functions $f_u : \mathbb{R} \rightarrow [0, +\infty)$ that satisfy (H). Then,*

$$(1.14) \quad \int_{\mathbb{R}^n} \exp \left(\int_{S^{n-1}} \log f_u(\langle x, u \rangle) \langle T_\nu^{-1} u, u \rangle \, d\nu(u) \right) dx \leq \gamma^{\frac{n}{2}} \exp \left(\int_{S^{n-1}} \log \left(\int_{\mathbb{R}} f_u \right) \langle T_\nu^{-1} u, u \rangle \, d\nu(u) \right).$$

Also, if h is a measurable function such that

$$(1.15) \quad h \left(\int_{S^{n-1}} \theta(u) u \, d\nu(u) \right) \geq \exp \left(\int_{S^{n-1}} \log f_u(\theta(u)) \langle T_\nu^{-1} u, u \rangle \, d\nu(u) \right)$$

for every integrable function θ , then

$$(1.16) \quad \gamma^{\frac{n}{2}} \int_{\mathbb{R}^n} h(y) \, dy \geq \exp \left(\int_{S^{n-1}} \log \left(\int_{\mathbb{R}} f_u \right) \langle T_\nu^{-1} u, u \rangle \, d\nu(u) \right).$$

The discrete analogues of Theorem 1.2 and Theorem 1.3 were recently obtained in [11] where it was shown that if an approximate John's decomposition is available by a set of $u_j \in S^{n-1}$ and $c_j > 0$ then we still have a Brascamp-Lieb inequality with a reasonable constant $D(\{u_j, c_j\}_{1 \leq j \leq m})$. We give the precise statements in Section 2.

In the last section of this article we apply Theorem 1.3 to obtain stability results for some classical positions of convex bodies. The main idea is to use the next fact.

Theorem 1.4. *Let ν be a finite Borel measure on S^{n-1} . Let $C(\nu)$ be the symmetric convex body whose support function is the cosine transform of ν*

$$(1.17) \quad C_\nu(x) = \int_{S^{n-1}} |\langle x, u \rangle| d\nu(u), \quad x \in \mathbb{R}^n,$$

and write $C^*(\nu)$ for the polar body of ν . Then,

$$(1.18) \quad \nu(S^{n-1})|C^*(\nu)|^{1/n} \geq \frac{n\omega_n^{\frac{n+1}{n}}}{2\omega_{n-1}} \geq c,$$

where $c > 0$ is an absolute constant. Conversely, if ν is a γ -approximation of an isotropic measure on S^{n-1} for some $\gamma > 1$, then

$$(1.19) \quad \nu(S^{n-1})|C^*(\nu)|^{1/n} \leq 2e\gamma^{\frac{3}{2}}.$$

The isotropic analogue of Theorem 1.4 has first appeared in [13], in the particular case where $\nu = \sigma_K$ is the area measure of a convex body K . Theorem 1.4 allows us to use it for approximate isotropic measures; as an example of application we show that if the area measure of a convex body K is almost isotropic then K has almost minimal surface area (this fact appears in [13] but we feel that the present argument is more transparent and may be the model for new applications of this type).

2 Notation and background

We work in \mathbb{R}^n , which is equipped with a Euclidean structure $\langle \cdot, \cdot \rangle$. We denote by $\|\cdot\|_2$ the corresponding Euclidean norm, and write B_2^n for the Euclidean unit ball and S^{n-1} for the unit sphere. Volume is denoted by $|\cdot|$. We write ω_n for the volume of B_2^n and σ for the rotationally invariant probability measure on S^{n-1} . We will denote by P_F the orthogonal projection from \mathbb{R}^n onto F . We also define $B_F = B_2^n \cap F$ and $S_F = S^{n-1} \cap F$.

The letters c, c', c_1, c_2 etc. denote absolute positive constants which may change from line to line. Whenever we write $a \simeq b$, we mean that there exist absolute constants $c_1, c_2 > 0$ such that $c_1 a \leq b \leq c_2 a$. Also, if $K, L \subseteq \mathbb{R}^n$ we will write $K \simeq L$ if there exist absolute constants $c_1, c_2 > 0$ such that $c_1 K \subseteq L \subseteq c_2 K$.

We refer to the book of Schneider [19] for basic facts from the Brunn-Minkowski theory and to the book of Artstein-Avidan, Giannopoulos and V. Milman [1] for basic facts from asymptotic convex geometry.

2.1. Convex bodies

A convex body in \mathbb{R}^n is a compact convex subset K of \mathbb{R}^n with non-empty interior. We say that K is symmetric if $x \in K$ implies that $-x \in K$, and that K is centered if its barycenter

$$(2.1) \quad \text{bar}(K) = \frac{1}{|K|} \int_K x dx$$

is at the origin. If $0 \in \text{int}(K)$ then the polar body K° of K is defined by

$$(2.2) \quad K^\circ := \{y \in \mathbb{R}^n : \langle x, y \rangle \leq 1 \text{ for all } x \in K\}.$$

The Blaschke-Santaló inequality states that for every centered convex body K in \mathbb{R}^n one has $|K||K^\circ| \leq \omega_n^2$, with equality if and only if K is an ellipsoid. The reverse Santaló inequality of Bourgain and V. Milman [9] states that there exists an absolute constant $c > 0$ such that

$$(2.3) \quad (|K||K^\circ|)^{1/n} \geq c/n,$$

where $c > 0$ is an absolute constant, for every convex body K in \mathbb{R}^n which contains 0 in its interior.

The support function of a convex body K in \mathbb{R}^n is defined by

$$(2.4) \quad h_K(x) = \max\{\langle x, y \rangle : y \in K\}.$$

The mean width of K is the quantity

$$(2.5) \quad w(K) = \int_{S^{n-1}} h_K(x) d\sigma(x).$$

We say that K has *minimal mean width* if $w(K) \leq w(TK)$ for every $T \in SL(n)$. It was proved in [12] that a smooth enough convex body K in \mathbb{R}^n has minimal mean width if and only if

$$(2.6) \quad \int_{S^{n-1}} h_K(x) \langle x, \theta \rangle^2 d\sigma(x) = \frac{w(K)}{n}$$

for every $\theta \in S^{n-1}$. Equivalently, if the measure ν_K on S^{n-1} with density h_K with respect to σ is a scalar multiple of an isotropic measure. Moreover, this minimal mean width position is unique up to orthogonal transformations.

The area measure σ_K of a convex body K in \mathbb{R}^n is defined on S^{n-1} and corresponds to the usual surface measure on K via the Gauss map: for every Borel $A \subseteq S^{n-1}$, we set

$$(2.7) \quad \sigma_K(A) = \lambda(\{x \in \text{bd}(K) : \text{the outer normal to } K \text{ at } x \text{ is in } A\}),$$

where λ is the $(n-1)$ -dimensional surface measure on K . The surface area $\partial(K)$ of K is obviously equal to $\sigma_K(S^{n-1})$. We say that K has minimal surface area if $\partial(K) \leq \partial(TK)$ for every $T \in SL(n)$. A characterization of the minimal surface area position through the area measure was given by Petty in [18] (see also [13]): a convex body K has minimal surface area if and only if σ_K is a scalar multiple of an isotropic measure. Moreover, this minimal surface area position is unique up to orthogonal transformations. The projection body ΠK of K is the symmetric convex body whose support function is defined by $h_{\Pi K}(\theta) = |P_{\theta^\perp}(K)|$, $\theta \in S^{n-1}$. We write $\Pi^* K$ for its polar body (the polar projection body of K).

A convex body K of volume 1 in \mathbb{R}^n is called isotropic if it is centered and there exists a constant $L_K > 0$ such that

$$(2.8) \quad \int_K \langle x, \theta \rangle^2 dx = L_K^2$$

for all $\theta \in S^{n-1}$. Moreover, the isotropic position is unique up to orthogonal transformations. Writing (2.8) in polar coordinates we get

$$(2.9) \quad \frac{n\omega_n}{n+2} \int_{S^{n-1}} \langle u, \theta \rangle^2 \rho_K^{n+2}(u) d\sigma(u) = L_K^2$$

for all $\theta \in S^{n-1}$, where $\rho_K(u) = \max\{t > 0 : tu \in K\}$ is the radial function of K . Therefore, K is isotropic if and only if the measure λ_K on S^{n-1} with density ρ_K^{n+2} with respect to σ is a scalar multiple of an isotropic measure.

We say that a convex body K is in John's position if the ellipsoid of maximal volume inscribed in K is the Euclidean unit ball B_2^n . John's theorem [14] states that K is in John's position if and only if $B_2^n \subseteq K$ and there exist $u_1, \dots, u_m \in \text{bd}(K) \cap S^{n-1}$ (contact points of K and B_2^n) and positive real numbers c_1, \dots, c_m such that

$$(2.10) \quad \sum_{j=1}^m c_j u_j = 0$$

and the identity operator I_n is decomposed in the form

$$(2.11) \quad I_n = \sum_{j=1}^m c_j u_j \otimes u_j.$$

2.2. Discrete approximate decompositions of the identity

It was mentioned in the introduction that if $u_1, \dots, u_m \in S^{n-1}$ and $c_1, \dots, c_m > 0$ satisfy (2.10) then $D(\{u_j, c_j\}_{1 \leq j \leq m}) = 1$. The next result from [11] provides a substitute of this fact in the case of an approximate John's decomposition.

Theorem 2.1 (Brazitikos). *Let $\gamma > 1$. If $u_1, \dots, u_m \in S^{n-1}$ and $c_1, \dots, c_m > 0$ satisfy*

$$(2.12) \quad I_n \preceq A := \sum_{j=1}^m c_j u_j \otimes u_j \preceq \gamma I_n$$

then

$$(2.13) \quad \gamma^n \det \left(\sum_{j=1}^m \kappa_j \lambda_j u_j \otimes u_j \right) \geq \det \left(\sum_{j=1}^m c_j \lambda_j u_j \otimes u_j \right) \geq \prod_{j=1}^m \lambda_j^{\kappa_j}$$

for all $\lambda_1, \dots, \lambda_m > 0$, where $\kappa_j = c_j \langle A^{-1} u_j, u_j \rangle > 0$, $1 \leq j \leq m$.

As a direct consequence one has the next approximate geometric Brascamp-Lieb inequality and its reverse counterpart.

Theorem 2.2 (Brazitikos). *Let $\gamma > 1$. Assume that $u_1, \dots, u_m \in S^{n-1}$ and $c_1, \dots, c_m > 0$ satisfy (2.6) and set $\kappa_j = c_j \langle A^{-1} u_j, u_j \rangle > 0$, $1 \leq j \leq m$. If $f_1, \dots, f_m : \mathbb{R} \rightarrow [0, +\infty)$ are integrable functions then*

$$(2.14) \quad \int_{\mathbb{R}^n} \prod_{j=1}^m f_j^{\kappa_j}(\langle x, u_j \rangle) dx \leq \gamma^{\frac{n}{2}} \prod_{j=1}^m \left(\int_{\mathbb{R}} f_j(t) dt \right)^{\kappa_j}.$$

Also, if $w, h_1, \dots, h_m : \mathbb{R} \rightarrow [0, \infty)$ are integrable functions and $w(x) \geq \sup \{ \prod_{j=1}^m h_j^{\kappa_j}(\theta_j) : \theta_j \in \mathbb{R}, x = \sum_{j=1}^m \theta_j c_j u_j \}$, then

$$(2.15) \quad \int_{\mathbb{R}^n} w(x) dx \geq \gamma^{-\frac{n}{2}} \prod_{j=1}^m \left(\int_{\mathbb{R}} h_j(t) dt \right)^{\kappa_j}.$$

2.3. Isotropic measures

A Borel measure ν on S^{n-1} is called isotropic if

$$(2.16) \quad I_n = \int_{S^{n-1}} u \otimes u d\nu(u).$$

Note that $u_j \in S^{n-1}$ and $c_j > 0$ satisfy (2.11) if and only if the measure ν with $\text{supp}(\nu) = \{u_1, \dots, u_m\}$ and $\nu(\{u_j\}) = c_j$, $j = 1, \dots, m$ is an isotropic measure on S^{n-1} .

For any non-negative and finite Borel measure ν on S^{n-1} we define the symmetric positive semi-definite $n \times n$ matrix

$$(2.17) \quad T_\nu = \int_{S^{n-1}} u \otimes u d\nu(u).$$

Equivalently, we have

$$(2.18) \quad \langle T_\nu x, x \rangle = \int_{S^{n-1}} \langle x, u \rangle^2 d\nu(u)$$

for all $x \in \mathbb{R}^n$. Note that

$$(2.19) \quad \text{tr}(T_\nu) = \nu(S^{n-1}).$$

In particular, for any John's decomposition (2.11) we have that

$$(2.20) \quad \sum_{j=1}^m c_j = \text{tr}(I_n) = n \quad \text{and} \quad \sum_{j=1}^m c_j \langle u_j, z \rangle^2 = 1$$

for any $z \in S^{n-1}$.

The cosine transform of a finite Borel measure ν on S^{n-1} is defined by

$$(2.21) \quad C_\nu(x) = \int_{S^{n-1}} |\langle x, u \rangle| d\nu(u), \quad x \in \mathbb{R}^n.$$

If ν is not concentrated on a great subsphere then C_ν is the support function of a symmetric convex body in \mathbb{R}^n , which we denote by $C(\nu)$. We also write $C^*(\nu)$ for the polar body of $C(\nu)$. For example, if K is a convex body in \mathbb{R}^n then

$$(2.22) \quad C_{\sigma_K}(\theta) = \int_{S^{n-1}} |\langle \theta, u \rangle| d\sigma_K(u) = 2|P_{\theta^\perp}(K)|$$

for every $\theta \in S^{n-1}$, and hence $C(\sigma_K) = 2\Pi K$.

2.4. Mixed discriminants

For the proof of Theorem 1.2 we shall modify the argument that Lutwak, Yang and Zhang used in [15] for the isotropic case, which employs some basic properties of mixed discriminants. Recall that if T_1, \dots, T_m are positive semi-definite $n \times n$ matrices then the determinant of $a_1 T_1 + \dots + a_m T_m$ can be expanded as a homogeneous polynomial of degree n in $a_1, \dots, a_m \geq 0$; one has

$$(2.23) \quad \det(a_1 T_1 + \dots + a_m T_m) = \sum_{1 \leq i_1, \dots, i_n \leq m} D(T_{i_1}, \dots, T_{i_n}) a_{i_1} \dots a_{i_n},$$

where the coefficient $D(T_{i_1}, \dots, T_{i_n})$ depends only on i_1, \dots, i_n and is invariant under permutations of the i_j 's. This coefficient is the mixed discriminant of T_{i_1}, \dots, T_{i_n} .

We shall use a number of properties of mixed discriminants (see [5] for a proof).

Lemma 2.3. *If S, T, T_i, T'_i are positive semi-definite $n \times n$ matrices, then:*

- (i) $D(T_1, \dots, T_n) \geq 0$.
- (ii) $D(T, T, \dots, T) = \det(T)$. In particular, $D(I_n, \dots, I_n) = 1$.
- (iii) $nD(T, I_n, \dots, I_n) = \text{tr}(T)$.
- (iv) $D(aT_1 + bT'_1, T_2, \dots, T_n) = aD(T_1, T_2, \dots, T_n) + bD(T'_1, T_2, \dots, T_n)$ for all $a, b \geq 0$.
- (v) $D(T_1 S, T_2 S, \dots, T_n S) = |\det(S)| D(T_1, \dots, T_n)$ and $D(ST_1, ST_2, \dots, ST_n) = |\det(S)| D(T_1, \dots, T_n)$.

We will be interested in n -tuples $(T_{\mu_1}, \dots, T_{\mu_n})$ where μ_1, \dots, μ_n are non-negative finite Borel measures on S^{n-1} . We will use the fact that

$$(2.24) \quad D(T_{\mu_1}, \dots, T_{\mu_n}) = \frac{1}{n!} \int_{S^{n-1}} \dots \int_{S^{n-1}} [u_1, \dots, u_n]^2 d\mu_1(u_1) \dots d\mu_n(u_n),$$

where $[u_1, \dots, u_n]$ denotes the volume of parallelotope defined by u_1, \dots, u_n (see [15] for a proof).

Note also that if $u \in S^{n-1}$ and δ_u is the probability measure supported by $\{u\}$, then

$$(2.25) \quad \frac{1}{n} = \frac{\text{tr}(T_{\delta_u})}{n} = D(T_{\delta_u}, I_n, \dots, I_n).$$

3 Proof of the main results

We start with the proof of Theorem 1.2. Let ν be a Borel measure on S^{n-1} which is a γ -approximation of an isotropic measure; in other words,

$$(3.1) \quad I_n \preceq T_\nu = \int_{S^{n-1}} u \otimes u \, d\nu(u) \preceq \gamma I_n$$

for some $\gamma > 1$. Given a continuous function $t : \text{supp}(\nu) \rightarrow (0, \infty)$ we will show that

$$(3.2) \quad \begin{aligned} \gamma^n \det \left(\int_{S^{n-1}} t(u) \langle T_\nu^{-1} u, u \rangle u \otimes u \, d\nu(u) \right) &\geq \det \left(\int_{S^{n-1}} t(u) u \otimes u \, d\nu(u) \right) \\ &\geq \exp \left[\int_{S^{n-1}} \log t(u) \langle T_\nu^{-1} u, u \rangle \, d\nu(u) \right]. \end{aligned}$$

Proof of (3.2). Applying (2.24) for the measures $\mu_1 = \dots = \mu_n = \mu$ where $d\mu = t \, d\nu$, we get

$$(3.3) \quad \begin{aligned} \det \left(\int_{S^{n-1}} t(u) u \otimes u \, d\nu(u) \right) &= \det(T_\mu) = D(T_\mu, \dots, T_\mu) \\ &= \frac{1}{n!} \int_{S^{n-1}} \dots \int_{S^{n-1}} t(u_1) \dots t(u_n) [u_1, \dots, u_n]^2 \, d\nu(u_1) \dots d\nu(u_n) \\ &= \det(T_\nu) \frac{1}{n! \det(T_\nu)} \int_{S^{n-1}} \dots \int_{S^{n-1}} t(u_1) \dots t(u_n) [u_1, \dots, u_n]^2 \, d\nu(u_1) \dots d\nu(u_n). \end{aligned}$$

Replacing t by the constant function $\mathbf{1}$ in the equation above, we see that

$$(3.4) \quad \frac{1}{n! \det(T_\nu)} \int_{S^{n-1}} \dots \int_{S^{n-1}} [u_1, \dots, u_n]^2 \, d\nu(u_1) \dots d\nu(u_n) = 1.$$

Then, from Jensen's inequality we see that

$$(3.5) \quad \begin{aligned} \det \left(\int_{S^{n-1}} t(u) u \otimes u \, d\nu(u) \right) &\geq \det(T_\nu) \exp \left[\frac{1}{n! \det(T_\nu)} \int_{S^{n-1}} \dots \int_{S^{n-1}} \log(t(u_1) \dots t(u_n)) [u_1, \dots, u_n]^2 \, d\nu(u_1) \dots d\nu(u_n) \right] \\ &= \det(T_\nu) \exp \left[\frac{1}{n! \det(T_\nu)} \sum_{j=1}^n \int_{S^{n-1}} \dots \int_{S^{n-1}} \log t(u_j) [u_1, \dots, u_n]^2 \, d\nu(u_1) \dots d\nu(u_n) \right]. \end{aligned}$$

Note that for every $u \in S^{n-1}$ we also have that

$$(3.6) \quad \begin{aligned} \int_{S^{n-1}} \dots \int_{S^{n-1}} [u, u_2, \dots, u_n]^2 \, d\nu(u_2) \dots d\nu(u_n) &= n! D(T_{\delta_u}, T_\nu, \dots, T_\nu) \\ &= n! \det(T_\nu) D(T_\nu^{-1} T_{\delta_u}, I_n, \dots, I_n) \\ &= (n-1)! \det(T_\nu) \text{tr}(T_\nu^{-1} T_{\delta_u}) \\ &= (n-1)! \det(T_\nu) \langle T_\nu^{-1} u, u \rangle. \end{aligned}$$

It follows that

$$(3.7) \quad \begin{aligned} \int_{S^{n-1}} \dots \int_{S^{n-1}} \log t(u_1) [u_1, \dots, u_n]^2 \, d\nu(u_1) \dots d\nu(u_n) \\ = (n-1)! \det(T_\nu) \int_{S^{n-1}} \log t(u) \langle T_\nu^{-1} u, u \rangle \, d\nu(u), \end{aligned}$$

and then, (3.3) and (3.5) give:

$$(3.8) \quad \det \left(\int_{S^{n-1}} t(u) u \otimes u \, d\nu(u) \right) \geq \det(T_\nu) \exp \left[\int_{S^{n-1}} \log t(u) \langle T_\nu^{-1} u, u \rangle \, d\nu(u) \right].$$

Finally, using the fact that $\gamma \langle T_\nu^{-1} u, u \rangle \geq 1$ we see that

$$(3.9) \quad \gamma^n \det \left(\int_{S^{n-1}} t(u) \langle T_\nu^{-1} u, u \rangle u \otimes u \, d\nu(u) \right) \geq \det \left(\int_{S^{n-1}} t(u) u \otimes u \, d\nu(u) \right),$$

which completes the proof. \square

Remark 3.1. Let $u_1, \dots, u_m \in S^{n-1}$ and $c_1, \dots, c_m > 0$ satisfy

$$(3.10) \quad I_n \preceq A := \sum_{j=1}^m c_j u_j \otimes u_j \preceq \gamma I_n$$

for some $\gamma > 1$. Applying (3.2) to the discrete measure ν with $\nu(\{u_j\}) = c_j$ and to the function $t : \{u_1, \dots, u_m\} \rightarrow (0, \infty)$ with $t(u_j) = \lambda_j$, we get

$$(3.11) \quad \gamma^n \det \left(\sum_{j=1}^m c_j \lambda_j \langle A^{-1} u_j, u_j \rangle u_j \otimes u_j \right) \geq \exp \left[\sum_{j=1}^m \log(\lambda_j) c_j \langle A^{-1} u_j, u_j \rangle \right] \\ = \prod_{j=1}^m \lambda_j^{c_j \langle A^{-1} u_j, u_j \rangle}$$

for all $\lambda_1, \dots, \lambda_m > 0$. This shows that Theorem 1.2 generalizes Theorem 2.1.

We pass to the proof of Theorem 1.3. In the next lemma, which is essentially the main lemma in [8], $(f_u), (g_u), u \in S^{n-1}$ are two families of functions $f_u, g_u : \mathbb{R} \rightarrow [0, +\infty)$ that satisfy condition (H):

- There exist two continuous functions $F, G : S^{n-1} \times \mathbb{R} \rightarrow (0, +\infty)$ and functions a, b, c, d on S^{n-1} with $a < b$ and $c < d$ (a, b, c, d are either real-valued continuous or constant with value $\pm\infty$) such that for all $(u, t) \in S^{n-1} \times \mathbb{R}$

$$(3.12) \quad f_u(t) = \mathbf{1}_{a(u) \leq t \leq b(u)} F(u, t) \quad \text{and} \quad g_u(t) = \mathbf{1}_{c(u) \leq t \leq d(u)} G(u, t).$$

- There exist two functions $U, V \in L_1(\mathbb{R}) \cap L_\infty(\mathbb{R})$ such that $0 \leq f_u \leq U$ and $0 \leq g_u \leq V$ for all $u \in S^{n-1}$.

Lemma 3.2. Let $\gamma > 1$ and let ν be a γ -approximation of an isotropic measure on S^{n-1} . Let μ be the measure on S^{n-1} with $d\mu(u) = \langle T_\nu^{-1} u, u \rangle d\nu(u)$. If $(f_u), (g_u), u \in S^{n-1}$ are two families of functions that satisfy (H) then

$$(3.13) \quad \exp \left(\int_{S^{n-1}} \log \left(\int_{\mathbb{R}} g_u \right) d\mu(u) \right) \int_{\mathbb{R}^n} \exp \left(\int_{S^{n-1}} \log f_u(\langle x, u \rangle) d\mu(u) \right) dx \\ \leq \exp \left(\int_{S^{n-1}} \log \left(\int_{\mathbb{R}} f_u \right) d\mu(u) \right) \int_{\mathbb{R}^n}^* \sup_{y=f \theta(u) u d\nu(u)} \exp \left(\int_{S^{n-1}} \log g_u(\theta(u)) d\mu(u) \right) dy.$$

Proof. We sketch the proof, following Barthe's argument, just in order to make the necessary modifications. We assume that the left hand side of (3.13) is positive and note that $\exp \left(\int_{S^{n-1}} \log f_u(\langle x, u \rangle) d\mu(u) \right)$ is equal to zero outside the closed and convex set

$$(3.14) \quad S = \{x \in \mathbb{R}^n : a(u) \leq \langle x, u \rangle \leq b(u) \text{ for all } u \in \text{supp}(\nu)\}.$$

For every $u \in S^{n-1}$ we define $T_u : (a(u), b(u)) \longrightarrow (c(u), d(u))$ by the equation

$$(3.15) \quad \frac{\int_{a(u)}^t f_u}{\int f_u} = \frac{\int_{c(u)}^{T_u(t)} g_u}{\int g_u}.$$

Note that the map $(u, t) \mapsto T_u(t)$ is continuous on the open set $\{(u, t) : u \in S^{n-1}, a(u) < t < b(u)\}$ and that for every u the function T_u is strictly increasing and differentiable, with

$$(3.16) \quad f_u(t) \int_{\mathbb{R}} g_u = g_u(T_u(t)) T'_u(t) \int_{\mathbb{R}} f_u$$

for all $t \in (a(u), b(u))$. It follows that the functions $u \mapsto T_u(\langle x, u \rangle)$ and $u \mapsto T'_u(\langle x, u \rangle)$ are continuous on S^{n-1} for every $x \in \text{int}(S)$.

We define $T : \text{int}(S) \longrightarrow \mathbb{R}^n$ by

$$(3.17) \quad T(x) = \int_{S^{n-1}} T_u(\langle x, u \rangle) u \, d\nu(u).$$

Then,

$$(3.18) \quad dT(x) = \int_{S^{n-1}} T'_u(\langle x, u \rangle) u \otimes u \, d\nu(u).$$

Using Theorem 1.2 we see that

$$(3.19) \quad \det(dT(x)) \geq \exp \left(\int_{S^{n-1}} \log T'_u(\langle x, u \rangle) \, d\mu(u) \right).$$

In particular, this implies that T is injective. Then, for any measurable function h that satisfies

$$(3.20) \quad h \left(\int_{S^{n-1}} \theta(u) u \, d\nu(u) \right) \geq \exp \left(\int_{S^{n-1}} \log g_u(\theta(u)) \, d\mu(u) \right)$$

for every integrable function θ , we may write

$$\begin{aligned} (3.21) \quad & \exp \left(\int_{S^{n-1}} \log \left(\int_{\mathbb{R}} f_u \right) d\mu(u) \right) \int_{\mathbb{R}^n} h(y) \, dy \\ & \geq \exp \left(\int_{S^{n-1}} \log \left(\int_{\mathbb{R}} f_u \right) d\mu(u) \right) \int_{\text{int}(S)} h(T(x)) \det(dT(x)) \, dx \\ & \geq \exp \left(\int_{S^{n-1}} \log \left(\int_{\mathbb{R}} f_u \right) d\mu(u) \right) \int_{\text{int}(S)} \exp \left(\int_{S^{n-1}} \log g_u(T_u(\langle x, u \rangle)) \, d\mu(u) \right) \det(dT(x)) \, dx \\ & \geq \int_{\text{int}(S)} \exp \left(\int_{S^{n-1}} \log \left(g_u(T_u(\langle x, u \rangle)) \int_{\mathbb{R}} f_u \right) d\mu(u) \right) dx \times \exp \left(\int_{S^{n-1}} \log T'_u(\langle x, u \rangle) \, d\mu(u) \right) dx \\ & = \int_{\text{int}(S)} \exp \left(\int_{S^{n-1}} \log \left(g_u(T_u(\langle x, u \rangle)) T'_u(\langle x, u \rangle) \int_{\mathbb{R}} f_u \right) d\mu(u) \right) dx \\ & = \int_{\text{int}(S)} \exp \left(\int_{S^{n-1}} \log \left(f_u(\langle x, u \rangle) \int_{\mathbb{R}} g_u \right) d\mu(u) \right) dx \\ & = \exp \left(\int_{S^{n-1}} \log \left(\int_{\mathbb{R}} g_u \right) d\mu(u) \right) \int_{\mathbb{R}^n} \exp \left(\int_{S^{n-1}} \log f_u(\langle x, u \rangle) d\mu(u) \right) dx \end{aligned}$$

as claimed. □

Proof of Theorem 1.3. Let $\gamma > 1$ and let ν be a γ -approximation of an isotropic Borel measure on S^{n-1} . Let μ be the measure on S^{n-1} with $d\mu(u) = \langle T_\nu^{-1}u, u \rangle d\nu(u)$ and let (f_u) , $u \in S^{n-1}$ be a family of functions $f_u : \mathbb{R} \rightarrow [0, +\infty)$ that satisfy (H).

Recall that $\gamma^{-1} \leq \langle T_\nu^{-1}u, u \rangle \leq 1$ for all $u \in S^{n-1}$. Note that if $y = \int \theta(u)u d\nu(u)$ then

$$\begin{aligned}
(3.22) \quad \|y\|_2^2 &= \left\langle y, \int_{S^{n-1}} \theta(u)u d\nu(u) \right\rangle = \int_{S^{n-1}} \theta(u) \langle y, u \rangle d\nu(u) \\
&\leq \left(\int_{S^{n-1}} \theta^2(u) d\nu(u) \right)^{1/2} \left(\int_{S^{n-1}} \langle y, u \rangle^2 d\nu(u) \right)^{1/2} \\
&= \left(\int_{S^{n-1}} \theta^2(u) d\nu(u) \right)^{1/2} \sqrt{\langle T_\nu y, y \rangle} \\
&\leq \left(\int_{S^{n-1}} \theta^2(u) \gamma \langle T_\nu^{-1}u, u \rangle d\nu(u) \right)^{1/2} \sqrt{\gamma} \|y\|_2 \\
&= \gamma \left(\int_{S^{n-1}} \theta^2(u) d\mu(u) \right)^{1/2} \|y\|_2.
\end{aligned}$$

Therefore,

$$(3.23) \quad \int_{S^{n-1}} \theta^2(u) d\mu(u) \geq \gamma^{-2} \|y\|_2^2.$$

We consider the functions $g_u(t) = \exp(-\pi t^2)$, $u \in S^{n-1}$. Then,

$$\begin{aligned}
(3.24) \quad &\int_{\mathbb{R}^n}^* \sup_{y=\int \theta(u)u d\nu(u)} \exp \left(\int_{S^{n-1}} \log g_u(\theta(u)) d\mu(u) \right) dy \\
&= \int_{\mathbb{R}^n}^* \sup_{y=\int \theta(u)u d\nu(u)} \exp \left(- \int_{S^{n-1}} \pi \theta^2(u) d\mu(u) \right) dy \\
&\leq \int_{\mathbb{R}^n} \exp(-\pi \gamma^{-2} \|y\|_2^2) dy = \gamma^n.
\end{aligned}$$

Applying Lemma 3.2 we get

$$(3.25) \quad \int_{\mathbb{R}^n} \exp \left(\int_{S^{n-1}} \log f_u(\langle x, u \rangle) d\mu(u) \right) dx \leq \gamma^n \exp \left(\int_{S^{n-1}} \log \left(\int_{\mathbb{R}} f_u \right) d\mu(u) \right).$$

For the second assertion of the theorem we apply Lemma 3.2 with $f_u(t) = \exp(-\pi t^2)$. Then,

$$\begin{aligned}
\int_{\mathbb{R}^n} \exp \left(\int_{S^{n-1}} \log f_u(\langle x, u \rangle) d\mu(u) \right) dx &= \int_{\mathbb{R}^n} \exp \left(\int_{S^{n-1}} -\pi \langle x, u \rangle^2 d\mu(u) \right) dx \\
&= \int_{\mathbb{R}^n} \exp \left(\int_{S^{n-1}} -\pi \langle x, u \rangle^2 \langle T_\nu^{-1}u, u \rangle d\nu(u) \right) dx \\
&\geq \int_{\mathbb{R}^n} \exp \left(\int_{S^{n-1}} -\pi \langle x, u \rangle^2 d\nu(u) \right) dx \\
&\geq \int_{\mathbb{R}^n} e^{-\pi \gamma \|x\|_2^2} dx = \gamma^{-n/2},
\end{aligned}$$

and hence

$$\begin{aligned}
(3.26) \quad \exp \left(\int_{S^{n-1}} \log \left(\int_{\mathbb{R}} g_u \right) d\mu(u) \right) &\leq \gamma^{n/2} \int_{\mathbb{R}^n}^* \sup_{y=f(\theta(u)u)d\nu(u)} \exp \left(\int_{S^{n-1}} \log g_u(\theta(u)) d\mu(u) \right) dy \\
&\leq \gamma^{n/2} \int_{\mathbb{R}^n}^* \sup_{y=f(\theta(u)u)d\nu(u)} h \left(\int_{S^{n-1}} \theta(u)u d\nu(u) \right) dy \\
&= \gamma^{n/2} \int_{\mathbb{R}^n} h(y) dy.
\end{aligned}$$

This completes the proof. \square

Remark 3.3. Our proof of the first assertion of Theorem 1.3 provides the constant γ^n in the “approximate Brascamp-Lieb inequality”, while the corresponding constant in the discrete analogue (Theorem 2.2) is $\gamma^{\frac{n}{2}}$. However, one can prove the continuous version with constant $\gamma^{\frac{n}{2}}$, using Theorem 2.2 and an approximation argument which we sketch below.

Let ν be a γ -approximation of an isotropic Borel measure on S^{n-1} . For any $\varepsilon > 0$ we consider a maximal ε -net N_ε in S^{n-1} and a partition $(C_u)_{u \in N_\varepsilon}$ of S^{n-1} into Borel sets $C_u \subseteq B(u, \varepsilon)$. Then, we consider the measure

$$(3.27) \quad \nu_\varepsilon = \sum_{u \in N_\varepsilon} \nu(C_u) \delta_u.$$

Note that, for every $z \in S^{n-1}$,

$$(3.28) \quad |\langle T_{\nu_\varepsilon} z, z \rangle - \langle T_\nu z, z \rangle| = \sum_{u \in N_\varepsilon} \int_{C_u} [\langle z, y \rangle^2 - \langle z, u \rangle^2] d\nu(y) \leq \sum_{u \in N_\varepsilon} \int_{C_u} 2\varepsilon d\nu(y) = 2\varepsilon \nu(S^{n-1}).$$

More generally, for any continuous function $\theta : S^{n-1} \rightarrow \mathbb{R}$ we have that

$$(3.29) \quad \left| \int_{S^{n-1}} \theta(u) d\nu - \int_{S^{n-1}} \theta(u) d\nu_\varepsilon(u) \right| \leq \nu(S^{n-1}) \omega_\theta(\varepsilon),$$

where ω_θ is the modulus of continuity of θ . In other words, $\nu_\varepsilon \rightarrow \nu$ weakly as $\varepsilon \rightarrow 0^+$. In particular, from (3.28) we see that if $\varepsilon > 0$ is small enough then

$$(3.30) \quad a_\varepsilon I_n \preceq T_{\nu_\varepsilon} \preceq b_\varepsilon I_n$$

for some $b_\varepsilon > a_\varepsilon > 0$, and we have

$$(3.31) \quad T_{\nu_\varepsilon} \longrightarrow T_\nu \quad \text{and} \quad T_{\nu_\varepsilon}^{-1} \longrightarrow T_\nu^{-1}$$

as $\varepsilon \rightarrow 0^+$, and

$$(3.32) \quad \lim_{\varepsilon \rightarrow 0^+} \frac{b_\varepsilon}{a_\varepsilon} \leq \gamma.$$

Let (f_u) , $u \in S^{n-1}$ be a family of functions $f_u : \mathbb{R} \rightarrow [0, +\infty)$ that satisfy condition (H). Then, Theorem 2.2, applied to the measure $\frac{1}{a_\varepsilon} \nu_\varepsilon$, shows that

$$(3.33) \quad \int_{\mathbb{R}^n} \prod_{u \in N_\varepsilon} f_u^{\kappa_\varepsilon(u)}(\langle x, u \rangle) dx \leq \left(\frac{b_\varepsilon}{a_\varepsilon} \right)^{\frac{n}{2}} \prod_{u \in N_\varepsilon} \left(\int_{\mathbb{R}} f_u(t) dt \right)^{\kappa_\varepsilon(u)},$$

where $\kappa_\varepsilon(u) = \nu(C_u)\langle T_{\nu_\varepsilon}^{-1}u, u \rangle$. Equivalently,

$$(3.34) \quad \int_{\mathbb{R}^n} \exp \left(\int_{S^{n-1}} \log f_u(\langle x, u \rangle) \langle T_{\nu_\varepsilon}^{-1}u, u \rangle d\nu_\varepsilon(u) \right) dx \\ \leq \left(\frac{b_\varepsilon}{d_\varepsilon} \right)^{\frac{n}{2}} \exp \left(\int_{S^{n-1}} \log \left(\int_{\mathbb{R}} f_u \right) \langle T_{\nu_\varepsilon}^{-1}u, u \rangle d\nu_\varepsilon(u) \right).$$

From (3.31) we see that $\langle T_{\nu_\varepsilon}^{-1}u, u \rangle \rightarrow \langle T_\nu^{-1}u, u \rangle$ uniformly on S^{n-1} as $\varepsilon \rightarrow 0^+$. Since (f_u) , $u \in S^{n-1}$ satisfies condition (H), the weak convergence of ν_ε to ν and (3.32) imply that

$$(3.35) \quad \int_{\mathbb{R}^n} \exp \left(\int_{S^{n-1}} \log f_u(\langle x, u \rangle) \langle T_\nu^{-1}u, u \rangle d\nu(u) \right) dx \leq \gamma^{\frac{n}{2}} \exp \left(\int_{S^{n-1}} \log \left(\int_{\mathbb{R}} f_u \right) \langle T_\nu^{-1}u, u \rangle d\nu(u) \right).$$

4 Approximate classical positions of convex bodies

Let ν be a finite Borel measure on S^{n-1} . Consider the cosine transform C_ν and the symmetric convex body $C(\nu)$ defined by

$$(4.1) \quad h_{C(\nu)}(x) = C_\nu(x) = \int_{S^{n-1}} |\langle x, u \rangle| d\nu(u).$$

Note that

$$(4.2) \quad w(C(\nu)) = \int_{S^{n-1}} h_{C(\nu)}(x) d\sigma(x) = \int_{S^{n-1}} \int_{S^{n-1}} |\langle x, u \rangle| d\nu(u) d\sigma(x) \\ = \int_{S^{n-1}} \left(\int_{S^{n-1}} |\langle x, u \rangle| d\sigma(x) \right) d\nu(u) = c_n \nu(S^{n-1}),$$

where

$$(4.3) \quad c_n = \int_{S^{n-1}} |\langle x, u \rangle| d\sigma(x) = \frac{2\omega_{n-1}}{n\omega_n} \simeq \frac{1}{\sqrt{n}}.$$

Combining this fact with the inequality

$$(4.4) \quad \frac{1}{w(C(\nu))} \leq \text{vrad}(C^*(\nu)) := \left(\frac{|C^*(\nu)|}{\omega_n} \right)^{1/n}$$

which follows immediately by expressing the volume of $C^*(\nu)$ in polar coordinates, we get the next simple fact.

Proposition 4.1. *Let ν be a finite Borel measure on S^{n-1} . Then,*

$$(4.5) \quad \nu(S^{n-1})|C^*(\nu)|^{1/n} \geq \frac{n\omega_n^{\frac{n+1}{n}}}{2\omega_{n-1}} \geq c,$$

where $c > 0$ is an absolute constant.

The next proposition establishes a reverse inequality for approximate isotropic measures.

Proposition 4.2. *Let $\gamma > 1$ and let ν be a γ -approximation of an isotropic measure on S^{n-1} . Then,*

$$(4.6) \quad \nu(S^{n-1})|C^*(\nu)|^{1/n} \leq 2e\gamma^{\frac{3}{2}}.$$

Proof. Using Fubini's theorem we check that

$$(4.7) \quad n!|C^*(\nu)| = \int_{\mathbb{R}^n} e^{-h_{C(\nu)}(x)} dx = \int_{\mathbb{R}^n} \exp\left(-\int_{S^{n-1}} |\langle x, u \rangle| d\nu(u)\right) dx.$$

Then, applying Theorem 1.3 with $f_u(t) = e^{-|t|}$ we see that

$$(4.8) \quad \begin{aligned} \int_{\mathbb{R}^n} \exp\left(-\int_{S^{n-1}} |\langle x, u \rangle| d\nu(u)\right) dx &\leq \int_{\mathbb{R}^n} \exp\left(-\int_{S^{n-1}} |\langle x, u \rangle| \langle T_\nu^{-1} u, u \rangle d\nu(u)\right) dx \\ &= \int_{\mathbb{R}^n} \exp\left(\int_{S^{n-1}} \log(f_u(\langle x, u \rangle)) \langle T_\nu^{-1} u, u \rangle d\nu(u)\right) dx \\ &\leq \gamma^{\frac{n}{2}} \exp\left(\int_{S^{n-1}} \log\left(\int_{\mathbb{R}} f_u\right) \langle T_\nu^{-1} u, u \rangle d\nu(u)\right) \\ &= \gamma^{\frac{n}{2}} \exp\left(\int_{S^{n-1}} (\log 2) \langle T_\nu^{-1} u, u \rangle d\nu(u)\right) \\ &= 2^n \gamma^{\frac{n}{2}}, \end{aligned}$$

where in the last step we use the observation that

$$(4.9) \quad \begin{aligned} \int_{S^{n-1}} \langle T_\nu^{-1} u, u \rangle d\nu(u) &= \int_{S^{n-1}} \text{tr}(T_\nu^{-1}(u \otimes u)) d\nu(u) = \text{tr}\left[T_\nu^{-1}\left(\int_{S^{n-1}} u \otimes u d\nu(u)\right)\right] \\ &= \text{tr}(T_\nu^{-1} T_\nu) = \text{tr}(I_n) = n. \end{aligned}$$

Since $\langle T_\nu^{-1} u, u \rangle \geq \gamma^{-1}$ for all $u \in S^{n-1}$, we also have

$$(4.10) \quad n = \int_{S^{n-1}} \langle T_\nu^{-1} u, u \rangle d\nu(u) \geq \gamma^{-1} \nu(S^{n-1}).$$

It follows that

$$(4.11) \quad \nu(S^{n-1})|C^*(\nu)|^{1/n} \leq \gamma n \frac{2\sqrt{\gamma}}{(n!)^{1/n}} \leq 2e\gamma^{\frac{3}{2}},$$

as claimed. □

From (4.7) and (4.8) we see that if ν is a γ -approximation of an isotropic measure on S^{n-1} then

$$(4.12) \quad \int_{S^{n-1}} h_{C(\nu)}^{-n} d\sigma(\theta) = \frac{|C^*(\nu)|}{\omega_n} \leq \frac{2^n \gamma^{\frac{n}{2}}}{n! \omega_n},$$

and an application of Markov's inequality shows that a random $\theta \in S^{n-1}$ satisfies

$$(4.13) \quad h_{C(\nu)}(\theta) \geq \frac{(n! \omega_n)^{\frac{1}{n}}}{4\sqrt{\gamma}} \geq \frac{c\sqrt{n}}{\sqrt{\gamma}}$$

with probability greater than $1 - 2^{-n}$, where $c > 0$ is an absolute constant. On the other hand, from the Cauchy-Schwarz inequality we have

$$(4.14) \quad \begin{aligned} h_{C(\nu)}(\theta) &= \int_{S^{n-1}} |\langle u, \theta \rangle| d\nu(u) \leq \left(\int_{S^{n-1}} \langle u, \theta \rangle^2 d\nu(u)\right)^{\frac{1}{2}} \sqrt{\nu(S^{n-1})} \\ &\leq \sqrt{\gamma} \sqrt{\nu(S^{n-1})} \leq \gamma \sqrt{n} \end{aligned}$$

In other words, with probability greater than $1 - 2^{-n}$ we have

$$(4.15) \quad \frac{c}{\sqrt{\gamma}} \sqrt{n} \leq \int_{S^{n-1}} |\langle u, \theta \rangle| d\nu(u) \leq \gamma \sqrt{n}.$$

This observation applies to all the classical positions of a convex body that we discussed in Section 2:

Fact 4.3. *Let K be a convex body in \mathbb{R}^n .*

(i) *If σ_K is a γ -approximation of an isotropic measure then*

$$(4.16) \quad \frac{c}{\sqrt{\gamma}} \frac{\partial(K)}{\sqrt{n}} \leq 2|P_{\theta^\perp}(K)| = \int_{S^{n-1}} |\langle u, \theta \rangle| d\sigma_K(u) \leq \gamma \frac{\partial(K)}{\sqrt{n}}.$$

with probability greater than $1 - 2^{-n}$ on S^{n-1} .

(ii) *If ν_K is a γ -approximation of an isotropic measure then*

$$(4.17) \quad \frac{c}{\sqrt{\gamma}} \frac{w(K)}{\sqrt{n}} \leq \int_{S^{n-1}} |\langle u, \theta \rangle| h_K(u) d\sigma(u) \leq \gamma \frac{w(K)}{\sqrt{n}}$$

with probability greater than $1 - 2^{-n}$ on S^{n-1} .

(iii) *If λ_K is a γ -approximation of an isotropic measure then*

$$(4.18) \quad \frac{c}{\sqrt{\gamma}} \sqrt{n} L_K^2 \leq \int_K |\langle x, \theta \rangle| \|x\|_2 dx \leq \gamma \sqrt{n} L_K^2$$

with probability greater than $1 - 2^{-n}$ on S^{n-1} .

Remark 4.4. Note that $(t\nu)(S^{n-1})|C^*(t\nu)|^{1/n} = \nu(S^{n-1})|C^*(\nu)|^{1/n}$ for every Borel measure ν on S^{n-1} and every $t > 0$.

As an application of Proposition 4.1 and Proposition 4.2 we provide an alternative proof of a result from [13] on the stability of the minimal surface area position.

Theorem 4.5. *Let K be a convex body of volume 1 in \mathbb{R}^n such that*

$$(4.19) \quad I_n \preceq \frac{1}{\alpha} \int_{S^{n-1}} u \otimes u d\sigma_K(u) \preceq \gamma I_n$$

for some $\gamma > 1$ and $\alpha > 0$. Then,

$$(4.20) \quad \partial(TK) \leq \partial(K) \leq c\gamma^{3/2}\partial(TK),$$

where $c > 0$ is an absolute constant, $T \in SL(n)$ and TK is in the minimal surface area position.

Proof. Recall that $\Pi^*K = 2C^*(\sigma_K)$ and $\Pi^*(TK) = 2C^*(\sigma_{TK})$. Applying Proposition 4.1 and Proposition 4.2 to suitable multiples of the measure σ_K and σ_{TK} (also, taking into account Remark 4.4) we get

$$(4.21) \quad \sigma_K(S^{n-1})|\Pi^*K|^{1/n} \leq 4e\gamma^{3/2} \leq \frac{2e}{c}\gamma^{3/2}\sigma_{TK}(S^{n-1})|\Pi^*(TK)|^{1/n}.$$

Now, we use the observation of Petty [18] that

$$(4.22) \quad \Pi^*(TK) = T(\Pi^*K)$$

(this holds true for every convex body K and every $T \in SL(n)$) and hence, $|\Pi^*(TK)| = |\Pi^*K|$. Going back to (4.21) we conclude that

$$(4.23) \quad \partial(K) = \sigma_K(S^{n-1}) \leq \frac{2e}{c}\gamma^{3/2}\sigma_{TK}(S^{n-1}) = \frac{2e}{c}\gamma^{3/2}\partial(TK).$$

The inequality $\partial(TK) \leq \partial(K)$ is obvious since TK has minimal surface area. □

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