Taylor Series, Universality and Potential Theory

Stephen J. Gardiner

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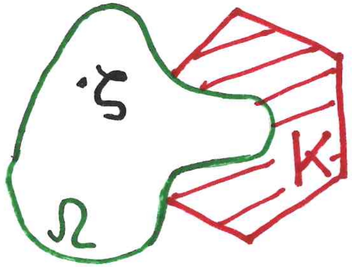
Notⁿ: $\Omega \subset \mathbb{C}$: a domain ; $\zeta \in \Omega$; $D(\zeta, r) = \{z : |z-\zeta| < r\}$.

$H(\Omega)$: holomorphic functions on Ω (topology of local uniform convergence).

$f \in H(\Omega)$;
$$S_N(f,\zeta)(z) = \sum_{n=0}^{N} \frac{f^{(n)}(\zeta)}{n!}(z-\zeta)^n.$$

Defⁿ: We say that f has a universal Taylor series about ζ if:



$\left.\begin{array}{l} \forall \text{ compact } K \subset \Omega^c \text{ with } K^c \text{ connected} \\ \forall \text{ polynomials } g \end{array}\right\}$ $\exists$ subsequence $S_{N_k}(f,\zeta) \to g$ uniformly on K.

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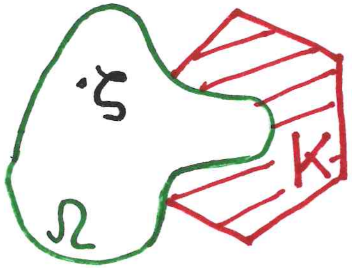

$\forall$ compact $K \subset \Omega^c$ with K^c connected $\}$ $\exists$ subsequence

$\forall$ polynomials g $\}$ $S_{N_k}(f,\zeta) \to g$ uniformly on K.

We then write $f \in \mathcal{U}(\Omega, f)$.

<u>Theorem</u> (Nestoridis 1996/7) If Ω is simply connected, then $\mathcal{U}(\Omega,\zeta)$ is a dense G_δ subset of $H(\Omega)$.

Thus almost all (Baire category) holomorphic functions on a s.c. Ω are in $\mathcal{U}(\Omega,\zeta)$.

Some basic questions:

1) Existence – on what kinds of domain do UTS exist (in addition to s.c. domains)?

2) Boundary behaviour – how must f behave if it has a UTS?

3) New for old – what new light do UTS shed on classical theory of power series?

4) Conformal maps – is possession of a UTS a conformally invariant property?

5) Dependence on centre of expansion – how does $\mathcal{U}(\Omega,\zeta)$ vary with ζ?

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Some key tools:

A) harmonic and subharmonic functions;

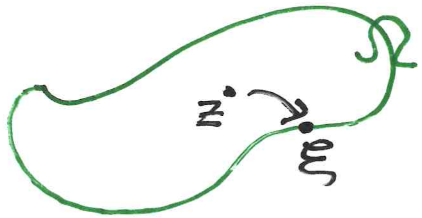
B) the Dirichlet problem and harmonic measure;

C) 'thinness' of a set at a point;

D) Martin boundary theory for the representation of positive harmonic functions.

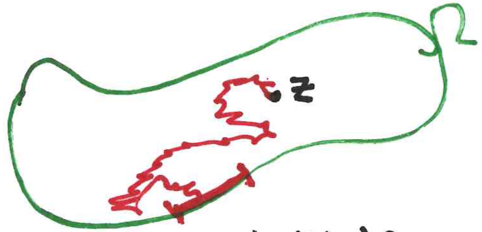
A LOT OF POTENTIAL THEORY!

Dirichlet Problem



Given $f \in C(\partial\Omega)$, find a harmonic function h_f on Ω s.t.

$$h_f(z) \to f(\xi) \quad \text{as} \quad z \to \xi \quad \text{for all } \xi \in \partial\Omega.$$



The solution has a representation of the form

$$h_f(z) = \int_{\partial\Omega} f \, d\mu_z$$

where μ_z is a probability measure on $\partial\Omega$: "harmonic measure". ('Hitting probability' for BM.)

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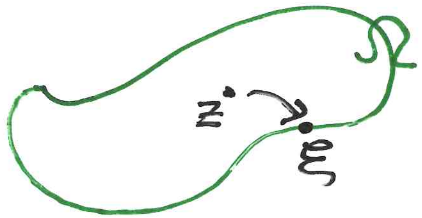

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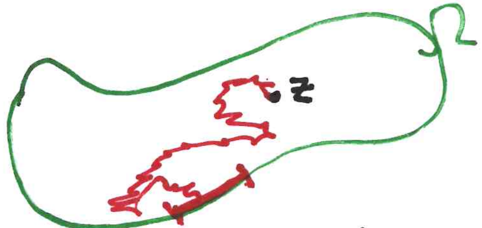

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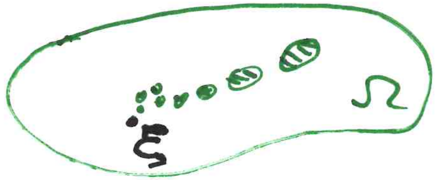
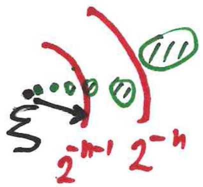


However, we must sacrifice the desired boundary behaviour at 'irregular' boundary points $\xi \in \partial\Omega$ where Ω^c is thin.

Thinness of a set $A \subset \mathbb{C}$ at ξ is characterized by

$$\sum_{n=1}^{\infty} \frac{n}{\operatorname{Log}(1/c(A_n))} < \infty \qquad \text{(Wiener's criterion)},$$



where $c(\cdot)$ is logarithmic capacity and $A_n = \{2^{-n-1} \leq |z-\xi| < 2^{-n}\}$.

Thinness at ∞ is defined by means of inversion.

Existence of universal Taylor series

Theorem (Melas, 2001) If Ω^c is compact and connected and $\zeta \in \Omega$, then $\mathcal{U}(\Omega, \zeta)$ is a dense G_δ subset of $H(\Omega)$.

Theorem (Müller, Vlachou, Yavrian, 2006) If Ω is multiply connected and Ω^c is non-thin at ∞, then $\mathcal{U}(\Omega, \zeta) = \emptyset$ for all $\zeta \in \Omega$.

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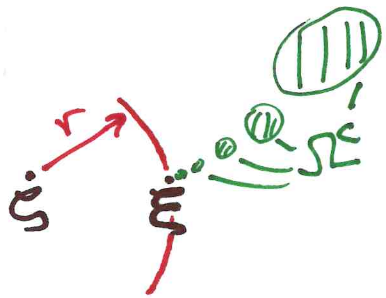


Theorem (G, 2012) Let $r = \operatorname{dist}(\zeta, \Omega^c)$ and suppose that $\Omega^c \setminus \overline{D}(\zeta, r)$ is non-polar (positive capacity). If Ω^c is thin at some $\xi \in \partial\Omega \cap \partial D(\zeta, r)$, then $\mathcal{U}(\Omega, \zeta) = \emptyset$.

Thus:

1) thinness of Ω^c at ∞ is necessary for the existence of UTS on multiply connected Ω;

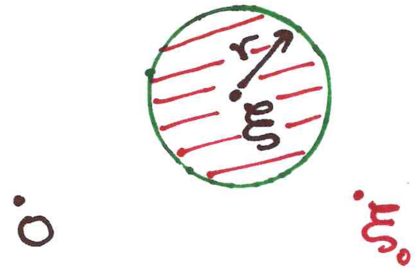
2) non-thinness of Ω^c at closest points of $\partial\Omega$ to ζ is also necessary.

However, thinness doesn't completely solve the existence problem...

Test case for triply connected domains:

Let $\Omega = \left(\bar{D}(\xi, r) \cup \{\xi_0\}\right)^c$, where $\begin{cases} 0 \in \Omega \\ \xi_0 \notin \bar{D}(\xi, r). \end{cases}$

Is $\mathcal{U}(\Omega, 0) \neq \emptyset$?



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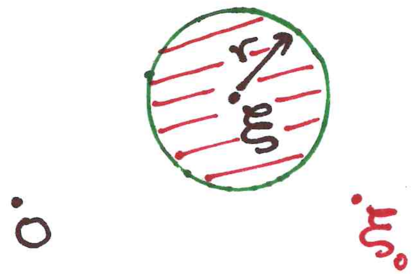


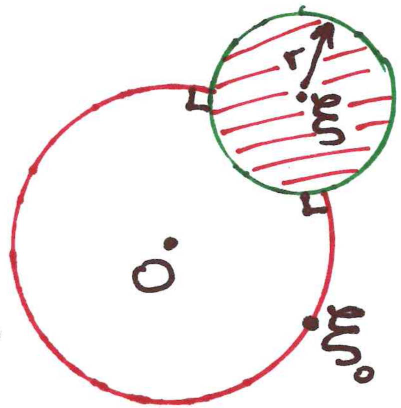
Known cases:

- YES, if $|\xi_0| \geq |\xi| + r$ (Costakis & Vlachou 2006, Tsirivas & Vlachou 2010),
- NO, if $|\xi_0| < \sqrt{|\xi|^2 - r^2}$ (G-Tsirivas 2010, G 2012).

Open question: What if $\sqrt{|\xi|^2 - r^2} \leq |\xi_0| < |\xi| + r$?

Underlying potential theory: an inequality between

- harmonic measure for $D(0, |\xi_0|) \setminus \bar{D}(\xi, r)$ and the point 0;
- harmonic measure for $[\bar{D}(0, |\xi_0|) \cup \bar{D}(\xi, r)]^c$ and ∞.



'First open case'

Boundary behaviour

Notation: $\mathbb{D} = D(0,1)$; $\mathbb{T} = \partial\mathbb{D}$.

Known (Bayart, 2005): $f \in \mathcal{U}(\mathbb{D}, 0) \Rightarrow f$ is unbounded near each point of $\mathbb{T}$.

Question: For simply connected, must every f in $\mathcal{U}(\Omega, \zeta)$ be unbounded?

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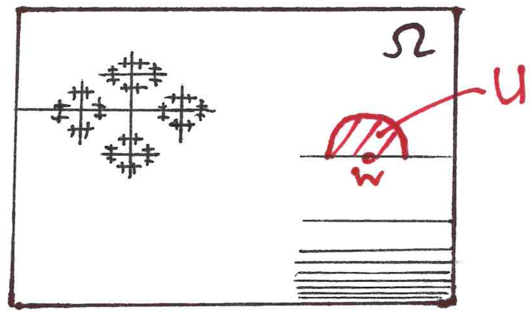
$f(U)^c$ has zero logarithmic capacity ($\Rightarrow$ Hausdorff dim 0).



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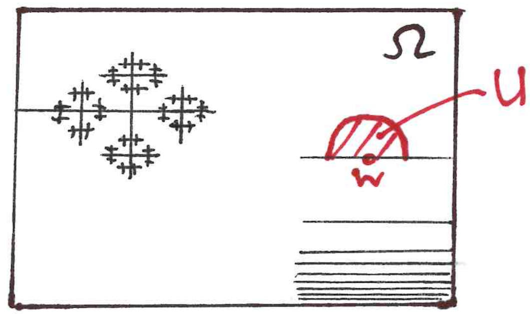

Remark: What is actually shown is that the subharmonic function $\log|f|$ cannot have a positive harmonic majorant on any such component U.

If $c(f(U)^c) > 0$, then (Myrberg's theorem) $\log^+|z|$ has a harmonic majorant on $f(U)$, whence $\log|f|$ would have the positive harmonic majorant $h\circ f$ on U.

For the unit disc we can say even more ...

Known (Costakis & Melas, 2000): $f \in \mathcal{U}(\mathbb{D},0) \Rightarrow f$ assumes every complex value, with at most one exception, infinitely often in $\mathbb{D}$. ("Picard-type" property.)

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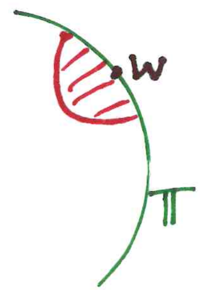

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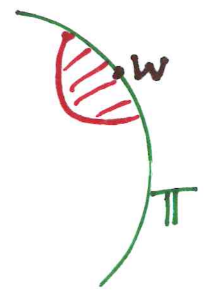


Thus: "universal Taylor series have universal boundary behaviour."

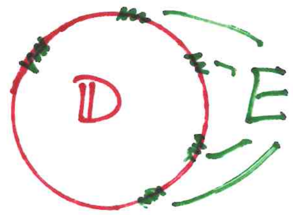
Something can even be said for completely arbitrary domains...

Theorem (G-Manolaki, 2014): For any domain Ω and any point $\zeta \in \Omega$, every function in $\mathcal{U}(\Omega, \zeta)$ is unbounded.

The proof depends on Martin boundary theory (a generalization of Carathéodory's theory of prime ends).

New light on an old topic

Def^n: Let $E \subset \mathbb{T}$ be closed. We call E a Dirichlet set if some subsequence of (z^n) tends to 1 uniformly on E.



Theorem (Beise, Meyrath, Müller, 2015) Let Ω be a domain, and suppose:

- each component of $\hat{\mathbb{C}} \setminus \Omega$ meets $\mathbb{T}$;
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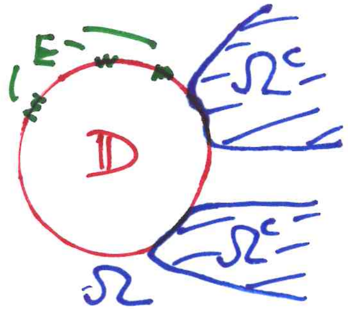

Then 'almost all' $f \in H(\Omega)$ have the property that

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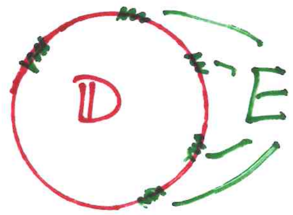

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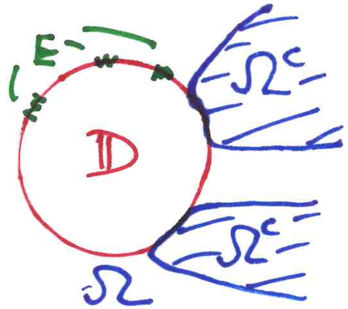

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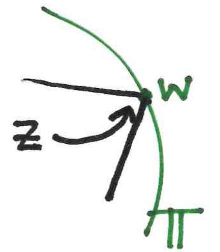
Remarks (i) This universal approximation occurs within the domain where f is holomorphic. Contrast – functions in $\mathcal{U}(\mathbb{D})$ have no holomorphic extension beyond $\mathbb{D}$.

(ii) Dirichlet sets can have Hausdorff dimension 1, but not positive arclength.

Question: Can such universal approximation occur on sets of positive arclength where f is holomorphic?

No! — not even where f merely has non-tangential limits (nt lim).



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Then $S = f$ a.e. on $E \cap F$ (w.r.t. arclength).

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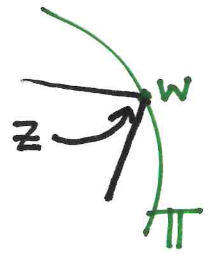

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Then $S = f$ a.e. on $E \cap F$ (w.r.t. arclength).

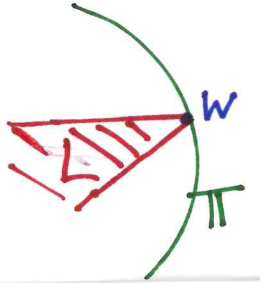
This is proved via a new convergence theorem for sequences of harmonic measures.

It complements the well-known result...

Abel's Limit Theorem: If $(S_m(w))$ converges for some $w \in \mathbb{T}$, then $\operatorname{nt}\lim_{z\to w} f(z)$ exists, and the two limits agree.

(Abel's Limit Theorem fails if we replace (S_m) by some subsequence (S_{m_k}) – cf UTS.)

An application to universal Taylor series —



Corollary (G, 2014) Let $f \in \mathcal{U}(\mathbb{D}, 0)$. Then, at almost every $w \in \mathbb{T}$, $\overline{f(L)} = \mathbb{C}$ for every Stolz angle L at w.

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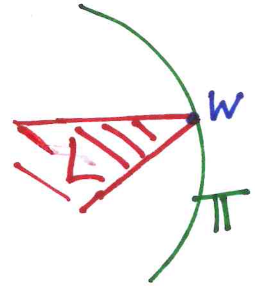

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Proof Plessner's Theorem says that, at almost every $w \in \mathbb{T}$,

either $\quad \operatorname{nt}\lim_{z \to w} f(z)$ exists

or $\quad \overline{f(L)} = \mathbb{C}$ for every Stolz angle L.

The preceding Theorem says that $\operatorname{nt}\lim_{z \to w} f(z)$ 'controls' $\lim_{k \to \infty} S_{m_k}$ a.e. on $\mathbb{T}$.

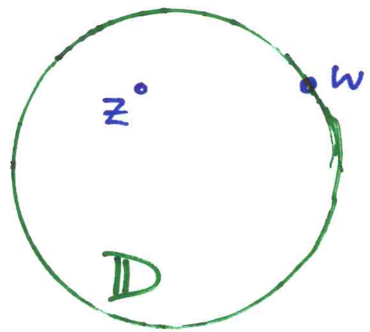
Thus, if $f \in \mathcal{U}(\mathbb{D}, 0)$, the first possibility in Plessner's Theorem fails a.e. on $\mathbb{T}$.

Another tool from potential theory

'Minimal thinness' – the notion of a set in $\mathbb{D}$ being 'thin' at a point of $\mathbb{T}$.

Recall: the Poisson kernel for $\mathbb{D}$ is

$$P_w(z) = \frac{1-|z|^2}{|z-w|^2} \qquad (z \in \mathbb{D},\ w \in \mathbb{T}).$$



Def^n: A set $E \subset \mathbb{D}$ is called minimally thin at $w \in \mathbb{T}$ if there is a positive superharmonic function v on $\mathbb{D}$ such that $\inf_E \frac{v}{P_w} > \inf_{\mathbb{D}} \frac{v}{P_w}$.

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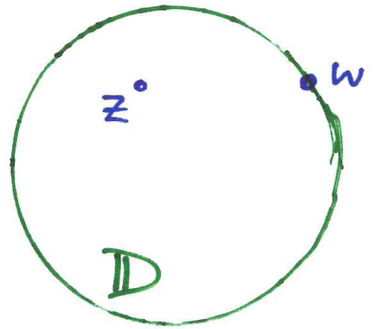

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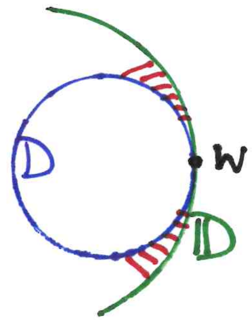
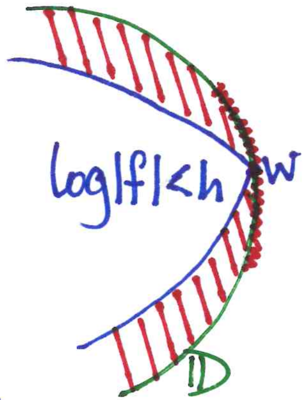
Example: If $D \subset \mathbb{D}$ is a disc internally tangent to $\mathbb{T}$ at w, then $\mathbb{D} \setminus D$ is minimally thin at w.



Reason: $D = \{P_w > c\}$ for some $c > 0$. Also,

$v = \min\{P_w, c\}$ is a positive superharmonic function on $\mathbb{D}$.

Another link between the boundary behaviour of f and the behaviour of (S_{m_k}) on $\mathbb{T}$...



Theorem (G, 2014) Let $f \in H(\mathbb{D})$, and $h > 0$ be harmonic on $\mathbb{D}$. If

(a) (S_{m_k}) is uniformly bounded on an arc containing $w \in \mathbb{T}$,

(b) $\{e^{-h}|f| \geq 1\}$ is minimally thin at w,

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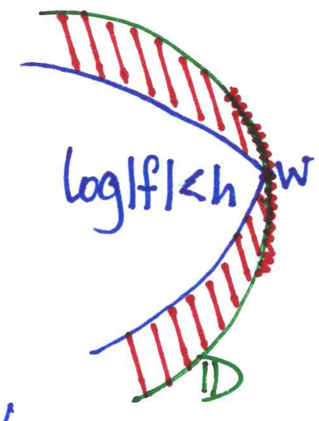

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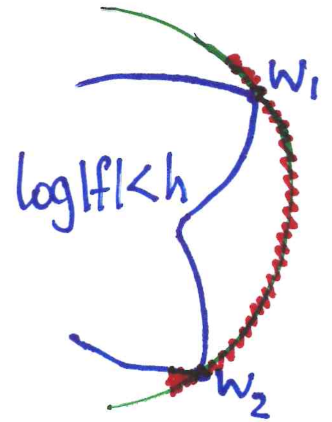
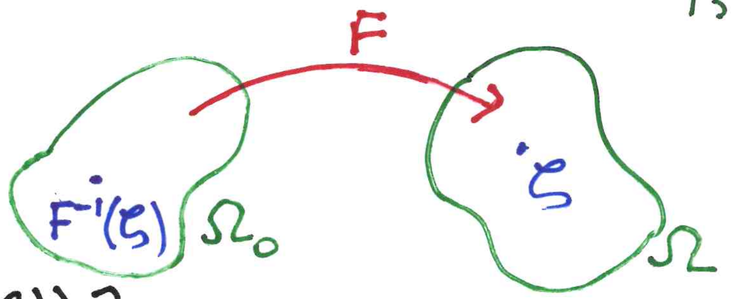
Observations (i) It is possible for $f \in \mathcal{U}(\mathbb{D}, 0)$ to satisfy (b) above at one point $w \in \mathbb{T}$, provided $h(z) \to \infty$ as $z \to w$.

(ii) It is impossible for $f \in \mathcal{U}(\mathbb{D}, 0)$ to satisfy (b) at two points $w_1, w_2 \in \mathbb{T}$. For the maximum principle on the sector Ow_1w_2 would then yield $\log|f| < h$ there, forcing f to have finite non-tangential limits a.e. on the arc w_1w_2.



Conformal invariance?



Let $F:\Omega_0 \to \Omega$ be a conformal mapping.

Question: If $f \in \mathcal{U}(\Omega, \zeta)$, does $f \circ F \in \mathcal{U}(\Omega_0, F^{-1}(\zeta))$?

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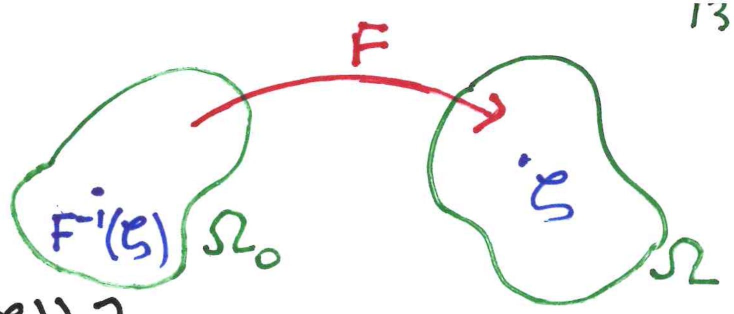

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Theorem (G, 2014) Let $S = \{-1 < \mathrm{Re}\, z < 1\}$. There exists $f \in \mathcal{U}(S, 0)$ s.t., for any conformal mapping $F: \mathbb{D} \to S$,

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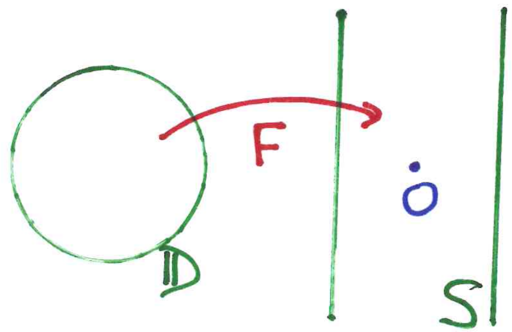

Conformal invariance?

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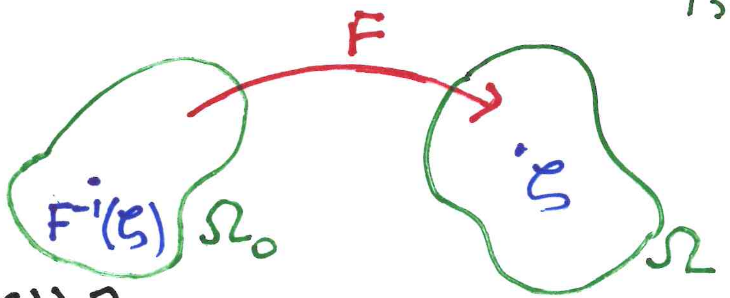

Question: If $f \in \mathcal{U}(\Omega,\zeta)$, does $f\circ F \in \mathcal{U}(\Omega_0, F^{-1}(\zeta))$?

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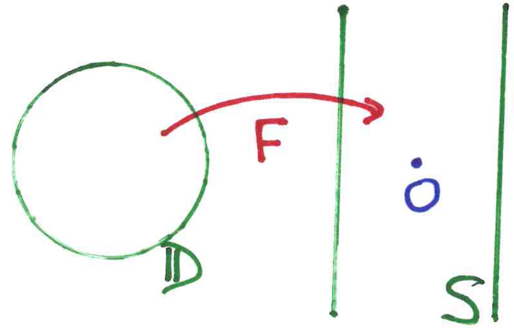
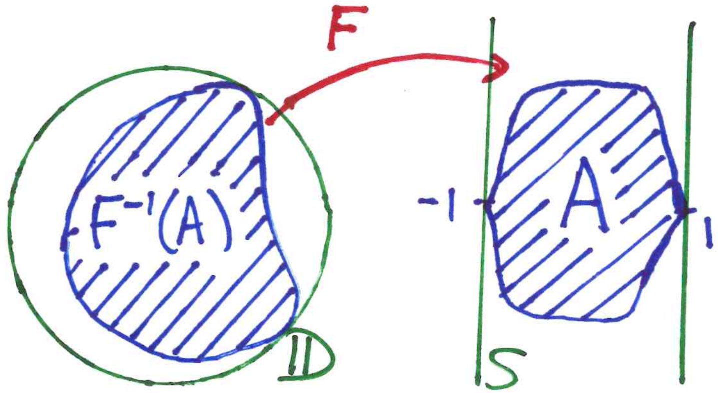


Reason One can construct $f\in\mathcal{U}(S,0)$ and a harmonic function $h>0$ on S such that

$\log|f| < h$ on the set A (illustrated).

Hence $\log|f\circ F| \le h\circ F$ on $F^{-1}(A)\subset\mathbb{D}$.

This is incompatible with $f\circ F$ being in $\mathcal{U}(\mathbb{D}, F^{-1}(0))$.



Dependence of $\mathcal{U}(\Omega, \zeta)$ on ζ?

Recall that $\mathcal{U}(\Omega, \zeta)$ is a dense G_δ subset of $H(\Omega)$ in two important cases:

(i) Ω is simply connected;

(ii) Ω^c is compact and connected.

Theorem (Müller, Vlachou & Yavrian, 2006): If Ω is simply connected, then $\mathcal{U}(\Omega, \zeta)$ does not depend on ζ.

Dependence of $\mathcal{U}(\Omega,\zeta)$ on ζ?

Recall that $\mathcal{U}(\Omega,\zeta)$ is a dense G_δ subset of $H(\Omega)$ in two important cases:

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Theorem (Müller, Vlachou & Yavrian, 2006): If Ω is simply connected, then $\mathcal{U}(\Omega,\zeta)$ does not depend on ζ.

Theorem (Bayart 2005, cf Costakis 2005): If Ω^c is compact and connected, then $\bigcap_{\zeta\in\Omega} \mathcal{U}(\Omega,\zeta)$ is residual in $H(\Omega)$.

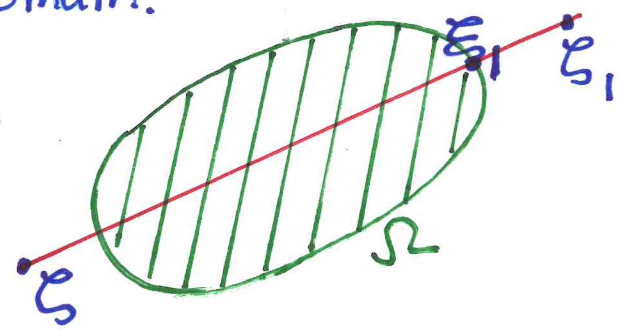
Question: Does $\mathcal{U}(\Omega,\zeta)$ depend on ζ in this second case?

Def^n: An exterior Dini domain is the exterior of a Jordan curve Γ with parametrization $\alpha(t)$, where $\alpha'(t)$ is Dini continuous (this holds if Γ is $C^{1,\varepsilon}$).

Theorem (G-Manolaki, 2014): Let Ω be an exterior Dini domain.

Then, $\forall \zeta \in \Omega \ \exists \zeta_1 \in \Omega$ s.t. $\mathcal{U}(\Omega,\zeta) \setminus \mathcal{U}(\Omega,\zeta_1) \neq \emptyset$.

Further, if ξ_1 is a furthest point of $\partial\Omega$ from ζ, then we can take $\zeta_1 = \xi_1 + t(\xi_1 - \zeta) \quad (t > 0)$.



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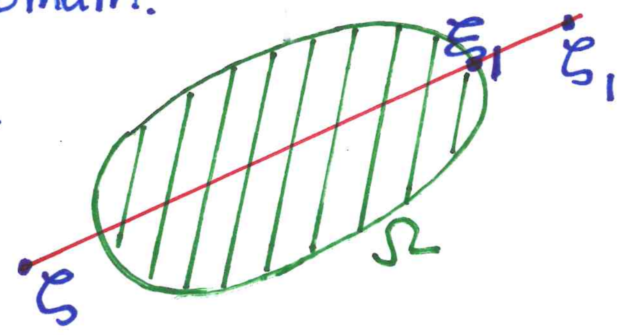

Idea: (i) As with $\mathbb{D}$, we can construct $f \in \mathcal{U}(\Omega,\zeta)$ with controlled growth on a set A 'fat' at some $\xi \in \partial\Omega$.

(ii) Additionally, we can control the growth of f near ξ_1.

(iii) Control on the growth of f is impossible at two points of $\partial\Omega$ that are close to ζ (as for $\mathbb{D}$).