

Bounded and unbounded generalizations of C^* -algebras

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Mathematical Analysis in Athens - Katavolos and Nestoridis
Athens, 16 December 2017

PART I

Symmetric Banach $*$ -algebras

For convenience we shall always assume that our complex algebras have an identity element denoted by e . The Russian mathematicians Gelfand and Naimark defined, studied and classified C^* -algebras, in 1943, resulting to the known commutative and noncommutative Gelfand–Naimark theorems. In 1946 another Russian mathematician D.A. Raikov, introduced in the context of Banach $*$ -algebras the concept of symmetry aiming to the replacement of the C^* -property by symmetry, in order to obtain Gelfand–Naimark type theorems. Symmetry for a $*$ -algebra $\mathcal{A}$, means that every element of the form $e + x^*x, x \in \mathcal{A}$, is invertible.

In fact, Gelfand and Naimark, in 1943, had posed on a C^* -algebra the preceding condition too (but without naming it) in order to get their results; though they suspected that this property was not necessary. Indeed this is true and follows easily: since $\mathcal{A}$ is a C^* -algebra, every element of the form $e + x^*x, x \in \mathcal{A}$, is greater than or equal to e , therefore it is invertible, consequently **every C^* - algebra is symmetric**. Raikov in his research for symmetric Banach * -algebras, he proved a very useful criterion for symmetry, but not Gelfand–Naimark type theorems. Nevertheless, later such theorems appeared in the literature.

For instance, every semisimple symmetric Banach $*$ -algebra $\mathcal{A}[\|\cdot\|]$ admits a continuous faithful $*$ -representation π on a Hilbert space $\mathcal{H}$. Additionally, symmetric Banach $*$ -algebras attain nontrivial irreducible $*$ -representations and nontrivial states. Since every symmetric Banach $*$ -algebra $\mathcal{A}[\|\cdot\|]$ is hermitian and vice versa (Shirali–Ford theorem), according to Pták’s theory (1972) for hermitian Banach $*$ -algebras, symmetry is equivalent to the inequality

$$r_{\mathcal{A}}(x) \leq p_{\mathcal{A}}(x), \forall x \in \mathcal{A}, \text{ where } p_{\mathcal{A}}(x) := r_{\mathcal{A}}(x^*x)^{\frac{1}{2}}, x \in \mathcal{A}$$

and to the fact that $p_{\mathcal{A}}(\cdot)$ is a C^* -seminorm on $\mathcal{A}$.

Pták remarks that the above inequality is, in fact, an algebraic analogue of the C^* -property. This, he explains, follows by taking the equivalent to the C^* -property inequality

$$\|x^*\| \|x\| \leq \|x^*x\|, x \in \mathcal{A}$$

and replace $\|\cdot\|$ with $r_{\mathcal{A}}$. Then, obviously one obtains Pták's inequality. Moreover, concerning the preceding faithful $*$ -representation π that a semisimple symmetric Banach $*$ -algebra $\mathcal{A}$ admits, one can prove that

$$\|\pi(x)\| = p_{\mathcal{A}}(x), \forall x \in \mathcal{A},$$

where $p_{\mathcal{A}}(x) \leq \|x\|$, for every $x \in \mathcal{A}$. So, π is continuous with respect to the norm of $\mathcal{A}$ and an isometry with respect to the smaller topology that the C^* -norm $p_{\mathcal{A}}$ defines on $\mathcal{A}$. Even more, using Raikov's criterion for symmetry, we can prove that a Banach $*$ -algebra is symmetric iff $p_{\mathcal{A}}$ is a maximal continuous C^* -seminorm on $\mathcal{A}$.

Some examples. (1) Let X be a compact Hausdorff space, $\mathcal{A}$ an arbitrary symmetric Banach $*$ -algebra and $C(X, \mathcal{A}) = C(X) \hat{\otimes} \mathcal{A}$ the Banach $*$ -algebra of all $\mathcal{A}$ -valued continuous functions on X . Since $C(X)$ is commutative and both $\mathcal{A}$ and $C(X)$ are symmetric, $C(X, \mathcal{A})$ is a (non-commutative) symmetric Banach $*$ -algebra.

(2) The group algebra of an abelian or compact group is symmetric.

(3) The algebra $\mathcal{C}^\infty[0, 1]$, under its usual topology (uniform convergence in all derivatives) is symmetric, but not a C^* -algebra, since under the preceding topology cannot be a Banach algebra.

PART II

Generalized B^* -algebras, shortly GB^* -algebras

The algebras in the title of Part II are locally convex $*$ -algebras. They were initiated by the British mathematician G.R. Allan, in 1967.

In 1970, [P.G. Dixon](#) (student of Allan) extended the concept of a GB^* -algebra, in order to include also examples that are topological $*$ -algebras but not locally convex. GB^* -algebras generalize C^* -algebras and Dixon proved that they are algebras of unbounded operators.

G.R. Allan reached at the definition of GB^* -algebras, trying to express the C^* -condition in a given Banach $*$ -algebra, not using its norm, but its properties as a locally convex $*$ -algebra. He solved this problem as follows: Having a Banach $*$ -algebra $\mathcal{A}[\|\cdot\|]$, he considered a collection $\mathfrak{B}_{\mathcal{A}}^*$ consisting of all subsets B in $\mathcal{A}[\|\cdot\|]$, with the following properties:

- (1) B is absolutely convex;
- (2) $B^2 \subset B$ and $B^\circ = B$, where $B^\circ = \{x^* \in B, \forall x \in B\}$;
- (3) B is bounded and closed.

After proving a series of lemmas he showed the following characterization:

Theorem. Let $\mathcal{A}[\|\cdot\|]$ be a Banach $*$ -algebra. Then, $\mathcal{A}$ is a C^* -algebra under a C^* -norm equivalent to the given norm $\|\cdot\|$, if and only if, the following conditions hold:

- (i) The collection $\mathfrak{B}_{\mathcal{A}}^*$ has a greatest member;
- (ii) $\mathcal{A}$ is symmetric, in the sense we mentioned in Part I.

A proper definition of the spectrum of an element, for this purpose, came from the spectral theory of a closed operator T on a Banach space E , where the spectrum of T is given by those complex numbers λ , such that the operator $\lambda I - T$ has no bounded inverse, where I is the restriction of the identity operator of E on the domain of T . Thus, a suitable definition of a bounded element in a locally convex algebra had to be given.

Let us see, how these definitions read.

Let $\mathcal{A}[\tau]$ be a *locally convex algebra*. An element $x \in \mathcal{A}$ is called (Allan-) **bounded** if there is a non-zero complex number λ , such that the set

$$\{(\lambda x)^n : n = 1, 2, \dots\} \quad \text{is bounded in } \mathcal{A}[\tau].$$

The set of all bounded elements of $\mathcal{A}[\tau]$ is denoted by $\mathcal{A}_b$. Every element of a normed algebra is bounded in the previous sense. Also if $\mathcal{A}[\tau]$ has an identity e , then e is bounded.

In general, $\mathcal{A}_b$ is not even a subspace of $\mathcal{A}[\tau]$, but if $\mathcal{A}[\tau]$ is commutative and complete, then $\mathcal{A}_b$ is a subalgebra.

Now, if $\mathcal{A}[\tau]$ has an identity element e , the **spectrum** of an element $x \in \mathcal{A}$ is denoted by $\sigma_{\mathcal{A}}(x)$ and is defined as follows

$$\sigma_{\mathcal{A}}(x) := \{\lambda \in \mathbb{C} : \lambda e - x \text{ has no inverse in } \mathcal{A}_b\} \\ \cup \{\infty, \text{ for } x \notin \mathcal{A}_b\}.$$

If $\mathcal{A}[\tau]$ has a continuous inversion, and $sp_{\mathcal{A}}(x)$ is the usual spectrum of $x \in \mathcal{A}$,

$$sp_{\mathcal{A}}(x) \subset \sigma_{\mathcal{A}}(x) \subset \overline{sp_{\mathcal{A}}(x)}.$$

Suppose now that $\mathcal{A}[\tau]$ is a locally convex $*$ -algebra with an identity e . Denote by $\mathfrak{B}_{\mathcal{A}[\tau]}^*$ the collection of all subsets B of $\mathcal{A}[\tau]$ such that:

- (1) B is absolutely convex;
- (2) $e \in B$, $B^2 \subseteq B$ and $B^* = B$;
- (3) B is bounded and closed.

For any $B \in \mathfrak{B}_{\mathcal{A}[\tau]}^*$, let $\mathcal{A}[B]$ be the $*$ -subalgebra of $\mathcal{A}$ generated by B . Then,

$$\mathcal{A}[B] = \{\lambda x : \lambda \in \mathbb{C}, x \in B\}.$$

If $\mathcal{A}[\tau]$ is complete, then $\mathcal{A}[B]$ becomes a Banach $*$ -algebra under the gauge function of B

$$\|x\|_B := \inf\{\lambda > 0 : x \in \lambda B\}, x \in \mathcal{A}[B].$$

Definition. A complete locally convex $*$ -algebra $\mathcal{A}[\tau]$ is called a **GB $*$ -algebra**, if

- (1) $\mathfrak{B}_{\mathcal{A}[\tau]}^*$ has a greatest member denoted by B_0 ;
- (2) for each $x \in \mathcal{A}$, $(e + x^*x)^{-1}$ exists in $\mathcal{A}[\tau]$ and it belongs to $\mathcal{A}_b$.

It is then usual to say that $\mathcal{A}[\tau]$ is a **GB $*$ -algebra over B_0** .

The Banach $*$ -algebra $\mathcal{A}[B_0]$ corresponding to the maximal element B_0 of $\mathfrak{B}_{\mathcal{A}[\tau]}^*$ is a C^* -algebra and has very nice properties. For instance, it is dense in $\mathcal{A}[\tau]$ and, when the GB $*$ -algebra is commutative, $\mathcal{A}[B_0]$ coincides with $\mathcal{A}_b$. Moreover, $\mathcal{A}[B_0]$ is the pillar on which the theory of GB $*$ -algebras is based.

Examples: (1) Every C^* -algebra $\mathcal{A}[\|\cdot\|]$ is a GB^* -algebra over its unit ball.

(2) Let $\mathcal{A}$ be the algebra $\mathcal{C}(\mathbb{R}) = \varprojlim_n \mathcal{C}[-n, n]$, of all $\mathbb{C}$ -valued continuous functions on $\mathbb{R}$, with the topology of uniform convergence on $[-n, n]$'s and $\mathcal{C}_b(\mathbb{R})$, the C^* -algebra of all bounded continuous functions on $\mathbb{R}$. Then, $\mathcal{C}(\mathbb{R})$ is a GB^* -algebra over the unit ball of $\mathcal{C}_b(\mathbb{R})$.

(3) The Arens algebra $L^\omega[0, 1] = \bigcap_{1 \leq p < \infty} L^p[0, 1]$ endowed with the topology of all L^p -norms, $1 \leq p < \infty$, is a GB^* -algebra over the unit ball of $L^\infty[0, 1]$.

(4) The algebra $\mathcal{M}[0, 1]$ of all measurable functions on $[0, 1]$ (modulo equality a.e.), endowed with the topology of convergence in measure, is also a GB*-algebra. More precisely, $\mathcal{M}[0, 1]$ is a complete and metrizable non-locally convex GB*-algebra, with jointly continuous multiplication with B_0 the unit ball of the C*-algebra $L^\infty[0, 1]$.

Suppose now that $\mathcal{A}[\tau]$ is a commutative GB*-algebra. Then the C*-algebra $\mathcal{A}[B_0] = \mathcal{A}_b$ has a non-empty Gelfand space $\mathfrak{M}_0$. Thus, taking an element $\varphi \in \mathfrak{M}_0$, this is uniquely extended to a '*-homomorphism' φ' on $\mathcal{A}$ with values in the one point compactification $\mathbb{C}^*$ of $\mathbb{C}$.

In particular, for a given $x \in \mathcal{A}$, the set $\{\varphi \in \mathfrak{M}_0 : \varphi'(x) = \infty\}$, is a closed, nowhere dense subset of $\mathfrak{M}_0$. Denote by $\widehat{\mathcal{A}}$ the $*$ -algebra of all $\mathbb{C}^*$ -valued continuous functions on $\mathfrak{M}_0$.

Theorem 1 (Allan; algebraic commutative Gelfand–Naimark type theorem).

For a commutative GB $*$ -algebra $\mathcal{A}[\tau]$ there exists an algebraic $*$ -isomorphism of $\mathcal{A}$ onto the $*$ -algebra $\widehat{\mathcal{A}}$ of continuous $\mathbb{C}^*$ -valued functions $\widehat{x}$ on $\mathfrak{M}_0$ with $\widehat{x}(\varphi) := \varphi'(x)$, $x \in \mathcal{A}$. The map $\mathcal{A} \rightarrow \widehat{\mathcal{A}} : x \mapsto \widehat{x}$ extends the Gelfand representation of the $\mathbb{C}^*$ -algebra $\mathcal{A}[B_0]$, and so $\widehat{\mathcal{A}}$ contains all continuous $\mathbb{C}$ -valued functions on $\mathfrak{M}_0$.

The statement now of the non-commutative Gelfand Naimark type theorem for GB^* -algebras is the following:

Theorem 2 (Dixon; algebraic non-commutative Gelfand Naimark type theorem for GB^* -algebras). Every GB^* -algebra $\mathcal{A}[\tau]$ admits a faithful $*$ -representation $(\pi, \mathcal{H}_\pi)$ as a $*$ -algebra of closed operators on the domain $\mathcal{D}_\pi$ of π . Furthermore, the $*$ -representation π can be chosen such that

$$\pi(B_0) = \{T \in \pi(\mathcal{A}) \cap \mathcal{B}(\mathcal{H}_\pi) : \|T\| \leq 1\},$$

where $\mathcal{H}_\pi$ is the completion of the pre-Hilbert space $\mathcal{D}_\pi$.

We should note that π is constructed by making no use of the existing topological structure of $\mathcal{A}[\tau]$. In particular, π is such that $\pi(B_0) = \mathcal{U}_{\pi(\mathcal{A})}^1$, where $\mathcal{U}_{\pi(\mathcal{A})}^1$ is the unit ball of the $*$ -subalgebra $\pi(\mathcal{A})$, endowed with the operator norm from $\mathcal{B}(\mathcal{H}_\pi)$. Consequently, π and particularly the set $\pi^{-1}(\mathcal{U}_{\pi(\mathcal{A})}^1)$ depends only upon the algebraic structure of $\mathcal{A}[\tau]$. Therefore, for any two topologies τ_1, τ_2 on a $*$ -algebra $\mathcal{A}$, making it into a GB $*$ -algebra, one has that

$$B_0(\tau_1) = \pi^{-1}(\mathcal{U}_{\pi(\mathcal{A})}^1) = B_0(\tau_2).$$

Thus, we obtain

Theorem 3 (Allan, Dixon). Let $\mathcal{A}$ be a $*$ -algebra and τ_1, τ_2 two topologies on $\mathcal{A}$ making it a GB^* -algebra. Then, τ_1, τ_2 are ‘equivalent’, in the sense that $B_0(\tau_1) = B_0(\tau_2)$.

This is an analogous situation with the uniqueness of the C^* -norm in a C^* -algebra. More precisely, having two topologies τ_1, τ_2 making a $*$ -algebra $\mathcal{A}$ into a GB^* -algebra, you obtain that $\mathcal{A}[B_0(\tau_1)] = \mathcal{A}[B_0(\tau_2)]$, which is quite natural and what exactly you need.

Now a helpful characterization for certain GB^* -algebras is the following:

Theorem (F., Inoue, Kürsten). If $\mathcal{A}_0[\|\cdot\|_0]$ is a C^* -algebra with identity and τ a locally convex $*$ -algebra topology on $\mathcal{A}_0$ making its involution continuous, its multiplication jointly continuous and $\tau \prec \|\cdot\|_0$, then the following statements are equivalent:

- (i) $\widetilde{\mathcal{A}_0[\tau]}$ is a GB^* -algebra over the unit ball U_0 of $\mathcal{A}_0[\|\cdot\|_0]$;
- (ii) U_0 is τ -closed.

PART III

Locally convex quasi C^* -algebras

The necessity of introducing locally convex quasi $*$ -algebras comes from physics. This new structure is defined as follows:

Definition (Bagarello, F., Inoue, Trapani) Let $\mathcal{A}_0[\|\cdot\|_0]$ be a C^* -algebra and τ a locally convex topology on $\mathcal{A}_0$, defined by an upwards directed family $\{p_\lambda\}_{\lambda \in \Lambda}$ of $*$ -seminorms, in such a way that the following conditions hold:

(T1) $\mathcal{A}_0[\tau]$ is a locally convex $*$ -algebra with separately continuous multiplication;

(T2) $\tau \preceq \|\cdot\|_0$;

(T3) For each $\lambda \in \Lambda$, there exists $\lambda' \in \Lambda$, such that $p_\lambda(xy) \leq \|x\|_0 p_{\lambda'}(y)$, $\forall x, y \in \mathcal{A}_0$ with $xy = yx$;

(T4) The set $\mathcal{U}(\mathcal{A}_0)_+ \equiv \{x \in (\mathcal{A}_0)_+ : \|x\|_0 \leq 1\}$ is τ -closed.

Furthermore, if $\mathcal{A}$ is a vector subspace of $\tilde{A}_0[\tau]$, invariant under involution and containing $\mathcal{A}_0$, then we say that the pair $(\mathcal{A}[\tau], \mathcal{A}_0)$ is a *locally convex quasi C^* -algebra* (over $\mathcal{A}_0$).

- If $\mathcal{A}[\tau]$ be a GB^* -algebra over B_0 , $\mathcal{A}_0 \equiv \mathcal{A}[B_0]$ and the locally convex topology τ fulfils the condition (T3), then the pair $(\mathcal{A}[\tau], \mathcal{A}_0)$ is a locally convex quasi C^* -algebra over $\mathcal{A}_0$. The GB^* -algebra $\mathcal{C}_c(\mathbb{R})$ fulfils all conditions T1–T4.

Let now $\mathcal{D}$ be a dense subspace in a Hilbert space $\mathcal{H}$. Let $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ denote all linear operators T from $\mathcal{D}$ to $\mathcal{H}$, such that the domain of T^* contains $\mathcal{D}$. Then, under usual linear operations $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$ is a linear space endowed with a linear involution defined by $T^\dagger := T^*|_{\mathcal{D}}$, $T \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$.

Endow $\mathcal{D}$ with the graph topology $t_{\mathcal{D}}$ defined by the family of seminorms

$$\|\xi\|_T := \|\xi\| + \|T\xi\|, T \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H}) \xi \in \mathcal{D}.$$

By taking a suitable bounded subset B of $\mathcal{D}[t_{\mathcal{D}}]$ the following locally convex topology τ_{s^*} is defined on $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})$:

$$p_\dagger^B(T) := \sup\{\|T\xi\| + \|T^\dagger\xi\| : \xi \in B\}, T \in \mathcal{L}^\dagger(\mathcal{D}, \mathcal{H}).$$

Now a [Gelfand–Naimark type theorem](#) for locally convex quasi C^* -algebras can be stated as follows:

Theorem (Bagarello, F., Inoue, Trapani) Let $(\mathcal{A}[\tau], \mathcal{A}_0)$ be a locally convex quasi C^* -algebra. The following statements are equivalent:

- (1) There exists a faithful, (τ, τ_{s^*}) -continuous $*$ -representation π of $(\mathcal{A}[\tau], \mathcal{A}_0)$ in $\mathcal{L}^\dagger(\mathcal{D}, \mathcal{H})[\tau_{s^*}]$;
- (2) the set of all continuous positive invariant sesquilinear forms φ on $\mathcal{A} \times \mathcal{A}$, is sufficient, in the sense that if $a \in \mathcal{A}$ is such that $\varphi(a, a) = 0$, for all forms φ as before, then $a = 0$.

In fact, sufficiency of the previous set is equivalent to the $*$ -semisimplicity of $(\mathcal{A}, \mathcal{A}_0)$.

THANK YOU VERY MUCH FOR YOUR ATTENTION