ON SECTIONS OF CONVEX BODIES IN JOHN'S POSITION AND OF GENERALISED B_p^n BALLS

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Abstract. We revisit an ingenious argument of K. Ball to provide sharp estimates for the volume of sections of a convex body in John's position. Our technique combines the geometric Brascamp-Lieb inequality with a generalised Parseval-type identity. This lets us complement some earlier results of the first two named authors, as well as generalise the classical estimates of Meyer-Pajor and Koldobsky regarding extremal sections of B_p^n balls to a broader family of norms induced by a John's decomposition of the identity in $\mathbb{R}^n$.

1. Introduction

The study of hyperplane sections and projections of convex bodies is a classical and actively developing area of modern convex geometry, with deep connections to functional analysis, geometric tomography, and high-dimensional probability. Extremal questions about k-dimensional sections — for example, determining maximal or minimal volumes of such sections — encode subtle quantitative information about the geometry of a body and are related to central problems of the field, cf. for example the Busemann–Petty problem [BP] and the slicing problem (recently settled in [KL]). We refer the reader to the recent survey [NT] as well as the monographs [K2], [BGVV] for a detailed account of history, related works and advances in the field as well as numerous applications.

The present work further explores and complements some of the results of the recent paper [AB], where the first two named authors initiated a systematic study of sections of convex bodies placed in *John's position*. John's position is a canonical normalisation: if the maximal inscribed Euclidean ball of a convex body $K \subset \mathbb{R}^n$ is the unit ball B_2^n, then John [J] exhibited contact points $v_1, \dots, v_m \in \partial K \cap S^{n-1}$ and positive weights $c_1, \dots, c_m$ that satisfy

$$\mathrm{Id}_n = \sum_{j=1}^m c_j\, v_j \otimes v_j, \qquad \sum_{j=1}^m c_j v_j = 0, \qquad \sum_{j=1}^m c_j = n, \tag{1}$$

where Id_n denotes the identity in $\mathbb{R}^n$. These identities provide a flexible algebraic framework that allows one to reduce multidimensional volume problems to a combination of one-dimensional estimates and functional inequalities.

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Recall that if $v_1, \ldots, v_m \in S^{n-1}$ an $c_1, \ldots, c_m \in (0, \infty)$ are as in (1), then for any integrable functions $g_j : \mathbb{R} \to [0, \infty)$,

$$\int_{\mathbb{R}^n} \prod_{j=1}^m g_j^{c_j}(\langle x, v_j \rangle)\, dx \leqslant \prod_{j=1}^m \left(\int_{\mathbb{R}} g_j(t)\, dt \right)^{c_j}. \tag{2}$$

The above so-called geometric form of the classical Brascamp-Lieb inequality [BL] was suggested by Ball (see [B2, Lemma 2]) who originally applied it to efficiently estimate the volume of k-dimensional sections of the unit cube B_∞^n in $\mathbb{R}^n$, showing in particular that

$$\mathrm{vol}_k(B_\infty^n \cap H) \leqslant \left(\frac{n}{k}\right)^{\frac{k}{2}} \mathrm{vol}_k(B_\infty^k), \tag{3}$$

for every k-dimensional linear subspace H of $\mathbb{R}^n$. We remark that this estimate is optimal if and only if k divides n. Using again (2) in conjunction with a Fourier-analytic argument, Ball established in the same work the inequality

$$\mathrm{vol}_k(B_\infty^n \cap H) \leqslant 2^{\frac{n-k}{2}} \mathrm{vol}_k(B_\infty^k) \tag{4}$$

which is an optimal estimate whenever $k \geqslant n/2$. We stress that Ball's approach in this case relies heavily on the product structure of the cube: the Fourier transform of the indicator of a product body factorizes as a product of one-dimensional Fourier transforms, and this factorization is the key to obtaining sharp bounds. In the absence of such coordinate independence — i.e. for general bodies in John's position — the direct product decomposition is no longer available, and the Fourier method appears a priori inapplicable.

One of the results in [AB] was a sharp generalisation of Ball's inequality (3) for k-dimensional sections of the unit cube in the regime $k \leqslant n/2$. Our first main result complements this estimate; we show in Section 3 below that in the case $k \geqslant n/2$, the section $K \cap H$ of an arbitrary centrally symmetric convex body K in John's position in $\mathbb{R}^n$ with a k-dimensional subspace H has volume at most equal to the right hand side in (4), provided that *at least one* of the orthogonal projections of $\sqrt{c_j} v_j$ onto H has length strictly less than $1/2$, where (c_j, v_j) is the John decomposition associated to K. When $k \geqslant n/2$ and all projections $P_H(\sqrt{c_j} v_j)$ are *large*, Ball's $2^{\frac{n+k}{2}}$ bound on the volume of sections is not correct in general; we showcase this by constructing convex polytopes L in John's position with $\mathrm{vol}_k(L \cap \mathbb{R}^k) = 2^k (n/k)^{k/2}$ (see Theorem 5 below). We also provide a general estimate in this case that takes into account the geometry of K, resulting to even finer estimates than (4) in certain special cases (cf. Remark 7).

The main conceptual advance of the present work is to show how the product-structure requirement in Ball's argument for the cube can be circumvented by combining a Parseval-type identity with the Brascamp–Lieb inequality. In our setting, this Parseval representation (Proposition 1) plays the role of the product decomposition in the cube case: the volume is represented as an integral of a product of univariate factors, but now the exponents and prefactors reflect the geometric data of John's decomposition rather than coordinate independence. Once the Parseval representation is in place, a Brascamp–Lieb reduction (Corollary 3) converts the multidimensional integral into a product of one-dimensional integrals with exponents determined by the Euclidean norms of the projections of our vectors. Therefore, whenever we need to bound the volume (or the volume of a section) of a symmetric polytope, there are two equivalent representations. One is the integral of the product of the indicator functions, and the other is the integral (over the complementary subspace) of the Fourier transform. After passing to the Fourier representation we apply the

Brascamp–Lieb inequality; the choice of exponents is governed by the sizes of the projections of the original vectors onto the complementary subspace. This procedure is explained in Section 2.

In parallel to the study in [AB], we have been also concerned with the extent at which our methods can provide meaningful estimates for the Wills functional of sections of convex bodies in John's position (see Section 4 for definitions and background). We provide an estimate that eventually yields a second proof of our aforementioned result on sections of centrally symmetric convex bodies, as well as a result on the mean width of sections, already witnessed in [AB].

Our second result extends the known bounds for sections of the ℓ_p^n–balls, $p \in [1,2]$ (due to Meyer–Pajor [MP] and Koldobsky [K1]). Notably, our proof shows that the same method used for cube-slicing applies here as well. We first express the relevant integral via Parseval's identity, then apply the Brascamp–Lieb inequality, and finally invoke a one-dimensional estimate for γ_p, the Fourier transform of $e^{-|x|^p}$. The technique allows us to consider a family of generalised ℓ_p norms of the form $\left(\sum_{j=1}^m \alpha_j |\langle \cdot, v_j\rangle|^p\right)^{1/p}$, given a John decomposition (c_j, v_j) in $\mathbb{R}^n$ and any positive scalars $(\alpha_j)_{j=1}^m$. We show for example that, for any $p \in [1,2]$, if K_p is the closed unit ball for this norm then

$$\mathrm{vol}_k(K_p \cap H) \leqslant \prod_{j=1}^m \left(\frac{\sqrt{c_j}}{\alpha_j^{1/p}}\right)^{1-\pi_j} \mathrm{vol}_k(B_p^k)$$

for every k-dimensional subspace H of $\mathbb{R}^n$, where $\pi_j = \|P_H(\sqrt{c_j}v_j)\|_2^2$. In the classical case, our approach yields an intermediate bound for arbitrary sections of the ℓ_1^n–ball, that depends only on the Euclidean norms of the projections of the standard basis vectors onto the subspace. Consequently, the resulting bound is more sensitive than that of Meyer–Pajor and Koldobsky. Our results in Section 5 should be compared to some of the results of [Bar] (see Section 3.3 therein, where a sharp, in the case $k \mid n$, estimate is also obtained for $\mathrm{vol}_k(B_p^n \cap H)$ for $p \geqslant 2$).

Our third principal result addresses *hyperplane* sections of a convex body in John's position, without imposing any symmetry assumptions. In this framework, we establish an improvement over the earlier result of [AB], specifically in the regime where all projections onto the given subspace have norm greater than $1/\sqrt{2}$. In this range, our method yields a sharp inequality, thereby refining the known bounds and highlighting a new threshold phenomenon for such sections. The details and results for the non-symmetric setting can be found in Section 6.

2. Parseval's identity and consequences

In this section we present the aforementioned Parseval-type representation and how, when combined with an application of the Brascamp-Lieb inequality, it provides an upper bound for the integral of the product of functions evaluated at the scalar products of the variable against the vectors that form a decomposition of the identity. We will also compare this upper bound with the upper bound we would obtain by directly applying Brascam-Lieb inequality and show that in some cases we obtain a tighter estimate (see Remark 4 below).

Our starting point is an appropriate general form of Parseval's identity. For a function $f : \mathbb{R}^n \to \mathbb{R}$, we denote by $\widehat{f}$ the Fourier transform

$$\widehat{f}(y) = \int_{\mathbb{R}^n} f(x) e^{i\langle x,y\rangle} dx, \qquad y \in \mathbb{R}^n.$$

We denote by $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space of all C^∞ functions $f : \mathbb{R}^n \to \mathbb{R}$ with derivatives that are rapidly decreasing. To overcome any integrability issues and for the sake of clarity, we formulate the results of this Section for functions in $\mathcal{S}(\mathbb{R}^n)$. For our purposes however, we will need them to be applicable for families of characteristic functions on bounded intervals. We have included the technical details that justify how such an extension is possible in an Appendix in the end of the manuscript.

Proposition 1 (Parseval)**.** *Let $m \in \mathbb{N}$, $n_1, \ldots, n_m \in \mathbb{N}$ and set $N = n_1 + \ldots + n_m$. For every linear subspace $H \in G_{N,k}$ and every family of functions $(f_j)_{j=1}^m$ such that $f_j \in \mathcal{S}(\mathbb{R}^{n_j})$, $j = 1, \ldots, m$,*

$$\int_H \prod_{j=1}^m f_j(P_{\mathbb{R}^{n_j}} y)\, dy = \frac{1}{(2\pi)^{N-k}} \int_{H^\perp} \prod_{j=1}^m \widehat{f_j}(P_{\mathbb{R}^{n_j}} z)\, dz.$$

Proof. By definition of the Fourier transform, we have that for every $1 \leqslant j \leqslant m$ and every $z_j \in \mathbb{R}^{n_j}$

$$\widehat{f_j}(z_j) = \int_{\mathbb{R}^{n_j}} f_j(x_j) e^{i\langle x_j, z_j\rangle} dx_j.$$

Therefore,

$$\begin{aligned}
\int_{H^\perp} \prod_{j=1}^m \widehat{f_j}(P_{\mathbb{R}^{n_j}} z) dz &= \int_{H^\perp} \int_{\mathbb{R}^{n_1}} \cdots \int_{\mathbb{R}^{n_m}} f_1(x_1) \ldots f_m(x_m) e^{i\sum_{j=1}^m \langle x_j, P_{\mathbb{R}^{n_j}} z\rangle} dx_m, \ldots dx_1 dz \\
&= \int_{H^\perp} \int_{\mathbb{R}^{n_1}} \cdots \int_{\mathbb{R}^{n_m}} f_1(x_1) \ldots f_m(x_m) e^{i\sum_{j=1}^m \langle x_j, z\rangle} dx_m, \ldots dx_1 dz \\
&= \int_{H^\perp} \int_{\mathbb{R}^N} \prod_{j=1}^m f_j(P_{\mathbb{R}^{n_j}}(x)) e^{i\sum_{j=1}^m \langle P_{\mathbb{R}^{n_j}}(x), z\rangle} dx dz \\
&= \int_{H^\perp} \int_{\mathbb{R}^N} \prod_{j=1}^m f_j(P_{\mathbb{R}^{n_j}}(x)) e^{i\langle x, z\rangle} dx dz.
\end{aligned}$$

Now, by Fubini's theorem,

$$\begin{aligned}
&\int_{H^\perp} \int_{\mathbb{R}^N} \prod_{j=1}^m f_j(P_{\mathbb{R}^{n_j}}(x)) e^{i\langle x, z\rangle} dx dz = \\
&\qquad = \int_{H^\perp} \int_H \int_{H^\perp} \prod_{j=1}^m f_j(P_{\mathbb{R}^{n_j}}(x_H + x_{H^\perp})) e^{i\langle x_H + x_{H^\perp}, z\rangle} dx_{H^\perp} dx_H dz \\
&\qquad = \int_{H^\perp} \int_H \int_{H^\perp} \prod_{j=1}^m (f_j \circ P_{\mathbb{R}^{n_j}})(x_H + x_{H^\perp}) e^{i\langle x_{H^\perp}, z\rangle} dx_{H^\perp} dx_H dz
\end{aligned}$$

Calling, for every $x_H \in H$, $F_{x_H} : H^\perp \to \mathbb{R}$ the function

$$F_{x_H}(x_{H^\perp}) = \prod_{j=1}^m (f_j \circ P_{\mathbb{R}^{n_j}})(x_H + x_{H^\perp}),$$

we have that for every $z \in H^\perp$

$$\int_{H^\perp} \prod_{j=1}^m (f_j \circ P_{\mathbb{R}^{n_j}})(x_H + x_{H^\perp}) e^{i\langle x_{H^\perp}, z\rangle} dx_{H^\perp} = \widehat{F_{x_H}}(z)$$

and then, by Fubini's theorem and Fourier's inversion formula,

$$\int_{H^\perp} \int_H \int_{H^\perp} \prod_{j=1}^m (f_j \circ P_{\mathbb{R}^{n_j}})(x_H + x_{H^\perp}) e^{i\langle x_{H^\perp}, z\rangle} dx_{H^\perp} dx_H dz$$

$$
\begin{aligned}
&= \int_{H^\perp}\int_H \widehat{F_{x_H}}(z)dx_H dz = \int_H\int_{H^\perp} \widehat{F_{x_H}}(z)dz dx_H = (2\pi)^{N-k}\int_H F_{x_H}(0)dx_H \\
&= (2\pi)^{N-k}\int_H \prod_{j=1}^m (f_j\circ P_{\mathbb{R}^{n_j}})(x_H)dx_H = (2\pi)^{N-k}\int_H \prod_{j=1}^m f_j(P_{\mathbb{R}^{n_j}}y)dy,
\end{aligned}
$$

which completes the proof. □

The following proposition is well known, see for example [Iv, Lemma 2.1]. Nevertheless, we include a proof for the sake of completeness.

Proposition 2. *Let $1\leqslant k\leqslant m$. Let $H\in G_{m,k}$ and let $(u_j)_{j=1}^m\subseteq S^{m-1}\cap H$ and $(c_j)_{j=1}^m\subseteq(0,\infty)$ such that*

$$\mathrm{Id}_H=\sum_{j=1}^m c_j u_j\otimes u_j.$$

Then there exists $(x_j)_{j=1}^m$, an orthonormal basis of $\mathbb{R}^m$ and $(w_j)_{j=1}^m\subseteq S^{m-1}\cap H^\perp$ such that

$$P_H x_j=\sqrt{c_j}u_j \qquad \text{and} \qquad P_{H^\perp}x_j=\sqrt{1-c_j}w_j$$

for every $1\leqslant j\leqslant m$. Moreover,

$$\mathrm{Id}_{H^\perp}=\sum_{j=1}^m(1-c_j)w_j\otimes w_j.$$

Proof. We can assume, without loss of generality, that $H=\mathrm{span}\{e_1,\dots,e_k\}\subseteq\mathbb{R}^m$, where $(e_j)_{j=1}^m$ denotes the canonical basis of $\mathbb{R}^m$. Calling $\tilde{u}_j=\sqrt{c_j}u_j$ we have that for every $1\leqslant i\leqslant k$,

$$1=\|e_i\|_2^2=\sum_{j=1}^m\langle e_i,\tilde{u}_j\rangle^2.$$

Besides, for every $1\leqslant i_1<i_2\leqslant k$ we have that

$$0=\langle e_{i_1},e_{i_2}\rangle=\sum_{j=1}^m\langle e_{i_1},\tilde{u}_j\rangle\langle e_{i_2},\tilde{u}_j\rangle.$$

Therefore, calling, for $1\leqslant j\leqslant k$, $v_j\in\mathbb{R}^m$ the j-th row of the $k\times m$ matrix $[\tilde{u}_1,\dots,\tilde{u_m}]$ we have that $\{v_j\,:\,1\leqslant j\leqslant k\}$ is a set of k orthonormal vectors in $\mathbb{R}^m$. Completing this set to an orthormal basis of $\mathbb{R}^m$, $\{v_j\,:\,1\leqslant j\leqslant m\}$ and taking, for $1\leqslant j\leqslant m$, $x_j\in\mathbb{R}^m$ the j-th column of the matrix $[v_1,\dots,v_m]^t$, we have that

- $(x_j)_{j=1}^m\subseteq\mathbb{R}^m$ is an orthonormal basis of $\mathbb{R}^m$,
- $P_Hx_j=\tilde{u}_j=\sqrt{c_j}u_j$ for every $1\leqslant j\leqslant m$.

Besides, if $H^\perp=\mathrm{span}\{e_{k+1}\dots e_m\}$ is the orthogonal complement of H in $\mathbb{R}^m$, we have that for every $1\leqslant j\leqslant m$

$$\|P_{H^\perp}x_j\|_2^2=\|x_j\|_2^2-\|P_Hx_j\|_2^2=1-\|\sqrt{c_j}u_j\|_2^2=1-c_j$$

and, therefore, there exist $(w_j)_{j=1}^m\in S^{m-1}\cap H^\perp$ such that $P_{H^\perp}x_j=\sqrt{1-c_j}w_j$. Moreover, since

$$I_m=\sum_{j=1}^m x_j\otimes x_j,$$

we have that

$$Id_{H^\perp} = \sum_{j=1}^m P_{H^\perp} x_j \otimes P_{H^\perp} x_j = \sum_{j=1}^m (1-c_j) w_j \otimes w_j. \qquad \square$$

As a consequence of, combining the above proposition with Proposition 1, we obtain the following corollary:

Corollary 3. *Let $1 \leqslant n \leqslant m$, $(v_j)_{j=1}^m \subseteq S^{n-1}$ and $(c_j)_{j=1}^m \subseteq (0,\infty)$ such that*

$$I_n = \sum_{j=1}^m c_j v_j \otimes v_j.$$

Then, for every $(f_j)_{j=1}^m \in \mathcal{S}(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} \prod_{j=1}^m f_j(\sqrt{c_j}\langle x, v_j\rangle) dx \leqslant \frac{1}{(2\pi)^{m-n}} \prod_{j=1}^m \left(\int_{\mathbb{R}} |\widehat{f_j}^{\frac{1}{1-c_j}}(\sqrt{1-c_j} t)| dt \right)^{1-c_j}.$$

Proof. Let us identify $\mathbb{R}^n$ with $H := \operatorname{span}\{e_j : 1 \leqslant j \leqslant n\} \subseteq \mathbb{R}^m$, where $(e_j)_{j=1}^m$ denotes the canonical basis in $\mathbb{R}^m$. By Proposition 2 there exists $(x_j)_{j=1}^m$, an orthonormal basis of $\mathbb{R}^m$ and $(w_j)_{j=1}^m \subseteq S^{m-1} \cap H^\perp$ such that

$$P_H x_j = \sqrt{c_j} v_j \qquad \text{and} \qquad P_{H^\perp} x_j = \sqrt{1-c_j} w_j$$

for every $1 \leqslant j \leqslant m$. Besides,

$$\mathrm{Id}_{H^\perp} = \sum_{j=1}^m (1-c_j) w_j \otimes w_j.$$

Writing coordinates with respect to the orthonormal basis of $\mathbb{R}^m$ given by $(x_j)_{j=1}^m$, we have that for every $x \in H$

$$\langle x, x_j\rangle = \langle x, \sqrt{c_j} u_j\rangle \quad \forall 1 \leqslant j \leqslant m$$

and for every $y \in H^\perp$

$$\langle y, x_j\rangle = \langle y, \sqrt{1-c_j} w_j\rangle, \quad \forall 1 \leqslant j \leqslant m$$

By Proposition 1 we have that

$$\int_{\mathbb{R}^n} \prod_{j=1}^m f_j(\sqrt{c_j}\langle x, v_j\rangle) dx = \frac{1}{(2\pi)^{m-n}} \int_{H^\perp} \prod_{j=1}^m \widehat{f_j}(\sqrt{1-c_j}\langle y, w_j\rangle) dy.$$

Since

$$\mathrm{Id}_{H^\perp} = \sum_{j=1}^m (1-c_j) w_j \otimes w_j,$$

by the geometric Brascamp-Lieb inequality we have that

$$\begin{aligned} \int_{H^\perp} \prod_{j=1}^m \widehat{f_j}(\sqrt{1-c_j}\langle y, w_j\rangle) dy &\leqslant \int_{H^\perp} \prod_{j=1}^m |\widehat{f_j}(\sqrt{1-c_j}\langle y, w_j\rangle)| dy \\ &\leqslant \prod_{j=1}^m \int_{\mathbb{R}} \left(|\widehat{f_j}(\sqrt{1-c_j} t)|^{\frac{1}{1-c_j}} dt \right)^{1-c_j}. \end{aligned}$$

Therefore,

$$\int_{\mathbb{R}^n} \prod_{j=1}^m f_j(\sqrt{c_j}\langle x, v_j\rangle) dx \leqslant \frac{1}{(2\pi)^{m-n}} \prod_{j=1}^m \left(\int_{\mathbb{R}} |\widehat{f_j}^{\frac{1}{1-c_j}}(\sqrt{1-c_j} t)| dt \right)^{1-c_j}. \qquad \square$$

Let us compare the upper bound obtained in the previous corollary with the upper bound obtained by directly applying Brascamp-Lieb inequality.

Remark 4. Notice that if for every $1 \leqslant j \leqslant m$ we have that $\frac{1}{1-c_j} \geqslant 2$, which happens if and only if $c_j \geqslant \frac{1}{2}$, then, by Beckner's sharp Haussdorff-Young inequality [Be], we have that for every $1 \leqslant j \leqslant m$,

$$\begin{aligned}
\left(\int_{\mathbb{R}} |\widehat{f_j}^{\frac{1}{1-c_j}}(\sqrt{1-c_j}t)|dt\right)^{1-c_j} &= \left(\frac{1}{\sqrt{1-c_j}}\right)^{1-c_j}\left(\int_{\mathbb{R}}|\widehat{f_j}(s)|^{\frac{1}{1-c_j}}ds\right)^{1-c_j} \\
&\leqslant (2\pi)^{1-c_j}\left(\frac{1}{\sqrt{1-c_j}}\right)^{1-c_j}\left(\frac{\left(\frac{1}{\sqrt{c_j}}\right)^{c_j}}{\left(\frac{1}{\sqrt{1-c_j}}\right)^{1-c_j}}\right)\left(\int_{\mathbb{R}}|f_j(s)|^{\frac{1}{c_j}}ds\right)^{c_j} \\
&= (2\pi)^{1-c_j}\left(\frac{1}{\sqrt{c_j}}\right)^{c_j}\left(\int_{\mathbb{R}}|f_j(s)|^{\frac{1}{c_j}}ds\right)^{c_j}
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_{\mathbb{R}^n}\prod_{j=1}^m f_j(\sqrt{c_j}\langle x, v_j\rangle)dx &\leqslant \frac{1}{(2\pi)^{m-n}}\prod_{j=1}^m (2\pi)^{1-c_j}\left(\frac{1}{\sqrt{c_j}}\right)^{c_j}\left(\int_{\mathbb{R}}|f_j(s)|^{\frac{1}{c_j}}ds\right)^{c_j} \\
&= \prod_{j=1}^m\left(\frac{1}{\sqrt{c_j}}\right)^{c_j}\left(\int_{\mathbb{R}}|f_j(s)|^{\frac{1}{c_j}}ds\right)^{c_j}.
\end{aligned}$$

On the other hand, applying directly Brascamp-Lieb inequality we obtain

$$\begin{aligned}
\int_{\mathbb{R}^n}\prod_{j=1}^m f_j(\sqrt{c_j}\langle x, v_j\rangle)dx &\leqslant \prod_{j=1}^m\left(\int_{\mathbb{R}}|f_j(\sqrt{c_j}t)|^{\frac{1}{c_j}}dt\right)^{c_j} \\
&= \prod_{j=1}^m\left(\frac{1}{\sqrt{c_j}}\right)^{c_j}\left(\int_{\mathbb{R}}|f_j(t)|^{\frac{1}{c_j}}dt\right)^{c_j}.
\end{aligned}$$

Therefore, the estimate we obtain applying Brascamp-Lieb inequality to the Fourier transforms is better than the one we obtain applying Brascamp-Lieb inequality to the functions itself whenever $c_j \geqslant \frac{1}{2}$ for every $1 \leqslant j \leqslant m$.

3. Centrally Symmetric Case

We start with a convex body K in John's position with associated decomposition of the identity (c_j, v_j), i.e. we assume that $v_1, \dots, v_m \in S^{n-1} \cap \partial K$ and $c_1, \dots, c_m > 0$ are such that

$$\mathrm{Id}_n = \sum_{j=1}^m c_j v_j \otimes v_j.$$

Let $H \in G_{n,k}$ and $J = \{1 \leqslant j \leqslant m : P_H v_j \neq 0\}$. We set $m_0 := \sharp J$ and, for every $j \in J$, let

- $u_j = \dfrac{P_H v_j}{\|P_H v_j\|_2}$,
- $\tilde{c}_j = c_j\|P_H v_j\|_2^2$,
- $t_j = \|P_H v_j\|_2^{-1}$.

It follows that

$$\mathrm{Id}_H = \sum_{j\in J} \tilde{c}_j u_j \otimes u_j.$$

Let us consider the symmetric convex polytope

$$L = \{x \in \mathbb{R}^n \,:\, |\langle x, v_j\rangle| \leqslant 1\}.$$

It is easy to see that $K \subseteq L$. We will in fact estimate the volume of $K \cap H$ by the volume of

$$L \cap H = \{x \in H : |\langle x, u_j\rangle| \leqslant t_j \text{ for every } j \in J\}\,. \tag{5}$$

In general, it is not true that we can obtain the analogous bound for the k-dimensional sections of L that Ball obtained for the cube, i.e. that for every $1 \leqslant k \leqslant n$ with $k \geqslant n/2$,

$$\mathrm{vol}_k(L \cap H) \leqslant 2^{\frac{n-k}{2}}\, \mathrm{vol}_k(B_\infty^k).$$

This is showcased by the following example.

Theorem 5. *There exists $k \in \mathbb{N}$ arbitrarily large such that for every $n \in \mathbb{N}$ with $n/2 \leqslant k \leqslant n$ there exists a convex body L in $\mathbb{R}^n$ which is in John's position, while*

$$\mathrm{vol}_k(L \cap \mathbb{R}^k) = \left(\frac{n}{k}\right)^{\frac{k}{2}} \mathrm{vol}_k(B_\infty^k).$$

Proof. Let $k \in \mathbb{N}$ be such that there exists a $k \times k$ Hadamard matrix (for instance, let k be any power of 2) and take any $k \leqslant n \leqslant 2k$. Let $m = 2k \geqslant n$ and define unit vectors $v_1, \ldots, v_m$ as follows: Start with an arbitrary $k \times k$ Hadamard matrix H_k with columns $\eta_1, \ldots, \eta_k$, that is, the components of each η_j are ± 1 and $\langle \eta_i, \eta_j\rangle = 0$ for every $i \neq j$. The matrix

$$H_m = \begin{pmatrix} H_k & H_k \\ H_k & -H_k \end{pmatrix}$$

is then a $m \times m$ Hadamard matrix. Let B be the upper $n \times m$ submatrix of H_m. If $\beta_1, \ldots, \beta_m$ are the columns of B, we define $v_j := n^{-\frac{1}{2}} \beta_j$ for every $j = 1, \ldots, m$. Clearly, the v_j's are unit vectors in $\mathbb{R}^n$ and if we let $c_j = n/(2k)$ for every $j = 1, \ldots, m$ we can check that

$$\sum_{j=1}^m c_j v_j \otimes v_j = \frac{1}{2k} \sum_{j=1}^m \beta_j \otimes \beta_j = \mathrm{Id}_n.$$

This shows that (c_j, v_j) induce a decomposition of the identity in $\mathbb{R}^n$. The symmetric convex polytope $L = \{x \in \mathbb{R}^n : |\langle x, v_j\rangle| \leqslant 1, j = 1, \ldots, m\}$ is then in John's position and note that

$$\begin{aligned} L \cap \mathbb{R}^k &= \left\{x \in \mathbb{R}^k : |\langle x, P_{\mathbb{R}^k} v_j\rangle| \leqslant 1,\, j = 1, \ldots, m\right\} \\ &= \left\{x \in \mathbb{R}^k : |\langle x, P_{\mathbb{R}^k} \beta_j\rangle| \leqslant \sqrt{n},\, j = 1, \ldots, m\right\} \\ &= \left\{x \in \mathbb{R}^k : |\langle x, \eta_j\rangle| \leqslant \sqrt{n},\, j = 1, \ldots, k\right\}, \end{aligned}$$

since, by our construction, $P_{\mathbb{R}^k} \beta_j = P_{\mathbb{R}^k} \beta_{k+j} = \eta_j$ for every $j = 1, \ldots, k$. Note also that if we let $W = k^{-\frac{1}{2}} H_k$, then we have that

$$W^T W = \frac{1}{k} H_k^T H_k = \mathrm{Id}_k.$$

We can ultimately write

$$\begin{aligned}\mathrm{vol}_k(L\cap\mathbb{R}^k)&=\mathrm{vol}_k(\{x\in\mathbb{R}^k:\|H_k^Tx\|_\infty\leqslant\sqrt{n}\})\\&=\mathrm{vol}_k\left(\left\{Wy\in\mathbb{R}^k:\|y\|_\infty\leqslant\sqrt{\frac{n}{k}}\right\}\right)|=\mathrm{vol}_k\left(W\left(\sqrt{\frac{n}{k}}B_\infty^k\right)\right)=\left(\frac{n}{k}\right)^{\frac{k}{2}}\mathrm{vol}_k(B_\infty^k),\end{aligned}$$

since $\det(W^TW)=1$, and the proof is complete. $\square$

Note that in the construction carried out in the preceding proof, we have $\tilde{c}_j=c_j\|P_Hv_j\|_2=1/2$ for every $j=1,\dots,m$. The following is the main result of the present section.

Theorem 6. *For every $1\leqslant k\leqslant n$ with $k\geqslant n/2$, the following holds:*

- *If $\tilde{c}_j\geqslant\frac{1}{2}$ for all j, then*

$$\mathrm{vol}_k(L\cap H)\leqslant 2^{\frac{m_0+k}{2}}\prod_{j=1}^{m_0}c_j^{\frac{c_j}{2t_j^2}}. \tag{6}$$

  In particular,

$$\mathrm{vol}_k(L\cap H)\leqslant 2^k\left(\frac{n-2k+m_0}{m_0-k}\right)^{\frac{m_0-k}{2}}. \tag{7}$$

- *Otherwise,* $\mathrm{vol}_k(L\cap H)\leqslant 2^{\frac{n-k}{2}}\mathrm{vol}_k(B_\infty^k)$.

Proof. Let us identify $\mathbb{R}^n$ with $\mathrm{span}\{e_j:1\leqslant j\leqslant n\}\subseteq\mathbb{R}^{m_0}$, where $(e_j)_{j=1}^{m_0}$ denotes the canonical basis in $\mathbb{R}^{m_0}$. By Proposition 2, there exits $(x_j)_{j=1}^{m_0}$, an orthonormal basis of $\mathbb{R}^{m_0}$ and $(w_j)_{j=1}^{m_0}\subseteq S^{m-1}\cap H^\perp$, where $H^\perp$ denotes the orthogonal linear subspace to H in $\mathbb{R}^{m_0}$ such that

$$P_Hx_j=\sqrt{\tilde{c}_j}u_j\qquad\text{and}\qquad P_{H^\perp}x_j=\sqrt{1-\tilde{c}_j}w_j$$

for every $1\leqslant j\leqslant m_0$. Writing coordinates with respect to the orthonormal basis of $\mathbb{R}^{m_0}$ given by $(x_j)_{j=1}^{m_0}$, we have that for every $x\in H$

$$\langle x,x_j\rangle=\langle\sqrt{\tilde{c}_j}u_j,x\rangle\quad\forall 1\leqslant j\leqslant m_0$$

and for every $y\in H^\perp$

$$\langle y,x_j\rangle=\langle\sqrt{1-\tilde{c}_j}w_j,y\rangle,\quad\forall 1\leqslant j\leqslant m_0$$

and the convex set

$$C_1=\left\{\sum_{j=1}^{m_0}a_jx_j\ :\ |a_j|\leqslant\sqrt{\tilde{c}_j}t_j,\ \forall 1\leqslant j\leqslant m_0\right\}$$

verifies that $C_1\cap H=L\cap H$. We distinguish two cases:

Case I: Assume that for every unit vector $\xi\in H^\perp$, $\left|\langle\sqrt{1-\tilde{c}_j}w_j,\xi\rangle\right|\leqslant 1/\sqrt{2}$ for every $1\leqslant j\leqslant m$. Note that in particular, considering $\xi=w_j$, this hypothesis implies that $\sqrt{1-\tilde{c}_j}\leqslant 1/\sqrt{2}$ or, equivalently, $\tilde{c}_j\geqslant\frac{1}{2}$ for every $1\leqslant j\leqslant m_0$. Applying Proposition 1 with $n_1=\ldots=n_{m_0}=1$ and $f_j:\mathbb{R}\to\mathbb{R}$ given by $f_j(x)=\mathbb{1}_{[-\sqrt{\tilde{c}_j}t_j,\sqrt{\tilde{c}_j}t_j]}$ we have that writing coordinates in $\mathbb{R}^{m_0}$ with respect to $(x_j)_{j=1}^{m_0}$,

$$\begin{aligned}
\mathrm{vol}_k(L\cap H) &= \int_H \prod_{j=1}^{m_0} \mathbb{1}_{[-\sqrt{\tilde{c}_j}t_j,\sqrt{\tilde{c}_j}t_j]}\left(\langle\sqrt{\tilde{c}_j}u_j,x\rangle\right)dx\\
&= \frac{1}{(2\pi)^{m_0-k}}\int_{H^\perp}\prod_{j=1}^{m_0}\widehat{\mathbb{1}}_{[-\sqrt{\tilde{c}_j}t_j,\sqrt{\tilde{c}_j}t_j]}\left(\langle\sqrt{1-\tilde{c}_j}w_j,y\rangle\right)dy\\
&= \frac{1}{(2\pi)^{m_0-k}}\int_{H^\perp}\prod_{j=1}^{m_0}\frac{2\sin\left(\sqrt{\tilde{c}_j}t_j\sqrt{1-\tilde{c}_j}\langle w_j,y\rangle\right)}{\sqrt{1-\tilde{c}_j}\langle w_j,y\rangle}dy,
\end{aligned}$$

where we have used that for any $c>0$, $\widehat{\mathbb{1}}_{[-c,c]}(t)=\frac{2\sin(ct)}{t}$. By the geometric Brascamp-Lieb inequality applied on $H^\perp$ to the vectors $(w_j)_{J=1}^{m_0}$ and the non-negative numbers $((1-\tilde{c}_j))_{j=1}^{m_0}$, followed by a change of variables, we get

$$\begin{aligned}
\mathrm{vol}_k(L\cap H) &\leqslant \frac{1}{(2\pi)^{m_0-k}}\prod_{j=1}^{m_0}\left(\int_{\mathbb{R}}\left|\frac{2\sin\left(\sqrt{\tilde{c}_j}t_j\sqrt{1-\tilde{c}_j}u\right)}{\sqrt{1-\tilde{c}_j}u}\right|^{\frac{1}{1-\tilde{c}_j}}du\right)^{1-\tilde{c}_j}\\
&= \frac{2^k}{\pi^{m_0-k}}\prod_{j=1}^{m_0}\sqrt{\tilde{c}_j}t_j\left(\frac{1}{t_j\sqrt{\tilde{c}_j(1-\tilde{c}_j)}}\right)^{1-\tilde{c}_j}\left(\int_{\mathbb{R}}\left|\frac{\sin(\pi x_j)}{\pi x_j}\right|^{\frac{1}{1-\tilde{c}_j}}dx_j\right)^{1-\tilde{c}_j}. \qquad (8)
\end{aligned}$$

Since $\sqrt{1-\tilde{c}_j}\leqslant 1/\sqrt{2}$ for every $1\leqslant j\leqslant m$ we can apply Ball's integral inequality [B1], to obtain

$$\mathrm{vol}_k(L\cap H) \leqslant 2^k\prod_{j=1}^{m_0}(\sqrt{\tilde{c}_j}t_j)^{\tilde{c}_j}\left(\frac{1}{\sqrt{(1-\tilde{c}_j)}}\right)^{1-\tilde{c}_j}\left(\sqrt{2(1-\tilde{c}_j)}\right)^{1-\tilde{c}_j}=2^{\frac{m_0+k}{2}}\prod_{j=1}^{m_0}c_j^{\frac{c_j}{2t_j^2}}.$$

Now, we will find an upper bound for the function

$$f(\tilde{c},c)=\sum_{j=1}^{m_0}\tilde{c}_j\log c_j.$$

This is linear with respect to $\tilde{c}$ and it is defined in the set $1/2\leqslant\tilde{c}_j\leqslant 1$ intersected with the set $\sum\tilde{c}_j=k$. This has maximum when $\tilde{c}_j\in\{1/2,1\}$ for all $j\in J$. In order to have $\sum\tilde{c}_j=k$, the set

$$I=\{j\in J: c_j=1\}$$

should have cardinality $2k-m_0$. Its complement, I^c, has cardinality $2m_0-2k$. Therefore,

$$\begin{aligned}
f(\tilde{c},c) &\leqslant \sum_{j\in I}\log c_j+\sum_{j\notin I}\frac{1}{2}\log c_j=\\
&= \sum_{j\notin I}\frac{1}{2}\log c_j\\
&\leqslant \frac{|I^c|}{2}\log\left(\frac{\sum_{j\notin I}c_j}{|I^c|}\right)\\
&= (m_0-k)\log\left(\frac{\sum_{j\in J}c_j-\sum_{j\in I}\log c_j}{2m_0-2k}\right)\\
&\leqslant (m_0-k)\log\left(\frac{n-(2k-m_0)}{2m_0-2k}\right),
\end{aligned}$$

where we used Jensen's inequality for the concave function $\log x$ and the fact that $\sum_{j\in J}c_j\leqslant n$.

Case II: We will prove this by induction. Assume that the result is true for any subspace Q of $\mathbb{R}^n$, with $\dim(Q)=n-1$ and any k dimensional subspace. This means that we assume that if C is any convex body in John's position in Q and H is a k dimensional subspace, then $\mathrm{vol}_k(C\cap H)\leqslant(\sqrt{2})^{n-1+k}$.

Suppose that there is a unit vector $\xi\in H^{\perp}$, such that $\left|\langle\sqrt{1-\tilde{c}_j}w_j,\xi\rangle\right|>1/\sqrt{2}$ for some $1\leqslant j\leqslant m$. Without loss of generality we can assume that $j=1$. Moreover, since $\xi\in H^{\perp}$, the latter is equivalent to $|\langle x_1,\xi\rangle|>1/\sqrt{2}$. Consider the set

$$C_2=\left\{\lambda x_1+\sum_{j=2}^{m_0}a_jx_j\ :\ |a_j|\leqslant\sqrt{\tilde{c}_jt_j},\,\forall\ 2\leqslant j\leqslant m_0\ \text{ and }\ \lambda\in\mathbb{R}\right\}.$$

It is clear that $C_1\subseteq C_2$, therefore

$$\mathrm{vol}_k(L\cap H)=\mathrm{vol}_k(C_1\cap H)\leqslant\mathrm{vol}_k(C_2\cap H)=\mathrm{vol}_k(P_H(C_2\cap H)).$$

Since $H\subseteq\xi^{\perp}$, we obtain that $\mathrm{vol}_k(P_H(C_2\cap H))\leqslant\mathrm{vol}_k(P_{\xi^\perp}(C_2\cap H))$. Note that if ξ is different than x_1, the subspaces $\xi^{\perp}$ and $x_1^{\perp}$ intersect in an $m-2$-dimensional subspace of $\mathbb{R}^m$, therefore we can find two orthonormal bases, for $\xi^{\perp}$ and $x_1^{\perp}$, respectively, such that they differ to only one element. Then we can easily find that the linear transformation T such that $TP_{x_1^\perp}=P_{\xi^\perp}$ has determinant $\langle x_1,\xi\rangle$. Therefore,

$$\mathrm{vol}_k(P_{\xi^\perp}(C_2\cap H))=\frac{1}{|\langle x_1,\xi\rangle|}\,\mathrm{vol}_k(P_{x_1^\perp}(C_2\cap H))<\sqrt{2}\,\mathrm{vol}_k(P_{x_1^\perp}(C_2\cap H)).$$

Finally, we write $\mathrm{vol}_k(P_{x_1^\perp}(C_2\cap H))=\mathrm{vol}_k(P_{\mathbb{R}^n}P_{x_1^\perp}(C_2\cap H))$ and we would like to express $P_{\mathbb{R}^n}P_{x_1^\perp}(C_2)$ back in $\mathbb{R}^n$. Setting $W=P_{\mathbb{R}^n}P_{x_1^\perp}(\mathbb{R}^m)$, we observe that

$$I_W=\sum_{j=2}^{m}c_jv_j\otimes v_j\quad\text{and}\quad P_{\mathbb{R}^n}P_{x_1^\perp}(C_2)=\{x\in W:|\langle x,v_i\rangle|\leqslant 1,j\geqslant 2\}.$$

By John's theorem $P_{\mathbb{R}^n}P_{x_1^\perp}(C_2)$ is in John's position in W and W has dimension $n-1$, since $v_1\notin W$. Therefore, by the inductive hypothesis, we get that

$$\mathrm{vol}_k(P_{\mathbb{R}^n}P_{x_1^\perp}(C_2\cap H))\leqslant(\sqrt{2})^{n-1+k}.$$

Putting everything together, we get $\mathrm{vol}_k(L\cap H)\leqslant(\sqrt{2})^{n+k}$, which is the desired result. □

Remark 7. It is noteworthy that in the case $m_0\leqslant n$, the upper bound (7) in Theorem 6 is better than Ball's general $2^{\frac{n+k}{2}}$ estimate. This is because, for $\alpha=\frac{n-k}{m_0-k}$, Bernoulli's inequality $\alpha+1\leqslant 2^{\alpha}$ is true for any integer $\alpha\geqslant 1$, equivalently $n\geqslant m_0$.

3.1. **A comparison between estimates.** Under the notation we have set above, it was proved in [AB, Theorem 1.1] that if $K\subseteq\mathbb{R}^n$ is a centrally symmetric convex body in John's position and $H\in G_{n,k}$ is a k-dimensional linear subspace, then

$$\mathrm{vol}_k\left(K\cap H\right)\leqslant 2^k\prod_{j\in J_0}\left(\frac{c_j}{\tilde{c}_j}\right)^{\frac{\tilde{c}_j}{2}},\tag{9}$$

where

$$I_n=\sum_{j=1}^{m}c_jv_j\otimes v_j$$

is the decomposition of the identity associated to K. We have now obtained in (6) that if for every $j \in J_0$ we have that $\tilde{c}_j \geqslant \frac{1}{2}$, then

$$\mathrm{vol}_k\,(K \cap H) \leqslant 2^{\frac{m_0+k}{2}} \prod_{j\in J_0} c_j^{\frac{\tilde{c}_j}{2}}. \tag{10}$$

Let us see that in some cases, the estimate given by (10) can be better than the estimate given by (9): Notice that

$$2^{\frac{m_0+k}{2}} \prod_{j\in J_0} c_j^{\frac{\tilde{c}_j}{2}} \leqslant 2^k \prod_{j\in J_0} \left(\frac{c_j}{\tilde{c}_j}\right)^{\frac{\tilde{c}_j}{2}}$$

is equivalent to

$$2^{\frac{m_0-k}{2}} \leqslant \prod_{j\in J_0} \left(\frac{1}{\tilde{c}_j}\right)^{\frac{\tilde{c}_j}{2}},$$

which is equivalent to

$$\prod_{j\in J_0} (\tilde{c}_j)^{\tilde{c}_j} \leqslant \left(\frac{1}{2}\right)^{m_0-k}$$

and, therefore, to

$$\sum_{j\in J_0} \tilde{c}_j \log \tilde{c}_j \leqslant (m_0-k)\log\frac{1}{2}. \tag{11}$$

Consider the case in which $k \geqslant \frac{m_0}{2}$ let $K = \left\{(x_j)_{j\in J_0} \,:\, \frac{1}{2} \leqslant x_j \leqslant 1\,,\, \sum_{j\in J_0} x_j = k\right\}$, which is convex and invariant under permutations of coordinates, and let $f : K \to \mathbb{R}$ be the function given by

$$f(x) = \sum_{j\in J_0} x_j \log x_j.$$

Notice that f is convex function and that for every $(x_j)_{j\in J_0}$ we have that

$$(x_j)_{j\in J_0} \prec \left(1,\dots,1,\frac{1}{2},\dots,\frac{1}{2}\right) = (y_j)_{j\in J_0},$$

where there are $2k-m_0$ 1's and $2(m_0-k)$ $\frac{1}{2}$'s. That is,

- $\sum_{j\in J_0} x_j = k = \sum_{j\in J_0} y_j$
- For every $k \in J_0$, $\sum_{j\in J_0, j\leqslant k} x_j^* \leqslant \sum_{j\in J_0, j\leqslant k} y_j^*$, where (z_j^*) denotes the nonincreasing rearrangement of (z_j).

Then, by Karamata's inequality we have that for every $x \in K$

$$f(x) \leqslant f(y) = m_0 - k\log\frac{1}{2},$$

which implies (11).

This shows that even after applying Ball's inequality, the bound we obtain when $\tilde{c}_j \geqslant \frac{1}{2}$ for every $j \in J_0$ if we first apply Parseval's identity and then the Brascamp-Lieb inequality is better than the one we obtain if we apply Brascamp-Lieb directly.

4. The Wills functional of sections of centrally symmetrc convex bodies in John's position

In this section we will use the method developed in Section 2 in order to give an upper bound for the Wills functional of sections of centrally symmetric convex bodies in John's position. We will also show that from such upper bound, one can recover the upper bound for the volume of sections provided by (8) and the upper bound for the mean width of sections provided in the proof of [AB, Theorem 1.5].

We first carry out some preliminary work towards the results of this section, starting with the following lemma.

Lemma 8. *Let $\alpha > 0$ and $f : \mathbb{R} \to \mathbb{R}$ be the function*

$$f(x) = e^{-\pi d^2(x,[-\alpha,\alpha])}.$$

Then $\widehat{f} : \mathbb{R} \to \mathbb{R}$ is given by

$$\widehat{f}(z) = \begin{cases} \frac{2\sin(\alpha z)}{z} + \cos(\alpha z)e^{-\frac{z^2}{4\pi}} - 2\sin(\alpha z)\int_0^\infty e^{-\pi y^2}\sin(yz)dy & \text{if } z \neq 0 \\ 2\alpha + 1 & \text{if } z = 0. \end{cases}$$

Proof. For every $z \in \mathbb{R}$ we have that

$$\begin{aligned} \widehat{f}(z) &= \int_{\mathbb{R}} e^{-\pi d^2(x,[-\alpha,\alpha])} e^{ixz} dx = 2\int_0^\infty e^{-\pi d^2(x,[-\alpha,\alpha])}\cos(xz)dx \\ &= 2\int_0^\alpha \cos(xz)dx + 2\int_\alpha^\infty e^{-\pi(x-\alpha)^2}\cos(xz)dx \\ &= 2\int_0^\alpha \cos(xz)dx + 2\int_0^\infty e^{-\pi y^2}\cos((y+\alpha)z)dy \\ &= 2\int_0^\alpha \cos(xz)dx + 2\cos(\alpha z)\int_0^\infty e^{-\pi y^2}\cos(yz)dy - 2\sin(\alpha z)\int_0^\infty e^{-\pi y^2}\sin(yz)dy \\ &= 2\int_0^\alpha \cos(xz)dx + \cos(\alpha z)e^{-\frac{z^2}{4\pi}} - 2\sin(\alpha z)\int_0^\infty e^{-\pi y^2}\sin(yz)dy. \end{aligned}$$

Since

$$\int_0^\alpha \cos(xz)dx = \begin{cases} \frac{\sin(\alpha z)}{z} & \text{if } z \neq 0 \\ \alpha & \text{if } z = 0 \end{cases}$$

we obtain the result. □

We will estimate from above the Wills functional of the section of a convex body K in John's position applying the geometric Brascamp-Lieb inequality, in the spirit of Section 2. In the next statement note that, in view of Lemma 8,

$$h_{\lambda,j}(t) = \left|\widehat{f}_{\lambda\sqrt{c_j}}(\sqrt{1-\tilde{c}_j}t)\right|^{\frac{1}{2-\tilde{c}_j}}.$$

Proposition 9. *Let $K \subseteq \mathbb{R}^n$ be a centrally symmetric convex body in John's position and let $H \in G_{n,k}$. Then, for any $\lambda > 0$ we have that*

$$\mathcal{W}(\lambda(K \cap H)) \leqslant \frac{1}{(2\pi)^{m_0-k}} \prod_{j\in J} \left(\int_{\mathbb{R}} h_{\lambda,j}(t)dt\right)^{1-\tilde{c}_j},$$

where, for any $t \in \mathbb{R} \setminus \{0\}$

$$\begin{aligned} h_{\lambda,j}(t) &= \left| \frac{2\sin(\lambda\sqrt{\tilde{c}_j}t_j\sqrt{1-\tilde{c}_j}t)}{\sqrt{1-\tilde{c}_j}t} + \cos(\lambda\sqrt{\tilde{c}_j}t_j\sqrt{1-\tilde{c}_j}t)e^{-\frac{(1-\tilde{c}_j)t^2}{4\pi}} \right. \\ &- \left. 2\sin(\lambda\sqrt{\tilde{c}_j}t_j\sqrt{1-\tilde{c}_j}t)\int_0^\infty e^{-\pi y^2}\sin(y\sqrt{1-\tilde{c}_j}t)dy \right|^{\frac{1}{1-\tilde{c}_j}} \end{aligned}$$

and

$$h_{\lambda,j}(0) = (2\lambda_j\sqrt{\tilde{c}_j}t_j + 1)^{\frac{1}{1-\tilde{c}_j}}.$$

Proof. Let $C := L \cap H$ be defined as in (5). We have that for every $j \in J$

$$P_{\langle u_j \rangle}(\lambda C) \subseteq [-\lambda t_j, \lambda t_j]u_j.$$

Therefore, for every $x \in H$,

$$d(\langle x, u_j\rangle u_j, P_{\langle u_j\rangle}(\lambda C)) \geqslant d(\langle x, u_j\rangle u_j, [-\lambda t_j, \lambda t_j]u_j) = d(\langle x, u_j\rangle, [-\lambda t_j, \lambda t_j]).$$

Since for every $x_0 \in \lambda C$, every $x \in H$ and every $j \in J$ we have that

$$d^2(\langle x, u_j\rangle u_j, P_{\langle u_j\rangle}(\lambda C)) \leqslant d^2(\langle x, u_j\rangle u_j, \langle x_0, u_j\rangle u_j) = \langle x - x_0, u_j\rangle^2,$$

we have that for every $x_0 \in \lambda C$ and every $x \in H$,

$$\sum_{j\in J}\tilde{c}_j d^2(\langle x, u_j\rangle u_j, P_{\langle u_j\rangle}(\lambda C)) \leqslant \sum_{j\in J}\tilde{c}_j\langle x - x_0, u_j\rangle^2 = \|x - x_0\|_2^2$$

and, taking infimum in $x_0 \in \lambda C$, we have that

$$d^2(x, \lambda C) \geqslant \sum_{j\in J}\tilde{c}_j d^2(\langle x, u_j\rangle u_j, P_{\langle u_j\rangle}(\lambda C)) \geqslant \sum_{j\in J}\tilde{c}_j d^2(\langle x, u_j\rangle, [-\lambda t_j, \lambda t_j]).$$

Therefore, calling $f_j(t) = e^{-\pi d^2(t,[-\lambda\sqrt{\tilde{c}_j}t_j, \lambda\sqrt{\tilde{c}_j}t_j])}$ for every $j \in J$, we have that

$$\begin{aligned} \mathcal{W}(\lambda C) = \int_H e^{-\pi d^2(x,\lambda C)}dx &\leqslant \int_H e^{-\sum_{j\in J}\tilde{c}_j\pi d^2(\langle x,u_j\rangle,[-\lambda t_j,\lambda t_j])}dx \\ &= \int_H e^{-\sum_{j\in J}\pi d^2(\sqrt{\tilde{c}_j}\langle x,u_j\rangle,[-\lambda\sqrt{\tilde{c}_j}t_j,\lambda\sqrt{\tilde{c}_j}t_j])}dx \\ &= \int_H \prod_{j\in J} f_j(\sqrt{\tilde{c}_j}\langle x, u_j\rangle)dx. \end{aligned}$$

By Corollary 3 and Lemma 8 it follows then that,

$$\begin{aligned} \mathcal{W}(\lambda C) &\leqslant \frac{1}{(2\pi)^{m_0-k}}\prod_{j\in J}\left(\int_{\mathbb{R}}\left|\widehat{f_j}^{\frac{1}{1-\tilde{c}_j}}(\sqrt{1-\tilde{c}_j}t)\right|dt\right)^{1-\tilde{c}_j} \\ &= \frac{1}{(2\pi)^{m_0-k}}\prod_{j\in J}\left(\int_{\mathbb{R}} h_{\lambda,j}(t)dt\right)^{1-\tilde{c}_j}. \end{aligned}$$

□

Along this section, for a fixed index $j \in J$ set, for any $\lambda > 0$, we will denote

$$\tilde{c} = \tilde{c}_j \in (0,1), \qquad p := p_j = \frac{1}{1-\tilde{c}} > 1, \qquad \alpha := \alpha_j = \lambda\sqrt{\tilde{c}}t_j.$$

We will also define

$$a_\alpha(s) = \begin{cases} \frac{2\sin(\alpha s)}{s} & \text{if } s \neq 0 \\ 2\alpha & \text{if } s = 0 \end{cases}, \qquad b_\alpha(s) = \cos(\alpha s)e^{-s^2/(4\pi)}, \qquad I(s) = \int_0^\infty e^{-\pi y^2}\sin(ys)\,dy,$$

$$c_\alpha(s) = -2\sin(\alpha s) I(s), \qquad A_\alpha(s) = a(s) + b(s) + c(s).$$

Making the change of variables $s = \sqrt{1-\tilde{c}}\, t$ (so $dt = ds/\sqrt{1-\tilde{c}}$). Then

$$\int_{\mathbb{R}} h_{\lambda,j}(t)\, dt = \frac{1}{\sqrt{1-\tilde{c}}} \int_{\mathbb{R}} |A_\alpha(s)|^p\, ds.$$

The following lemma concerns the behavior of $I(s)$.

Lemma 10. *There exists $M > 0$ such that for every $s \in \mathbb{R}$,*

$$\big|1 - sI(s)\big| \leqslant \frac{M}{1+s^2}.$$

Proof. Set $f(y) = e^{-\pi y^2}$. We first integrate by parts once in the definition of $I(s)$. Taking $dv = \sin(ys)\, dy$ and $u = f(y)$ we get $v = -\frac{\cos(ys)}{s}$ and $u' = f'(y) = -2\pi y f(y)$. Hence

$$I(s) = \Big[-\frac{f(y)\cos(ys)}{s}\Big]_0^\infty + \frac{1}{s}\int_0^\infty f'(y)\cos(ys)\, dy.$$

Using $f(\infty) = 0,\ f(0) = 1$ and $f'(y) = -2\pi y f(y)$ we obtain

$$I(s) = \frac{1}{s} - \frac{2\pi}{s}\int_0^\infty y f(y)\cos(ys)\, dy.$$

Therefore, letting

$$J(s) := \int_0^\infty y f(y)\cos(ys)\, dy, \qquad \text{we have} \qquad 1 - sI(s) = 2\pi J(s).$$

We now show that $J(s) = O(1/s^2)$ as $|s| \to \infty$. Integrate $J(s)$ by parts twice. Put $u(y) = yf(y)$ and $dv = \cos(ys)\, dy$. Then $v = \frac{\sin(ys)}{s}$ and

$$u'(y) = \frac{d}{dy}\big(ye^{-\pi y^2}\big) = (1 - 2\pi y^2)e^{-\pi y^2} =: g(y).$$

Thus

$$J(s) = \Big[\frac{yf(y)\sin(ys)}{s}\Big]_0^\infty - \frac{1}{s}\int_0^\infty g(y)\sin(ys)\, dy = -\frac{1}{s}\int_0^\infty g(y)\sin(ys)\, dy,$$

because the boundary terms vanish.

Integrate the remaining integral by parts again, with $dv = \sin(ys)\, dy$ and $v = -\frac{\cos(ys)}{s}$. This gives

$$\int_0^\infty g(y)\sin(ys)\, dy = \Big[-\frac{g(y)\cos(ys)}{s}\Big]_0^\infty + \frac{1}{s}\int_0^\infty g'(y)\cos(ys)\, dy.$$

The boundary term at infinity vanishes because g is a polynomial times a Gaussian. Hence

$$\int_0^\infty g(y)\sin(ys)\, dy = \frac{g(0)}{s} + \frac{1}{s}\int_0^\infty g'(y)\cos(ys)\, dy,$$

and therefore

$$J(s) = -\frac{g(0)}{s^2} - \frac{1}{s^2}\int_0^\infty g'(y)\cos(ys)\, dy.$$

From this representation we obtain, for $s \neq 0$,

$$|J(s)| \leqslant \frac{|g(0)|}{s^2} + \frac{1}{s^2}\int_0^\infty |g'(y)|\, dy = \frac{C}{s^2},$$

where

$$C := |g(0)| + \int_0^\infty |g'(y)|\, dy,$$

as it can be verified through a direct calculation that the integral on the right hand side is finite.

Consequently, for $s \neq 0$,

$$|1 - sI(s)| = 2\pi |J(s)| \leqslant \frac{2\pi C}{s^2}.$$

To obtain a single inequality valid for all $s \in \mathbb{R}$, including small $|s|$, define

$$M_0 := \sup_{|s| \leqslant 1} |1 - sI(s)| < \infty$$

(which is finite because the integrand defining $I(s)$ is continuous in s), and set

$$M := \max\{\, 2M_0,\ 4\pi C \,\}.$$

If $|s| \leqslant 1$ then $1 + s^2 \leqslant 2$ and so

$$\frac{M}{1+s^2} \geqslant \frac{2M_0}{2} = M_0 \geqslant |1 - sI(s)|.$$

If $|s| \geqslant 1$ then $1/s^2 \leqslant 2/(1+s^2)$, hence from the bound above

$$|1 - sI(s)| \leqslant \frac{2\pi C}{s^2} \leqslant \frac{4\pi C}{1+s^2} \leqslant \frac{M}{1+s^2}.$$

Thus for every $s \in \mathbb{R}$ we have

$$\big|1 - sI(s)\big| \leqslant \frac{M}{1+s^2},$$

as desired. □

Finally, we recall that the Wills functional relates to the volume and the mean width in the following way:

Lemma 11. *Let $K \subseteq \mathbb{R}^n$ be a convex body and let $H \in G_{n,k}$. Then,*

$$\mathrm{vol}_k(K \cap H) = \lim_{\lambda \to \infty} \frac{\mathcal{W}(\lambda(K \cap H))}{\lambda^k} \quad \text{and} \quad V_1(K \cap H) = \frac{k\omega_k}{\omega_{k-1}} w(K \cap H) = \lim_{\lambda \to 0^+} \frac{\mathcal{W}(\lambda(K \cap H)) - 1}{\lambda},$$

where $w(L)$ denotes the mean width of L and $\omega_m = \mathrm{vol}_m(B_2^m)$.

Proof. By the definition of the Wills functional, we have that for any $\lambda > 0$

$$\mathcal{W}(\lambda(K \cap H)) = \sum_{i=0}^{k} V_i(\lambda(K \cap H)) = \sum_{i=0}^{k} \lambda^i V_i(K \cap H),$$

where $V_i(L)$ denotes the i-th intrinsic volume of L. Taking into account that $V_k(K \cap H) = \mathrm{vol}_k(K \cap H)$ we obtain the first identity, and taking into account that $V_0(K \cap H) = 1$ and $V_1(K \cap H) = \frac{k\omega_k}{\omega_{k-1}} w(K \cap H)$ we obtain the second identity. □

4.1. **The volume of sections via the Wills functional.** In this Section we show how, from Proposition 9, we can recover the estimate (6).

Theorem 12. *Let $K \subseteq \mathbb{R}^n$ be a centrally symmetric convex body in John's position and let $I_n = \sum_{i=1}^{m} c_j v_j \otimes v_j$ be its associated decomposition of the identity. Let $H \in G_{n,k}$ be a k-dimensional subspace and let $J = \{1 \leqslant j \leqslant m : P_H v_j \neq 0\}$. Then*

$$\mathrm{vol}_k(K \cap H) \leqslant \frac{2^k}{\pi^{m_0 - k}} \prod_{j \in J} \left((1 - \tilde{c}_j)^{-1/(2p_j)}\, I_{p_j}^{1/p_j} \left(\sqrt{\tilde{c}_j} t_j\right)^{\tilde{c}_j} \right),$$

where, for any $p > 1$, $I_p = \displaystyle\int_{\mathbb{R}} \left|\frac{\sin x}{x}\right|^p dx$, and for any $j \in J$, $\tilde{c}_j = c_j \|P_H v_j\|_2^2$, $p_j = \frac{1}{1-\tilde{c}_j}$, and $t_j = \frac{1}{\|P_H v_j\|_2}$.

In particular, if $\tilde{c}_j \geqslant 2$ for every $j \in J$,

$$\mathrm{vol}_k(K \cap H) \leqslant 2^{\frac{m_0+k}{2}} \prod_{j \in J} c_j^{\frac{c_j \|P_H v_j\|_2^2}{2}}.$$

Proof. Let, for any fixed $j \in J$ and $\alpha > 0$, $a_\alpha(s), b_\alpha(s), c_\alpha(s)$ and $A_\alpha(s)$ be defined as before. By Minkowski's inequality in $L^p(\mathbb{R})$, we have that

$$\left(\int_{\mathbb{R}} |A_\alpha(s)|^p ds\right)^{\frac{1}{p}} \leqslant \left(\int_{\mathbb{R}} |a_\alpha(s)|^p\, ds\right)^{\frac{1}{p}} + \left(\int_{\mathbb{R}} |b_\alpha(s)|^p\, ds\right)^{\frac{1}{p}} + \left(\int_{\mathbb{R}} |c_\alpha(s)|^p\, ds\right)^{\frac{1}{p}}$$

Let us estimate each term:

- By the scaling $u = \alpha s$,
$$\left(\int_{\mathbb{R}} |a_\alpha(s)|^p\, ds\right)^{1/p} = 2 I_p^{1/p} |\alpha|^{(p-1)/p} = 2 I_p^{1/p} |\alpha|^{\tilde{c}}.$$
- For the second term,
$$\left(\int_{\mathbb{R}} |b_\alpha(s)|^p\, ds\right)^{1/p} \leqslant \left(\int_{\mathbb{R}} |b_0(s)|^p\, ds\right)^{1/p} = \left(\int_{\mathbb{R}} e^{-ps^2/(4\pi)}\, ds\right)^{1/p} = \left(\frac{2\pi}{\sqrt{p}}\right)^{1/p}.$$
- For the third term, notice that if $|s| \leqslant 1$ we have
$$|I(s)| = \left|\int_0^\infty e^{-\pi y^2} \sin(ys)\, dy\right| \leqslant \int_0^\infty e^{-\pi y^2}\, dy = \frac{1}{2} \leqslant \frac{1}{1+s} \leqslant \frac{4}{1+s},$$
and that, if $|s| \geqslant 1$,
$$\left|\frac{2\pi}{s} \int_0^\infty y e^{-\pi y^2} \cos(ys)\, dy\right| \leqslant \frac{2\pi}{|s|} \int_0^\infty y e^{-\pi y^2}\, dy = \frac{1}{|s|}$$
and then
$$|I(s)| = \left|\frac{1}{s} - \frac{2\pi}{s} \int_0^\infty y e^{-\pi y^2} \cos(ys)\, dy\right| \leqslant \frac{2}{|s|} \leqslant \frac{4}{1+|s|}.$$
Therefore,
$$\left(\int_{\mathbb{R}} |c_\alpha(s)|^p\, ds\right)^{1/p} \leqslant 8 \left(\int_{\mathbb{R}} \frac{ds}{(1+|s|)^p}\right)^{1/p} = 8 \frac{2^{1/p}}{(p-1)^{1/p}}.$$

As a consequence, we have that for every fixed $j \in J$ there exists $M_j : [0,\infty) \to [0,\infty)$ such that $M_j(\alpha) \sim 2 I_{p_j}^{1/p_j} \alpha^{\tilde{c}_j}$, as $\alpha \to \infty$, and

$$\left(\int_{\mathbb{R}} |A_\alpha(s)|^p ds\right)^{\frac{1}{p}} \leqslant M_j(\alpha).$$

Thus, the function $M(\lambda) := \prod_{j \in J} M_j(\lambda \sqrt{\tilde{c}_j} t_j)$ satisfies that

$$M(\lambda) \sim 2^{m_0} \prod_{j \in J} I_{p_j}^{1/p_j} \tilde{c}_j^{\frac{\tilde{c}_j}{2}} t_j^{\tilde{c}_j} \lambda^k, \quad \lambda \to \infty$$

and then, using the estimate granted by Proposition 9,

$$\begin{aligned}\mathcal{W}(\lambda(K\cap H)) &\leqslant \frac{1}{(2\pi)^{m_0-k}}\prod_{j\in J}\left(\int_{\mathbb{R}} h_{\lambda,j}(t)dt\right)^{1/p_j}\\ &\leqslant \frac{2^k}{\pi^{m_0-k}}\prod_{j\in J}\left((1-\tilde{c}_j)^{-1/(2p_j)}\, I_{p_j}^{1/p_j}\,(\sqrt{\tilde{c}_j}t_j)^{\tilde{c}_j}\right)\lambda^k, \quad \lambda\to\infty.\end{aligned}$$

Therefore, by Lemma 11,

$$\mathrm{vol}_k(K\cap H)=\lim_{\lambda\to\infty}\frac{\mathcal{W}(K\cap H)}{\lambda^k}\leqslant \frac{2^k}{\pi^{m_0-k}}\prod_{j\in J}\left((1-\tilde{c}_j)^{-1/(2p_j)}\, I_{p_j}^{1/p_j}\,(\sqrt{\tilde{c}_j}t_j)^{\tilde{c}_j}\right).$$

If $\tilde{c}_j\leqslant 2$ then $p_j\geqslant 2$ and then, applying Ball's inequality [B1]

$$I_{p_j}=\int_{\mathbb{R}}\left|\frac{\sin u}{u}\right|^{p_j}du\leqslant \sqrt{2}\pi\, p_j^{-1/2},$$

we obtain

$$\mathrm{vol}_k(K\cap H)\leqslant 2^{\frac{m_0+k}{2}}\prod_{j\in J}(\sqrt{\tilde{c}_j}t_j)^{\tilde{c}_j}=2^{\frac{m_0+k}{2}}\prod_{j\in J}c_j^{\frac{c_j\|P_Hv_j\|_2^2}{2}}.$$ □

4.2. **The mean width of sections via the Wills functional.** In this section we are going to showcase how, from the estimate in Proposition 9, we can recover the following estimate for $V_1(K\cap H)$ which is included in the proof of [AB, Theorem 3.3].

Proposition 13. *Let $K\subseteq\mathbb{R}^n$ be a centrally symmetric convex body in John's position and let (c_j,v_j) be its associated decomposition of the identity. Let $H\in G_{n,k}$ be a k-dimensional subspace and let $J=\{1\leqslant j\leqslant m\,:\,P_Hv_j\neq 0\}$. Then*

$$V_1(K\cap H)\leqslant 2\sum_{j\in J}c_j\|P_Hv_j\|_2.$$

For the proof we will rely on the following technical lemma, whose proof we postpone until the end of the section.

Lemma 14. *For each fixed j, we have that*

$$\int_{\mathbb{R}}|A_\alpha(s)|^p\,ds=S_0+\alpha\,S_1+o(\alpha),\quad \alpha\to 0^+,$$

where

$$S_0=\int_{\mathbb{R}}b(s)^p\,ds=\frac{2\pi}{\sqrt{p}},\qquad S_1=4\pi\sqrt{p-1}.$$

Proof of Proposition 13. Let, for every $j\in J$, $\tilde{c}_j=c_j\|P_Hv_j\|_2^2$, $p_j=\frac{1}{1-\tilde{c}_j}>1$, and $t_j=\frac{1}{\|P_Hv_j\|_2}$. By Lemma 14 we have that for every $j\in J$ and every $\lambda>0$, calling $\alpha_j=\lambda\sqrt{\tilde{c}_j}t_j$

$$\begin{aligned}\int_{\mathbb{R}}h_{\lambda,j}(t)dt &= \frac{1}{\sqrt{1-\tilde{c}_j}}\left(\frac{2\pi}{\sqrt{p_j}}+4\pi\sqrt{p_j-1}\alpha_j+o(\alpha_j)\right)=2\pi+\alpha\cdot 4\pi\sqrt{p_j(p_j-1)}+o(\alpha_j)\\ &= 2\pi\left(1+2\alpha_j\sqrt{p_j(p_j-1)}+o(\alpha_j)\right),\quad \alpha_j\to 0^+.\end{aligned}$$

Thus,

$$\begin{aligned}\left(\int_{\mathbb{R}}h_{\lambda,j}(t)dt\right)^{1/p_j} &= (2\pi)^{1/p_j}\left(1+\tfrac{2}{p_j}\alpha_j\sqrt{p_j(p_j-1)}+o(\alpha_j)\right),\\ &= (2\pi)^{1/p_j}\left(1+2\sqrt{\tilde{c}_j}\alpha_j+o(\alpha_j)\right),\quad \alpha_j\to 0^+.\end{aligned}$$

Equivalently, taking into account that $\alpha_j = \lambda\sqrt{\tilde{c}_j}t_j$,

$$\left(\int_{\mathbb{R}} h_j(t)\,dt\right)^{1-\tilde{c}_j} = (2\pi)^{1-\tilde{c}_j}\left(1+2\lambda\tilde{c}_j t_j + o(\lambda)\right), \lambda \to 0^+.$$

Multiplying over $j \in J$ and taking into account that $\sum_{j\in J}(1-\tilde{c}_j) = m_0 - k$ we obtain, by Proposition 9,

$$\mathcal{W}(\lambda(K\cap H)) \leqslant 1 + 2\lambda\sum_{j\in J}\tilde{c}_j t_j + o(\lambda), \quad \lambda \to 0^+.$$

Therefore, by Lemma 11,

$$V_1(K\cap H) = \lim_{\lambda\to 0^+}\frac{\mathcal{W}(\lambda(K\cap H)) - 1}{\lambda} \leqslant 2\sum_{j\in J}\tilde{c}_j t_j = 2\sum_{j\in J} c_j\|P_H v_j\|_2. \qquad \square$$

It remains to justify the estimate in Lemma 14.

Proof of Lemma 14. Let $g : [0,\infty) \to \mathbb{R}$ be the function

$$g(\alpha) = \int_{\mathbb{R}} |A_\alpha(s)|^p\,ds.$$

Let us prove that g is differentiable at 0 with $g'(0) = S_1 = 4\pi\sqrt{p-1}$. Therefore, since

$$g(0) = \int_{\mathbb{R}} |A_0(s)|^p ds = \int_{\mathbb{R}} b_0(s) ds = \int_{\mathbb{R}} e^{-p\frac{s^2}{4\pi}}\,ds = \frac{2\pi}{\sqrt{p}},$$

we will have the result.

Since $p > 1$, we have that the function $|x|^p$ is differentiable on $\mathbb{R}$ and, by the mean value theorem, for every $s \in \mathbb{R}$ there exists $\xi_\alpha(s)$ between $A_\alpha(s)$ and $b_0(s)$ such that

$$|A_\alpha(s)|^p - b_0^p(s) = p|\xi_\alpha(s)|^{p-1}\mathrm{sign}(\xi)(A_\alpha(s) - b_0(s)).$$

Therefore, taking into account that for any $s \in \mathbb{R}$ $\lim_{\alpha\to 0^+} A_\alpha(s) = b_0(s) > 0$ and then $\lim_{\alpha\to 0^+}\xi_\alpha(s) = b_0(s)$, we have that for every $s \in \mathbb{R}$,

$$\begin{aligned}
\lim_{\alpha\to 0^+}\frac{|A_\alpha(s)|^p - b_0^p(s)}{\alpha} &= \lim_{\alpha\to 0^+}\frac{p|\xi_\alpha(s)|^{p-1}\mathrm{sign}(\xi_\alpha(s))(A_\alpha(s) - b_0(s))}{\alpha}\\
&= \lim_{\alpha\to 0^+}\frac{p|\xi_\alpha(s)|^{p-1}(a_\alpha(s) + b_\alpha(s) - b_0(s) + c_\alpha(s))}{\alpha}\\
&= pb_0(s)^{p-1}(2 - 2sI(s)).
\end{aligned}$$

Moreover, for every $0 < \alpha < 1$ and every $s \in \mathbb{R}$,

$$\begin{aligned}
\left|\frac{|A_\alpha(s)|^p - b_0^p(s)}{\alpha}\right| &= \frac{p|\xi_\alpha(s)|^{p-1}|A_\alpha(s) - b_0(s)|}{\alpha}\\
&= \frac{p|\xi_\alpha(s)|^{p-1}|a_\alpha(s) + b_\alpha(s) - b_0(s) + c_\alpha(s)|}{\alpha}\\
&\leqslant p\max\{|A_\alpha(s)|^{p-1}, b_0^{p-1}(s)\}\left(\frac{|a_\alpha(s) + c_\alpha(s)|}{\alpha} + \frac{|b_\alpha(s) - b_0(s)|}{\alpha}\right)
\end{aligned}$$

On the one hand, notice that for every $0 < \alpha < 1$ and every $s \in \mathbb{R}$, by the mean value theorem there exists $0 < \beta_\alpha(s) < \alpha s$ such that

$$\frac{|a_\alpha(s) + c_\alpha(s)|}{\alpha} = 2|\cos(\beta_\alpha(s))||1 - sI(s)| \leqslant 2|1 - sI(s)| \leqslant \frac{2M}{1+s^2},.$$

where M is the constant given by Lemma 10.

On the other hand, for every $0<\alpha<1$ and every $s\in\mathbb{R}$, by the mean value theorem there exists $0<\gamma_\alpha(s)<\alpha s$ such that

$$\frac{|b_\alpha(s)-b_0(s)|}{\alpha}=|\sin(\gamma_\alpha(s))||s|e^{-\frac{s^2}{4\pi}}\leqslant |s|e^{-\frac{s^2}{4\pi}}\leqslant \frac{C_1}{1+s^2},$$

where $C_1>0$ is an absolute constant. Therefore, there exists an absolute constant $C_2>0$ such that for every $0<\alpha<1$ and every $s\in\mathbb{R}$

$$\left|\frac{|A_\alpha(s)|^p-b_0^p(s)}{\alpha}\right|\leqslant \frac{C_2 p}{1+s^2}\max\{|A_\alpha(s)|^{p-1},b_0^{p-1}(s)\}$$

Now, notice that for every $p>1$ we have that, for every $0<\alpha<1$ and every $s\in\mathbb{R}$,

$$\begin{aligned}|A_\alpha(s)|^{p-1} &\leqslant 3^{p-1}\max\{|a_\alpha(s)|^{p-1}+|b_\alpha(s)|^{p-1}+|c_\alpha(s)|^{p-1}\}\\ &\leqslant 3^{p-1}\max\{|a_\alpha(s)|^{p-1}+|b_0(s)|^{p-1}+|c_\alpha(s)|^{p-1}\}\end{aligned}$$

and then

$$\max\{|A_\alpha(s)|^{p-1},b_0^{p-1}(s)\}\leqslant 3^{p-1}\max\{|a_\alpha(s)|^{p-1}+|b_0(s)|^{p-1}+|c_\alpha(s)|^{p-1}\}.$$

Since for any $p>1$, $0<\alpha<1$, and every $s\in\mathbb{R}$,

$$|a_\alpha(s)|^{p-1}\leqslant 2^{p-1}\min\left\{1,\frac{1}{|s|^{p-1}}\right\},\quad |b_0(s)|^{p-1}=e^{-(p-1)\frac{s^2}{4\pi}}\leqslant\frac{C_3}{|s|^{p-1}},$$

where $C_3>0$ is an absolute constant, and there exists an absolute constant $C_4>0$ such that

$$|c_\alpha(s)|^p\leqslant 2^{p-1}|I(s)|^{p-1}\leqslant C_4^{p-1}\min\left\{1,\frac{1}{|s|^{p-1}}\right\}.$$

we have that for every $p>1$ there exists an absolute constant $C>0$ such that for every $0<\alpha<1$ and every $s\in\mathbb{R}$

$$\left|\frac{|A_\alpha(s)|^p-b_0^p(s)}{\alpha}\right|\leqslant C\min\left\{1,\frac{1}{|s|^{p+1}}\right\}\in L^1(\mathbb{R}).$$

Therefore, by the dominated convergence theorem,

$$\begin{aligned}\lim_{\alpha\to0^+}\frac{g(\alpha)-g(0)}{\alpha} &= \lim_{\alpha\to0^+}\int_{\mathbb{R}}\frac{|A_\alpha(s)|^p-b_0^p(s)}{\alpha}ds=\int_{\mathbb{R}}pb_0^{p-1}(s)(2-2sI(s))ds\\ &= 2p\int_{\mathbb{R}}b_0^{p-1}(s)ds-2p\int_{\mathbb{R}}sb_0^{p-1}(s)I(s))ds\end{aligned}$$

The first integral is

$$\int_{\mathbb{R}}b_0^{p-1}(s)\,ds=\frac{2\pi}{\sqrt{p-1}}.$$

For the second one, by Fubini's theorem and the Gaussian–Fourier identity, one computes

$$\int_{\mathbb{R}}sI(s)b_0^{p-1}(s)\,ds=\frac{2\pi}{p\sqrt{p-1}}.$$

Hence,

$$S_1=g'(0)=2p\left(\frac{2\pi}{\sqrt{p-1}}-\frac{2\pi}{p\sqrt{p-1}}\right)=4\pi\sqrt{p-1}.\qquad\square$$

5. Sections of Generalised ℓ_p^n Balls

Assume that $(v_j)_{j=1}^m \subseteq S^{n-1}$ and $(c_j)_{j=1}^m \subseteq (0,\infty)$ are such that

$$\mathrm{Id}_n = \sum_{j=1}^m c_j v_j \otimes v_j.$$

Given $p \in [1,2]$ and $(\alpha_j)_{j=1}^m \subseteq (0,\infty)$ we define K_p as the symmetric convex body in $\mathbb{R}^n$ with norm given by

$$\|x\|_{K_p} = \left(\sum_{j=1}^m \alpha_j |\langle x, v_j\rangle|^p\right)^{\frac{1}{p}}.$$

Integrating in polar coordinates we can check that, for any $H \in G_{n,k}$,

$$\mathrm{vol}_k(K_p \cap H) = \Gamma\left(1+\frac{k}{p}\right)^{-1} I_p, \tag{12}$$

where $I_p := \int_H e^{-\|x\|_{K_p}^p}\, dx$. As usual, in the sequel we let $J = \{j : P_H v_j \neq 0\}$, $m_0 = \sharp J$ and $u_j = P_H v_j \|P_H v_j\|_2$ and $\tilde{c}_j = c_j \|P_H v_j\|_2^2$, for every $j \in J$. Then

$$\mathrm{Id}_H = \sum_{j\in J} \tilde{c}_j u_j \otimes u_j.$$

Without loss of generality, assume that $J = [m_0]$ and let us identify $\mathbb{R}^n$ with $\mathrm{span}\{e_j : 1 \leqslant j \leqslant n\} \subseteq \mathbb{R}^{m_0}$, where $(e_j)_{j=1}^{m_0}$ denotes the canonical basis in $\mathbb{R}^{m_0}$. By Proposition 2, there exits $(x_j)_{j=1}^{m_0}$, an orthonormal basis of $\mathbb{R}^{m_0}$ and $(w_j)_{j=1}^{m_0} \subseteq S^{m_0-1} \cap H^\perp$, where $H^\perp$ denotes the orthogonal linear subspace to H in $\mathbb{R}^{m_0}$ such that

$$P_H x_j = \sqrt{\tilde{c}_j}\, u_j \qquad \text{and} \qquad P_{H^\perp} x_j = \sqrt{1-\tilde{c}_j}\, w_j$$

for every $1 \leqslant j \leqslant m_0$. Writing coordinates with respect to the orthonormal basis of $\mathbb{R}^{m_0}$ given by $(x_j)_{j=1}^{m_0}$, we have that for every $x \in H$

$$\langle x, x_j\rangle = \langle \sqrt{\tilde{c}_j}\, u_j, x\rangle \quad \forall 1 \leqslant j \leqslant m_0$$

and for every $y \in H^\perp$

$$\langle y, x_j\rangle = \langle \sqrt{1-\tilde{c}_j}\, w_j, x\rangle, \quad \forall 1 \leqslant j \leqslant m_0.$$

For every $p \in [1,2]$, we denote by

$$\gamma_p(y) = \int_{-\infty}^{+\infty} e^{-|x|^p} e^{ixy}\, dx,$$

the Fourier transform of $e^{-|x|^p}$. If $f_{\alpha,p}(t) = e^{-\alpha|t|^p}$, we can then check that $\widehat{f_{\alpha,p}}(y) = \alpha^{-1/p}\gamma_p(\alpha^{-1/p}y)$, for any $\alpha > 0$.

Using the notation introduced above, we can verify that the following identity holds.

Lemma 15. *For any $p \in [1,2]$, $1 \leqslant k \leqslant n-1$ and $H \in G_{n,k}$,*

$$I_p = \frac{1}{(2\pi)^{m_0-k}} \prod_{j\in J} \frac{\sqrt{c_j}}{\alpha_j^{1/p}} \int_{H^\perp} \prod_{j\in J} \gamma_p\left(\frac{\sqrt{c_j}}{\alpha_j^{1/p}}\langle y, x_j\rangle\right) dy. \tag{13}$$

Proof. We note that

$$\begin{aligned}\int_H e^{-\sum_{j=1}^m \alpha_j |\langle x, v_j\rangle|^p} dx &= \int_H e^{-\sum_{j=1}^m \alpha_j |\langle x, P_H v_j\rangle|^p} dx = \int_H e^{-\sum_{j\in J} \alpha_j |\langle x, P_H v_j\rangle|^p} dx \\ &= \int_H e^{-\sum_{j\in J} \alpha_j \|P_H v_j\|_2^p |\langle x, u_j\rangle|^p} dx = \int_H e^{-\sum_{j\in J} \frac{\alpha_j}{c_j^{p/2}} |\langle x, \sqrt{\tilde{c}_j} u_j\rangle|^p} dx \\ &= \int_H e^{-\sum_{j\in J} \frac{\alpha_j}{c_j^{p/2}} |\langle x, x_j\rangle|^p} dx.\end{aligned}$$

By Proposition 1,

$$\int_H e^{-\sum_{j\in J} \frac{\alpha_j}{c_j^{p/2}} |\langle x, x_j\rangle|^p} dx = \int_H \prod_{j\in J} e^{-\frac{\alpha_j}{c_j^{p/2}} |\langle x, x_j\rangle|^p} dx = \frac{1}{(2\pi)^{m_0-k}} \int_{H^\perp} \prod_{j\in J} \widehat{f}_{\beta_j,p}(y, x_j)\, dy,$$

where $\beta_j = \frac{\alpha_j}{c_j^{p/2}}$, so the wanted identity follows from the aforementioned calculation of $\widehat{f}_{\alpha,p}$. $\square$

5.1. **Sections of K_1.** Having (13) as a starting point, we will provide upper and lower bounds for $|K_p \cap H|$. We treat the case $p = 1$ separately; the fact that $\gamma_1(y)$ can be computed explicitly, allows us to provide sharper estimates for section volumes of K_1, summarised in the following theorem.

Theorem 16. *Let $(\alpha_j)_{j=1}^m \subseteq (0,\infty)$, $1 \leqslant k \leqslant n-1$ and $H \in G_{n,k}$. Then*

$$\mathrm{vol}_k(K_1 \cap H) \geqslant \frac{m_0^{m_0}}{\pi^{\frac{m_0-k}{2}} \left(\sum_{j\in J} \frac{\alpha_j^2}{c_j}\right)^{\frac{m_0+k}{2}}} \prod_{j\in J} \left(\frac{\alpha_j}{\sqrt{c_j}}\right) \frac{\Gamma\left(\frac{m_0+k}{2}\right)}{\Gamma(m_0)} \cdot |B_1^k|.$$

On the other hand,

$$\mathrm{vol}_k(K_1 \cap H) \leqslant \mathrm{vol}_k(B_1^k) \prod_{j=1}^m \left(\frac{\sqrt{c_j}}{\alpha_j}\right)^{c_j \|P_H v_j\|_2^2}.$$

Remark 17. The case $\alpha = \sqrt{c_i}$ is included in [MP, Lemma 3.7].

We start with the following straightforward computation.

Lemma 18. *Let $\alpha > 0$ and $f_{\alpha,1} : \mathbb{R} \to \mathbb{R}$ the function given by $f_{\alpha,1}(x) = e^{-\alpha|x|}$. Then $\widehat{f}_{\alpha,1} : \mathbb{R} \to \mathbb{R}$ is given by*

$$\widehat{f}_{\alpha,1}(y) = \frac{2\alpha}{\alpha^2 + y^2}.$$

Proof. For every $y \in \mathbb{R}$ we have that

$$\begin{aligned}\widehat{f}_{\alpha,1}(y) &= \int_{\mathbb{R}} f_{\alpha,1}(x) e^{ixy} dx = \int_{\mathbb{R}} e^{-\alpha|x|} \left(\cos(xy) + i\sin(xy)\right) dx = \int_{\mathbb{R}} e^{-\alpha|x|} \cos(xy) dx \\ &= 2\int_0^\infty e^{-\alpha x} \cos(xy) dx.\end{aligned}$$

Since, integrating by parts twice,

$$\begin{aligned}\int_0^\infty e^{-\alpha x} \cos(xy) dx &= \frac{1}{\alpha} - \frac{y}{\alpha} \int_0^\infty e^{-\alpha x} \sin(xy) dx \\ &= \frac{1}{\alpha} - \frac{y^2}{\alpha^2} \int_0^\infty e^{-\alpha x} \cos(xy) dx,\end{aligned}$$

we have that

$$\left(1 + \frac{y^2}{\alpha^2}\right) \int_0^\infty e^{-\alpha x} \cos(xy) dx = \frac{1}{\alpha} \Leftrightarrow \int_0^\infty e^{-\alpha x} \cos(xy) dx = \frac{1}{\alpha} \frac{1}{1 + \frac{y^2}{\alpha^2}} = \frac{\alpha}{\alpha^2 + y^2}$$

and then

$$\widehat{f}_{\alpha,1}(y) = 2\int_0^\infty e^{-\alpha x}\cos(xy)dx = \frac{2\alpha}{\alpha^2+y^2}. \qquad \square$$

Combining Lemma 15 and Lemma 18 we have that

$$I_1 = \frac{1}{(2\pi)^{m_0-k}} \prod_{j\in J}\left(\frac{2\alpha_j}{\sqrt{c_j}}\right)\int_{H^\perp}\prod_{j\in J}\frac{1}{\frac{\alpha_j^2}{c_j}+\langle y,x_j\rangle^2}dy. \tag{14}$$

Let us now establish the lower bound in Theorem 16. This is a direct consequence of (12) and the following.

Lemma 19. *Under the notation introduced above,*

$$I_1 \geqslant \frac{m_0^{m_0}\omega_{m_0-k}}{(2\pi)^{m_0-k}\left(\sum_{j\in J}\frac{\alpha_j^2}{c_j}\right)^{\frac{m_0+k}{2}}}\prod_{j\in J}\left(\frac{2\alpha_j}{\sqrt{c_j}}\right)\frac{\Gamma\left(1+\frac{m_0-k}{2}\right)\Gamma\left(\frac{m_0+k}{2}\right)}{\Gamma(m_0)},$$

where $\omega_m = \mathrm{vol}_m(B_2^m)$.

Proof. We lower bound the integrand in (14): By the arithmetic-geometric mean inequality, we have that

$$\prod_{j\in J}\left(\frac{\alpha_j^2}{c_j}+\langle y,x_j\rangle^2\right)^{\frac{1}{m_0}} \leqslant \frac{1}{m_0}\sum_{j\in J}\left(\frac{\alpha_j^2}{c_j}+\langle y,x_j\rangle^2\right) = \frac{1}{m_0}\sum_{j\in J}\frac{\alpha_j^2}{c_j}+\frac{\|y\|_2^2}{m_0}$$

and then

$$\begin{aligned}
\int_{H^\perp}\prod_{j\in J}\frac{1}{\frac{\alpha_j^2}{c_j}+\langle y,x_j\rangle^2}dy &\geqslant \int_{H^\perp}\frac{m_0^{m_0}}{\left(\sum_{j\in J}\frac{\alpha_j^2}{c_j}+\|y\|_2^2\right)^{m_0}}dy\\
&= m_0^{m_0}(m_0-k)\omega_{m_0-k}\int_0^\infty \frac{r^{m_0-k-1}}{\left(\sum_{j\in J}\frac{\alpha_j^2}{c_j}+r^2\right)^{m_0}}dr\\
&= \frac{m_0^{m_0}(m_0-k)\omega_{m_0-k}}{\left(\sum_{j\in J}\frac{\alpha_j^2}{c_j}\right)^{\frac{m_0+k}{2}}}\int_0^\infty\frac{s^{\frac{m_0-k}{2}-1}}{2\,(1+s)^{m_0}}ds.
\end{aligned}$$

The proof is complete; it only remains to check that the integral in the last expression is exactly equal to $\frac{\Gamma\left(1+\frac{m_0-k}{2}\right)\Gamma\left(\frac{m_0+k}{2}\right)}{(m_0-k)\Gamma(m_0)}$. $\square$

We proceed to the upper bound in Theorem 16. This is the essence of the next proposition.

Proposition 20. *Under the notation introduced above,*

$$I_1 \leqslant 2^k\prod_{j=1}^m\left(\frac{\sqrt{c_j}}{\alpha_j}\right)^{c_j\|P_Hv_j\|_2^2}.$$

We will make use of the following auxiliary estimate, whose proof we defer.

Lemma 21. *Let* $J_1 = \{j\in J : \tilde{c}_j \neq 1\}$. *Then*

$$\prod_{j\in J_1}\left(\frac{\Gamma\left(\frac{1}{1-\tilde{c}_j}-\frac{1}{2}\right)}{\sqrt{1-\tilde{c}_j}\,\Gamma\left(\frac{1}{1-\tilde{c}_j}\right)}\right)^{1-\tilde{c}_j} \leqslant \pi^{\frac{m_0-k}{2}}.$$

Proof of Proposition 20. We first manipulate (14) as follows,

$$\begin{aligned}
I_1 &= \frac{1}{(2\pi)^{m_0-k}} \prod_{j\in J} \left(\frac{2\alpha_j}{\sqrt{c_j}}\right) \int_{H^\perp} \prod_{j\in J} \frac{1}{\frac{\alpha_j^2}{c_j} + \langle y, x_j\rangle^2} dy \\
&= \frac{1}{(2\pi)^{m_0-k}} \prod_{j\in J} \left(\frac{2\sqrt{c_j}}{\alpha_j}\right) \int_{H^\perp} \prod_{j\in J} \frac{1}{1 + \frac{c_j(1-\tilde{c}_j)}{\alpha_j^2}\langle y, w_j\rangle^2} dy \\
&= \frac{1}{(2\pi)^{m_0-k}} \prod_{j\in J} \left(\frac{2\sqrt{c_j}}{\alpha_j}\right) \int_{H^\perp} \prod_{j\in J_1} \frac{1}{1 + \frac{c_j(1-\tilde{c}_j)}{\alpha_j^2}\langle y, w_j\rangle^2} dy,
\end{aligned}$$

where $J_1 = \{j \in J : \tilde{c}_j \neq 1\} = \{j \in J : x_j \notin H\} = \{j \in J : P_{H^\perp} x_j \neq 0\}$. Next, we apply the geometric Brascamp-Lieb inequality in $H^\perp$ to get

$$\begin{aligned}
\int_{H^\perp} \prod_{j\in J_1} \frac{1}{1 + \frac{c_j(1-\tilde{c}_j)}{\alpha_j^2}\langle y, w_j\rangle^2} dy &\leqslant \prod_{j\in J_1} \left(\int_{\mathbb{R}} \frac{1}{\left(1 + \frac{c_j(1-\tilde{c}_j)}{\alpha_j^2} x^2\right)^{\frac{1}{1-\tilde{c}_j}}} dx \right)^{1-\tilde{c}_j} \\
&= \prod_{j\in J_1} \left(\frac{\alpha_j}{\sqrt{c_j(1-\tilde{c}_j)}} \int_{\mathbb{R}} \frac{1}{(1+x^2)^{\frac{1}{1-\tilde{c}_j}}} dx \right)^{1-\tilde{c}_j}.
\end{aligned}$$

Since

$$\int_{\mathbb{R}} \frac{1}{(1+x^2)^{\frac{1}{1-\tilde{c}_j}}} dx = \frac{\sqrt{\pi}\,\Gamma\left(\frac{1}{1-\tilde{c}_j} - \frac{1}{2}\right)}{\Gamma\left(\frac{1}{1-\tilde{c}_j}\right)},$$

we arrive at

$$I_1 \leqslant \frac{2^k}{\pi^{\frac{m_0-k}{2}}} \prod_{j\in J} \left(\frac{\sqrt{c_j}}{\alpha_j}\right)^{\tilde{c}_j} \prod_{j\in J_1} \left(\frac{\Gamma\left(\frac{1}{1-\tilde{c}_j} - \frac{1}{2}\right)}{\sqrt{1-\tilde{c}_j}\,\Gamma\left(\frac{1}{1-\tilde{c}_j}\right)} \right)^{1-\tilde{c}_j}, \tag{15}$$

and then Lemma 21 concludes the proof. □

The proof of Lemma 21 is the final missing piece.

Proof of Lemma 21. Notice that the wanted inequality is equivalent to

$$\sum_{j\in J_1} (1-\tilde{c}_j) \log \left(\frac{\Gamma\left(\frac{1}{1-\tilde{c}_j} - \frac{1}{2}\right)}{\sqrt{1-\tilde{c}_j}\,\Gamma\left(\frac{1}{1-\tilde{c}_j}\right)} \right) \leqslant (m_0 - k) \log \Gamma\left(\frac{1}{2}\right).$$

Let $f : [0,1] \to \mathbb{R}$ be the function given by

$$f(x) = x \log \left(\frac{\Gamma\left(\frac{1}{x} - \frac{1}{2}\right)}{\sqrt{x}\,\Gamma\left(\frac{1}{x}\right)} \right), \qquad x \in (0,1]$$

and $f(0) = \lim_{x\to 0^+} f(x) = 0$. We have that f is continuous on $[0,1]$. Besides, for every $x \in (0,1)$ we have that

$$f''(x) = -\frac{x^2 - 2\psi'\left(\frac{1}{x} - \frac{1}{2}\right) + 2\psi'\left(\frac{1}{x}\right)}{2x^3},$$

where ψ denotes the logarithmic derivative of the Gamma function.

Convexity of ψ' implies that

$$2\left(\psi'\left(\frac{1}{x} - \frac{1}{2}\right) - \psi'\left(\frac{1}{x}\right)\right) \geqslant -\psi''\left(\frac{1}{x}\right) \geqslant x^2 + x^3,$$

for any $x \in (0,1)$, where the latter inequality follows from known bounds on polygamma functions (see for example [A]). This implies that $f'' \geqslant 0$, hence f is convex on $(0,1)$. Then the function $F : K \to \mathbb{R}$ given by

$$F(x) = \sum_{j\in J} f(x_j),$$

where $K = \{(x_j)_{j\in J} : 0 \leqslant x_j \leqslant 1 \forall j \in J, \sum_{j\in J} x_j = m_0 - k\}$, is convex and invariant under permutations of coordinates. Besides, for every $(x_j)_{j\in J} \in K$ we have that

$$(x_j)_{j\in J} \prec (1,\dots,1,0,\dots,0) = (y_j)_{j\in J},$$

where there are $m_0 - k$ 1's and k 0's. Therefore, by Karamata's inequality, for every $(x_j)_{j\in J} \in K$ we have

$$\sum_{j\in J} f(x_j) \leqslant \sum_{j\in J} f(y_j) = (m_0 - k)\Gamma\left(\frac{1}{2}\right).$$

In particular, since $(1-\tilde{c}_j)_{j\in J} \in K$ and

$$\sum_{j\in J} f(1-\tilde{c}_j) = \sum_{j\in J_1} f(1-\tilde{c}_j) = \sum_{j\in J_1} (1-\tilde{c}_j)\log\left(\frac{\Gamma\left(\frac{1}{1-\tilde{c}_j}-\frac{1}{2}\right)}{\sqrt{1-\tilde{c}_j}\,\Gamma\left(\frac{1}{1-\tilde{c}_j}\right)}\right),$$

we get the wanted upper bound. □

We stress that the method of the proof provides an intermediate bound that improves upon the general estimate, e.g. for sections of the cross-polytope, in view of an extra term taking into account the lengths of the projections of the vertices e_j onto the complementary subspace $H^\perp$. Note that the following is merely a consequence of (15) for the case $K_1 = B_1^n$.

Corollary 22. *For any $H \in G_{n,k}$ such that $e_j \notin H \cup H^\perp$ for every $1 \leqslant j \leqslant n$, we have that*

$$\mathrm{vol}_k(B_1^n \cap H) \leqslant \mathrm{vol}_k(B_1^k)\frac{1}{\pi^{\frac{n-k}{2}}}\prod_{j=1}^{n}\left(\frac{\Gamma\left(\frac{1}{\|P_{H^\perp}e_j\|_2^2}-\frac{1}{2}\right)}{\|P_{H^\perp}e_j\|_2\Gamma\left(\frac{1}{\|P_{H^\perp}e_j\|_2^2}\right)}\right)^{\|P_{H^\perp}e_j\|_2^2} \leqslant \mathrm{vol}_k(B_1^k).$$

5.2. **Sections of K_p, $p \in [1,2]$.** For the case of general $p \in [1,2]$, our result reads as follows.

Theorem 23. *Let $p \in [1,2]$, $(\alpha_j)_{j=1}^m \subset (0,\infty)$, $1 \leqslant k \leqslant n-1$ and $H \in G_{n,k}$. Then, one has the following explicit lower bound*

$$\mathrm{vol}_k(K_p \cap H) \geqslant \frac{\prod_{j\in J}\frac{\sqrt{c_j}}{\alpha_j^{1/p}}}{(2\pi)^{m_0-k}}\cdot\pi^\beta(m_0-k)^\beta\frac{1}{\Gamma(\beta)}\int_0^\infty t^{\beta-1}\prod_{j\in J}\gamma_p\left(\sqrt{t\frac{c_j}{\alpha_j^{2/p}}(1-\tilde{c}_j)}\right)dt,$$

where $\beta = (m_0-k)/2$. For the upper bound,

$$\mathrm{vol}_k(K_p \cap H) \leqslant \prod_{j\in J}\left(\frac{\sqrt{c_j}}{\alpha_j^{1/p}}\right)^{\tilde{c}_j}\mathrm{vol}_k(B_p^k),$$

which is sharp for B_p^n.

The upper bound in Theorem 23 is a direct consequence of the Brascamp-Lieb inequality and the following lemma about the Fourier transform, γ_p, of the function $e^{-|x|^p}$.

Lemma 24. *Let $0 < p < 2$ and $f(x) = e^{-|x|^p}$. Set*

$$\gamma_p(y) := \int_{\mathbb{R}} e^{ixy} f(x)\, dx.$$

Then, for all $s \geqslant 1$ we have

$$\int_{\mathbb{R}} \gamma_p\Big(\frac{t}{\sqrt{s}}\Big)^s\, dt \leqslant 2\pi\big(2\Gamma(1+\tfrac{1}{p})\big)^{s-1}.$$

Proof. It is known that there is a positive measure μ on $(0,\infty)$ such that

$$\gamma_p(\xi) = \int_0^\infty e^{-\xi^2 v}\, d\mu(v), \qquad \mu((0,\infty)) = \gamma_p(0) = \int_{\mathbb{R}} f(x)\, dx = 2\Gamma\big(1+\tfrac{1}{p}\big) := A_p.$$

Set $\nu := \mu/A_p$ and write $\gamma_p(\xi) = A_p \int_0^\infty e^{-\xi^2 v}\, d\nu(v)$. Application of Hölder's inequality

$$\Big(\int g\, d\nu\Big)^s \leqslant \int g^s\, d\nu,$$

gives us

$$\Big(\int_0^\infty e^{-\xi^2 v/s}\, d\nu(v)\Big)^s \leqslant \int_0^\infty e^{-\xi^2 v}\, d\nu(v).$$

Multiplying by A_p^s and integrating with respect to $\xi \in \mathbb{R}$ we get

$$\int_{\mathbb{R}} \gamma_p\Big(\frac{t}{\sqrt{s}}\Big)^s\, dt \leqslant A_p^{s-1} \int_{\mathbb{R}} \gamma_p(t)\, dt.$$

To this end, note

$$f(0) = \frac{1}{2\pi} \int_{\mathbb{R}} \gamma_p(t)\, dt,$$

therefore $\int_{\mathbb{R}} \gamma_p(t)\, dt = 2\pi f(0) = 2\pi$ and the lemma follows. □

We now proceed to the proof of Theorem 23.

Proof of Theorem 23, upper bound. Starting from the identity (13), we apply the geometric Brascamp–Lieb inequality in $H^\perp$ to get

$$\begin{aligned}
I_p &= \frac{1}{(2\pi)^{m_0-k}} \prod_{j\in J} \frac{\sqrt{c_j}}{\alpha_j^{1/p}} \int_{H^\perp} \prod_{j\in J_1} \gamma_p\left(\frac{\sqrt{c_j(1-\tilde{c}_j)}}{\alpha_j^{1/p}}\langle y, w_j\rangle\right) dy \\
&\leqslant \frac{1}{(2\pi)^{m_0-k}} \prod_{j\in J} \frac{\sqrt{c_j}}{\alpha_j^{1/p}} \prod_{j\in J_1} \left(\int_{\mathbb{R}} \gamma_p\left(\frac{\sqrt{c_j}}{\alpha_j^{1/p}}\sqrt{1-\tilde{c}_j}\, t\right)^{\frac{1}{1-\tilde{c}_j}} dt\right)^{1-\tilde{c}_j} \\
&= \frac{1}{(2\pi)^{m_0-k}} \prod_{j\in J} \frac{\sqrt{c_j}}{\alpha_j^{1/p}} \prod_{j\in J_1} \left(\frac{\alpha_j^{1/p}}{\sqrt{c_j}}\right)^{1-\tilde{c}_j} \left(\int_{\mathbb{R}} \gamma_p\left(\sqrt{1-\tilde{c}_j}\, t\right)^{\frac{1}{1-\tilde{c}_j}} dt\right)^{1-\tilde{c}_j}
\end{aligned}$$

Application of Lemma 24 gives

$$\Big(\int_{\mathbb{R}} \gamma_p(\sqrt{1-\tilde{c}_j}\, t)^{\frac{1}{1-\tilde{c}_j}}\, dt\Big)^{1-\tilde{c}_j} \leqslant (2\pi)^{1-\tilde{c}_j} A_p^{\tilde{c}_j},$$

for every $j \in J_1$. Ultimately, using the fact that $\sum_{j\in J_1} \tilde{c}_j = \sum_{j\in J} \tilde{c}_j = k$, we get

$$I_p \leqslant \prod_{j\in J_1} \left(\frac{\sqrt{c_j}}{\alpha_j^{1/p}}\right)^{\tilde{c}_j} \cdot A_p^k,$$

which completes the proof. □

Finally, we justify the lower bound in Theorem 23.

Proof of Theorem 23, lower bound. Use the Gaussian–mixture representation of γ_p:

$$\gamma_p(t) = \int_0^\infty e^{-t^2 s}\, d\mu(s), \qquad \mu((0,\infty)) = A_p.$$

Hence the product inside the y–integral equals

$$\prod_{j\in J} \gamma_p\left(\frac{\sqrt{c_j}}{\alpha_j^{1/p}}\langle y, x_j\rangle\right) = \int_{(0,\infty)^{m_0}} \exp\left(-\sum_{j\in J} \tfrac{c_j}{a_j^{2/p}} s_j \langle y, x_j\rangle^2\right) d\mu^{\otimes m_0}(s).$$

Using Fubini's theorem we get

$$I = \frac{\prod_{j\in J}\sqrt{c_j}/\alpha_j^{1/p}}{(2\pi)^{m_0-k}} \int_{(0,\infty)^{m_0}} \left(\int_{H^\perp} e^{-y^T S(s) y}\, dy\right) d\mu^{\otimes m_0}(s),$$

where the symmetric positive operator on $H^\perp$ is

$$S(s) = \sum_{j\in J} \tfrac{c_j}{a_j^{2/p}} s_j\, (x_j \otimes x_j)\Big|_{H^\perp}.$$

The inner y–integral is Gaussian and equals $\pi^{(m_0-k)/2}\det(S(s))^{-1/2}$. Using the determinant–trace estimate for positive operators in dimension $m_0 - k$,

$$\det(S(s)) \leqslant \left(\frac{\operatorname{tr} S(s)}{m_0 - k}\right)^{m_0-k},$$

we obtain

$$\int_{H^\perp} e^{-y^T S(s) y}\, dy \geqslant \pi^{\frac{m_0-k}{2}} (m_0-k)^{\frac{m_0-k}{2}} \big(\operatorname{tr} S(s)\big)^{-\frac{m_0-k}{2}}.$$

Since $\operatorname{tr}(x_j \otimes x_j|_{H^\perp}) = \|P_{H^\perp} x_j\|_2^2 = 1 - \tilde{c}_j$, we have

$$\operatorname{tr} S(s) = \sum_{j\in J} \tfrac{c_j}{a_j^{2/p}} s_j\, (1 - \tilde{c}_j).$$

Combining the above gives the stated lower bound

$$I \geqslant \frac{\prod_{j\in J}\sqrt{c_j}/\alpha_j^{1/p}}{(2\pi)^{m_0-k}}\, \pi^{\frac{m_0-k}{2}} (m_0-k)^{\frac{m_0-k}{2}} \int_{(0,\infty)^{m_0}} \left(\sum_{j\in J} \tfrac{c_j}{a_j^{2/p}} s_j (1-\tilde{c}_j)\right)^{-\frac{m_0-k}{2}} d\mu^{\otimes m_0}(s).$$

To obtain the desired bound we will use the formula:

$$x^{-\beta} = \frac{1}{\Gamma(\beta)} \int_0^\infty t^{\beta-1} e^{-tx}\, dt, \qquad \text{for } x > 0, \beta > 0.$$

We set $\beta = (m_0 - k)/2$ and use the exact expression for the trace,

$$x = \operatorname{tr} S(s) = \sum_{j\in J} \tfrac{c_j}{a_j^{2/p}} (1 - \tilde{c}_j) s_j.$$

Substituting this into the integral over the variables $(s_j)_{j\in J}$ and then applying Fubini's theorem to exchange the order of integration, we get:

$$\int_{(0,\infty)^{m_0}} (\operatorname{tr} S(s))^{-\beta}\, d\mu^{\otimes m_0}(s) =$$

$$= \frac{1}{\Gamma(\beta)} \int_0^\infty t^{\beta-1} \left(\int_{(0,\infty)^{m_0}} \exp\left(-t \sum_{j\in J} \tfrac{c_j}{a_j^{2/p}} (1-\tilde{c}_j) s_j \right) d\mu^{\otimes m_0}(s) \right) dt$$

$$= \frac{1}{\Gamma(\beta)} \int_0^\infty t^{\beta-1} \prod_{j\in J} \left(\int_0^\infty \exp\left(-t \left[\tfrac{c_j}{a_j^{2/p}} (1-\tilde{c}_j) \right] s_j \right) d\mu(s_j) \right) dt.$$

From the definition $\gamma_p(y) = \int_0^\infty e^{-y^2 v} d\mu(v)$, each inner integral is equal to γ_p evaluated at the square root of the coefficient of s_j. This gives:

$$\frac{1}{\Gamma(\beta)} \int_0^\infty t^{\beta-1} \prod_{j\in J} \gamma_p \left(\sqrt{t \tfrac{c_j}{a_j^{2/p}} (1-\tilde{c}_j)} \right) dt.$$

Inserting this back into the expression for I_p yields the stated lower bound. □

Remark 25. We should remark that the proof of Theorem 23 can not provide an intermediate bound in the likes of (15), and thus we do not have a result similar to Corollary 22 for $p > 1$. This is a natural consequence of the fact that in contrast to the case $p = 1$, we can not explicitly compute γ_p for general $p > 1$.

6. Non-Symmetric Case

Let K be a not necessarily origin-symmetric convex body in $\mathbb{R}^n$ in John's position. We can then find $u_1, \ldots, u_m \in \partial K \cap S^{n-1}$ and positive scalars $c_1, \ldots, c_m$ such that $\sum_{j=1}^m c_j u_j = 0$, $\sum_{j=1}^m c_j = n$ and

$$\mathrm{Id}_n = \sum_{j=1}^m c_j u_j \otimes u_j.$$

Let

$$C := \{x \in \mathbb{R}^n : \langle x, u_j \rangle \leqslant 1 \text{ for every } j = 1, \ldots, m\}.$$

It is easy to see that $K \subseteq C$. We also set

$$v_j = \sqrt{\frac{n}{n+1}} \left(-u_j, \frac{1}{\sqrt{n}} \right) \qquad \text{and} \qquad \delta_j = \frac{n+1}{n} c_j.$$

for every $j = 1, \ldots, m$. With this setup, we have that $\sum_{j=1}^m \delta_j v_j = (0, \sqrt{n+1})$, $\sum_{j=1}^m \delta_j = n+1$ and

$$\mathrm{Id}_{n+1} = \sum_{j=1}^m \delta_j v_j \otimes v_j$$

Finally, given $F \in G_{n,k}$ we let $H = \operatorname{span}\{(x, \sqrt{n}) : x \in F\} \in G_{n+1,k+1}$. Denote $J = \{j \in [m] : P_H v_j \neq 0\}$ and set

$$w_j = \frac{P_H v_j}{\|P_H v_j\|_2} \qquad \text{and} \qquad \kappa_j = \delta_j \|P_H v_j\|_2^2$$

for every $j \in J$ so that

$$\mathrm{Id}_H = \sum_{j\in J} \kappa_j w_j \otimes w_j \qquad \text{and} \qquad \sum_{j\in J} \kappa_j = k+1.$$

For any $k \in \mathbb{N}$, we denote $\Delta_k = \operatorname{conv}\{e_1, \ldots, e_k\}$ and $S_k = \sqrt{k(k+1)}\Delta_k$.

Under the notation introduced above, in this section we establish the following estimate.

Theorem 26. *Let K be a convex body in $\mathbb{R}^n$ in John's position and let $a \in S^{n-1}$. If $\kappa_j \geqslant 1/2$ for every $j \in J$, then*

$$\mathrm{vol}_{n-1}(K \cap a^{\perp}) \leqslant \frac{1}{\sqrt{2}} \sqrt{\frac{n+1}{n}} \left(\frac{n+1}{n-1} \right)^{\frac{n-1}{2}} \mathrm{vol}_{n-1}(S_{n-1}).$$

Note that in the case of central sections this upper bound is sharp, attained in the case that $K = S_n$ and $a = \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}}, 0, \ldots, 0 \right)$.

Let us first provide the following intermediate estimate.

Lemma 27. *Assume that* $\min \kappa_j \geqslant 1/2$. *Then for any* $1 \leqslant k \leqslant n-1$,

$$\frac{\mathrm{vol}_k(C \cap F)}{\mathrm{vol}_k(S_k)} \leqslant 2^{\frac{k+1-|J|}{2}} \frac{n^{\frac{k}{2}} (n+1)^{\frac{k+1}{2}}}{k^{\frac{k}{2}} (k+1)^{\frac{k+1}{2}}} \prod_{j \in J} \delta_j^{-\frac{\kappa_j}{2}}.$$

Proof. Our starting point is the formula, established in [AB, Section 4],

$$\frac{k^{\frac{k}{2}} (k+1)^{\frac{k+1}{2}}}{n^{\frac{k}{2}} (n+1)^{\frac{k+1}{2}}} \frac{\mathrm{vol}_k(C \cap F)}{\mathrm{vol}_k(S_k)} = \int_{L \cap H} e^{-\sum_{j \in J} \sqrt{\delta_j \kappa_j} \langle y, w_j \rangle} \, dy,$$

where $L = \{ y \in \mathbb{R}^{n+1} : \langle y, v_j \rangle \geqslant 0 \text{ for every } j \in J \}$. We will upper bound the integral on the right hand side above using Parseval's identity and the Brascamp-Lieb inequality. Following the notation $\tilde{w}_j = \frac{P_{H^\perp} v_j}{\|P_{H^\perp} v_j\|_2}$, we write

$$\begin{aligned}
\int_H e^{-\sum_{j \in J} \sqrt{\delta_j \kappa_j} \langle y, w_j \rangle} \cdot \mathbb{1}_L(y) \, dy &= \int_H \prod_{j \in J} e^{-\sqrt{\delta_j \kappa_j} \langle y, w_j \rangle} \cdot \mathbb{1}_{(0,\infty)}(\langle y, w_j \rangle) \, dy \\
&= \int_{H^\perp} \prod_{j \in J} \frac{1}{\sqrt{\delta_j} + 2\pi i \sqrt{1-\kappa_j} \langle y, \tilde{w}_j \rangle} \, dy \\
&\leqslant \int_{H^\perp} \prod_{j \in J_0} \left(\frac{1}{\delta_j + 4\pi^2 (1-\kappa_j) \langle y, w_j \rangle^2} \right)^{\frac{1}{2}} dy \\
&\leqslant \prod_{j \in J} \left(\int_{\mathbb{R}} \left(\frac{1}{\delta_j + 4\pi^2 (1-\kappa_j) t^2} \right)^{\frac{1}{2(1-\kappa_j)}} dt \right)^{1-\kappa_j},
\end{aligned}$$

where the last equality comes from an application of Proposition 1 and the last inequality follows from the Brascamp-Lieb inequality. To finish the proof, we apply the inequality

$$\left(\frac{1}{1+\alpha\beta} \right)^{\frac{1}{2\beta}} \leqslant \frac{1}{1+\frac{\alpha}{2}},$$

valid for all $\alpha > 0$ ad $0 < \beta \leqslant 1/2$. It follows that if $1 - \kappa_j \leqslant 1/2$ for every $j \in J$ then

$$\begin{aligned}
\prod_{j \in J} \left(\int_{\mathbb{R}} \left(\frac{1}{\delta_j + 4\pi^2 (1-\kappa_j) t^2} \right)^{\frac{1}{2(1-\kappa_j)}} dt \right)^{1-\kappa_j} &\leqslant \prod_{j \in J} \delta_j^{-\frac{1}{2}} \left(\int_{\mathbb{R}} \frac{1}{1 + 2\pi^2 \delta_j^{-2} t^2} \right)^{1-\kappa_j} \\
&= \prod_{j \in J} \delta_j^{-\frac{1}{2}} \left(\sqrt{\frac{\delta_j}{2}} \right)^{1-\kappa_j} = 2^{\frac{k+1-|J|}{2}} \prod_{j \in J} \delta_j^{-\frac{\kappa_j}{2}},
\end{aligned}$$

which is the desired upper bound. $\square$

Proof of Theorem 26. Recalling that $K \subseteq C$, a straightforward application of Lemma 27 yields

$$\mathrm{vol}_{n-1}(K \cap a^\perp) \leqslant 2^{\frac{n-|J|}{2}} \sqrt{\frac{n+1}{n}} \left(\frac{n+1}{n-1}\right)^{\frac{n-1}{2}} \left(\prod_{j\in J} \delta_j^{-\frac{\kappa_j}{2}}\right) \mathrm{vol}_{n-1}(S_{n-1}),$$

so our goal is to establish the bound $\prod_{j\in J} \delta_j^{-\frac{\kappa_j}{2}} \leqslant 2^{\frac{|J|-n-1}{2}}$. Denote $x_j = \|P_{a^\perp} v_j\|_2$, $S = \{(x,\delta) \in [1/2,1]^J \times [1/2,1]^J : \sum_{j\in J} \delta_j x_j = k+1\}$ and consider the function $F : S \to \mathbb{R}_+$ defined by

$$F(x,\delta) = \frac{1}{2}\sum_{j\in J} \delta_j x_j \log\frac{1}{\delta_j}.$$

By continuity, F attains its maximum on S. let $(\tilde{x},\tilde{\delta}) = \mathrm{argmax} F(x,\delta)$ and assume, without loss of generality, that $J = \{1,2,\dots,m_0\}$ and $\tilde{\delta}_1 \geqslant \tilde{\delta}_2 \geqslant \dots \geqslant \tilde{\delta}_{m_0}$. Keeping $\tilde{\delta}$ fixed, consider the function $f_{\tilde{\delta}} : \prod_{j\in J}[1/(2\tilde{\delta}_j), 1/\tilde{\delta}_j] \to \mathbb{R}_+$ given by $f_{\tilde{\delta}}(x) = F(x,\tilde{\delta})$. It is easy to check that $\tilde{x} = \mathrm{argmax} f_{\tilde{\delta}}$ and that $\tilde{x}$ has to be an extreme point of the set $\prod_{j\in J}[1/(2\tilde{\delta}_j), 1/\tilde{\delta}_j] \cap \{\sum_{j\in J} \tilde{\delta}_j x_j = k+1\}$. It follows that $\tilde{\delta}_j\tilde{x}_j \in \{1/2, 1\}$ for every $j \in J$. Since $\log\frac{1}{\tilde{\delta}_1} \leqslant \log\frac{1}{\tilde{\delta}_1} \leqslant \dots \leqslant \log\frac{1}{\tilde{\delta}_{m_0}}$, by a rearrangement argument it follows that there is some $m_1 \in [1, m_0]$ such that

$$\tilde{\delta}_j\tilde{x}_j = \begin{cases} \frac{1}{2}, & \text{if } 1 \leqslant j \leqslant m_1 \\ 1, & \text{if } m_1 < j \leqslant m_0. \end{cases}$$

Note also that for every $m_1 < j \leqslant m_0$, since $\tilde{\delta}_j\tilde{x}_j = 1$ and $\max \tilde{\delta}_j, \tilde{x}_j \leqslant 1$, it follows that $\tilde{\delta}_j = \tilde{x}_j = 1$. This argument shows that there is some $1 \leqslant m_1 \leqslant m_0$ such that

$$F(x,\delta) \leqslant \frac{1}{2}\sum_{j=m_1+1}^{m_0} \frac{1}{2}\log\frac{1}{\tilde{\delta}_j},$$

and the constraint $\frac{1}{2}\cdot m_1 + m_0 - m_1 = k+1 = n$ actually gives us $m_1 = 2n - m_0$. To find the maximiser $\tilde{\delta}$ we invoke the convexity of $x \mapsto \log\frac{1}{x}$ together with the fact that

$$\sum_{j=m_1+1}^{m_0} \tilde{\delta}_j = \sum_{j=1}^{m_0} \tilde{\delta}_j - \sum_{j=1}^{m_1} \tilde{\delta}_j = n+1-m_1.$$

Since $\tilde{\delta}_{m_1+1} \geqslant \dots \geqslant \tilde{\delta}_m$, it follows that $(\tilde{\delta}_{m_1+1}, \dots, \tilde{\delta}_m)$ is majorised by a vector of the form $(1,\dots,1,\frac{1}{2},\dots,\frac{1}{2})$. If m_2 is the cardinality of 1's, the maximiser has to satisfy

$$1\cdot m_2 + \frac{1}{2}\cdot(m_0 - m_1 - m_2) = n+1-m_1,$$

which is equivalent to $m_2 = 2$. We eventually get that

$$F(x,\delta) \leqslant \frac{1}{4}(m_0 - m_1 - 2)\log 2 = \frac{m_0-n-1}{2},$$

which yields the desired bound $\prod_{j\in J} \delta_j^{-\frac{\kappa_j}{2}} \leqslant 2^{\frac{|J|-n-1}{2}}$. □

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Appendix A. Parseval's identity for characteristic functions

We include here a proof of the following statement, that justifies the validity of Proposition 1 for the case of characteristic functions on bounded intervals. In what follows, we identify $\mathbb{R}^n$ with $H = \operatorname{span}\{e_1, \ldots, e_n\} \subset \mathbb{R}^m$, let $H^\perp$ denote the orthogonal complement, and let $I_j \subset \mathbb{R}^{n_j}$ be bounded measurable sets.

Theorem 28. *Assume that the linear map $L : H^\perp \longrightarrow \bigoplus_{j=1}^m \mathbb{R}^{n_j}$ given by $L(z) = (P_{\mathbb{R}^{n_j}} z)_{j=1}^m$ has image of dimension $d = \dim H^\perp$ (equivalently: L has full rank d). Then the identity*

$$\int_H \prod_{j=1}^m \mathbb{1}_{I_j}(P_{\mathbb{R}^{n_j}} y)\, dy = \frac{1}{(2\pi)^{m-n}} \int_{H^\perp} \prod_{j=1}^m \widehat{\mathbb{1}_{I_j}}(P_{\mathbb{R}^{n_j}} z)\, dz \tag{16}$$

holds provided that

(a) *The inequality*

$$\left|\widehat{\mathbb{1}_{I_j}}(\xi)\right| \leqslant C_j\, (1 + \|\xi\|_2)^{-1} \tag{17}$$

holds for every $\xi \in \mathbb{R}^{n_j}$, and

(b) *The number m_0 of indices j for which the corresponding projection factor is nontrivial (i.e. for which the multiplier on the j-th Fourier variable is nonzero) satisfies $m_0 > d$.*

Remark 29. The estimate

$$\left|\widehat{\mathbb{1}_{I_j}}(\xi)\right| \leqslant C_j(1+\|\xi\|_2)^{-1}, \qquad \xi \in \mathbb{R}^{n_j},$$

is very general and holds all bounded sets of finite perimeter (for example convex bodies, Lipschitz domains, intervals etc.)

We begin with the next general integrability criterion; the polynomial decay estimate (17) is a sufficient condition for integrability.

Lemma 30. *Let $L : H^\perp \to \bigoplus_{j=1}^m \mathbb{R}^{n_j}$ be the linear map $L(z) = (P_{\mathbb{R}^{n_j}} z)_{j=1}^m$, and suppose* $\operatorname{rank} L = d = \dim H^\perp$*. Suppose for each j there exist constants $C_j > 0$ and $\beta_j > 0$ such that*

$$\left|\widehat{\mathbb{1}_{I_j}}(\xi)\right| \leqslant C_j \left(1+\|\xi\|_2\right)^{-1}$$

for every $\xi \in \mathbb{R}^{n_j}$. If $m > d$, then the function

$$G(z) := \prod_{j=1}^m \left|\widehat{\mathbb{1}_{I_j}}(P_{\mathbb{R}^{n_j}} z)\right|$$

belongs to $L^1(H^\perp)$.

Proof. Since L has rank d, its image $S := L(H^\perp)$ is a d-dimensional linear subspace of $\bigoplus_{j=1}^m \mathbb{R}^{n_j}$. The map $L : H^\perp \to S$ is surjective and linear; choosing orthonormal coordinates on $H^\perp$ and on S we may identify L with an invertible linear map $\tilde{L} : \mathbb{R}^d \to \mathbb{R}^d$. In particular the change of variables $u = L(z)$ yields (up to the constant factor $|\det(\tilde{L}^{-1})|$) an equivalence of integrals:

$$\int_{H^\perp} G(z)\,dz \simeq \int_{u\in S} \prod_{j=1}^m |\widehat{\mathbb{1}_{I_j}}(u^{(j)})|\,du,$$

where $u^{(j)}$ denotes the $\mathbb{R}^{n_j}$-block of u corresponding to index j. Using the uniform bounds,

$$\prod_{j=1}^m |\widehat{\mathbb{1}_{I_j}}(u^{(j)})| \leqslant \Big(\prod_{j=1}^m C_j\Big) \prod_{j=1}^m (1+\|u^{(j)}\|_2)^{-1}.$$

Now, on a d-dimensional linear subspace S the asymptotic behaviour at infinity is controlled by the Euclidean norm $\|u\|_2$. There exists $C' > 0$ and $R > 0$ such that for $\|u\|_2 > R$,

$$\prod_{j=1}^m (1+\|u^{(j)}\|_2)^{-1} \leqslant C'(1+\|u\|_2)^{-m}.$$

(Indeed each $\|u^{(j)}\|_2 \leqslant \|u\|_2$ and therefore $(1+\|u^{(j)}\|_2)^{-1} \leqslant (1+\|u\|_2)^{-1}$). Thus the tail integral is bounded by a multiple of

$$\int_{|u|>R} (1+\|u\|_2)^{-m}\,du.$$

The latter converges precisely when $m > d$. The integral over the compact ball of radius R is finite because the integrand is bounded there. This proves $G \in L^1(H^\perp)$. $\square$

We now proceed to the proof of Theorem 28.

Proof of Theorem 28. We proceed by approximating characteristic functions by functions in the Schwarz space. Take a mollifier $\phi \in \mathcal{S}(\mathbb{R}^{n_j})$ with $\int \phi = 1$ and set $\phi_k(x) = k^{n_j}\phi(kx)$. Let $g_{j,k} = \mathbb{1}_{I_j} * \phi_k$. Then $g_{j,k} \in \mathcal{S}(\mathbb{R}^{n_j})$, $g_{j,k} \to \mathbb{1}_{I_j}$ a.e. and in $L^1(\mathbb{R}^{n_j})$, and $\|g_{j,k}\|_\infty \leqslant 1$. Applying Proposition 1 to $g_{j,k}$ we have that for every k,

$$\int_H \prod_{j=1}^m g_{j,k}(P_{\mathbb{R}^{n_j}} y)\, dy = \frac{1}{(2\pi)^{m-n}} \int_{H^\perp} \prod_{j=1}^m \widehat{g_{j,k}}(P_{\mathbb{R}^{n_j}} z)\, dz.$$

By dominated convergence and the L^1-approximation of $\mathbb{1}_{I_j}$ by $g_{j,k}$, the left-hand side converges to the desired $\int_H \prod_j \mathbb{1}_{I_j}(P_{\mathbb{R}^{n_j}} y)\, dy$. It remains to justify passing to the limit on the right-hand side.

Since $\widehat{g_{j,k}} = \widehat{\mathbb{1}_{I_j}}\, \widehat{\phi_k}$ and $\widehat{\phi_k}(\xi) = \widehat{\phi}(\xi/k) \to 1$ for each fixed ξ, it follows that for each fixed $z \in H^\perp$,

$$\lim_{k\to\infty} \prod_{j=1}^m \widehat{g_{j,k}}(P_{\mathbb{R}^{n_j}} z) = \prod_{j=1}^m \widehat{\mathbb{1}_{I_j}}(P_{\mathbb{R}^{n_j}} z).$$

Thus pointwise convergence of the integrand on $H^\perp$ is established. From $|\widehat{\phi_k}| \leqslant \sup|\widehat{\phi}| =: C_\phi$ we get the uniform bound

$$\Big|\prod_{j=1}^m \widehat{g_{j,k}}(P_{\mathbb{R}^{n_j}} z)\Big| \leqslant C_\phi^m \prod_{j=1}^m \Big|\widehat{\mathbb{1}_{I_j}}(P_{\mathbb{R}^{n_j}} z)\Big|.$$

Hence, the problem reduces to proving that

$$G(z) := \prod_{j=1}^m \Big|\widehat{\mathbb{1}_{I_j}}(P_{\mathbb{R}^{n_j}} z)\Big|$$

belongs to $L^1(H^\perp)$, which is precisely the statement of Lemma 30, the hypotheses of which are granted by (a) and (b) in the statement of Theorem 28. It follows that $G \in L^1(H^\perp)$, and the dominated convergence theorem yields

$$\lim_{k\to\infty} \frac{1}{(2\pi)^{m-n}} \int_{H^\perp} \prod_{j=1}^m \widehat{g_{j,k}}(P_{\mathbb{R}^{n_j}} z)\, dz = \frac{1}{(2\pi)^{m-n}} \int_{H^\perp} \prod_{j=1}^m \widehat{\mathbb{1}_{I_j}}(P_{\mathbb{R}^{n_j}} z)\, dz. \qquad \square$$

Finally, the following lemma shows that in the setting of our concern the conditions about m_0 and L are indeed met.

Lemma 31. *Following the notation of Proposition 2 and moreover denoting by $s = \#\{j : c_j = 1\}$ and $m_0 := m - s = \#\{j : c_j < 1\}$ (the number of indices with nontrivial projection to $H^\perp$), we have that*

(a) *The vectors $(w_j)_{j=1}^m$ span $H^\perp$, and therefore the linear map*

$$L : H^\perp \longrightarrow \mathbb{R}^m, \qquad L(y) = (\langle y, w_j\rangle)_{j=1}^m,$$

has image of dimension $d = \dim H^\perp$ (i.e. L has full rank d).

(b) *We have the inequality $m_0 \geqslant d$. In particular $m_0 > d$, except in the degenerate situation $s = n$ (equivalently: $m = n$ and all $c_j = 1$), in which case $d = 0$.*

Proof. (a) The operator identity $\mathrm{Id}_{H^\perp} = \sum_{j=1}^m (1-c_j) w_j \otimes w_j$ implies immediately that the linear span of the vectors $(w_j)_{j=1}^m$ equals $H^\perp$; otherwise the right-hand side would have rank strictly less than d and could not equal the identity on $H^{\perp}$. Hence the map L has image equal to all coordinate functionals on $H^{\perp}$ and so $\operatorname{rank} L = d$.

(b) Take traces on the identity $\mathrm{Id}_{H^{\perp}} = \sum_{j=1}^{m}(1-c_j)w_j \otimes w_j$. The trace of the left-hand side is d. The trace of the right-hand side equals $\sum_{j=1}^{m}(1-c_j)$. Thus

$$d = \sum_{j=1}^{m}(1-c_j).$$

Split the sum into indices with $c_j = 1$ and $c_j < 1$:

$$d = \sum_{j:c_j<1}(1-c_j) + \sum_{j:c_j=1}(1-1) = \sum_{j:c_j<1}(1-c_j) \leqslant \sum_{j:c_j<1} 1 = m_0.$$

Hence $m_0 \geqslant d$. Equality $m_0 = d$ forces every $1-c_j$ with $c_j < 1$ to equal 1, hence $c_j = 0$ for those j — but $c_j > 0$ by hypothesis, therefore the only way to have $m_0 = d$ in this weighted identity is the borderline/degenerate situation where $d = 0$ (equivalently $H^{\perp} = \{0\}$, which occurs exactly when $m = n$ and all $c_j = 1$). Thus, in the nondegenerate geometric configurations one gets $m_0 > d$. $\square$

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