ON THE MAXIMAL NUMBER OF FACETS OF 0/1 POLYTOPES

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Abstract. We show that there exist 0/1 polytopes in $\mathbb{R}^n$ whose number of facets exceeds $\left(\frac{cn}{\log n}\right)^{n/2}$, where $c > 0$ is an absolute constant.

1. Introduction

Let P be a polytope with non-empty interior in $\mathbb{R}^n$. We write $f_{n-1}(P)$ for the number of its $(n-1)$-dimensional faces. Consider the class of 0/1 polytopes in $\mathbb{R}^n$; these are the convex hulls of subsets of $\{0,1\}^n$. In this note we obtain a new lower bound for the quantity

$$g(n) := \max\left\{f_{n-1}(P_n) : P_n \text{ is a } 0/1 \text{ polytope in } \mathbb{R}^n\right\}. \tag{1.1}$$

The problem of determining the correct order of growth of $g(n)$ as $n \to \infty$ was posed by Fukuda and Ziegler (see [4], [10]). It is currently known that $g(n) \leq 30(n-2)!$ if n is large enough (see [3]). In the other direction, Bárány and Pór in [1] determined that $g(n)$ is superexponential in n: they obtained the lower bound

$$g(n) \geq \left(\frac{cn}{\log n}\right)^{n/4}, \tag{1.2}$$

where $c > 0$ is an absolute constant. In [5] we showed that

$$g(n) \geq \left(\frac{cn}{\log^2 n}\right)^{n/2}. \tag{1.3}$$

A more recent observation allows us to remove one logarithmic factor from the estimate in (1.3).

Theorem 1.1. *There exists a constant $c > 0$ such that*

$$g(n) \geq \left(\frac{cn}{\log n}\right)^{n/2}. \tag{1.4}$$

The method of proof of Theorem 1.1 is probabilistic and has its origin in the work of Dyer, Füredi and McDiarmid [2]. The proof is essentially the same with the one in [5], which in turn is based on [1], with the exception of a different approach to one estimate, summarized in Proposition 3.1 below. We consider random ± 1 polytopes (i.e., polytopes whose vertices are independent and uniformly distributed vertices

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$\vec{X}_i$ of the unit cube $C = [-1,1]^n$). We fix $n < N \leq 2^n$ and consider the random polytope

$$K_N = \mathrm{conv}\{\vec{X}_1, \dots, \vec{X}_N\}. \tag{1.5}$$

Our main result is a lower bound on the expectation $\mathbb{E}[f_{n-1}(K_N)]$ of the number of facets of K_N.

Theorem 1.2. *There exist two positive constants a and b such that: for all sufficiently large n, and all N satisfying $n^a \leq N \leq \exp(bn)$, one has that*

$$\mathbb{E}[f_{n-1}(K_N)] \geq \left(\frac{\log N}{a \log n}\right)^{n/2}. \tag{1.6}$$

The same result was obtained in [5] under the restriction $N \leq \exp(bn/\log n)$. This had a direct influence on the final estimate obtained, leading to (1.3).

The note is organized as follows. In Section 2 we briefly describe the method (the presentation is not self-contained and the interested reader should consult [1] and [5]). In Section 3 we present the new technical step (it is based on a more general lower estimate for the measure of the intersection of a symmetric polyhedron with the sphere, which might be useful in similar situations). In Section 4 we use the result of Section 3 to extend the range of N's for which Theorem 1.2 holds true. Theorem 1.1 easily follows.

We work in $\mathbb{R}^n$ which is equipped with the inner product $\langle \cdot, \cdot \rangle$. We denote by $\|\cdot\|_2$ the Euclidean norm and write B_2^n for the Euclidean unit ball and S^{n-1} for the unit sphere. Volume, surface area, and the cardinality of a finite set, are all denoted by $|\cdot|$. We write $\partial(F)$ for the boundary of F. All logarithms are natural. Whenever we write $a \simeq b$, we mean that there exist absolute constants $c_1, c_2 > 0$ such that $c_1 a \leq b \leq c_2 a$. The letters c, c', c_1, c_2 etc. denote absolute positive constants, which may change from line to line.

2. The method

The method makes essential use of two families (Q^β) and (F^β) $(0 < \beta < \log 2)$ of convex subsets of the cube $C = [-1,1]^n$, which were introduced by Dyer, Füredi and McDiarmid in [2]. We briefly recall their definitions. For every $\vec{x} \in C$, set

$$q(\vec{x}) := \inf\big\{\mathrm{Prob}\big(\vec{X} \in H\big) \colon \vec{x} \in H,\ H \text{ is a closed halfspace}\big\}. \tag{2.1}$$

The β-center of C is the convex polytope

$$Q^\beta = \{\vec{x} \in C \colon q(\vec{x}) \geq \exp(-\beta n)\}. \tag{2.2}$$

Next, define $f : [-1,1] \to \mathbb{R}$ by

$$f(x) = \tfrac{1}{2}(1+x)\log(1+x) + \tfrac{1}{2}(1-x)\log(1-x) \tag{2.3}$$

if $x \in (-1,1)$ and $f(\pm 1) = \log 2$, and for every $\vec{x} = (x_1, \dots, x_n) \in C$ set

$$F(\vec{x}) = \frac{1}{n}\sum_{i=1}^n f(x_i). \tag{2.4}$$

Then, F^β is defined by

$$F^\beta = \{\vec{x} \in C : F(\vec{x}) \leq \beta\}. \tag{2.5}$$

Since f is a strictly convex function on $(-1,1)$, F^β is convex.

When $\beta \to \log 2$ the convex bodies Q^β and F^β tend to C. The main tool for the proof of Theorem 1.2 is the fact that the two families (Q^β) and (F^β) are very close, in the following sense.

Theorem 2.1. (i) *$Q^\beta \cap (-1,1)^n \subseteq F^\beta$ for every $\beta > 0$.*
(ii) *There exist $\gamma \in \left(0, \frac{1}{10}\right)$ and $n_0 = n_0(\gamma) \in \mathbb{N}$ with the following property: If $n \geq n_0$ and $4\log n/n \leq \beta < \log 2$, then*

$$F^{\beta-\varepsilon} \cap \gamma C \subseteq Q^\beta \tag{2.6}$$

for some $\varepsilon \leq 3\log n/n$. □

Part (i) of Theorem 2.1 was proved in [2]. Part (ii) was proved in [5] and strengthens a previous estimate from [1].

Fix $n^8 \leq N \leq 2^n$ and define $\alpha = (\log N)/n$. The family (Q^β) is related to the random polytope K_N through a lemma from [2] (the estimate for ε claimed below is checked in [5]): If n is sufficiently large, one has that

$$\mathrm{Prob}\big(K_N \supseteq Q^{\alpha-\varepsilon}\big) > 1 - 2^{-(n-1)} \tag{2.7}$$

for some $\varepsilon \leq 3\log n/n$.

Combining (2.7) with Theorem 2.1, one gets the following.

Lemma 2.2. *Let $n^8 \leq N \leq 2^n$ and $n \geq n_0(\gamma)$. Then,*

$$\mathrm{Prob}\big(K_N \supseteq F^{\alpha-\varepsilon} \cap \gamma C\big) > 1 - 2^{-(n-1)} \tag{2.8}$$

for some $\varepsilon \leq 6\log n/n$. □

Bárány and Pór proved that K_N is weakly sandwithced between $F^{\alpha-\varepsilon} \cap \gamma C$ and $F^{\alpha+\delta}$ in the sense that $K_N \supseteq F^{\alpha-\varepsilon} \cap \gamma C$ and most of the surface area of $F^{\alpha+\delta} \cap \gamma C$ is outside K_N for small positive values of δ (the estimate for δ given below is checked in [5]).

Lemma 2.3. *If $n \geq n_0$ and $\alpha < \log 2 - 12n^{-1}$, then*

$$\mathrm{Prob}\big(\,|\partial(F^{\alpha+\delta}) \cap \gamma C \cap K_N| \geq \tfrac{1}{2}|\partial(F^{\alpha+\delta}) \cap \gamma C|\,\big) \leq \tfrac{1}{100}. \tag{2.9}$$

for some $\delta \leq 6/n$. □

We will also need the following geometric lemma from [1].

Lemma 2.4. *Let $\gamma \in \left(0, \frac{1}{10}\right)$ and assume that $\beta + \zeta < \log 2$. Then,*

$$|\partial(F^{\beta+\zeta}) \cap \gamma C \cap H| \leq (3\zeta n)^{(n-1)/2}|S^{n-1}| \tag{2.10}$$

for every closed halfspace H whose interior is disjoint from $F^\beta \cap \gamma C$. □

The strategy of Bárány and Pór (which is also followed in [5] and in the present note) is that for a random K_N and for each halfspace H_A which is defined by a facet A of K_N and has interior disjoint from K_N, we also have that H_A has interior disjoint from $F^{\alpha-\varepsilon} \cap \gamma C$ (from Lemma 2.2) and hence cuts a small amount (independent from A) of the surface of $\partial(F^{\alpha+\delta}) \cap \gamma C$ (from Lemma 2.4). Since the surface area of $\partial(F^{\alpha+\delta}) \cap \gamma C$ is mostly outside K_N (from Lemma 2.3) we see that the number of facets of K_N must be large, depending on the total surface of $\partial(F^{\alpha+\delta}) \cap \gamma C$. We will describe these steps more carefully in the last Section. First, we give a new lower bound for $|\partial(F^\beta) \cap \gamma C|$.

3. An additional lemma

The new element in our argument is the next Proposition.

Proposition 3.1. *There exists $r > 0$ with the following property: for every $\gamma \in (0,1)$ and for all $n \geq n_0(\gamma)$ and $\beta < c(\gamma)/r$ one has that*

$$|\partial(F^\beta) \cap \gamma C| \geq c(\gamma)^{n-1}(2\beta n)^{(n-1)/2}|S^{n-1}|, \tag{3.1}$$

where $c(\gamma) > 0$ is a constant depending only on γ.

Proof. We first estimate the product curvature $\kappa(\vec{x})$ of the surface $F(\vec{x}) = \beta$: in [5] it is proved that if $\beta < \log 2$ and $\vec{x} \in \gamma C$ with $F(\vec{x}) = \beta$, then

$$\frac{1}{\kappa(\vec{x})} \geq \left(1-\gamma^2\right)^{n-1}(2\beta n)^{(n-1)/2}. \tag{3.2}$$

Let $\vec{\theta} \in S^{n-1}$ and write $\vec{x}(\vec{\theta},\beta)$ for the point on the boundary of F^β for which $n\,\nabla F\big(\vec{x}(\vec{\theta},\beta)\big)$ is a positive multiple of $\vec{\theta}$. This point is well-defined and unique if $0 < \beta < \big|\mathrm{supp}\,\vec{\theta}\big|(\log 2)/n$ (see [1, Lemma 6.2]).

Let $r > 0$ be an absolute constant (which will be suitably chosen) and set

$$M_r = \big\{\vec{\theta} \in S^{n-1} : \sqrt{n/r}\,\vec{\theta} \in C\big\}. \tag{3.3}$$

The argument given in [1, Lemma 6.3] shows that if $\beta < c_1(\gamma)/r$, then for every $\vec{\theta} \in M_r$ we have $\vec{x}(\vec{\theta},\beta) \in \gamma C$. Also, we easily check that for every $\vec{\theta} \in M_r$ the condition $\big|\mathrm{supp}\,\vec{\theta}\big| \geq n/r$ is satisfied, and hence, if $\beta < c_1(\gamma)/r$ then $\vec{x}(\vec{\theta},\beta)$ is well-defined and unique. We will estimate the measure of M_r.

Lemma 3.2. *There exists $r > 0$ such that: if $n \geq 3$ then*

$$|M_r| \geq e^{-n/2}|S^{n-1}|. \tag{3.4}$$

Proof. Write γ_n for the standard Gaussian measure on $\mathbb{R}^n$ and σ_n for the rotationally invariant probability measure on S^{n-1}. We use the following fact.

Fact 3.3. *If K is a symmetric convex body in $\mathbb{R}^n$ then*

$$\tfrac{1}{2}\,\sigma_n\big(S^{n-1} \cap \tfrac{1}{2}K\big) \leq \gamma_n(\sqrt{n}K) \leq \sigma_n(S^{n-1} \cap eK) + e^{-n/2}. \tag{3.5}$$

Proof of Fact 3.3. A proof appears in [7]. We sketch the proof of the right hand side inequality (which is the one we need). Observe that

$$\sqrt{n}K \subseteq \big(\tfrac{1}{e}\sqrt{n}B_2^n\big) \cup C\,\big(\tfrac{1}{e}\sqrt{n}S^{n-1} \cap \sqrt{n}K\big) \tag{3.6}$$

where, for $A \subseteq \frac{1}{e}\sqrt{n}S^{n-1}$, we write $C(A)$ for the positive cone generated by A. It follows that

$$\gamma_n(\sqrt{n}K) \leq \gamma_n\big(\tfrac{1}{e}\sqrt{n}B_2^n\big) + \sigma\big(\tfrac{1}{e}\sqrt{n}S^{n-1} \cap \sqrt{n}K\big) \tag{3.7}$$

where σ denotes the rotationally invariant probability measure on $\frac{1}{e}\sqrt{n}S^{n-1}$. Now

$$\sigma\big(\tfrac{1}{e}\sqrt{n}S^{n-1} \cap \sqrt{n}K\big) = \sigma_n(S^{n-1} \cap eK), \tag{3.8}$$

and a direct computation shows that

$$\gamma_n\big(\rho\sqrt{n}B_2^n\big) \leq (\rho\sqrt{e})^n e^{-\rho^2 n/2} \tag{3.9}$$

for all $0 < \rho \leq 1$. It follows that

$$\gamma_n\big(\tfrac{1}{e}\sqrt{n}B_2^n\big) \leq \exp(-n/2). \tag{3.10}$$

From (3.7)–(3.10) we get the Fact. □

Proof of Lemma 3.2. Observe that

$$M_r = S^{n-1} \cap e\left(\sqrt{r/(e^2 n)}\, C\right). \tag{3.11}$$

Hence

$$\begin{aligned} \frac{|M_r|}{|S^{n-1}|} = \sigma_n(M_r) &= \sigma_n\Big(S^{n-1} \cap e\big(\sqrt{r/(e^2 n)}\, C\big)\Big) \\ &\geq \gamma_n\big((\sqrt{r}/e)\, C\big) - e^{-n/2} \\ &= d\left(\sqrt{r}/e\right)^n - e^{-n/2}, \end{aligned}$$

where

$$d(s) := \frac{1}{\sqrt{2\pi}} \int_{-s}^{s} e^{-t^2/2} dt. \tag{3.12}$$

Observe that $2e^{-n/2} < e^{-n/4}$ for $n \geq 3$. Choose $r > 0$ so that

$$d\left(\sqrt{r}/e\right) > e^{-1/4}; \tag{3.13}$$

this is possible, since $\lim_{s\to+\infty} d(s) = 1$. Then,

$$d\left(\sqrt{r}/e\right)^n > 2e^{-n/2} \tag{3.14}$$

for $n \geq 3$, which completes the proof. □

We can now finish the proof of Proposition 3.1. Writing $\vec{x}$ for $\vec{x}\big(\vec{\theta}, \beta\big)$ and expressing surface area in terms of product curvature (cf. [8, Theorem 4.2.4]), we can write

$$|\partial(F^\beta) \cap \gamma C| \geq \int_{M_r} \frac{1}{\kappa(\vec{x})}\, d\vec{\theta} \geq e^{-n/2}\big(1-\gamma^2\big)^{n-1}(2\beta n)^{(n-1)/2}|S^{n-1}|, \tag{3.15}$$

and the result follows. □

A general version of Lemma 3.2. The method of proof of Lemma 3.2 provides a general lower estimate for the measure of the intersection of an arbitrary symmetric polyhedron with the sphere. Let $\vec{u}_1, \ldots, \vec{u}_m$ be non-zero vectors in $\mathbb{R}^n$ and consider the symmetric polyhedron

$$T = \bigcap_{j=1}^{m} \big\{x : |\langle x, \vec{u}_i\rangle| \leq 1\big\}. \tag{3.16}$$

The following theorem of Sidák (see [9]) gives an estimate for $\gamma_n(T)$.

Fact 3.4 (Sidák's lemma)**.** *If T is the symmetric polyhedron defined by* (3.16) *then*

$$\gamma_n(T) \geq \prod_{i=1}^{m} \gamma_n(\{x : |\langle x, \vec{u}_i\rangle| \leq 1\}) = \prod_{i=1}^{m} d\left(\frac{1}{\|\vec{u}_i\|_2}\right). \tag{3.17}$$

We will also use an estimate which appears in [6].

Fact 3.5. *There exists an absolute constant $\lambda > 0$ such that, for every $t_1, \ldots, t_m > 0$,*

$$\prod_{i=1}^{m} d\left(\frac{1}{t_i}\right) \geq \exp\left(-\lambda \sum_{i=1}^{m} t_i^2\right). \tag{3.18}$$

Consider the parameter $R = R(T)$ defined by

$$R^2(T) = \sum_{i=1}^{m} \|\vec{u}_i\|_2^2. \tag{3.19}$$

Let $s > 0$. Fact 3.4 shows that

$$\gamma_n(sT) \geq \prod_{i=1}^{m} d\left(\frac{s}{\|\vec{u}_i\|_2}\right). \tag{3.20}$$

Then, Fact 3.5 shows that

$$\gamma_n(sT) \geq \exp\left(-\lambda R^2(T)/s^2\right) \geq e^{-n/4} \geq 2e^{-n/2}, \tag{3.21}$$

provided that $n \geq 3$ and

$$s \geq \frac{2\sqrt{\lambda}R(T)}{\sqrt{n}}. \tag{3.22}$$

We then apply Fact 3.3 for the polyhedron $K = (s/\sqrt{n})T$ to get

$$\sigma_n\left(S^{n-1} \cap \frac{es}{\sqrt{n}}T\right) \geq \exp\left(-\lambda R^2(T)/s^2\right) - \exp(-n/2) \geq \tfrac{1}{2}\exp\left(-\lambda R^2(T)/s^2\right). \tag{3.23}$$

In other words, we have proved the following.

Proposition 3.6. *Let $n \geq 3$ and let $\vec{u}_1, \ldots, \vec{u}_m$ be non-zero vectors in $\mathbb{R}^n$. Consider the symmetric polyhedron*

$$T = \bigcap_{j=1}^{m} \left\{x : |\langle x, \vec{u}_i\rangle| \leq 1\right\},$$

and define

$$R^2(T) = \sum_{i=1}^{m} \|\vec{u}_i\|_2^2.$$

Then, for all $t \geq cR(T)/\sqrt{n}$ we have that

$$\sigma_n\left(S^{n-1} \cap (t/\sqrt{n})\, T\right) \geq \tfrac{1}{2}\exp\left(-cR^2(T)/t^2\right), \tag{3.24}$$

where $c > 0$ is an absolute constant. □

4. Proof of the theorems

Proof of Theorem 1.2. Let $\gamma \in (0,1)$ be the constant in Theorem 2.1. Assume that n is large enough and set $b = c(\gamma)/(2r)$, where $c(\gamma) > 0$ is the constant in Proposition 3.1.

Given N with $n^8 \leq N \leq \exp(bn)$, let $\alpha = (\log N)/n$. From Lemma 2.2 there exists $\varepsilon \leq 6\log n/n$ such that

$$K_N \supseteq F^{\alpha-\varepsilon} \cap \gamma C \tag{4.1}$$

with probability greater than $1 - 2^{-n+1}$, and from Lemma 2.3 there exists $\delta \leq 6/n$ such that

$$|(\partial(F^{\alpha+\delta}) \cap \gamma C) \setminus K_N| \geq \tfrac{1}{2}|\partial(F^{\alpha+\delta}) \cap \gamma C| \tag{4.2}$$

with probability greater than $1 - 10^{-2}$. We assume that K_N satisfies both (4.1) and (4.2) (this holds with probability greater than $\frac{1}{2}$).

We apply Lemma 2.4 with $\beta = \alpha - \varepsilon$ and $\zeta = \varepsilon + \delta$: If A is a facet of K_N and H_A is the corresponding halfspace which has interior disjoint from K_N, then

$$|\partial(F^{\alpha+\delta}) \cap \gamma C \cap H_A| \leq \big(3n(\varepsilon+\delta)\big)^{(n-1)/2} |S^{n-1}|. \tag{4.3}$$

It follows that

$$\begin{aligned} f_{n-1}(K_N)\,\big(3n(\varepsilon+\delta)\big)^{(n-1)/2}|S^{n-1}| &\geq \sum_A |\partial(F^{\alpha+\delta}) \cap \gamma C \cap H_A| \\ &\geq \big|\big(\partial(F^{\alpha+\delta}) \cap \gamma C\big) \setminus K_N\big| \\ &\geq \tfrac{1}{2}|\partial(F^{\alpha+\delta}) \cap \gamma C|. \end{aligned}$$

Since $\alpha \leq b = c(\gamma)/(2r)$ and $\delta \leq 6/n$, we have $\alpha + \delta \leq c(\gamma)/r$ if n is large enough. Applying Proposition 3.1 with $\beta = \alpha + \delta$, we get

$$f_{n-1}(K_N)\,\big(3n(\varepsilon+\delta)\big)^{(n-1)/2} \geq \Big(c(\gamma)\sqrt{2\alpha n}\Big)^{n-1}, \tag{4.4}$$

for sufficiently large n. Since $\alpha n = \log N$ and $(\varepsilon + \delta)n \leq 12 \log n$, this shows that

$$f_{n-1}(K_N) \geq \left(\frac{c_1(\gamma)\log N}{\log n}\right)^{n/2} \tag{4.5}$$

with probability greater than $\frac{1}{2}$. □

Proof of Theorem 1.1. We can apply Theorem 1.2 with $N \geq \exp(bn)$ where $b > 0$ is an absolute constant. This shows that there exist 0/1 polytopes P in $\mathbb{R}^n$ with

$$f_{n-1}(P) \geq \left(\frac{cn}{\log n}\right)^{n/2}, \tag{4.6}$$

as claimed. □

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