

Many Facets in Random Polytopes from Product and Log-Concave Measures

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Abstract

We prove bounds of order $n^{n/2}e^{O(n)}$ for the expected number of facets of high-dimensional random polytopes. First, let μ be a non-degenerate compactly supported even probability measure on $\mathbb{R}$ satisfying $\mu([x^* - s, x^*]) \asymp s^\kappa$ near its right endpoint x^* . For every sufficiently small fixed $\alpha > 0$, the convex hull of $N = \lfloor e^{\alpha n} \rfloor$ independent points with law $\mu^{\otimes n}$ has at least $n^{n/2}e^{-C_{\mu,\alpha}n}$ expected facets; this includes all symmetric finite-alphabet distributions.

For every full-dimensional log-concave probability measure on $\mathbb{R}^n$, we prove that there exist $T \in [n, 2n]$ and $N = \lceil e^T n^{3/2} \rceil$ for which

$$n^{n/2}e^{-Cn} \leq \mathbb{E}f_{n-1}(P_N) \leq n^{n/2}e^{Cn}.$$

Thus the scale $n^{n/2}$, up to exponential factors, is universal for log-concave measures in this high-dimensional exponential regime. Finally, we construct a symmetric isotropic full-support non-log-concave counterexample with only $(1 + o(1))2^n$ expected facets.

1 Introduction

For an n -dimensional polytope P we write $f_{n-1}(P)$ for its number of facets. A classical extremal problem asks for the largest possible value

$$g(n) = \max\{f_{n-1}(P) : P \subset \mathbb{R}^n \text{ is a 0/1 polytope}\}.$$

The problem, associated with Fukuda and Ziegler and recorded in [30], has a long history. Fleiner, Kaibel and Rote proved the upper bound

$$g(n) \leq 30(n-2)!$$

for all sufficiently large n [14]. On the lower-bound side, Bárány and Pór obtained the first superexponential estimate

$$g(n) \geq \left(\frac{cn}{\log n}\right)^{n/4}$$

in [4]. Gatzouras, Giannopoulos and Markoulakis subsequently proved

$$g(n) \geq \left(\frac{cn}{(\log n)^2}\right)^{n/2}$$

in [18], and later improved this to

$$g(n) \geq \left(\frac{cn}{\log n}\right)^{n/2}$$

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in [19].

Our starting point is probabilistic rather than extremal. We study the number of facets of the convex hull

$$P_N = \text{conv}\{X_1, \dots, X_N\}$$

of independent random points in high dimension, in a regime where the number N of points grows exponentially with the dimension. The main theme of the paper is that, under two rather different structural assumptions—product structure and log-concavity—the scale

$$n^{n/2}$$

emerges naturally for the expected number of facets, up to factors exponential in n . In the log-concave setting this scale is in fact universal throughout the class.

The first part of the paper concerns product measures and is genuinely non-log-concave. Let μ be a non-degenerate compactly supported even probability measure on $\mathbb{R}$, and put

$$x^* = \sup\{x > 0 : \mu([x, \infty)) > 0\}.$$

We assume that, for some $\kappa \geq 0$,

$$cs^\kappa \leq \mu([x^* - s, x^*]) \leq Cs^\kappa, \quad 0 < s \leq s_0. \quad (1)$$

This includes measures having an atom at the endpoint, corresponding to $\kappa = 0$, and hence all non-degenerate symmetric finitely supported distributions. It also includes many purely atomic measures with infinitely many atoms accumulating at the endpoints.

For every sufficiently small fixed $\alpha > 0$, if

$$N = \lfloor e^{\alpha n} \rfloor$$

and $X_1, \dots, X_N$ are independent with law $\mu^{\otimes n}$, we prove

$$\mathbb{E}f_{n-1}(P_N) \geq n^{n/2} e^{-C_{\mu, \alpha} n}.$$

Thus the many-facets phenomenon is not confined to log-concave distributions or to smooth probability measures. In particular, for the symmetric Bernoulli measure the random polytope is a sign polytope and is affinely equivalent to a 0/1 polytope. We therefore recover, as a special case,

$$g(n) \geq (cn)^{n/2}.$$

After the results of the present paper had been obtained, we learned of a recent independent work of Friedland [16], who proved the same log-free estimate

$$g(n) \geq (cn)^{n/2}$$

by a direct argument for random sign polytopes. The two approaches are different. Friedland's argument is tailored to the discrete cube, whereas our result is obtained as part of a general theorem for compactly supported even product measures satisfying (1).

The probabilistic study of random 0/1 polytopes is closely related to threshold phenomena for random convex hulls. An early source of this circle of ideas is the work of Dyer, Füredi and McDiarmid [13]. In a more general product setting, Pafis expressed the relevant thresholds in terms of the one-dimensional Cramér transform [27]. For log-concave measures, connections between the Cramér transform, half-space depth and random convex hulls were developed in [10, 11, 9, 20]; the atomic case and some limitations of discrete log-concavity were investigated in [12]. We shall also use the general convex-hull estimates of Hayakawa, Lyons and Oberhauser [22].

Before turning to the second theorem, we give a simple full-support symmetric isotropic example showing that without product structure or log-concavity the expected number of facets may remain only exponential in n .

The second part of the paper leaves the product setting altogether. For every full-dimensional log-concave probability measure μ on $\mathbb{R}^n$, with no symmetry or product assumption, we prove the upper bound

$$\mathbb{E}f_{n-1}(P_N) \leq C^n N n^{(n-3)/2},$$

when $N \geq 2n$. Consequently, throughout every fixed exponential window

$$N \leq e^{bn} n^{3/2},$$

one has

$$\mathbb{E}f_{n-1}(P_N) \leq C_b^n n^{n/2}.$$

We complement this estimate by a matching lower bound, up to exponential factors: for every such μ there exist $T \in [n, 2n]$ and

$$N = \lceil e^T n^{3/2} \rceil$$

such that

$$n^{n/2} e^{-Cn} \leq \mathbb{E}f_{n-1}(P_N) \leq n^{n/2} e^{Cn}.$$

Thus, in the genuinely high-dimensional exponential regime, the scale $n^{n/2}$ is universal for the expected number of facets of random polytopes generated by log-concave measures.

The point is that this conclusion is uniform over the entire class of log-concave measures, despite the very different facet asymptotics exhibited by particular models when the dimension is fixed and $N \rightarrow \infty$. The number of sample points is larger than in the product theorem; the product structure is what permits one to work at Cramér levels $B_t := \{\Lambda_\mu^* \leq t\}$ of size αn with arbitrarily small fixed α .

It is worth contrasting the preceding result with the classical theory of random polytopes. A substantial part of that theory concerns random points chosen from a fixed convex body in a fixed dimension, with the number of points tending to infinity. In this setting the asymptotic behaviour of the face numbers depends strongly on the geometry of the underlying body. For smooth convex bodies it is governed by boundary curvature and affine surface area, whereas for polytopal containers a quite different, logarithmic behaviour occurs; see, for instance, [3, 2, 28]. Exact formulae are also available for several highly symmetric models, including Gaussian and beta polytopes; see, e.g., [23, 24]. Thus there is no universal dependence on the sample size throughout the class of log-concave measures when the dimension is kept fixed and the number of points tends to infinity.

The situation is rather different in the genuinely high-dimensional exponential regime considered here. For Gaussian polytopes, Böröczky, Lugosi and Reitzner [7] proved, in the regime $N/n \rightarrow \infty$, the asymptotic formula

$$\mathbb{E}f_{n-1}(P_N) = \left((4\pi + o(1)) \log \frac{N}{n} \right)^{(n-1)/2}.$$

In particular, when $\log N$ is proportional to n , the expected number of facets is $n^{n/2}$ up to factors exponential in n . Bonnet and O'Reilly obtained a closely related high-dimensional picture for spherical random polytopes, including the exponential regime $\log N \asymp n$ [6]. The general log-concave result proved here shows that this $n^{n/2}$ scale is not a consequence of Gaussian or spherical symmetry. Up to exponential factors, it persists for every full-dimensional log-concave probability measure, without any assumption of smoothness, compact support, symmetry, product

structure, or an explicit form of the density. In this sense, the result is of a different nature from the classical fixed-dimensional stochastic approximation results: it is a high-dimensional universality statement for the number of facets in the window $N = e^{\Theta(n)}$. Outside this window one should not expect such a universal behaviour, as the classical smooth, polytopal, and Gaussian models already exhibit substantially different asymptotics.

We briefly describe the two proofs. In the product case, the endpoint assumption gives uniform estimates for moments under one-dimensional exponential tilts. Combined with a coarse half-space-depth estimate from [27] and Berry–Esseen, this yields the sharp local estimate

$$q_{\mu^{\otimes n}}(x) \gtrsim \frac{e^{-\Lambda_{\mu^{\otimes n}}^*(x)}}{\sqrt{n}}$$

on a fixed cube and on Cramér levels comparable with n , together with the matching upper estimate for the tangent half-space. A large portion of a Cramér level surface lies inside a smaller fixed cube. For every point of this portion, a finite set of at most D_L^n points at a slightly lower level meets every half-space that penetrates sufficiently far into the Cramér body. The finite set is obtained from the VC ε -net theorem [21, Corollary 3.7]. The depth estimate and [22, Proposition 13] show that all these points belong to the random polytope with overwhelmingly high probability. Finally, strong convexity shows that an exterior half-space which misses a lower Cramér body intersects the chosen level surface in a set of surface area at most $C_L^n |S^{n-1}|$. Comparing the total exposed area with the contribution of one facet gives the result.

The log-concave proof for the lower bound is shorter. After isotropic normalization, the solution of the slicing problem [25] gives the required uniform density bound for the original measure and all relevant exponential tilts. We select a Cramér level through the volume of the corresponding set of tilt parameters. On a substantial set of directions the variance of the tilted one-dimensional statistic is between a^2 and $4a^2$, where $c\sqrt{n} \leq a \leq Cn$. The density of a hyperplane section and the conditional log-concave density on that section can then be controlled directly. A simple random-simplex lemma converts the latter estimate into a lower bound for the weighted section-simplex integral. The Blaschke–Petkantschin formula [29] then gives the desired facet count.

For the upper bound, we make again use of the Blaschke–Petkantschin formula. Corresponding one-dimensional variance bounds for the tilted measures allow us to control the probability of certain half-spaces. Combining again with the random-simplex lemma yields the proof.

2 Compactly supported product measures

Throughout this section μ is a non-degenerate even Borel probability measure on $\mathbb{R}$ with support contained in $[-x^*, x^*]$, where $x^* > 0$, and we assume (1). Let X have distribution μ and for $t \in \mathbb{R}$ write

$$\Lambda_\mu(t) = \log \mathbb{E} e^{tX}.$$

Define the tilted probability measure

$$dP_t = e^{tX - \Lambda_\mu(t)} d\mu,$$

and denote expectation under P_t by $\mathbb{E}_t$. Then

$$\mathbb{E}_t X = \Lambda'_\mu(t), \quad \text{Var}_t(X) = \Lambda''_\mu(t).$$

In addition, if $t_1, \dots, t_n \in \mathbb{R}$ we also define the probability measure

$$P_{t_1, \dots, t_n} = P_{t_1} \otimes \dots \otimes P_{t_n}.$$

The function Λ_μ is even and strictly convex, and $\Lambda'_\mu : \mathbb{R} \rightarrow (-x^*, x^*)$ is a strictly increasing bijection. Its Legendre transform Λ_μ^* is even, strictly convex and smooth on $(-x^*, x^*)$, and

$$(\Lambda_\mu^*)'(x) = (\Lambda'_\mu)^{-1}(x), \quad (\Lambda_\mu^*)''(x) = \frac{1}{\Lambda_\mu''((\Lambda_\mu^*)'(x))}. \quad (2)$$

Since $\Lambda_\mu''(t) = \text{Var}_t(X) \leq (x^*)^2$, we have the global strong-convexity estimate

$$(\Lambda_\mu^*)''(x) \geq (x^*)^{-2}, \quad |x| < x^*. \quad (3)$$

For $\mu_n = \mu^{\otimes n}$ and $r > 0$ we write $B_r = \{x \in [-x^*, x^*]^n : \Lambda_{\mu_n}^*(x) \leq r\}$ and an important property in the product case is

$$\Lambda_{\mu_n}^*(x) = \sum_{i=1}^n \Lambda_\mu^*(x_i).$$

We also write

$$q_n(x) = \inf\{\mu_n(H) : H \text{ is a closed half-space containing } x\}.$$

for Tukey's half-space depth of μ_n .

2.1 One-dimensional estimates

The endpoint assumption is used only through the following elementary consequences.

Lemma 2.1. *For every fixed integer $m \geq 1$,*

$$\sup_{t \in \mathbb{R}} |t|^m \mathbb{E}_t |X - \Lambda'_\mu(t)|^m < \infty. \quad (4)$$

Moreover,

$$\frac{t^2 \Lambda_\mu''(t)}{\Lambda_\mu^*(\Lambda'_\mu(t))} \longrightarrow 2 \quad (t \rightarrow 0). \quad (5)$$

Finally,

$$\frac{-\log \mu([x, \infty))}{\Lambda_\mu^*(x)} \longrightarrow 1 \quad (x \uparrow x^*). \quad (6)$$

Proof. By symmetry it is enough to consider $t > 0$. Put $Y = x^* - X$ and let ν be its law. Notice that

$$|X - \mathbb{E}_t X| = |Y - \mathbb{E}_t Y|.$$

So a standard probabilistic argument implies that it suffices to prove that for every $m \in \mathbb{N}$

$$\sup_{t > 0} t^m \mathbb{E}_t Y^m < +\infty.$$

Let $t > 0$. Observe that

$$t^m \mathbb{E}_t Y^m = \frac{\int_0^{2x^*} (ty)^m e^{-ty} d\nu(y)}{\int_0^{2x^*} e^{-ty} d\nu(y)}. \quad (7)$$

If $t \leq s_0^{-1}$, then

$$t^m \mathbb{E}_t Y^m \leq \left(\frac{2x^*}{s_0} \right)^m.$$

We may therefore assume that $t \geq s_0^{-1}$. For the denominator of (7), we obtain

$$\begin{aligned} \int_0^{2x^*} e^{-ty} d\nu(y) &\geq \int_0^{1/t} e^{-ty} d\nu(y) \\ &\geq e^{-1} \nu([0, 1/t]) \\ &= e^{-1} \mu([x^* - 1/t, x^*]) \geq ce^{-1} t^{-\kappa}. \end{aligned} \quad (8)$$

For the numerator, write

$$\int_0^{2x^*} (ty)^m e^{-ty} d\nu(y) = \int_{[0, s_0]} (ty)^m e^{-ty} d\nu(y) + \int_{(s_0, 2x^*]} (ty)^m e^{-ty} d\nu(y) =: I_1 + I_2. \quad (9)$$

Clearly,

$$I_2 \leq (2tx^*)^m e^{-ts_0}. \quad (10)$$

For

$$J_k = \left[\frac{k}{t}, \frac{k+1}{t} \right), \quad k = 0, 1, \dots, \lfloor ts_0 \rfloor - 1,$$

condition (1) gives

$$\begin{aligned} \int_{J_k} (ty)^m e^{-ty} d\nu(y) &\leq (k+1)^m e^{-k} \nu([0, (k+1)/t]) \\ &\leq Ct^{-\kappa} (k+1)^{\kappa+m} e^{-k}. \end{aligned} \quad (11)$$

The remaining interval satisfies the same estimate:

$$\int_{[\lfloor ts_0 \rfloor / t, s_0]} (ty)^m e^{-ty} d\nu(y) \leq Ct^{-\kappa} (\lfloor ts_0 \rfloor + 1)^{\kappa+m} e^{-\lfloor ts_0 \rfloor}. \quad (12)$$

Consequently,

$$I_1 \leq Ct^{-\kappa} \sum_{k=0}^{\lfloor ts_0 \rfloor} (k+1)^{\kappa+m} e^{-k} \leq C_{\kappa, m} t^{-\kappa}. \quad (13)$$

Combining (7), (8), (10) and (13), we get

$$t^m \mathbb{E}_t Y^m \leq C_{\kappa, m} + C_m t^{\kappa+m} e^{-ts_0},$$

which is uniformly bounded for $t \geq s_0^{-1}$, and thus (4) is proved.

Since

$$\Lambda_\mu^*(\Lambda'_\mu(t)) = t\Lambda'_\mu(t) - \Lambda_\mu(t) = \int_0^t s\Lambda''_\mu(s) ds = t^2 \int_0^1 u\Lambda''_\mu(tu) du,$$

continuity of Λ''_μ at the origin gives (5). For (6) we refer to [17, Theorem 1.3]. $\square$

Let

$$r^* = \sup_{|x| < x^*} \Lambda_\mu^*(x) \in (0, \infty].$$

By (6), the product-measure depth estimate of [27, Theorem 3.3] applies: for every $\zeta > 0$, if n is sufficiently large, then

$$\inf_{y \in B_r} q_n(y) \geq \exp(-(1 + \zeta)r - 2\zeta n), \quad r > 0. \quad (14)$$

When μ has endpoint atoms, contact points on the boundary of $[-x^*, x^*]^n$ are interpreted through the normal-cone argument in [12, Theorem 4.9].

2.2 A local depth estimate

Theorem 2.2. *There exists*

$$0 < \eta < \min\{1, x^*/4\}$$

with the following property. For every fixed $0 < a < b < r^$ there are constants $c, C > 0$ such that, for all sufficiently large n and every $x \in [-\eta, \eta]^n$ satisfying*

$$an \leq \Lambda_{\mu_n}^*(x) \leq bn,$$

one has

$$q_n(x) \geq c \frac{e^{-\Lambda_{\mu_n}^*(x)}}{\sqrt{n}}. \quad (15)$$

Moreover, if

$$H_x^+ = \{y : \langle \nabla \Lambda_{\mu_n}^*(x), y - x \rangle \geq 0\},$$

then

$$\mu_n(H_x^+) \leq C \frac{e^{-\Lambda_{\mu_n}^*(x)}}{\sqrt{n}}. \quad (16)$$

Proof. Fix a sufficiently small absolute number $\varepsilon > 0$. By (5), continuity of the tilted second and third centered moments at 0, and (2), we may choose $\eta > 0$ so small that, whenever $|z| \leq 2\eta$ and $t = (\Lambda_\mu^*)'(z)$,

$$\Lambda_\mu^*(z) \leq t^2 \Lambda_\mu''(t) \leq 3\Lambda_\mu^*(z) \quad (17)$$

and

$$|t|^3 \mathbb{E}_t |X - z|^3 \leq \varepsilon t^2 \Lambda_\mu''(t). \quad (18)$$

The choice of η is independent of a and b .

Fix $0 < a < b < r^*$ and a closed half-space H containing x . If $0 \in H$, evenness gives $\mu_n(H) \geq 1/2$. Assume therefore that $0 \notin H$. Choose $z \in H \cap [-x^*, x^*]^n$ minimizing $\Lambda_{\mu_n}^*$ and put

$$r_H = \Lambda_{\mu_n}^*(z) \leq \Lambda_{\mu_n}^*(x).$$

Then $z \in \partial H$ and ∂H supports B_{r_H} at z .

Suppose first that $z \in \partial[-x^*, x^*]^n$. This is possible only when μ has an endpoint atom. The normal-cone description in [12, Lemma 4.8] then shows, after sign changes and permutation of coordinates, that the half-space contains the event on which a certain set of coordinates is fixed at the endpoint. Hence

$$\mu_n(H) \geq e^{-r_H} \geq e^{-\Lambda_{\mu_n}^*(x)},$$

which is stronger than the required estimate.

We may therefore assume that $z \in (-x^*, x^*)^n$. Then

$$H = \{y : \langle t, y - z \rangle \geq 0\}, \quad t = (t_1, \dots, t_n) = \nabla \Lambda_{\mu_n}^*(z),$$

and $z_i = \Lambda_\mu'(t_i)$. Put

$$\Delta = \Lambda_{\mu_n}^*(x) - r_H.$$

Choose $0 < \delta < a/2$ sufficiently small, and then set $\zeta = \delta/(b+3)$. If $\Delta \geq \delta n$, the coarse estimate (14) gives

$$\begin{aligned} \mu_n(H) &\geq q_n(z) \\ &\geq \exp(-(1+\zeta)r_H - 2\zeta n) \\ &\geq \exp(-\Lambda_{\mu_n}^*(x) + \zeta n), \end{aligned}$$

and we are done.

It remains to consider $\Delta < \delta n$. By the strong convexity (3) and the fact that $x \in H$,

$$\|x - z\|_2^2 \leq C\delta n.$$

Let

$$I = \{i : |z_i| \leq 2\eta\}.$$

Since $|x_i| \leq \eta$, every $i \notin I$ satisfies $|x_i - z_i| > \eta$, and hence

$$|I^c| \leq C_\eta \delta n. \quad (19)$$

We claim that

$$\sum_{i \in I} \Lambda_\mu^*(z_i) \geq c_{\eta,a} n. \quad (20)$$

Indeed, $t_i z_i \geq 0$ and, since $x \in H$,

$$\langle t, z \rangle \leq \langle t, x \rangle \leq \eta \sum_{i=1}^n |t_i|.$$

For $i \notin I$ we have $|z_i| > 2\eta$, so

$$\sum_{i \in I^c} |t_i| \leq \sum_{i \in I} |t_i|.$$

Moreover, $\Lambda_\mu \geq 0$ implies

$$\Lambda_\mu^*(z_i) = t_i z_i - \Lambda_\mu(t_i) \leq x^* |t_i|.$$

On the fixed interval $[-2\eta, 2\eta]$, the relation $\Lambda_\mu^*(\Lambda'_\mu(t)) \asymp_\eta t^2$ that follows by (5) and the positivity of Λ_μ'' , gives

$$\sum_{i \in I} |t_i| \leq C_\eta \sqrt{n \sum_{i \in I} \Lambda_\mu^*(z_i)}.$$

Since $r_H \geq (a - \delta)n \geq an/2$, we obtain

$$\frac{a}{2}n \leq \sum_{i \in I} \Lambda_\mu^*(z_i) + C_\eta x^* \sqrt{n \sum_{i \in I} \Lambda_\mu^*(z_i)},$$

which proves (20).

Under the product tilt $P_{t_1, \dots, t_n}$ set

$$S = \sum_{i=1}^n t_i (X_i - z_i),$$

and write

$$\sigma^2 = \sum_{i=1}^n t_i^2 \Lambda_\mu''(t_i), \quad \rho = \sum_{i=1}^n |t_i|^3 \mathbb{E}_{t_i} |X_i - z_i|^3.$$

The local estimates (17), (18), the lower bound (20), the global bounds (4) for $m = 2, 3$, and (19) give

$$c_{\eta,a} n \leq \sigma^2 \leq C_\mu n$$

and

$$\frac{\rho}{\sigma^2} \leq \varepsilon + C_{\eta,a} \delta.$$

After first fixing η and then choosing δ sufficiently small, the right-hand side is as small as desired. The change-of-measure identity is

$$\mu_n(H) = e^{-r_H} \mathbb{E}_{t_1, \dots, t_n} [e^{-S} \mathbf{1}_{\{S \geq 0\}}]. \quad (21)$$

Let G be a centered Gaussian random variable with variance σ^2 . Berry–Esseen gives

$$\sup_{s \in \mathbb{R}} |P_{t_1, \dots, t_n}(S \leq s) - \mathbb{P}(G \leq s)| \leq C \frac{\rho}{\sigma^3}.$$

The function $s \mapsto e^{-s} \mathbf{1}_{[0, \infty)}(s)$ has bounded variation, and the standard Mills estimate gives

$$\mathbb{E}[e^{-G} \mathbf{1}_{\{G \geq 0\}}] \asymp \frac{1}{\sigma}.$$

It follows that

$$\mathbb{E}_{t_1, \dots, t_n} [e^{-S} \mathbf{1}_{\{S \geq 0\}}] \geq \frac{c}{\sqrt{n}}.$$

Since $r_H \leq \Lambda_{\mu_n}^*(x)$, (21) proves (15).

For the tangent half-space at x , take $z = x$. Then every coordinate lies in $[-\eta, \eta]$, so (17) gives

$$an \leq \sigma^2 \leq 3bn,$$

and (18) gives $\rho \leq \varepsilon \sigma^2$. The same Berry–Esseen argument, using the upper Mills estimate, gives

$$\mathbb{E}[e^{-S} \mathbf{1}_{\{S \geq 0\}}] \leq \frac{C}{\sqrt{n}},$$

which proves (16). □

2.3 Three geometric estimates

From now on η is fixed as in Theorem 2.2. We use only the smaller cube

$$Q = [-\eta/2, \eta/2]^n,$$

while the larger cube will always be written explicitly as $[-\eta, \eta]^n$. This fixed gap is the only separation that is needed.

Lemma 2.3. *There are $\alpha_0, C > 0$, depending only on μ , such that for every $n \geq 2$ and every $0 < t \leq \alpha_0 n$,*

$$\mathcal{H}^{n-1}(\partial B_t \cap Q) \geq e^{-Cn} t^{(n-1)/2} |S^{n-1}|.$$

Proof. Choose $0 < m \leq M < \infty$ so that

$$\frac{m}{2} s^2 \leq \Lambda_\mu^*(s) \leq \frac{M}{2} s^2, \quad |s| \leq \eta/2. \quad (22)$$

Let

$$\Omega_n = \left\{ u \in S^{n-1} : \|u\|_\infty \leq \frac{2}{\sqrt{n}} \right\}.$$

There is an absolute constant $c_0 > 0$ such that $|\Omega_n| \geq c_0^n |S^{n-1}|$. Choose $\alpha_0 > 0$ so small that

$$2\sqrt{\frac{2\alpha_0}{m}} \leq \frac{\eta}{2}.$$

Let $u \in \Omega_n$ and $t \leq \alpha_0 n$. Then (22) implies that the ray $\{ru : r \geq 0\}$ meets ∂B_t at a point $r_t(u)u \in Q$, with

$$\sqrt{\frac{2t}{M}} \leq r_t(u) \leq \sqrt{\frac{2t}{m}}.$$

The radial surface element is at least $r_t(u)^{n-1} du$. Hence

$$\mathcal{H}^{n-1}(\partial B_t \cap Q) \geq c_0^n \left(\frac{2t}{M}\right)^{(n-1)/2} |S^{n-1}|,$$

as required. $\square$

Lemma 2.4. Fix $0 < a < b < r^*$. There exists $L_0 > 0$ such that, for every fixed $L \geq L_0$, there are constants $C_L, D_L > 1$ with the following property. For all sufficiently large n , if

$$an \leq t \leq bn, \quad x \in \partial B_t \cap Q,$$

then there is a finite set W_x with $|W_x| \leq D_L^n$ satisfying

$$W_x \subset \left\{ w \in [-\eta, \eta]^n : t - C_L \sqrt{n} \leq \Lambda_{\mu_n}^*(w) \leq t - \frac{L}{4} \right\}, \quad (23)$$

and every open half-space G such that

$$x \in G, \quad G \cap B_{t-2L} \cap [-\eta, \eta]^n \neq \emptyset \quad (24)$$

meets W_x . Moreover,

$$\log D_L = O_{\mu, \eta}(\log(1 + L)).$$

Proof. Put

$$m = \min_{|s| \leq \eta} (\Lambda_{\mu}^*)''(s) > 0, \quad M = \max_{|s| \leq \eta} (\Lambda_{\mu}^*)''(s) < \infty, \quad A = \max_{|s| \leq \eta} |(\Lambda_{\mu}^*)'(s)| < \infty.$$

Fix G satisfying (24), and let z minimize $\Lambda_{\mu_n}^*$ on $\overline{G} \cap [-\eta, \eta]^n$. Then $\Lambda_{\mu_n}^*(z) \leq t - 2L$. Since $x \in Q$, the segment $[z, x]$ lies in $[-\eta, \eta]^n$. Choose $y \in [z, x]$ with

$$\Lambda_{\mu_n}^*(y) = t - L.$$

The function $s \mapsto \Lambda_{\mu_n}^*((1-s)z + sx)$ is non-decreasing, so $\langle \nabla \Lambda_{\mu_n}^*(y), x - y \rangle \geq 0$. Strong convexity gives

$$\|x - y\|_2 \leq \sqrt{\frac{2L}{m}}.$$

Set $c = (x + y)/2$. Then $c \in G$,

$$c \in [-3\eta/4, 3\eta/4]^n, \quad \Lambda_{\mu_n}^*(c) \leq t - \frac{L}{2},$$

and by integrating

$$\Lambda_{\mu_n}^*(c) \geq t - A \sqrt{\frac{L}{2m}} \sqrt{n}.$$

Since $t \geq an$, this is at least $an/2$ for all sufficiently large n . The function $((\Lambda_{\mu}^*)')^2 / \Lambda_{\mu}^*$ has a positive continuous extension at 0, hence

$$c_1 \sqrt{n} \leq \|\nabla \Lambda_{\mu_n}^*(c)\|_2 \leq A \sqrt{n}.$$

Write

$$g = \|\nabla \Lambda_{\mu_n}^*(c)\|_2, \quad e = \frac{\nabla \Lambda_{\mu_n}^*(c)}{g}, \quad r_0 = \frac{\eta}{32}, \quad \rho = \frac{L}{16g}.$$

For fixed L and large n , we have $\rho \leq r_0$. Consider

$$E_G = c + \left\{ u + se : u \perp e, \frac{\|u\|_2^2}{r_0^2} + \frac{s^2}{\rho^2} \leq 1 \right\}.$$

The margin between c and the boundary of $[-\eta, \eta]^n$ shows that $E_G \subset [-\eta, \eta]^n$. Taylor's formula yields

$$\Lambda_{\mu_n}^*(c + u + se) \leq t - \frac{L}{2} + \frac{L}{16} + \frac{M}{2}(r_0^2 + \rho^2) \leq t - \frac{L}{4}$$

provided $L \geq L_0(\mu, \eta)$. Also

$$E_G \subset B(x, R_L), \quad R_L = \sqrt{\frac{L}{2m}} + 2r_0.$$

For $w \in E_G$, the gradient bound on $[-\eta, \eta]^n$ gives

$$\Lambda_{\mu_n}^*(w) \geq t - AR_L \sqrt{n}.$$

Define, independently of G ,

$$D_x = \{w \in [-\eta, \eta]^n : \|w - x\|_2 \leq R_L, \quad t - AR_L \sqrt{n} \leq \Lambda_{\mu_n}^*(w) \leq t - L/4\}.$$

Then $E_G \subset D_x$. Since E_G is centrally symmetric about $c \in G$, at least half of it lies in G . If $\omega_n = \text{vol}_n(B_2^n)$, then

$$\text{vol}_n(E_G) = \omega_n r_0^{n-1} \rho, \quad \text{vol}_n(D_x) \leq \omega_n R_L^n.$$

Since $g \leq A\sqrt{n}$,

$$\frac{\text{vol}_n(D_x \cap G)}{\text{vol}_n(D_x)} \geq \frac{1}{2} \left(\frac{r_0}{R_L} \right)^{n-1} \frac{L}{16AR_L \sqrt{n}} \geq B_L^{-n},$$

where $B_L = O_{\mu, \eta}(\sqrt{1+L})$. Apply the VC ε -net theorem for half-spaces [21] to normalized Lebesgue measure on D_x , with $\varepsilon = B_L^{-n}$. It gives a set $W_x \subset D_x$ meeting every such half-space and satisfying

$$|W_x| \leq C(n+1)B_L^n \log(B_L^n) \leq D_L^n.$$

This proves the lemma. □

Lemma 2.5. Fix $L > 0$. For all sufficiently large n , let $t > 2L$ and let H be a closed half-space satisfying

$$\text{int } H \cap (B_{t-2L} \cap [-\eta, \eta]^n) = \emptyset.$$

Then

$$\mathcal{H}^{n-1}(\partial B_t \cap Q \cap H) \leq \left(\frac{4L}{m} \right)^{(n-1)/2} |S^{n-1}|,$$

where

$$m = \min_{|s| \leq \eta} (\Lambda_\mu^*)''(s) > 0.$$

Proof. Enlarging H if necessary, assume that its boundary supports $B_{t-2L} \cap [-\eta, \eta]^n$ at a point z . Thus

$$H = \{y : \langle v, y - z \rangle \geq 0\}$$

for some $v \neq 0$. Since the origin belongs to the interior of both sets, the normal-cone sum rule gives

$$v = \lambda \nabla \Lambda_{\mu_n}^*(z) + w, \quad \lambda \geq 0, \quad w \in N_{[-\eta, \eta]^n}(z).$$

Let $y \in Q \cap H$. If $w \neq 0$, the fixed gap between the two cubes and the coordinate description of the normal cone of a cube, give

$$\langle w, y - z \rangle \leq -\frac{\eta}{2} \|w\|_1 < 0.$$

Hence $\lambda > 0$ and $\langle \nabla \Lambda_{\mu_n}^*(z), y - z \rangle \geq 0$. The same conclusion is immediate if $w = 0$. Since $\lambda > 0$, we also have $\Lambda_{\mu_n}^*(z) = t - 2L$. Therefore, if $y \in B_t \cap Q \cap H$, strong convexity gives

$$t \geq \Lambda_{\mu_n}^*(y) \geq t - 2L + \frac{m}{2} \|y - z\|_2^2.$$

Thus

$$B_t \cap Q \cap H \subset z + 2\sqrt{\frac{L}{m}} B_2^n.$$

By monotonicity of surface area under inclusion for convex bodies,

$$\mathcal{H}^{n-1}(\partial B_t \cap Q \cap H) \leq \left(\frac{4L}{m}\right)^{(n-1)/2} |S^{n-1}|.$$

□

2.4 The theorem for product measures

Theorem 2.6. *There exists $\alpha_0 = \alpha_0(\mu) > 0$ with the following property. For every fixed $0 < \alpha < \alpha_0$, set*

$$N = \lfloor e^{\alpha n} \rfloor,$$

let $X_1, \dots, X_N$ be independent random vectors with law μ_n , and set

$$P_N = \text{conv}\{X_1, \dots, X_N\}.$$

Then, for all sufficiently large n ,

$$\mathbb{E} f_{n-1}(P_N) \geq n^{n/2} e^{-C_{\mu, \alpha} n} \geq (c_{\mu, \alpha} n)^{n/2}.$$

Proof. Choose $\alpha_0 > 0$ so small that $2\alpha_0 < r^*$ and Lemma 2.3 applies for $t \leq 2\alpha_0 n$. Fix $0 < \alpha < \alpha_0$ and set

$$N = \lfloor e^{\alpha n} \rfloor, \quad t = \log N - \frac{3}{2} \log n. \quad (25)$$

For all sufficiently large n ,

$$\frac{\alpha}{2} n \leq t \leq \alpha n.$$

Let

$$\Sigma_t = \partial B_t \cap Q.$$

By Lemma 2.3,

$$\mathcal{H}^{n-1}(\Sigma_t) \geq e^{-C_1 n} t^{(n-1)/2} |S^{n-1}|. \quad (26)$$

Fix $x \in \Sigma_t$. Theorem 2.2 gives

$$p_x := \mu_n(H_x^+) \leq C \frac{e^{-t}}{\sqrt{n}} = C \frac{n}{N}.$$

If none of $X_1, \dots, X_N$ lies in H_x^+ , then $x \notin P_N$. Hence

$$\mathbb{P}(x \notin P_N) \geq (1 - p_x)^N \geq e^{-C_2 n}. \quad (27)$$

Fix $L \geq L_0$ as in Lemma 2.4. If $w \in W_x$, then (23) and the choice of t place w , for all large n , in the range of Theorem 2.2. Thus

$$q_n(w) \geq c \frac{e^{-\Lambda_{\mu_n}^*(w)}}{\sqrt{n}} \geq ce^{L/4} \frac{e^{-t}}{\sqrt{n}} = ce^{L/4} \frac{n}{N}.$$

Also $\Lambda_{\mu_n}^*(w) > 0$, which implies that $w \neq 0$; by symmetry $q_n(w) \leq 1/2$. Put

$$q = q_n(w), \quad r = \frac{Nq}{n}.$$

For L sufficiently large, $r \geq B := ce^{L/4} > 2$. Proposition 13 of [22] yields

$$\mathbb{P}(w \notin P_N) \leq [r \exp(A_q(1 + q - r))]^n, \quad A_q = \frac{1}{q} \log \frac{1}{1 - q}.$$

Since $A_q \geq 1$, $q \leq 1/2$ and the function $s \mapsto se^{2-s}$ is decreasing for $s \geq 1$,

$$\mathbb{P}(w \notin P_N) \leq (Be^{2-B})^n.$$

Taking the union over W_x and using $\log D_L = O(\log(1 + L))$, we may choose L so large that

$$\mathbb{P}(W_x \not\subset P_N) \leq e^{-(C_2+3)n}. \quad (28)$$

We also need full dimensionality. Set

$$p_\mu = \sup_{a \in \mathbb{R}} \mu(\{a\}) < 1.$$

Every proper affine hyperplane $H \subset \mathbb{R}^n$ satisfies $\mu_n(H) \leq p_\mu$: condition on all but one coordinate corresponding to a non-zero coefficient in the equation defining H . Therefore, until the affine span of the sampled points has dimension n , the next point increases its dimension with conditional probability at least $1 - p_\mu$. A binomial lower-tail estimate gives

$$\mathbb{P}(P_N \text{ is not full-dimensional}) \leq e^{-c_\mu N}. \quad (29)$$

For $x \in \Sigma_t$, let E_x be the event that $x \notin P_N$, that P_N is full-dimensional, and that every open half-space containing x and meeting $B_{t-2L} \cap [-\eta, \eta]^n$ also meets P_N . By (27), (28), (29), and Lemma 2.4,

$$\mathbb{P}(E_x) \geq \frac{1}{2} e^{-C_2 n} \quad (30)$$

for all sufficiently large n . The event is measurable; failure of the last condition is equivalent to

$$\max_{u \in S^{n-1}} (\min \{ \langle u, x \rangle, h_{B_{t-2L} \cap [-\eta, \eta]^n}(u) \} - h_{P_N}(u)) > 0.$$

Tonelli's theorem and (26) therefore give

$$\mathbb{E} \mathcal{H}^{n-1}\{x \in \Sigma_t : E_x\} \geq e^{-C_3 n t^{(n-1)/2}} |S^{n-1}|. \quad (31)$$

Let F be a facet of a full-dimensional realization of P_N , and let H_F be its closed exterior half-space. If E_x occurs and x strictly violates the facet inequality of F , then $x \in \text{int } H_F$. Since $\text{int } H_F \cap P_N = \emptyset$, the defining property of E_x implies

$$\text{int } H_F \cap (B_{t-2L} \cap [-\eta, \eta]^n) = \emptyset.$$

Lemma 2.5 gives

$$\mathcal{H}^{n-1}(\Sigma_t \cap H_F) \leq C_L^n |S^{n-1}|.$$

Every point counted on the left-hand side of (31) lies outside P_N and therefore strictly violates at least one facet inequality. Hence

$$\mathcal{H}^{n-1}\{x \in \Sigma_t : E_x\} \leq f_{n-1}(P_N) C_L^n |S^{n-1}|.$$

Taking expectations and using (31),

$$\mathbb{E} f_{n-1}(P_N) \geq e^{-C_4 n t^{(n-1)/2}}.$$

Since $t \geq \alpha n/2$, this is

$$\mathbb{E} f_{n-1}(P_N) \geq n^{n/2} e^{-C_{\mu, \alpha} n},$$

after changing the constant in the exponential. $\square$

Corollary 2.7 (The 0/1 case). *There is an absolute constant $c > 0$ such that, for every sufficiently large n , there exists a 0/1 polytope $P \subset \mathbb{R}^n$ with*

$$f_{n-1}(P) \geq (cn)^{n/2}.$$

Proof. Take $\mu = \frac{1}{2}(\delta_{-1} + \delta_1)$. The endpoint condition holds with $\kappa = 0$. Theorem 2.6 gives a realization of $P_N = \text{conv}\{X_1, \dots, X_N\}$ with at least $(cn)^{n/2}$ facets. The affine map

$$x \mapsto \frac{x+1}{2}$$

sends $\{-1, 1\}^n$ onto $\{0, 1\}^n$ and preserves the number of facets. $\square$

Corollary 2.8 (Mass at the endpoint). *Theorem 2.6 applies to every non-degenerate symmetric probability measure with mass at the endpoint. In particular, it applies to every symmetric bounded discrete log-concave probability measure.*

Proof. For all sufficiently small $s > 0$, we have that

$$\mu(\{x^*\}) \leq \mu([x^* - s, x^*]) \leq 1.$$

Thus (1) holds with $\kappa = 0$. $\square$

Remark 2.9 (Infinite atomic examples). The product theorem also contains purely atomic measures with no atom at the endpoint. For example, if $0 < q, r < 1$, then

$$\mu_{q,r} = \frac{1-r}{2} \sum_{k=1}^{\infty} r^{k-1} (\delta_{1-q^k} + \delta_{-(1-q^k)})$$

satisfies (1) with

$$\kappa = \frac{\log r}{\log q} > 0.$$

More generally, if $x_k \uparrow x^*$, the gaps $x^* - x_k$ have bounded multiplicative ratios, and

$$\sum_{j \geq k} p_j \asymp (x^* - x_k)^\kappa,$$

then the symmetric atomic measure assigning mass $p_k/2$ to each of $\pm x_k$ satisfies the endpoint assumption.

3 Why some structural assumption is needed

The preceding theorem uses product structure, while the next one uses log-concavity. Without either type of structure, a many-facets conclusion of the same order is false, even for symmetric isotropic probability measures with full support. The following elementary example will also clarify the role of log-concavity in the next section.

Proposition 3.1. *Fix $\alpha > 0$ and let $N = \lfloor e^{\alpha n} \rfloor$. There exists a symmetric isotropic probability measure ν_n on $\mathbb{R}^n$ with $\text{supp } \nu_n = \mathbb{R}^n$, which is not log-concave, such that for independent $X_1, \dots, X_N$ with law ν_n and*

$$P_N = \text{conv}\{X_1, \dots, X_N\},$$

one has

$$\mathbb{E}f_{n-1}(P_N) = (1 + o(1))2^n.$$

In particular,

$$\mathbb{E}f_{n-1}(P_N) = e^{O(n)} \ll n^{n/2} e^{-Cn}$$

for every fixed $C > 0$.

Proof. Let

$$\nu_n^0 = \frac{1}{2n} \sum_{j=1}^n (\delta_{\sqrt{n}e_j} + \delta_{-\sqrt{n}e_j}).$$

Then ν_n^0 is symmetric and isotropic. A sample from ν_n^0 chooses one of the $2n$ points $\pm\sqrt{n}e_j$ uniformly. Hence

$$\mathbb{P}(\text{some point } \pm\sqrt{n}e_j \text{ is missing}) \leq 2n \left(1 - \frac{1}{2n}\right)^N \leq 2ne^{-N/(2n)}.$$

If all $2n$ points occur among the samples, then

$$P_N = \text{conv}\{\pm\sqrt{n}e_1, \dots, \pm\sqrt{n}e_n\} = \sqrt{n}B_1^n,$$

and therefore $f_{n-1}(P_N) = 2^n$.

To obtain full support, let γ_n be the standard Gaussian measure on $\mathbb{R}^n$, put $\varepsilon_n = e^{-n^3}$, and define

$$\nu_n = (1 - \varepsilon_n)\nu_n^0 + \varepsilon_n\gamma_n.$$

Both measures in the mixture are symmetric and isotropic, so the same is true of ν_n , while the Gaussian component gives $\text{supp } \nu_n = \mathbb{R}^n$. The measure ν_n is not log-concave, since it has atoms and at the same time full-dimensional support.

The probability that at least one of the N samples comes from the Gaussian component is at most $N\varepsilon_n$. On the event that no Gaussian sample occurs and every point $\pm\sqrt{n}e_j$ occurs, the polytope is $\sqrt{n}B_1^n$. On the complementary event we use the elementary bound

$$f_{n-1}(P_N) \leq \binom{N}{n}.$$

Consequently,

$$|\mathbb{E}f_{n-1}(P_N) - 2^n| \leq 2 \left(2ne^{-N/(2n)} + Ne^{-n^3} \right) \binom{N}{n} = o(2^n),$$

because

$$\binom{N}{n} \leq \left(\frac{eN}{n} \right)^n = e^{O_\alpha(n^2)}.$$

□

4 General log-concave measures

We now turn to arbitrary log-concave probability measures. No product or symmetry assumption will be used.

4.1 A random simplex estimate

Lemma 4.1. *Let ν be a full-dimensional log-concave probability measure on $\mathbb{R}^d$ with density g , and let $Z_0, \dots, Z_d$ be independent random vectors with distribution ν . Then*

$$\frac{c^d}{\sqrt{d!} \|g\|_\infty} \leq \mathbb{E} \text{vol}_d(\text{conv}\{Z_0, \dots, Z_d\}) \leq \frac{C^d}{\sqrt{d!} \|g\|_\infty}, \quad (32)$$

where $c, C > 0$ are absolute constants.

Proof. We will first assume that ν is isotropic. Set

$$D = \det(Z_1 - Z_0, \dots, Z_d - Z_0).$$

Adjoining a first row of ones gives

$$D = \det \begin{pmatrix} 1 & \cdots & 1 \\ Z_0 & \cdots & Z_d \end{pmatrix}.$$

The columns $(1, Z_j)^T$ are independent and have second-moment matrix I_{d+1} . Expanding the square of the determinant, the independence of the columns and the fact that the second-moment matrix is the identity imply

$$\mathbb{E}D^2 = (d+1)!.$$

Since

$$\text{vol}_d(\text{conv}\{Z_0, \dots, Z_d\}) = \frac{|D|}{d!},$$

on the one hand Cauchy–Schwarz yields

$$\mathbb{E} \text{vol}_d(\text{conv}\{Z_0, \dots, Z_d\}) \leq \frac{\sqrt{\mathbb{E}D^2}}{d!} = \frac{\sqrt{d+1}}{\sqrt{d!}} \leq \frac{C^d}{\sqrt{d!}}$$

and on the other hand the fact that D is a polynomial of degree d of the log-concave random vector $(Z_0, \dots, Z_d)$ in $\mathbb{R}^{d(d+1)}$ gives

$$\mathbb{E} \operatorname{vol}_d(\operatorname{conv}\{Z_0, \dots, Z_d\}) \geq c^d \frac{\sqrt{\mathbb{E} D^2}}{d!} \geq \frac{c^d}{\sqrt{d!}},$$

by reverse Hölder inequality (see [5, Theorem 2]).

Now if m is the barycenter of ν and Σ its covariance matrix and $Y_i = \Sigma^{-1/2}(Z_i - m)$ for every $i = 0, \dots, d$, the random vectors Y_i are independent, log-concave and isotropic. Moreover,

$$\operatorname{vol}_d(\operatorname{conv}\{Y_0, \dots, Y_d\}) = \frac{1}{\sqrt{\det \Sigma}} \operatorname{vol}_d(\operatorname{conv}\{Z_0, \dots, Z_d\}).$$

To conclude, the universal lower bound for the isotropic constant, as well as the proof of the slicing conjecture, yield

$$c_1 \leq \|g\|_\infty^{1/d} (\det \Sigma)^{1/(2d)} \leq c_2,$$

for some absolute constants $c_1, c_2 > 0$. □

4.2 Cramér levels in the space of tilt parameters

Let μ be a centered full-dimensional log-concave probability measure on $\mathbb{R}^n$ with density f . We write

$$\Lambda_\mu(\theta) = \log \int_{\mathbb{R}^n} e^{\langle \theta, x \rangle} f(x) dx$$

and denote its Legendre transform by Λ_μ^* . The result is affine invariant, so we shall work in isotropic position.

We first argue under the additional assumption that f is smooth, strictly log-concave and that Λ_μ is finite and smooth on all of $\mathbb{R}^n$. The general case follows at the end by approximation.

For $\theta \in \mathbb{R}$ we consider the exponential tilt

$$d\mu_\theta(z) = e^{\langle \theta, z \rangle - \Lambda_\mu(\theta)} d\mu(z),$$

which is also log-concave with

$$\operatorname{bar}(\mu_\theta) = \nabla \Lambda_\mu(\theta), \quad \operatorname{Cov}(\mu_\theta) = \nabla^2 \Lambda_\mu(\theta).$$

For $t > 0$ set

$$\Theta_t = \{\theta \in \mathbb{R}^n : \Lambda_\mu^*(\nabla \Lambda_\mu(\theta)) \leq t\}$$

Observe that Θ_t is a star-shaped body and define

$$M(t) = \operatorname{vol}_n(\Theta_t).$$

For $u \in S^{n-1}$, let $s_t(u) > 0$ be determined by

$$x_t(u) = \nabla \Lambda_\mu(s_t(u)u), \quad \Lambda_\mu^*(x_t(u)) = t. \tag{33}$$

Thus s_t is the radial function of Θ_t , and if σ denotes the normalized spherical probability measure,

$$M(t) = \omega_n \int_{S^{n-1}} s_t(u)^n d\sigma(u), \quad \omega_n = \operatorname{vol}_n(B_2^n). \tag{34}$$

Define

$$\lambda_t(u)^2 = s_t(u)^2 \langle \nabla^2 \Lambda_\mu(s_t(u)u)u, u \rangle. \tag{35}$$

Lemma 4.2. *There is an absolute constant $C > 0$ such that*

$$M(n) \geq e^{-Cn}, \quad M(3n/2) \leq e^{Cn}.$$

Proof. For the lower bound, put

$$\mathcal{L}_n = \{\theta : \Lambda_\mu(\theta) \leq n, \Lambda_\mu(-\theta) \leq n\}.$$

The standard comparison between logarithmic Laplace sublevel sets and L_n -centroid bodies [26, Lemma 2.3], together with the volume estimates for centroid bodies and the reverse Santaló inequality [8], gives

$$\text{vol}_n(\mathcal{L}_n) \geq e^{-Cn}. \quad (36)$$

We claim that $\frac{1}{2}\mathcal{L}_n \subset \Theta_n$. Indeed, if $2\theta \in \mathcal{L}_n$ and $g(s) = \Lambda_\mu(s\theta)$, then $g \geq 0$ because μ is centered, and convexity gives

$$g'(1) \leq g(2) - g(1).$$

Therefore

$$\Lambda_\mu^*(\nabla \Lambda_\mu(\theta)) = g'(1) - g(1) \leq g(2) - 2g(1) \leq n.$$

This and (36) imply $M(n) \geq e^{-Cn}$.

For the upper bound, set

$$B_t = \{x : \Lambda_\mu^*(x) \leq t\}.$$

The comparison of Cramér sublevel sets with one-sided centroid bodies and the corresponding volume estimates imply

$$\text{vol}_n(B_{3n/2}) \leq e^{Cn}; \quad (37)$$

see, for example, [20, Proposition 2.7].

For $x \in B_{3n/2}$ let $\theta_x = \nabla \Lambda_\mu^*(x)$ and let g_x be the density of the exponential tilt μ_{θ_x} . By duality, its barycenter is x and its covariance is

$$\text{Cov}(g_x) = \nabla^2 \Lambda_\mu(\theta_x) = (\nabla^2 \Lambda_\mu^*(x))^{-1}.$$

By Fradelizi's inequality [15, Theorem 4],

$$\|g_x\|_\infty \leq e^n g_x(x) = e^{n+\Lambda_\mu^*(x)} f(x) \leq e^{5n/2} \|f\|_\infty.$$

The functional form of the slicing theorem, obtained from Ball's reduction [1] and the resolution of the slicing problem by Klartag and Lehec [25], gives

$$\|f\|_\infty^{1/n} \leq C$$

for isotropic μ . Hence $\|g_x\|_\infty \leq e^{Cn}$.

If $H_x = \nabla^2 \Lambda_\mu^*(x)$, the lower bound of the isotropic constant used in the proof of Lemma 4.1 gives

$$\|g_x\|_\infty \sqrt{\det \text{Cov}(g_x)} \geq c^n,$$

that is,

$$\sqrt{\det H_x} \leq e^{Cn}, \quad x \in B_{3n/2}.$$

The change of variables $x = \nabla \Lambda_\mu(\theta)$ gives

$$M(3n/2) = \int_{B_{3n/2}} \det \nabla^2 \Lambda_\mu^*(x) dx \leq e^{Cn} \text{vol}_n(B_{3n/2}),$$

and (37) completes the proof. $\square$

Lemma 4.3. *There is an absolute constant $C > 0$ such that*

$$\lambda_t(u) \leq C(1+t)$$

for all $t > 0$ and $u \in S^{n-1}$.

Proof. Write $s = s_t(u)$, $x = x_t(u)$, $h = \langle x, u \rangle$, and consider the exponential tilt with parameter su . If X is a random vector distributed according to μ_{su} , under this tilt, the random variable

$$Y = s(\langle X, u \rangle - h)$$

is centered, log-concave, and has variance $\lambda_t(u)^2$. Moreover,

$$\mathbb{E}_{su} e^{-Y} = e^t.$$

Write $Y = \lambda_t(u)Z$, where Z is centered, log-concave and has variance one. Standard one-dimensional log-concave estimates give absolute constants $c_0, c_1 > 0$ such that the density of Z is at least c_1 on $[-c_0, -c_0/2]$. Hence

$$e^t \geq c_1 \int_{-c_0}^{-c_0/2} e^{-\lambda_t(u)z} dz \geq c_1 e^{c_0 \lambda_t(u)}.$$

The assertion follows. □

Differentiating

$$\Lambda_\mu^*(\nabla \Lambda_\mu(su)) = s \langle \nabla \Lambda_\mu(su), u \rangle - \Lambda_\mu(su)$$

with respect to s , we get

$$\frac{dt}{ds} = s \langle \nabla^2 \Lambda_\mu(su) u, u \rangle = \frac{\lambda_t(u)^2}{s}.$$

Thus (34) implies

$$M'(t) = n\omega_n \int_{S^{n-1}} \frac{s_t(u)^n}{\lambda_t(u)^2} d\sigma(u). \quad (38)$$

Proposition 4.4. *There are an absolute $C > 0$, a level*

$$t \in [n, 3n/2],$$

a number

$$c\sqrt{n} \leq a \leq Cn,$$

and a measurable set $S \subset S^{n-1}$ such that

$$a \leq \lambda_t(u) < 2a, \quad u \in S,$$

and

$$\int_S s_t(u)^n d\sigma(u) \geq \frac{e^{-Cn}}{\omega_n}. \quad (39)$$

Proof. Lemma 4.2 gives

$$\int_n^{3n/2} \frac{M'(r)}{M(r)} dr = \log \frac{M(3n/2)}{M(n)} \leq Cn.$$

Hence there is $t \in [n, 3n/2]$ with

$$M'(t) \leq C_0 M(t)$$

for an absolute C_0 . Let

$$R = \left\{ u : \lambda_t(u)^2 \geq \frac{n}{2C_0} \right\}.$$

By (38),

$$\int_{S^{n-1} \setminus R} s_t(u)^n d\sigma(u) \leq \frac{M(t)}{2\omega_n}.$$

Together with (34),

$$\int_R s_t(u)^n d\sigma(u) \geq \frac{M(t)}{2\omega_n} \geq \frac{e^{-Cn}}{\omega_n},$$

since $M(t)$ is increasing. On R , Lemma 4.3 gives

$$c\sqrt{n} \leq \lambda_t(u) \leq Cn.$$

A dyadic decomposition of this interval has $O(\log n)$ pieces. One of them, $[a, 2a)$, satisfies (39), after changing the absolute constant in the exponential. $\square$

4.3 Lower bound for the expected facets

For $u \in S$ from Proposition 4.4, write

$$s = s_t(u), \quad x = x_t(u), \quad h = \langle x, u \rangle, \quad \lambda = \lambda_t(u).$$

If X is distributed according to μ_{su} , let g be the density of

$$Y = s(\langle X, u \rangle - h).$$

Then Y is centered and log-concave with variance λ^2 . Since $\lambda \geq c\sqrt{n}$, the standard one-dimensional estimates give absolute constants $c_0, c_1, C_1 > 0$ such that

$$g(y) \geq \frac{c_1}{\lambda} \quad (0 \leq y \leq c_0), \quad \|g\|_\infty \leq \frac{C_1}{\lambda}. \quad (40)$$

For $u \in S^{n-1}$ and $r \in \mathbb{R}$ we set

$$H(u, r) = \{z : \langle z, u \rangle = r\}, \quad H^+(u, r) = \{z : \langle z, u \rangle \geq r\}.$$

With this notation, for $0 \leq y \leq c_0$, let also

$$H_y = H\left(u, h + \frac{y}{s}\right), \quad H_y^+ = H^+\left(u, h + \frac{y}{s}\right).$$

If

$$p(u, r) = \int_{H(u, r)} f(z) d\mathcal{H}^{n-1}(z),$$

then the definition of the tilt gives

$$g(y) = \frac{1}{s} e^{t+y} p\left(u, h + \frac{y}{s}\right). \quad (41)$$

Consequently,

$$p\left(u, h + \frac{y}{s}\right) \geq c \frac{se^{-t}}{\lambda}, \quad 0 \leq y \leq c_0. \quad (42)$$

Also,

$$\begin{aligned} \mu(H_y^+) &= e^{-t} \mathbb{E}_{su} [e^{-Y} \mathbf{1}_{\{Y \geq y\}}] \\ &\leq C \frac{e^{-t}}{\lambda} \leq C \frac{e^{-t}}{a}. \end{aligned} \quad (43)$$

Theorem 4.5. *There exists an absolute constant $C > 0$ such that the following holds. For every sufficiently large n and every full-dimensional log-concave probability measure μ on $\mathbb{R}^n$, there exist $T \in [n, 2n]$ and*

$$N = \left\lceil e^T n^{3/2} \right\rceil$$

such that, if $X_1, \dots, X_N$ are independent with distribution μ and

$$P_N = \text{conv}\{X_1, \dots, X_N\},$$

then

$$\mathbb{E}f_{n-1}(P_N) \geq n^{n/2} e^{-Cn} \geq (cn)^{n/2}.$$

Proof. By affine invariance we work in isotropic position and use the smooth setting described above. Choose t, a, S as in Proposition 4.4. Fix a sufficiently large absolute constant $A > 0$ and set

$$N = \lceil Ae^t na \rceil. \quad (44)$$

For $u \in S$ and $0 \leq y \leq c_0$, (43) gives

$$N\mu(H_y^+) \leq Cn.$$

Since $t \geq n$, $\mu(H_y^+) = o(1)$, and therefore

$$(1 - \mu(H_y^+))^{N-n} \geq e^{-Cn}. \quad (45)$$

For $u \in S^{n-1}$ and $r > 0$ define

$$J(u, r) = \int_{H(u, r)^n} \mathcal{H}^{n-1}(\text{conv}\{z_1, \dots, z_n\}) \prod_{i=1}^n f(z_i) d\mathcal{H}^{n-1}(z_i).$$

Observe now that

$$J(u, r) = p(u, r)^n \mathbb{E} \mathcal{H}^{n-1}(\text{conv}\{Z_1, \dots, Z_n\}),$$

where $Z_1, \dots, Z_n$ are i.i.d $(n-1)$ -dimensional log-concave random vectors, with density

$$\frac{f(z)}{p(u, r)} \mathbf{1}_{H(u, r)}$$

Combining Lemma 4.1 with (42),

$$J\left(u, h + \frac{y}{s}\right) \geq \frac{e^{-Cn} e^{-tn} s^{n+1}}{\sqrt{(n-1)!} \lambda^{n+1}}, \quad 0 \leq y \leq c_0. \quad (46)$$

The affine Blaschke–Petkantschin formula [29] gives the following lower bound for the expected number of facets:

$$\mathbb{E}f_{n-1}(P_N) \geq \binom{N}{n} I,$$

where

$$I = n! \omega_n \int_{S^{n-1}} \int_0^\infty (1 - \mu(H^+(u, r)))^{N-n} J(u, r) dr d\sigma(u).$$

We restrict the integral to $u \in S$ and $r = h + y/s$, $0 \leq y \leq c_0$. Since $dr = dy/s$, equations (45) and (46) give

$$\begin{aligned} I &\geq \frac{e^{-Cn} n! \omega_n e^{-tn}}{\sqrt{(n-1)!}} \int_S \frac{s_t(u)^n}{\lambda_t(u)^{n+1}} d\sigma(u) \\ &\geq \frac{e^{-Cn} n! \omega_n e^{-tn}}{\sqrt{(n-1)!} a^{n+1}} \int_S s_t(u)^n d\sigma(u). \end{aligned}$$

By (39),

$$I \geq \frac{e^{-Cn} n! e^{-tn}}{\sqrt{(n-1)!} a^{n+1}}. \quad (47)$$

Since $N \gg n$, we have $\binom{N}{n} n! \geq (N/2)^n$. Using (44) and (47),

$$\mathbb{E}f_{n-1}(P_N) \geq e^{-Cn} \frac{n^n}{a \sqrt{(n-1)!}}.$$

Because $a \leq Cn$, Stirling's formula gives

$$\mathbb{E}f_{n-1}(P_N) \geq n^{n/2} e^{-Cn}.$$

Finally, from

$$n \leq t \leq \frac{3n}{2}, \quad c\sqrt{n} \leq a \leq Cn,$$

and the choice of the absolute constant A , we may write

$$N = \left\lceil e^T n^{3/2} \right\rceil$$

with $T \in [n, 2n]$ for all sufficiently large n .

For a general full-dimensional log-concave measure, truncate to large balls, regularize by a small Gaussian convolution, recenter, and apply an invertible linear map. The resulting measures converge in total variation to μ , and for fixed N the facet-count functional is continuous outside a set of μ^N -measure zero. Since for fixed n only finitely many integer values of N occur in the range $[e^n n^{3/2}, e^{2n} n^{3/2} + 1]$, a subsequence has constant N ; passing to the limit proves the theorem without the temporary regularity assumptions. $\square$

4.4 Upper bound for the expected facets

Theorem 4.6. *There exists an absolute constant $C > 0$ with the following property. For $n \geq 2$, let μ be a full-dimensional log-concave probability measure on $\mathbb{R}^n$ with density f , and $X_1, \dots, X_N$ be independent random vectors with law μ , where $N \geq 2n$. If*

$$P_N = \text{conv}\{X_1, \dots, X_N\},$$

then

$$\mathbb{E}f_{n-1}(P_N) \leq C^n N n^{(n-3)/2}. \quad (48)$$

In particular, for every fixed $b > 0$, if

$$N \leq e^{bn} n^{3/2},$$

then

$$\mathbb{E}f_{n-1}(P_N) \leq C_b^n n^{n/2}. \quad (49)$$

Proof. By affine invariance we may assume that μ is isotropic. As in the lower-bound argument, we first work under the same regularity assumptions on Λ_μ ; the general case is recovered by the same approximation at the end.

For $u \in S^{n-1}$ we write

$$\pi(u, r) := \mu(H^+(u, r)).$$

Blaschke–Petkantschin yields

$$\mathbb{E}f_{n-1}(P_N) = \binom{N}{n} I, \quad (50)$$

where

$$I := n! \omega_n \int_{S^{n-1}} \int_0^\infty ((1 - \pi(u, r))^{N-n} + \pi(u, r)^{N-n}) J(u, r) dr d\sigma(u).$$

Since μ is centered, Grünbaum's lemma gives

$$\pi(u, r) \leq \mu(\{z : \langle z, u \rangle \geq 0\}) \leq 1 - e^{-1}$$

for every $u \in S^{n-1}$ and every $r > 0$. Hence,

$$\begin{aligned} I &\leq e^{-c_1(N-n)} + n! \omega_n \int_{S^{n-1}} \int_0^\infty (1 - \pi(u, r))^{N-n} J(u, r) dr d\sigma(u) \\ &:= e^{-c_1(N-n)} + I_1. \end{aligned} \tag{51}$$

For every $r > 0$ and every $u \in S^{n-1}$, there is a unique $s = s(u, r) > 0$ such that

$$r = \langle \nabla \Lambda_\mu(su), u \rangle. \tag{52}$$

For this particular pair (u, r) set

$$x = x(u, r) := \nabla \Lambda_\mu(su), \quad t = t(u, r) := \Lambda_\mu^*(x) = sr - \Lambda_\mu(su), \tag{53}$$

and write

$$\theta_x := \nabla \Lambda_\mu^*(x) = su, \quad H_x := \nabla^2 \Lambda_\mu^*(x), \quad \lambda = \lambda(u, r) := \sqrt{\langle \theta_x, H_x^{-1} \theta_x \rangle}. \tag{54}$$

Observe that

$$\langle x, u \rangle = r, \quad t > 0.$$

Therefore, the point $x = x(u, r)$ belongs to the specific hyperplane $H(u, r)$. Moreover,

$$\lambda^2 = s^2 \langle \nabla^2 \Lambda_\mu(su) u, u \rangle. \tag{55}$$

Let μ_{θ_x} be the exponential tilt corresponding to θ_x , and put

$$Z_x = \langle \theta_x, Y - x \rangle, \quad Y \sim \mu_{\theta_x}.$$

Then Z_x is centered, log-concave, and has variance λ^2 . If g_x denotes its density, (41) implies

$$g_x(0) = \frac{e^t}{s} p(u, r). \tag{56}$$

Hence, (40) yields

$$p(u, r) \leq C \frac{se^{-t}}{\lambda}. \tag{57}$$

Moreover, a standard argument for centered log-concave random variables gives

$$\pi(u, r) = e^{-t} \mathbb{E}_{\mu_{\theta_x}} [e^{-Z_x} \mathbf{1}_{\{Z_x \geq 0\}}] \geq c \frac{e^{-t}}{1 + \lambda}, \tag{58}$$

where $c > 0$ is an absolute constant. We shall also use the following two elementary consequences of the same one-dimensional normalization:

Lemma 4.7. *We have that*

$$\lambda \leq C(1 + t).$$

In addition, there exist absolute constants $t_0, c_0 > 0$ such that

$$t \geq t_0 \implies \lambda \geq c_0.$$

Proof. The first part is proven exactly as Lemma 4.3. For the second part, the standard one-dimensional ψ_1 estimate gives

$$\mathbb{E}_{\mu_{\theta_x}} e^{a|Z_x|/\lambda} \leq C,$$

for some absolute constants $a, C > 0$. Thus, if $\lambda \leq a$

$$e^t = \mathbb{E}_{\theta_x} e^{-Z_x} \leq \mathbb{E}_{\theta_x} e^{a|Z_x|/\lambda} \leq C.$$

The proof follows. $\square$

If this Cramér level $t = t(u, r)$ satisfies $t < t_0$, then Lemma 4.7 and (58) give

$$\pi(u, r) \geq p_0 \quad (59)$$

for an absolute $p_0 > 0$. Hence,

$$\begin{aligned} I_1 &\leq e^{-c_2(N-n)} + n! \omega_n \iint_{\{(u,r): t(u,r) \geq t_0\}} (1 - \pi(u, r))^{N-n} J(u, r) dr d\sigma(u) \\ &:= e^{-c_2(N-n)} + I_2. \end{aligned} \quad (60)$$

If $u \in S^{n-1}$ and $r > 0$, observe that

$$J(u, r) = p(u, r)^n \mathbb{E} \mathcal{H}^{n-1}(\text{conv}\{W_1, \dots, W_n\}),$$

where $W_1, \dots, W_n$ are i.i.d $(n-1)$ -dimensional log-concave random vectors, with density

$$\frac{f(z)}{p(u, r)} \mathbf{1}_{H(u, r)}$$

Therefore Lemma 4.1 yields,

$$J(u, r) \leq \frac{C^n}{\sqrt{(n-1)!}} \frac{p(u, r)^{n+1}}{\|f \mathbf{1}_{H(u, r)}\|_\infty} \leq \frac{C^n}{\sqrt{(n-1)!}} \frac{p(u, r)^{n+1}}{f(x)},$$

since by (53), the point $x = x(u, r)$ lies on $H(u, r)$. Using (57), we obtain

$$J(u, r) \leq \frac{C^n}{\sqrt{(n-1)!}} \frac{s^{n+1} e^{-(n+1)t}}{f(x) \lambda^{n+1}}. \quad (61)$$

Lemma 4.8. *The parametrization $(u, r) \mapsto x(u, r)$ has Jacobian*

$$dr d\sigma(u) = \frac{1}{n\omega_n} \frac{\det H_x \lambda^2}{s^{n+1}} dx$$

Proof. The change of variables in the lemma is the composition

$$(u, r) \longleftrightarrow (u, s) \longleftrightarrow \theta = su \longleftrightarrow x = \nabla \Lambda(\theta).$$

We justify each step separately. As for fixed $u \in S^{n-1}$, we have defined

$$r = \langle \nabla \Lambda_\mu(su), u \rangle,$$

and $(u, s) \mapsto (u, r_u(s))$ leaves the first coordinate unchanged

$$dr d\sigma(u) = \langle \nabla^2 \Lambda_\mu(su)u, u \rangle ds d\sigma(u) = \frac{\lambda^2}{s^2} ds d\sigma(u),$$

Next, writing $\theta = su$ and recalling that σ is the probability measure on S^{n-1} , the polar-coordinate formula gives

$$d\theta = n\omega_n s^{n-1} ds d\sigma(u).$$

Finally, the map

$$x \longmapsto \theta_x = \nabla \Lambda_\mu^*(x)$$

is a diffeomorphism. Hence,

$$d\theta = \det H_x dx.$$

Combining the preceding identities, we obtain

$$\begin{aligned} dr d\sigma(u) &= \frac{\lambda^2}{s^2} ds d\sigma(u) \\ &= \frac{\lambda^2}{s^2} \frac{1}{n\omega_n s^{n-1}} d\theta \\ &= \frac{1}{n\omega_n} \frac{\det H_x \lambda^2}{s^{n+1}} dx, \end{aligned}$$

which proves the lemma. $\square$

We now need one pointwise estimate which is specific to the upper bound. Let g_x denote the density of the tilt μ_{θ_x} . Since x is the barycenter of μ_{θ_x} ,

$$g_x(x) = e^t f(x), \quad \text{Cov}(\mu_{\theta_x}) = H_x^{-1}.$$

Fradelizi's inequality gives

$$\|g_x\|_\infty \leq e^n g_x(x) = e^{n+t} f(x).$$

On the other hand, the universal lower bound for the isotropic constant of a log-concave density gives

$$c^n \leq \|g_x\|_\infty \sqrt{\det \text{Cov}(\mu_{\theta_x})} = \frac{\|g_x\|_\infty}{\sqrt{\det H_x}}.$$

Consequently,

$$\frac{\det H_x}{f(x)} \leq C^n e^{2t} f(x). \quad (62)$$

From (58) and $N \geq 2n$,

$$\begin{aligned} (1 - \pi(u, r))^{N-n} &\leq \exp(-(N-n)\pi(u, r)) \\ &\leq \exp\left(-c \frac{Ne^{-t}}{1+\lambda}\right). \end{aligned} \quad (63)$$

Inserting (61), the Jacobian Lemma 4.8, and finally (62) into (60), we obtain

$$\begin{aligned} I_2 &\leq \frac{C^n n!}{n\sqrt{(n-1)!}} \int_{\{x:t(x) \geq t_0\}} e^{-(n+1)t} \frac{\det H_x}{f(x)\lambda^{n-1}} \exp\left(-c \frac{Ne^{-t}}{1+\lambda}\right) dx \\ &\leq C^n \sqrt{(n-1)!} \int_{\{x:t(x) \geq t_0\}} f(x) \frac{e^{-(n-1)t}}{\lambda^{n-1}} \exp\left(-c \frac{Ne^{-t}}{1+\lambda}\right) dx. \end{aligned} \quad (64)$$

On $\{x : t(x) \geq t_0\}$, Lemma 4.7 gives $\lambda \geq c_0$. Introduce

$$y = \frac{Ne^{-t}}{1+\lambda}. \quad (65)$$

Then, pointwise, for $d = n - 1$

$$\begin{aligned} \frac{e^{-dt}}{\lambda^d} e^{-cy} &= N^{-d} y^d \left(\frac{1+\lambda}{\lambda} \right)^d e^{-cy} \\ &\leq C^d N^{-d} y^d e^{-cy} \\ &\leq C^d N^{-d} d^d, \end{aligned} \tag{66}$$

where in the last step we used

$$\sup_{y \geq 0} y^d e^{-cy} = \left(\frac{d}{ce} \right)^d \leq C^d d^d.$$

Substituting (66) into (64) and using $\int f = 1$, we get

$$I_2 \leq C^m \sqrt{(n-1)!} \frac{(n-1)^{n-1}}{N^{n-1}}. \tag{67}$$

Therefore, combining (51), (60) and (67)

$$\mathbb{E} f_{n-1}(P_N) \leq 2 \binom{N}{n} e^{-c(N-n)} + \binom{N}{n} C^m \sqrt{(n-1)!} \frac{(n-1)^{n-1}}{N^{n-1}}.$$

For $N \geq 2n$,

$$2 \binom{N}{n} e^{-c(N-n)} \leq \left(\frac{2eN}{n} \right)^n e^{-cN/2},$$

and the right-hand side is at most $e^{C_1 n}$ uniformly for $N/n \geq 2$. This error will be absorbed into the final estimate. Consequently,

$$\begin{aligned} \mathbb{E} f_{n-1}(P_N) &\leq e^{C_1 n} + C^m \frac{N^n}{n!} \sqrt{(n-1)!} \frac{(n-1)^{n-1}}{N^{n-1}} \\ &= e^{C_1 n} + C^m N \frac{(n-1)^{n-1}}{n \sqrt{(n-1)!}} \end{aligned} \tag{68}$$

By Stirling's formula,

$$\frac{(n-1)^{n-1}}{n \sqrt{(n-1)!}} \leq C^m n^{(n-3)/2}. \tag{69}$$

After increasing C , the error $e^{C_1 n}$ is also bounded by the right-hand side of (69), since $N \geq 2n$. This proves (48).

Finally, if $N \leq e^{bn} n^{3/2}$, then

$$C^m N n^{(n-3)/2} \leq (C e^b)^n n^{3/2} n^{(n-3)/2} = (C e^b)^n n^{n/2},$$

which proves (49) under the regularity assumptions.

The regularity assumptions are removed exactly as in the lower-bound argument. For fixed N , the approximating product measures converge in total variation, while

$$f_{n-1}(P_N) \leq \binom{N}{n}.$$

Hence the expectations converge as well. This completes the proof of Theorem 4.6. $\square$

Remark 4.9. The preceding estimate is genuinely an average-case statement and has no pointwise analogue, even for the standard Gaussian measure. Indeed, McMullen’s Upper Bound Theorem states that among all n -dimensional polytopes with N vertices the cyclic polytope $C(N, n)$ has the largest possible number of facets. Since the Gaussian density is strictly positive everywhere, and the combinatorial type of a fixed realization of $C(N, n)$ is preserved under sufficiently small perturbations, a Gaussian random polytope has, with positive probability, exactly this maximal number of facets.

$$f_{n-1}(C(N, n)) = \begin{cases} \binom{N-m}{m} + \binom{N-m-1}{m-1}, & n = 2m, \\ 2\binom{N-m-1}{m}, & n = 2m+1. \end{cases}$$

Thus, for fixed n and large N , the maximal number of facets is of order $N^{\lfloor n/2 \rfloor}$, whereas the expectation in the exponential regime considered above is only $n^{n/2}e^{O(n)}$. This is in sharp contrast with discrete models such as 0/1 polytopes, where one may hope for a corresponding bound on the maximal number of facets.

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