Symmetries of Differrential equations and Applications in Relativistic Physics.

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Preface

This thesis is part of the PhD program of the Department of Astronomy, Astrophysics and Mechanics of the Faculty of Physics of the University of Athens, Greece.

In memory of my grandmother Amalia

Abstract

In this thesis, we study the one parameter point transformations which leave invariant the differential equations. In particular we study the Lie and the Noether point symmetries of second order differential equations. We establish a new geometric method which relates the point symmetries of the differential equations with the collineations of the underlying manifold where the motion occurs. This geometric method is applied in order the two and three dimensional Newtonian dynamical systems to be classified in relation to the point symmetries; to generalize the Newtonian Kepler-Ermakov system in Riemannian spaces; to study the symmetries between classical and quantum systems and to investigate the geometric origin of the Type II hidden symmetries for the homogeneous heat equation and for the Laplace equation in Riemannian spaces. At last but not least, we apply this geometric approach in order to determine the dark energy models by use the Noether symmetries as a geometric criterion in modified theories of gravity.

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Introduction

1.1 Summary

In this thesis, we study the geometric properties of the Lie and the Noether point symmetries of second order differential equations. In particular, we find a connection between the point symmetries of some class of second order differential equations with the collineations of the underlying manifold where the "motion" occurs.

The novelty here is that we provide a geometrical method to determine the symmetries of dynamical systems. The importance of Lie and Noether symmetries is that they offer invariant functions which can be used to find analytic solutions of the dynamical system.

The above mentioned geometric method is applied in order the two and three dimensional Newtonian dynamical systems to be classified in relation to the point symmetries; to generalize the Newtonian Kepler-Ermakov system in Riemannian spaces; to study the symmetries between classical and quantum systems and to investigate the geometric origin of the Type II hidden symmetries for the homogeneous heat equation and for the Laplace equation in Riemannian spaces. At last but not least, we apply this geometric approach in order to determine the dark energy models by use the Noether symmetries as a geometric criterion in modified theories of gravity.

The plan of the thesis is as follows.

1.1.1 Summary of Part I: Introduction

In Part I we give the basic properties and definitions of one parameter point transformations.

In Chapter 2, we study the geometry of the one parameter point transformation, the properties of Lie algebras and the invariant functions. Moreover, the Lie and the Noether symmetries of ordinary and partial differential equations are analyzed and two schemes for using Lie symmetries to construct solutions are presented.

Furthermore, we study the action of point transformation on linear differential geometry object.

1.1.2 Summary of Part II: Symmetries of ODEs

In Part II we study the geometric origin of the Lie and the Noether point symmetries of second order ordinary differential equations.

In Chapter 3, we consider the set of autoparallels - not necessarily affinely parameterized - of a symmetric connection. We find that the major symmetry condition relates the Lie symmetries with the special projective symmetries of the connection. We derive the Lie symmetry conditions for a general system of second order ODE polynomial in the first derivatives and we apply these conditions in the special case of geodesic equations of Riemannian spaces. Furthermore we give the generic Lie symmetry vector of the geodesic equations in terms of the special projective collineations of the metric and their degenerates and the generic Noether symmetry vector of the geodesic Lagrangian in terms of the homothetic algebra of the Riemannian space. Finally we apply the results to various cases and eventually we give the Lie symmetries, the Noether symmetries and the associated conserved quantities of Einstein spaces, the Gödel spacetime, the Taub spacetime and the Friedman Robertson Walker spacetimes.

In Chapter 4, we generalize the results of the previous chapter in the case of the equations of motion of a particle moving in a Riemannian space under the action of a general force F^i. We apply these results in order to determine all two dimensional and and all three dimensional Newtonian dynamical systems which admit Lie and Noether point symmetries. We demonstrate the use of the results in two cases. The non-conservative Kepler - Ermakov system and the case of the Hènon Heiles type potentials.

In Chapter 5, we generalize the two-dimensional autonomous Hamiltonian–Kepler–Ermakov dynamical system to three dimensions using the $sl(2,R)$ invariance of Noether symmetries and determine all three-dimensional autonomous Hamiltonian–Kepler–Ermakov dynamical systems which are Liouville integrable via Noether symmetries. Subsequently, we generalize the autonomous Kepler–Ermakov system in a Riemannian space which admits a gradient homothetic vector by the requirements (a) that it admits a first integral (the Riemannian Ermakov invariant) and (b) it has $sl(2,R)$ invariance. We consider both the non-Hamiltonian and the Hamiltonian systems. In each case, we compute the Riemannian–Ermakov invariant and the equations defining the dynamical system. We apply the results in general relativity and determine the autonomous Hamiltonian-Riemannian–Kepler–Ermakov system in the spatially flat Friedman Robertson Walker spacetime. We consider a locally rotational symmetric spacetime of class A and discuss two cosmological models. The first cosmological model consists of a scalar field with an exponential potential and a perfect fluid with a stiff equation of state. The second cosmological model is the$f(R)$-modified gravity model of Λ_{bc}CDM. It is shown that in both applications the gravitational field equations reduce to those of the generalized autonomous Riemannian–Kepler–Ermakov dynamical system which is Liouville integrable via Noether integrals.

1.1.3 Summary of Part III: Symmetries of PDEs

In Part III we study the geometric origin of the Lie and the Noether point symmetries of second order ordinary differential equations.

In Chapter 6, we attempt to extend this correspondence of point symmetries and collineations of the space to the case of second order partial differential equations. We examine the PDE of the form $A^{ij}u_{ij} - F(x^i, u, u_i) = 0$,where $u = u\left(x^i\right)$ and u_{ij} stands for the second partial derivative. We find that if the coefficients A_{ij} are independent of u then the Lie point symmetries of the PDE form a subgroup of the conformal symmetries of the metric defined by the coefficients A_{ij}. We specialize the study to linear forms of $F(x^i, u, u_i)$ and write the Lie symmetry conditions for this case. We apply this result to two cases. The Poisson/Yamabe equation for which we derive the Lie symmetry vectors. Subsequently we consider the heat equation with a flux in an n-dimensional Riemannian space and show that the Lie symmetry algebra is a subalgebra of the homothetic algebra of the space. We discuss this result in the case of de Sitter space time and in flat space.

In Chapter 7, we determine the Lie point symmetries of the Schrödinger and the Klein Gordon equations in a general Riemannian space. It is shown that these symmetries are related with the homothetic and the conformal algebra of the metric of the space respectively. We consider the kinematic metric defined by the classical Lagrangian and show how the Lie point symmetries of the Schrödinger equation and the Klein Gordon equation are related with the Noether point symmetries of this Lagrangian. The general results are applied to two practical problems a. The classification of all two and three dimensional potentials in a Euclidian space for which the Schrödinger equation and the Klein Gordon equation admit Lie point symmetries and b. The application of Lie point symmetries of the Klein Gordon equation in the exterior Schwarzschild spacetime and the determination of the metric by means of conformally related Lagrangians.

In Chapter 8, we study the geometric origin of Type II hidden symmetries for the Laplace equation and for the homogeneous heat equation in certain Riemannian spaces. As concerns the homogeneous heat equation, we study the reduction of the heat equation in Riemannian spaces which admit a gradient Killing vector, a gradient homothetic vector and in Petrov Type D, N, II and Type III spacetimes. In each reduction we identify the source of the Type II hidden symmetries. More specifically we find that (a) if we reduce the heat equation by the symmetries generated by the gradient KV the reduced equation is a linear heat equation in the nondecomposable space. (b) If we reduce the heat equation via the symmetries generated by the gradient HV the reduced equation is a Laplace equation for an appropriate metric. In this case the Type II hidden symmetries are generated from the proper CKVs. (c) In the Petrov space–times the reduction of the heat equation by the symmetry generated from the nongradient HV gives PDEs which inherit the Lie symmetries hence no Type II hidden symmetries appear. For the reduction of the Laplace equation we consider Riemannian spaces which admit a gradient Killing vector, a gradient Homothetic vector and a special Conformal Killing vector. In each reduction we identify the source of Type II hidden symmetries. We find that in general the Type II hidden symmetries of the Laplace equation are directly related to the transition of the CKVs from the space where the original equation is defined to the space where the reduced equation resides. In particular we consider the reduction of the Laplace equation (i.e., the wave equation) in the Minkowski space and obtain the results of all previous studies in a straightforward manner. We consider the reduction of Laplace equation in spaces which admit Lie point symmetries generated from a non-gradient HV and a proper CKV and we show that the reduction with these vectors does not produce Type II hidden symmetries. We apply the results to general relativity and consider the reduction of Laplace equation in locally rotational symmetric space times (LRS) and in algebraically special vacuum solutions of Einstein's equations which admit a homothetic algebra acting simply transitively. In each case we determine the Type II hidden symmetries. We apply the general results to cases in which the initial metric is specified.

1.1.4 Summary of Part IV: Noether symmetries and theories of gravity

In Part IV, we apply the Noether symmetry approach as a geometric criterion, in order to probe the nature of dark energy in modified theories of gravity.

In Chapter 9, we discuss the conformal equivalence of Lagrangians for scalar fields in a Riemannian space of dimension 4 and n respectively. In particular we enunciate a theorem which proves that the field equations for a non-minimally coupled scalar field are the same at the conformal level with the field equations of the minimally coupled scalar field. The necessity to preserve Einstein's equations in the context of Friedmann Robertson Walker spacetime leads us to apply, the current general analysis to the scalar field (quintessence or phantom) in spatially flat FRW cosmologies. Furthermore, we apply the Noether symmetry approach in non minimally coupled scalar field in a spatially flat FRW spacetime and by using the Noether invariants we determine analytical solutions for the field equations. Moreover we apply the same procedure for a minimally coupled scalar field in a spatially flat FRW spacetime and in Biachi Class A homogeneous spacetimes.

In Chapter 10, a detailed study of the modified gravity, $f(R)$ models is performed, using that the Noether point symmetries of these models are geometric symmetries of the mini superspace of the theory. It is shown that the requirement that the field equations admit Noether point symmetries selects definite models in a self-consistent way. As an application in Cosmology we consider the Friedman -Robertson-Walker spacetime and show that the only cosmological model which is integrable via Noether point symmetries is the Λ_{bc}CDM model, which generalizes the Lambda Cosmology. Furthermore using the corresponding Noether integrals we compute the analytic form of the main cosmological functions.

In Chapter 11, we apply the Noether symmetry approach in the $f(T)$ modified theory of gravity in a spatially flat FRW spacetime and in static spherically symmetric spacetime. First, we present a full set of Noether symmetries for some minisuperspace models and we find that only the $f(T) = T^n$ model admits extra Noether symmetries. The existence of extra Noether integrals can be used in order to simplify the system of differential equations as well as to determine the integrability of the model. Then, we compute analytical solutions and find that spherically symmetric solutions in $f(T)$ gravity can be recast in terms of Schwarzschild-like solutions modified by a distortion function depending on a characteristic radius.

Finally, in Chapter 12 we discuss our results.

Chapter 2

Point transformations and Invariant functions

2.1 Introduction

Lie symmetry of a differential equation is a one parameter point transformation which leaves the differential equation invariant. Lie symmetries[1] is the main tool to study nonlinear differential equations. Indeed Lie symmetries provide invariant functions which can be used to construct analytic solutions for a differential equation. These solutions we call invariant solutions. One such example is the solution $u(x,y) = e^{k(t-x)}$ of the wave equation

$$u_{xx} - u_{tt} = 0$$

which is found by applying the Lie symmetry $X = \partial_x + ku\partial_u$.

The structure of the chapter is as follows. In section 2.2, we study the geometry of the one parameter point transformation, the properties of Lie algebras and the invariant functions. Invariant functions are functions which remain unchanged under the action of a point transformation. In section 2.3, the Lie symmetries of ordinary and partial differential equations are analyzed and two schemes for using Lie symmetries to construct solutions are presented. In section 2.6, Noether symmetries, a special class of Lie symmetries, are discussed. Noether symmetries are admited only by systems whose equation of motion result from a variational principle. Noether symmetries are important because they produce conservation laws and can be used to simplify the differential equations.

In section 2.7, the action of point transformation on linear differential geometry object is examined. Collineations are point transformations which do not leave necessary invariant a geometric object. In particular we study the

[1] In the following sections, by Lie symmetry we mean point symmetry. There are also generalized Lie symmetries which are not point symmetries.

collineations of the metric (Conformal motions) and of the Christoffel symbols (Projective collineations) of a Riemannian space.

2.2 Point Transformations

Let M be a manifold of class C^p with $p \succeq 2$ and let U be a neighborhood in M. Consider two points $P,Q \in U$ with coordinates (x_P, y_P) and $\left(x'_Q, y'_Q\right)$ respectively. A point transformation on U is a relation among the coordinates of the points $P,Q \in U$ which is defined by the transformation equations

$$x'_Q = x'\left(x_P, y_P\right) \quad , \quad y'_Q = y'\left(x_P, y_P\right)$$

where the functions $x'\left(x,y\right),\ y'\left(x,y\right)$ are C^{p-1} and

$$\det\left|\frac{\partial\left(x',y'\right)}{\partial\left(x,y\right)}\right| \neq 0. \tag{2.1}$$

Condition (2.1) means that the functions $x'\left(x,y\right),\ y'\left(x,y\right)$ are independent. A special class of point transformations are the one parameter point transformations defined as follows [1].

Definition 2.2.1 *The one parameter point transformations are point transformations that depend on one arbitrary parameter as follows*

$$x' = y'\left(x,y,\varepsilon\right) \ , \ y' = y'\left(x,y,\varepsilon\right) \tag{2.2}$$

where $\varepsilon \in \mathbb{R}$ and the transformation satisfies the following conditions.

a) They are well defined, that is, that if

$$x'\left(x_1,y_1,\varepsilon\right) = x'\left(x_2,y_2,\varepsilon\right),\ y'\left(x_1,y_1,\varepsilon\right) = y'\left(x_2,y_2,\varepsilon\right)$$

then $x_1 = y_1$ and $y_1 = y_2$.

b) They can be composed, that is, that if

$$x' = y'\left(x,y,\varepsilon\right) \quad , \quad y' = y'\left(x,y,\varepsilon\right)$$

and

$$x'' = x''\left(x',y',\varepsilon'\right) \quad , \quad y'' = y''\left(x',y',\varepsilon'\right)$$

are two successive one parameter point transformations, there is a one parameter point transformation parametrized by the real parameter $\varepsilon'' = \left(\varepsilon,\varepsilon'\right)$ so that $x'' = x''\left(x,y,\varepsilon''\right),\quad y'' = y''\left(x,y,\varepsilon''\right).$

c) They are invertible, that is, for each one parameter point transformation

$$x' = y'\left(x,y,\varepsilon\right) \quad , \quad y' = y'\left(x,y,\varepsilon\right)$$

there exists the inverse transformation

$$x''(x', y', \varepsilon_{inv}) = x \quad , \quad y''(x', y', \varepsilon_{inv}) = y$$

d) There is the identity transformation defined by the value[2] $\varepsilon = \varepsilon_0$, *that is*

$$x'(x, y, \varepsilon_0) = x \quad , \quad y'(x, y, \varepsilon_0) = y.$$

From the above it follows that the one parameter point transformations form a group. A group of one parameter point transformation defines a family of curves in M, which are parametrized by the parameter ε and are called the **orbits of the group of transformations**. These curves may be viewed as the integral curves of a differentiable vector field $X \in M$.

2.2.1 Infinitesimal Transformations

Let $\bar{x}(x, y, \varepsilon)$, $\bar{y}(x, y, \varepsilon)$ be the parametric equations of a group orbit through the point $P(x, y, 0)$. The tangent vector at the point $P = P(x, y, 0)$ is given by

$$X_P = \frac{\partial \bar{x}}{\partial \varepsilon}|_{\varepsilon \to 0} \partial_x|_P + \frac{\partial \bar{y}}{\partial \varepsilon}|_{\varepsilon \to 0} \partial_y|_P.$$

The vector X_P defines near the point $P(x, y, 0)$ a point transformation

$$\bar{x} = x + \varepsilon \xi_P \quad , \quad \bar{y} = y + \varepsilon \eta_P \tag{2.3}$$

where we have set

$$\xi_P = \frac{\partial \bar{x}}{\partial \varepsilon}|_{\varepsilon \to 0} \quad , \quad \eta = \frac{\partial \bar{y}}{\partial \varepsilon}|_{\varepsilon \to 0}. \tag{2.4}$$

The point transformation (2.3) is a one parameter a point transformation which is called an **infinitesimal point transformation**. The vector field X_P is called the **generator** of the infinitesimal transformation (2.3) along the orbit through the point P. Evidently, the infinitesimal transformation *moves* a point along the orbit of the group through that point.

Example 2.2.2 *Compute the generator of the infinitesimal transformation for the one parameter point transformation*

$$\begin{aligned} \bar{x} &= x \cos \varepsilon - y \sin \varepsilon \\ \bar{y} &= x \sin \varepsilon + y \cos \varepsilon. \end{aligned}$$

Solution: We have

$$\begin{aligned} \xi(x, y) &= \frac{\partial \bar{x}}{\partial \varepsilon}|_{\varepsilon \to 0} = -(x \sin \varepsilon + y \cos \varepsilon)|_{\varepsilon \to 0} = -y \\ \eta(x, y) &= \frac{\partial \bar{y}}{\partial \varepsilon}|_{\varepsilon \to 0} = (x \cos \varepsilon - y \sin \varepsilon)|_{\varepsilon \to 0} = x \end{aligned}$$

from which follows that the generator of the infinitesimal transformation is $X = -y\partial_x + x\partial_y$.

[2] Without abandoning generality, we can take $\varepsilon_0 = 0$.

It has been showed that a one parameter point transformation fixes an infinitesimal generator up to a constant depending on the parametrization of the group orbit. In the following section, it will be shown that the converse holds true, that is, for an infinitesimal generator there always exists a unique one parameter point transformation.

Integral curves

Consider a differentiable vector field $X \in M$ given by $X = X^i \partial_i$. At each point of $P \in M$, X determines a smooth curve $\gamma_X(\varepsilon, P) = c_X^i(\varepsilon, P)\,\partial_i$, where $\varepsilon \in J_\varepsilon$ and J_ε is an open intevral of $\mathbb{R}$, as follows

$$\frac{dc_X^i(\varepsilon, P)}{d\varepsilon} = X^i\left(c_X^i(\varepsilon, P)\right) \ , \ \ c_X^i(0, P) = x^i(P) \, . \tag{2.5}$$

The curve $\gamma_X(J_\varepsilon, P)$ is called the **integral curve** of X through P.

Equation (2.5) defines an autonomous system of ordinary differential equations (ODEs) with solutions $\gamma_X(t, P)$ subject to the initial conditions $c_X^i(0, P) = x^i(P)$. The existence and uniqueness of integral curves is given by, the following theorem [2].

Theorem 2.2.3 *Let $\gamma_1(J_1, P)$ and $\gamma_2(J_2, P)$ be two integral curves of the vector field X on M, with the same initial condition $x^i(P)$. Then $\gamma_1(J_1, P)$ and $\gamma_2(J_2, P)$ are equal on $J_1 \cap J_2$, where J_1, J_2 are two open intervals of $\mathbb{R}$.*

In the case where $\gamma_X(J_\varepsilon, P)$ defines a one parameter point transformation, the system (2.5) defines the generator of the infinitesimal transformation. Due to the uniqueness of the solution there exists only one parameter point transformation for each vector field X.

Example 2.2.4 *Consider the space $\mathbb{R}^2$ with coordinates (x, y) and the vector field $X = y\partial_x + x\partial_y$. Find the integral curve of X through the point $P = (x_0, y_0)$.*

Solution: Let $\gamma_P(\varepsilon) = (x(\varepsilon), y(\varepsilon))$ be the integral curve of X passing through P. The system of autonomous first order equations defining the integral curves are

$$\frac{dx}{d\varepsilon} = y \ \ , \ \ \frac{dy}{d\varepsilon} = x$$

with the initial condition $x(0) = x_0$, $y(0) = y_0$. The solution of this system is

$$x(\varepsilon) = x_0 \cosh\varepsilon + y_0 \sinh\varepsilon \tag{2.6}$$

$$y(\varepsilon) = x_0 \sinh\varepsilon + y_0 \cosh\varepsilon. \tag{2.7}$$

Equations (2.6),(2.7) define the rotation in the hyperbolic space.

2.2.2 Invariant Functions

Let $F(x,y)$ be a function in M. Under the one parameter point transformation

$$\bar{x} = \bar{x}(x,y,\varepsilon),\ \bar{y} = \bar{y}(x,y,\varepsilon)$$

the function becomes $\bar{F}(\bar{x},\bar{y})$.

Definition 2.2.5 *The function F is invariant under the one parameter point transformation if and only if $\bar{F}(\bar{x},\bar{y}) = 0$ when $F(x,y) = 0$ at all points where the one parameter point transformation acts. Equivalently, the generator X of the point transformation is a symmetry of the function F if*

$$X(F) = 0 \quad , \quad \operatorname{mod} F = 0 \tag{2.8}$$

The symmetry condition (2.8) is equivalent to the first order partial differential equation (PDE)

$$\xi \frac{\partial F}{\partial x} + \eta \frac{\partial F}{\partial y} = 0. \tag{2.9}$$

In order to determine all functions which are invariant under the infinitesimal generator X one has to solve the associated Lagrange system

$$\frac{dx}{\xi(x,y)} = \frac{dy}{\eta(x,y)}.$$

The characteristic function or zero order invariant W of X is defined as follows

$$dW = \frac{dx}{\xi(x,y)} - \frac{dy}{\eta(x,y)}. \tag{2.10}$$

The zero order invariant is indeed invariant under the X, that is $X(W) = 0$. Therefore, any function of the form $F = F(W)$, where W is the zero order invariant satisfies (2.9) and it is invariant under the one parameter point transformation with generator X.

2.2.3 Lie Algebras

In section 2.2 we considered the one parameter point transformation which depend on one parameter ε. However transformations can depend on more that one parameter, as follows

$$\bar{x} = \bar{x}(x,y,\mathbf{E}) \quad , \quad \bar{y} = \bar{y}(x,y,\mathbf{E}) \tag{2.11}$$

where $\mathbf{E} = \varepsilon^{\beta}\partial_{\beta}$, is a vector field in the $\mathbb{R}^{\kappa}$, $\beta = 1...\kappa$ with the same properties of definition 2.2.1, is a multi parameter point transformation,

For every parameter ε^{β} of the multi parameter point transformation (2.11) an infinitesimal generator can be defined

$$X_{\beta} = \xi_{\beta}(x,y)\partial_x + \eta_{\beta}(x,y)\partial_y$$

where

$$\xi_\beta = \frac{\partial \bar{x}}{\partial \varepsilon_\beta}|_{\varepsilon_\beta \to 0} \quad , \quad \eta_\beta = \frac{\partial \bar{y}}{\partial \varepsilon_\beta}|_{\varepsilon_\beta \to 0}.$$

Let $F(x,y)$ be a function in M which is invariant under a multiparameter point transformation. Since the multi parameter transformation can be described as m one parameter point transformations, F is invariant under m infinitesimal generators.

Definition 2.2.6 *A Lie algebra is a finite dimensional linear space G, in which a binary operator, denoted* $[\ ,\ ]$ *has been defined which has the following properties*

i) $[X,X]=0$ *for all* $X\in G$

ii) $[X,[Y,Z]]+[Y,[Z,X]]+[Z,[X,Y]]=0$ *for all* $X,Y,Z\in G$.

iii)If $X_A,X_B\in G$ *then* $[X_A,X_B]=C^C_{AB}X_C$*,* $X_C\in G$*. The quantities* C^C_{AB} *are constants and are called the* ***structure constants*** *of the Lie algebra.*

The operator $[\ ,\ ]$ is called the **commutator** and it is defined by the following expression

$$[X_A,X_B]=X_AX_B-X_BX_A=-[X_B,X_N]. \tag{2.12}$$

From the definition of the commutator (2.12) and from the requirements of the Lie algebra, it follows that the structure constants are antisymmetric in the two lower indices, i.e.

$$C^C_{AB}+C^C_{BA}=0 \tag{2.13}$$

and they have to satisfy the Jacobi identity

$$C^E_{AB}C^C_{DE}+C^E_{BD}C^C_{AE}+C^E_{DA}C^C_{BE}=0. \tag{2.14}$$

The structure constants characterize the Lie algebra because every set of constants C^C_{AB} which satisfy (2.13) and (2.14) defines *locally* a unique Lie group. An important property of the structure constants is that they do not change under a coordinate transformation. The structure constants do change under a transformation of the basis; this property is useful because it can be used to simplify the structure constants of the given group.

Definition 2.2.7 *Let* G,H *be closed Lie algebras with elements* $\{X_A\},\ \{Y_a\}$ *respectively. If* $\dim H\leq\dim G$ *and* $Y_a\in G$ *then* H *is called a Lie subalgebra of* G*.*

Suppose that the vector fields X,Y leave invariant a function $F=F(x,y)$. If $[X,Y]=Z$ with $Z\neq X,Y$, i.e. the generators X,Y do not form a closed Lie algebra, then F is also invariant under the action of Z. This process can be used to find extra symmetries.

Example 2.2.8 *The vector fields*

$$X_1=\sin\theta\partial_\phi+\cos\theta\cot\phi\partial_\theta \quad , \quad X_2=\cos\theta\partial_\phi-\sin\theta\cot\phi\partial_\theta \quad , \quad X_3=\partial_\theta$$

span the so (3) *Lie algebra with structure constants* $C_{12}^{3}=C_{31}^{2}=C_{23}^{1}=1$*, i.e. the commutators are*

$$[X_1, X_2] = X_3 \quad , \quad [X_3, X_1] = X_2 \quad , \quad [X_2, X_3] = X_1$$

If the function $F=F(\theta,\phi)$ *is invariant under the infinitesimal generators* X_1, X_3*, then it is also invariant under the action of* X_2*. In that case, it is easy to see that* $F=F_0$*, where* F_0 *is a constant.*

2.3 Lie symmetries of differential equations

Previously, we studied the case when a function $F\in M$ is invariant under the action of a one parameter point transformation. In the following we consider the case of differential equations (DEs) which are invariant under a group of one parameter point transformations.

2.3.1 Prolongation of point transformations

In order to study the action of a point transformation to a differential equation $H\left(x,y,y',...,y^{(n)}\right)$ where $y=y(x)$, we have to prolong the point transformation to the derivatives $y^{(n)}$. The infinitesimal transformation (2.3) in the jet space $B_M=\left\{x,y,y',...,y^{(n)}\right\}$ is

$$\begin{aligned}
\bar{x} &= x+\varepsilon\xi \\
\bar{y} &= x+\varepsilon\eta \\
\bar{y}^{(1)} &= y^{(1)}+\varepsilon\eta^{[1]} \\
&\ldots \\
\bar{y}^{(n)} &= y^{(n)}+\varepsilon\eta^{[n]}
\end{aligned}$$

where $y^{(n)}=\frac{d^n y}{dx^n}$, $\bar{y}^{(n)}=\frac{d^n\bar{y}}{d\bar{x}^n}$ and

$$\eta^{[1]}=\frac{\partial\bar{y}^{(1)}}{\partial\varepsilon} \quad ,..., \quad \eta^{[n]}=\frac{\partial\bar{y}^{(n)}}{\partial\varepsilon}.$$

That means that the variation equals the difference of the derivatives before and after the action of the one parameter transformation. For the first prolongation function $\eta^{[1]}$ we have

$$\eta^{[1]}\equiv\lim_{\varepsilon\to 0}\left[\frac{1}{\varepsilon}\left(\bar{y}^{(1)}-y^{(1)}\right)\right]=\frac{d\eta}{dx}-y^{(1)}\frac{d\xi}{dx}.$$

Similarly for $\eta^{[n]}$ we have the expression

$$\eta^{[n]}=\frac{d\eta^{n-1}}{dx}-y^{(n)}\frac{d\xi}{dx}=\frac{d^n}{dx^n}\left(\eta-y^{(1)}\xi\right)+y^{(n+1)}\xi. \tag{2.15}$$

Finally, the extension of the infinitesimal generator in the jet space B_M is

$$X^{[n]}=X+\eta^{[1]}\partial_{y^{(1)}}+...+\eta^{[n]}\partial_{y^{[n]}}.$$

The field $X^{[n]} \in B_M$ is called the **nth prolongation** of the generator X, where

$$X = \xi(x,y)\,\partial_x + \eta(x,y)\,\partial_y$$

is the infinitesimal point generator in the space $\{x, y\}$.

It is possible to write the prolongation coefficients in terms of the partial derivatives of the components $\xi(x,y)$, $\eta(x,y)$. The first and the second prolongation are expressed as follows

$$X^{[1]} = X + \left[\eta_{,x} + y^{(1)}\left(\eta_{,y} - \xi_{,x}\right) - {y^{(1)}}^2 \xi_{,y}\right]\partial_{y^{(1)}}$$

$$X^{[2]} = X^{[1]} + \left[\begin{array}{c} \eta_{,xx} + 2\left(\eta_{,xy} - \xi_{,xx}\right)y^{(1)} + \left(\eta_{,yy} - 2\xi_{,xy}\right){y^{(1)}}^2 + \\ -{y^{(1)}}^3 \xi_{,yy} + \left(\eta_{,y} - 2\xi_{,x} - 3\xi_{,y}y^{(1)}\right)y^{(2)} \end{array}\right]\partial_{y^{(2)}} \tag{2.16}$$

where the comma "," denotes partial derivative.

Some important observations [3] for the prolongation coefficient $\eta^{[n]}$ are:

(a) $\eta^{[n]}$ is linear in $y^{(n)}$

(b) $\eta^{[n]}$ is a polynomial in the derivatives $y^{(1)}, ..., y^{(n)}$ whose coefficients are linear homogeneous in the functions $\xi(x,y)$, $\eta(x,y)$ up to nth order partial derivatives.

Multiparameter prolongation

In the case the differential equation H depends on n independent variables $\{x^i : i = 1..n\}$ and m dependent variables $\{u^A : A = 1...m\}$, i.e. $H = H\left(x^i, u^A, u^A_{,i}, u^A_{,ij}, ..\right)$, we consider the one parameter point transformation

$$\bar{x}^i = \Xi^i\left(x^i, u^A, \varepsilon\right) \quad , \quad \bar{u}^A = \Phi^A\left(x^i, u^A, \varepsilon\right).$$

In this case the generating vector is

$$X = \xi^i\left(x^k, u^A\right)\partial_i + \eta^A\left(x^k, u^A\right)\partial_A \tag{2.17}$$

where

$$\xi^i\left(x^k, u^A\right) = \frac{\partial \Xi^i\left(x^i, u^A, \varepsilon\right)}{\partial \varepsilon}|_{\varepsilon\to 0} \quad , \quad \eta^A\left(x^k, u^A\right) = \frac{\partial \Phi\left(x^i, u^A, \varepsilon\right)}{\partial \varepsilon}|_{\varepsilon\to 0}.$$

To extend the generator vector in the jet space $\bar{B}_{\bar{M}} = \left\{x^i, u^A, u^A_{,i}, u^A_{,ij}, .., u^A_{ij...i_n}\right\}$ we apply the same procedure as in section 2.3.1. Therefore the vector field $X^{[n]} \in \bar{B}_{\bar{M}}$

$$X^{[n]} = X + \eta^A_i \partial_{u_i} + ... + \eta^A_{ij..i_n}\partial_{u_{ij..i_n}}$$

is defined as the nth prolongation of the generator (2.17), where[3]

$$\eta^A_i = D_i \eta^A - u^A_{,j} D_i \xi^j \tag{2.18}$$

[3] Where $D_i = \frac{\partial}{\partial x^i} + u^A_i \frac{\partial}{\partial u^A} + u^A_{ij}\frac{\partial}{\partial u^A_j} + ... + u^A_{ij..i_n}\frac{\partial}{\partial u^A_{jk..i_n}}$.

$$\eta^A_{ij..i_n} = D_{i_n}\eta^A_{ij..i_{n-1}} - u_{ij..k}D_{i_n}\xi^k. \tag{2.19}$$

In terms of the partial derivatives of the components $\xi^i\left(x^k,u^A\right)$, $\eta^A\left(x^k,u^A\right)$ of the generator vector (2.17), the first and the second prologations of (2.17) are expressed as follows

$$X^{[1]} = X + \left(\eta^A_{,i} + u^B_{,i}\eta^A_{,B} - \xi^j_{,i}u^A_{,j} - u^A_{,i}u^B_{,j}\xi^j_{,B}\right)\partial_{u^A_i} \tag{2.20}$$

$$X^{[2]} = X^{[1]} + \left[\begin{array}{c} \eta^A_{,ij} + 2\eta^A_{,B(i}u^B_{,j)} - \xi^k_{,ij}u^A_{,k} + \eta^A_{,BC}u^B_{,i}u^C_{,j} - 2\xi^k_{,(i|B|}u^B_{j)}u^A_{,k} + \\ -\xi^k_{,BC}u^B_{,i}u^A_{,j}u^A_{,k} + \eta^A_{,B}u^B_{,ij} - 2\xi^k_{,(j}u^A_{,i)k} + -\xi^k_{,B}\left(u^A_{,k}u^B_{,ij} + 2u^B_{(,j}u^A_{,i)k}\right)\end{array}\right]\partial_{u_{ij}}. \tag{2.21}$$

2.3.2 Lie symmetries of ODEs

In the previous sections we analyzed the invariance of functions under the action of a point transformation. In the following we define the invariance of ordinary differential equations (ODEs) under a one parameter point transformation.

Consider the $N-$dimensional system of ODEs[4]

$$x^{(n)i} = \omega^i\left(t, x^k, \dot{x}^k, \ddot{x}^k, ..., x^{(n-1)i}\right) \tag{2.22}$$

where $\dot{x}^i = \frac{dx^i}{dt}$, $x^{(n)} = \frac{d^n x}{dt^n}$ and the infinitesimal point transformation with infinitesimal generator X is

$$\bar{t} = t + \varepsilon\xi\left(t, x^k\right) \tag{2.23}$$

$$\bar{x}^i = x + \varepsilon\eta^i\left(t, x^k\right) \tag{2.24}$$

Theorem 2.3.1 *Let*

$$X = \xi\left(t, x^k\right)\partial_t + \eta^i\left(t, x^k\right)\partial_i \tag{2.25}$$

be the infinitesimal generator of point transformation (2.23)-(2.24) and

$$X^{[n]} = X + \eta^i_{[1]}\partial_{\dot{x}^i} + ... + \eta^i_{[n]}\partial_{x^{(n)i}}$$

be the nth prolongation of X, where $\eta^i_{[n]}$ is given by (2.19). We shall say that the $N-$dimensional system of ODEs (2.22) is invariant under the point transformation (2.23), (2.24) if and only if there exists a function λ such as the following condition holds

$$\left[X^{[n]}, A\right] = \lambda A \tag{2.26}$$

where[5]

$$A = \frac{\partial}{\partial t} + \dot{x}^i\frac{\partial}{\partial x^i} + ... + \omega^i\left(t, x^k, \dot{x}^k, \ddot{x}^k, ..., x^{(n-1)i}\right)\frac{\partial}{\partial x^{(n)i}}.$$

*In that case we say that X is a **Lie point symmetry** of the $N-$dimensional system of ODEs (2.22).*

[4] In the following equations, t is the independent parameter and $x^i = x^i(t)$ the dependent parameters.

[5] For Hamiltonian systems, the operator **A** is called **Hamiltonian vector field**.

If f^i is a solution of the system (2.22), i.e. $Af^i = 0$, then condition (2.26) becomes $X\left(Af^i\right) = 0,$ that is,

$$\eta^i_{[n]} = X^{[n-1]}\omega^i\left(t, x^k, \dot{x}^k, \ddot{x}^k, ..., x^{(n-1)i}\right). \tag{2.27}$$

Equations (2.27) are called the determining equations. The solution of the determining equations (2.27) gives the infinitesimal generators of the transformation (2.23)-(2.24).

In general, a function $H\left(t, \dot{x}^k, \ddot{x}^k, ..., x^{(n)k}\right) = 0$ is invariant under the transformation (2.23)-(2.24) if and only if

$$X^{[n]}(H) = \lambda H \ , \ \ \mathrm{mod}\, H = 0. \tag{2.28}$$

where λ is a function to be determined [4].

Below we give an example in which the Lie symmetries are calculated using the symmetry condition (2.27).

Example 2.3.2 *Find the Lie symmetries of the ODE $\ddot{x} = 0$.*

Solution: Condition (2.27) gives $\eta_{[2]} = 0$. From (2.16) the following condition is found

$$\eta_{,tt} + 2\left(\eta_{,tx} - \xi_{,tt}\right)\dot{x} + \left(\eta_{,xx} - 2\xi_{,tx}\right)\dot{x}^2 - \dot{x}^3\xi_{,xx} = 0 \tag{2.29}$$

since $\ddot{x} = 0$. Functions ξ, η are dependent only on the variables $\{t, x\}$, hence equation (2.29) is a polynomial of $\dot{x}$. This polynomial must vanish identically hence the coefficients of all powers of $\dot{x}$ must vanish. Therefore, we have the following determining equations

$$\begin{aligned} (\dot{x})^0 &: \ \eta_{,tt} = 0 \\ (\dot{x})^1 &: \ \eta_{,xy} - \xi_{,tt} = 0 \\ (\dot{x})^2 &: \ \eta_{,xx} - 2\xi_{,tx} = 0 \\ (\dot{x})^3 &: \ \xi_{,xx} = 0. \end{aligned}$$

whosw solution is

$$\begin{aligned} \xi(t, x) &= a_1 + a_2 t + a_3 t^2 + a_4 x + a_5 t x \\ \eta(t, x) &= a_6 + a_7 t + a_8 x + a_3 t x + a_5 x^2. \end{aligned}$$

We conclude that the second order ODE $\ddot{x} = 0$ has eight Lie pont symmetries generated by the generic vector field[6]

$$X = \left(a_1 + a_2 t + a_3 t^2 + a_4 x + a_5 t x\right)\partial_t + \left(a_6 + a_7 t + a_8 x + a_3 t x + a_5 x^2\right)\partial_x. \tag{2.30}$$

These vectors are the generators of the projective algebra $sl(3, R)$ of the 2-d Euclidian plane. Furthermore, this is the maximum number of symmetries that a single second order ODE (in one variable!) can have.

[6] As many as the unspecified constants in the expression of the generic symmetric vector.

Lie symmetries can be used to find invariants or to reduce the order of an ODE. Furthermore an ODE is characterized by the admitted algebra of Lie symmetries. For second order ODEs we have the following theorem.

Theorem 2.3.3 *If a second order ODE admits as Lie point symmetries the eight of $sl\left(3,R\right)$, then, there exists a transformation which brings the equation to the form $x^{*\prime\prime}=0$ and vice versa.*

For instance the equations

$$\ddot{x}+\frac{1}{x}\dot{x}^{2}=0$$

$$\ddot{x}+3x\dot{x}+x^{3}=0$$

$$\ddot{x}+\omega_{0}^{2}x+\sin t=0$$

$$\left(4+x^{2}+t^{2}\right)\ddot{x}-2t\dot{x}^{3}+2x\dot{x}^{2}-2t\dot{x}+2x=0$$

are equivalent to the equation of motion of a free particle [5, 6, 7], because they are invariant under the action of the Lie algebra $sl\left(3,R\right)$. The transformation where the admitted $sl\left(3,R\right)$ algebra is written in the form (2.30) is a coordinate transformation on $M:\left(t,x\right)\rightarrow\left(\tau,y\right)$ which transforms the ODE to the form $\frac{d^{2}y}{d\tau^{2}}=0$. This procedure is called linearization process (see [8, 9] and references therein).

Below we present two methods where the use of Lie symmetries reduces the order of an ODE.

Canonical coordinates

Definition 2.3.4 *Let X be a vector field with coordinates $X=\xi\left(t,x\right)\partial_{t}+\eta\left(t,x\right)\partial_{x}$. If under the coordinate transformation $\left\{t,x\right\}\rightarrow\left\{r,s\right\}$ holds that*

$$Xr=0\quad,\quad Xs=1 \tag{2.31}$$

*then, we say that $\left\{r,s\right\}$ are the **canonical coordinates** of X, i.e. $X=\partial_{s}$.*

Canonical coordinates can be used to reduce by one the order of an ODE. Consider the *nth* order ODE $(n\succeq 2)$

$$\frac{d^{n}s}{dr^{2}}=\bar{\omega}\left(r,s,\frac{ds}{dr},...,\frac{d^{n-1}s}{dr^{n-1}}\right). \tag{2.32}$$

Let $X_{C}=\partial_{s}$ be a Lie symmetry of (2.32) written in canonical coordinates. The nth prolongation of the symmetry vector is $X_{C}^{[n]}=X_{C},$ so condition (2.27) sets the constraint

$$\frac{\partial}{\partial s}\bar{\omega}\left(r,s,\frac{ds}{dr},...,\frac{d^{n-1}s}{dr^{n-1}}\right)=0.$$

This implies that the function $\bar{\omega}$ is independent of s, consequently (2.32) can be written in the form

$$\frac{d^{n-1}S}{dr^{n-1}}=\bar{\omega}\left(r,S,\frac{dS}{dr},...,\frac{d^{n-2}S}{dr}\right) \tag{2.33}$$

which is a $(n-1)$ order ODE, where S is defined by $S=\frac{ds}{dr}$.

Example 2.3.5 *Use the canonical coordinates of the Lie symmetry $X = x\partial_x - t\partial_t$ to reduce the order of the ODE*

$$\ddot{x} + x\dot{x} = 0 \tag{2.34}$$

Solution: The canonical coordinates of X are

$$x = e^s \quad , \quad t = re^{-s}.$$

For the first and the second derivative of $x(t)$ we compute

$$\dot{x} = \frac{e^{2s}s'}{1 - rs'}$$

$$\ddot{x} = \frac{e^{3s}}{(1 - rs')^3}\left[\left(2s'^2 + s''\right)(1 - rs') + s'(s' + rs'')\right]$$

where $s' = \frac{ds}{dr}$.

Replacing in 2.34, we find the reduced equation

$$\left(2S^2 + S'\right)(1 - rS) + S(S + rS') + S(1 - rS)^2 = 0 \tag{2.35}$$

where we have set $S = s'$. Equation (2.35) is an Abel equation of the first kind [10] invariant under $X = \frac{d}{ds}$.

Invariants

As in the case of functions in M (see section 2.2.2), condition (2.28) is equivalent to the following Lagrange system

$$\frac{dt}{\xi} = \frac{dx}{\eta} = \frac{d\dot{x}}{\eta_{[1]}} = ... = \frac{dx^{(n)}}{\eta_{[n]}}. \tag{2.36}$$

The Lagrange system (2.36) provides us with characteristic functions

$$W^{[0]}(t, x)\,,\ W^{[1]i}(t, x, \dot{x})\,,\ W^{[n]}\left(t, x, \dot{x}, ..., \dot{x}^{(n)}\right)$$

where $W^{[n]}$ is called the **nth order invariant** of the Lie symmetry vector. If (2.25) is a Lie symmetry for the ODE

$$x^{(n)} = \omega\left(t, \dot{x}, \ddot{x}, ..., x^{(n-1)}\right) \tag{2.37}$$

it follows that (2.37) can be written as a function of the characteristic functions $W^{[1]}, ..., W^{[n]}$, of (2.25).

Let $u = W^{[0]}$, $v = W^{[1]}$, where $W^{[0]}$, $W^{[1]}$ are the zero and the first order invariants of a Lie symmetry repetitively. From the invariants u, v, we define the differential invariants

$$\frac{dv}{du}\ , ..., \frac{d^{n-1}v}{du^{n-1}}. \tag{2.38}$$

where

$$\frac{dv}{du} = \frac{\frac{\partial v}{\partial t} + \frac{\partial v}{\partial x}\dot{x} + ... + \frac{\partial v}{\partial \dot{x}}\ddot{x}}{\frac{\partial v}{\partial t} + \frac{\partial v}{\partial x}\dot{x}}.$$

The differential invariants are functions of $x^{(n)}$, hence, it is feasible that (2.37) may be written in terms of the differential invariants (2.38), i.e.

$$\frac{d^{n-1}v}{du^{n-1}} = \Omega\left(u, v, \frac{dv}{du}, ...\frac{d^{n-1}v}{du^{n-1}}\right) \tag{2.39}$$

which is a $(n-1)$ order ODE.

Example 2.3.6 *Use the first order invariants of the Lie symmetry $X = x\partial_x - nt\partial_t$ to reduce the order of the Lane-Emden equation*

$$\ddot{x} + \frac{2}{t}\dot{x} + x^{2n+1} = 0$$

where $n \neq -\frac{1}{2}, 0$. The Lane-Emden equation arises in the study of equilibrium configurations of a spherical gas cloud acting under the mutual attractions of its molecules and subject to the laws of thermodynamics [11, 12].

Solution: The first prolongation of X is

$$X^{[1]} = x\partial_x - nt\partial_t + (n+1)\,\dot{x}\partial_{\dot{x}}$$

hence, the corresponding Lagrange system is

$$\frac{dt}{-nt} = \frac{dx}{x} = \frac{d\dot{x}}{(n+1)\,\dot{x}}.$$

The zero and the first order invariants are found to be

$$u = xt^{\frac{1}{n}} \quad , \quad v = \dot{x}t^{1+\frac{1}{n}}.$$

The differential invariant is defined as

$$\frac{dv}{du} = \frac{\left(1+\frac{1}{n}\right)\dot{x} + t\ddot{x}}{\frac{1}{n}t^{-1}x + \dot{x}}.$$

Substituting in the Lane-Emden equation we obtain the first order ODE

$$v'\,(u + nv) - (1-n)\,v + nu^{2n+1} = 0. \tag{2.40}$$

where $v' = \frac{dv}{du}$. Equation (2.40) is an Abel equation of the second kind [10].

An interesting application of the Lie invariants to the classical Kepler system can be found in [13] where the authors derive the Runge-Lenz vector using the first order invariants of a point transformation.

It is possible an ODE to admit many symmetries which span a Lie algebra G_m of dimension $m > 1$. The following theorem relates the Lie point symmetries of the reduced and of the original equation.

Theorem 2.3.7 *Consider an ODE which admits the Lie point symmetries $X_1,\ X_2$ which are such that $[X_1, X_2] = C^1_{12}X_1$. Then, the reduction by X_1 leads to a reduced equation which admits X_2 as a Lie symmetry whereas reduction of the ODE by X_2 leads to a reduced equation, which does not admit X_1 as a Lie symmetry.*

In case the generators X_1, X_2 form an Abelian Lie algebra, i.e. $[X_1, X_2] = 0$, then, the reduction preserves the Lie symmetries. Theorem 2.3.7 is important because it gives a hint as to which generator the reduction of an ODE must start in order to continue the reduction process with the reduced equation.

Since the reduced equation is different from the original equation, it is possible the reduced equation to admit extra Lie symmetries, which are not Lie symmetries of the original equation. These new Lie symmetries have been named **Type II hidden symmetries**. Type II hidden symmetries can be used to continue the reduction process and - if it is feasible - to find a solution of the original equation [14].

There are other methods to apply Lie symmetries; a quite intriguing application of Lie symmetries is to produce integrals or Lagrangian functions for a system of ODEs by the method of Jacobi's last multiplier (see [15, 16, 17] and references therein).

2.4 Lie symmetries of PDEs

A partial differential equation (PDE) is a function $H = H\left(x^i, u^A, u^A_{,i}, u^A_{,ij}, .., u^A_{ij...i_n}\right)$ in the jet space $\bar{B}_{\bar{M}}$, where x^i are the independent variables and u^A are the dependent variables. As in the case of ODEs we define the invariance of a PDE under the infinitesimal point transformation

$$\bar{x}^i = x^i + \varepsilon\xi^i\left(x^k, u^B\right) \tag{2.41}$$

$$\bar{u}^A = \bar{u}^A + \varepsilon\eta^A\left(x^k, u^B\right) \tag{2.42}$$

with generator X as follows.

Definition 2.4.1 *Let*

$$X = \xi^i\left(x^k, u^B\right)\partial_t + \eta^A\left(x^k, u^B\right)\partial_B \tag{2.43}$$

be the generator of the infinitesimal point transformation (2.41)-(2.42) and

$$X^{[n]} = X + \eta^A_{[i]}\partial_{\dot{x}^i} + ... + \eta^A_{[ij...i_n]}\partial_{u_{ij...i_n}}$$

be the nth prolongation vector, where $\eta^A_{[ij...i_n]}$ is given from (2.19). Then, the transformation (2.41)-(2.42) leaves invariant the PDE

$$H\left(x^i, u^A, u^A_{,i}, u^A_{,ij}, ..\right) = 0 \tag{2.44}$$

if there exist a function λ where the following condition holds,

$$X^{[n]}\left(H\right) = \lambda H \quad , \quad \mathrm{mod}\, H = 0. \tag{2.45}$$

The infinitesimal generator X is called a Lie point symmetry of the PDE (2.44).

In case the PDE (2.44) can we written in solved form, i.e.

$$u^A_{ij...i_n} = h^A\left(x^i, u^B, u^B_{,i}, u^B_{,ij}, .., u^B_{ij...i_{n-1}}\right)$$

then the Lie condition (2.45) is equivalent to the following system of equations

$$\eta^{A}_{[ij...i_n]} = X^{[n-1]}h^{A}\left(x^{i}, u^{B}, u^{B}_{,i}, u^{B}_{,ij}, .., u^{B}_{ij...i_{n-1}}\right). \tag{2.46}$$

Consequently the family of all solutions of (2.44) is invariant under (2.41)-(2.42) if condition (2.45) holds. In the example below we compute the Lie symmetries of the homogeneous heat equation.

Example 2.4.2 *Find the Lie symmetries of the 1+1 heat equation*

$$H\left(u_{,t}, u_{,xx}\right) : u_{,xx} - u_{,t} = 0. \tag{2.47}$$

Solution: Let

$$X = \xi^{t}\left(t,x,u\right)\partial_{t} + \xi^{x}\left(t,x,u\right)\partial_{x} + \eta\left(t,x,u\right)\partial_{u}$$

be a Lie symmetry of (2.47). Then condition (2.46) gives

$$\eta_{[xx]} - \eta_{[t]} = 0.$$

Substituting $\eta_{[t]}$, $\eta_{[xx]}$ from (2.20),(2.21) and collecting terms of derivatives of $u\left(t,x\right)$ we find the following determining equations

$$\xi^{t}_{,u} = 0 \ \ , \ \ \xi^{t}_{,x} = 0 \ \ , \ \ \xi^{x}_{,u} = 0 \ \ , \ \ \xi^{t}_{,ttt} = 0$$

$$\xi^{x}_{,t} = -2\eta_{,xu} \ \ , \ \ \xi^{x}_{,x} = \frac{1}{2}\xi^{t}_{,t} \ \ , \ \ \eta_{,uu} = 0$$

$$\eta_{,xx} = \eta_{,t} \ \ , \ \ 4\eta_{,tu} + \xi_{,tt} = 0.$$

The solution of the system of equations is

$$\begin{aligned} \xi^{t}\left(t,x,u\right) &= a_{1}t^{2} + 2a_{2}t + a_{3} \\ \xi^{x}\left(t,x,u\right) &= a_{1}tx + a_{2}x + a_{4}t + a_{5} \\ \eta\left(t,x,u\right) &= -\frac{a_{1}}{2}\left(t + \frac{1}{2}x^{2}\right)u - \frac{a_{4}}{2}xu + a_{6}u + b\left(t,x\right) \end{aligned}$$

where $b\left(t,x\right)$ is a solution of the heat equation. The generic vector of the infinitesimal transformation which leaves invariant the heat equation is

$$\begin{aligned} X = & \left(a_{1}t^{2} + 2a_{2}t + a_{3}\right)\partial_{t} + \left(a_{1}tx + a_{2}x + a_{4}t + a_{5}\right)\partial_{x} + \\ & + \left[-\frac{a_{1}}{2}\left(t + \frac{1}{2}x^{2}\right)u - \frac{a_{4}}{2}xu + a_{6}u + b\left(t,x\right)\right]\partial_{u}. \end{aligned}$$

Invariant solutions

In section 2.3.2, it has been explained how to use the canonical coordinates to reduce the order of an ODE. The infinitesimal generator $X_3 = \partial_t$ is a Lie symmetry of the heat equation (2.47), written in canonical coordinates. Nevertheless (2.47) cannot be reduced as it happened with the case of ODEs. However, one can calculate from the Lie symmetry X_3 the zero order invariants, which are $y = x$, $w = u$. We select y to be the independent variable and $w = w(y)$. Substituting the invariants in (2.47) we get the ODE $w_{yy} = 0$. Then, the following solution can be found easily

$$u(x) = c_1 x + c_2 \tag{2.48}$$

which is independent on the variable t. That is, the use of the Lie symmetry X_3 reduces by one the number of independent variables but not the order of the PDE. The solution (2.48) is called **an invariant solution**.

Definition 2.4.3 *The function $u = U(x^i)$ is an invariant solution corresponding to the infinitesimal generator (2.43), if and only if $U(x^i)$ is an invariant of the infinitesimal generator, i.e. $X(U) = 0$ and solves the PDE (2.44).*

By definition, a Lie symmetry maps a solution onto a solution. The action of the point transformation with infinitesimal generator X_3 on a solution u of (2.47) is $u(\bar{t}, \bar{x}) = u(t + \varepsilon, x)$, that is, $X_3 \bar{u} = 0$ or $\bar{u}_{,t} = 0$, hence the use of Lie point symmetries gives a constraint equation. However, if a solution of a PDE is already known we can apply the point transformation to obtain a family of solutions. This new solutions will depend on at most as many new parameters as there are in the symmetry transformation used.

Theorem 2.4.4 *Assume that the PDE (2.44) admits a Lie point symmetry and let $u = u(x^i)$ be a solution of (2.44) which is not invariant under this Lie symmetry. Then, under the transformation (2.41)-(2.42) the solution $u(x^i)$ defines a family of solutions of the PDE.*

For instance, the point transformation corresponding to the Lie symmetry

$$X_1 = t^2 \partial_t + tx\partial_x - \frac{1}{2}\left(t + \frac{1}{2}x^2\right) u\partial_u$$

of the heat equation (2.47) is

$$\bar{t} = t(1 - \varepsilon t)^{-1} \quad , \quad \bar{x} = x(1 - \varepsilon t)^{-1}$$

$$\bar{u} = u\sqrt{1 - \varepsilon t}e^{-\frac{\varepsilon x^2}{4(1-\varepsilon t)}}.$$

Then, if $\bar{u}_c = \bar{u}_c(\bar{t}, \bar{x})$ is a solution of (2.47), which is not an invariant solution, the function

$$u_c(x,t) = \frac{1}{\sqrt{1 - \varepsilon t}} e^{\frac{\varepsilon x^2}{4(1-\varepsilon t)}} \bar{u}_c\left(\frac{x}{1 - \varepsilon t}, \frac{t}{1 - \varepsilon t}\right)$$

is also a solution of (2.47).

2.5 Lie Bäcklund symmetries

So far we have considered point transformations which depend on the variables of the base manifold only. However there exist point transformations who are defined in the jet space and depend also on the derivatives.

The infinitesimal transformation

$$\bar{x}^i = x^i + \varepsilon \xi^i \left(x^i, u, u_{,i}, u_{,ij}...\right) \tag{2.49}$$

$$\bar{u} = u + \varepsilon \eta \left(x^i, u, u_{,i}, u_{,ij}...\right) \tag{2.50}$$

is called **Lie Bäcklund transformation**. This point transformation does not concern the present work but for the completeness we we give the basic definitions.

Definition 2.5.1 *The generator $X = \xi \left(x^i, u, u_{,i}, u_{,ij}...\right) \partial_i + \eta \left(x^i, u, u_{,i}, u_{,ij}...\right) \partial_u$ generates a **Lie Bäcklund symmetry** for the DE $H \left(x^i, u, u_{,i}, u_{,ij}...\right) = 0$, if and only if there exist a function $\lambda \left(x^i, u, u_{,i}, u_{,ij}...\right)$ such as the following condition holds.*

$$[X, H] = \lambda H \quad , \quad \mathrm{mod}\, H = 0.$$

It follows from the above definition that a Lie Bäcklund symmetry preserves the set of solutions u of the DE, i.e. $X(u) = Cu$, where C is a constant. A special class of Lie Bäcklund symmetries are the contact symmetries. **Contact symmetry** is a Lie Bäcklund symmetry where the generator depends only on the first derivative of $u_{,i}$, i.e. the generator of transformation (2.49)-(2.49) has the form $X = \xi \left(x^i, u, u_{,i}\right) \partial_i + \eta \left(x^i, u, u_{,i}\right) \partial_u$.

Proposition 2.5.2 *The generator $\bar{X} = \left(\eta - \xi^k u_{,k}\right) \partial_u$ is the canonical form of the operator $X = \xi \partial_i + \eta \partial_u$.*

The operator $D_i = \partial_i + u_{,i}\partial_u + u_{,ij}\partial_{u_i} + ...$ is always a Lie Bäcklund symmetry (the trivial one, since D_i in the canonical form is $\bar{D}_i = \left(u_{,i} - u_{,i}\right) \partial_u + ... = 0$), hence $f^i \left(x^i, u, u_{,i}, u_{,ij}...\right) D_i$ is also a Lie Bäcklund symmetry. If $X = \xi \partial_i + \eta \partial_u$ is a Lie Bäcklund symmetry, then the generator $\bar{X}$,

$$\bar{X} = X - f D_i = \left(\xi^k - f^k\right) \partial_k + \left(\eta - f^k u_{,k}\right) \partial_u + ...$$

is also a Lie Bäcklund symmetry. Since f^k is an arbitrary function we set $f^k = \xi^k$ and obtain

$$\bar{X} = \left(\eta - \xi^k u_{,k}\right) \partial_u.$$

We can always absorb the term $\xi^k u_k$ inside the η and have the final result that $\bar{X} = Z \left(x^i, u, u_{,i}, u_{,ij}...\right) \partial_u$ is the generator of a Lie Bäcklund symmetry.

For a PDE the generator of a Lie point symmetry is

$$X = \xi \left(x^i, u\right) \partial_i + \eta \left(x^i, u\right) \partial_u$$

then in this case the vector $\bar{X}$ can be written as

$$\bar{X} = \left(\eta \left(x^i, u\right) - \xi^k \left(x^i, u\right) u_{,k}\right) \partial_u$$

which is linear in the first derivative; that is, for PDEs, Lie symmetries are equivalent to contact symmetries that are linear in the first derivative u_k (*that property does not hold for ODEs*). For example, the Lie point symmetry $X = \frac{\partial}{\partial x}$ of a PDE is equivalent to the Lie Bäcklund symmetry $\bar{X} = -u_{,x}\frac{\partial}{\partial u}$.

2.6 Noether point symmetries

In the following sections we consider the Lie symmetries and the conservation laws of systems admitting a Lagrangian function, i.e. systems whose equations of motion follow from a variational principle (e.g. Hamilton principle).

2.6.1 Noether symmetries of ODEs

In Analytical Mechanics the Lagrangian $L = L\left(t, x^k, \dot{x}^k\right)$ is a function describing the dynamics of a system. The equations of motion of the dynamical system follow from the action of the Euler Lagrange vector E_i on the Lagragian L, i.e.

$$E_i\left(L\right) = 0. \tag{2.51}$$

where the **Euler Lagrange vector**

$$E_i = \frac{d}{dt}\frac{\partial}{\partial \dot{x}^i} - \frac{\partial}{\partial x^i} \tag{2.52}$$

If the Lagrangian is invariant under the action of the transformation (2.23)-(2.24), then, it is easy to observe that the Euler Lagrange equations (2.51) are invariant under the transformation (2.23)-(2.24). E. Noether proved that if the action of a one parameter point transformation leaves invariant the Euler Lagrange equations (2.51) then there exists a conserved quantity corresponding to the point transformation.

Theorem 2.6.1 *Let*

$$X = \xi\left(t, x^k\right)\partial_t + \eta^i\left(t, x^k\right)\partial_i \tag{2.53}$$

be the infinitesimal generator of transformation (2.23)-(2.24) and

$$L = L\left(t, x^k, \dot{x}^k\right) \tag{2.54}$$

be a Lagrangian describing the dynamical system (2.51). The action of the transformation (2.23)-(2.24) on (2.54)[7] *leaves the Euler Lagrange equations (2.51) invariant, if and only if there exist a function $f = f\left(t, x^k\right)$ such that the following condition holds*

$$X^{[1]}L + L\frac{d\xi}{dt} = \frac{df}{dt} \tag{2.55}$$

where $X^{[1]}$ is the first prolongation of (2.53).

[7] For the application of Noether theorem in higher order Lagrangians see [18]

If the generator (2.53) satisfies (2.55), the generator (2.53) is a Noether symmetry of the dynamical system described by the Lagrangian (2.54). Noether symmetries form a Lie algebra called the **Noether algebra**. If the dynamical system (2.51) admits Lie symmetries which span a Lie algebra G_m of dimension $m \succeq 1$ then the Noether symmetries of (2.51) form a Lie algebra G_h , $h \geq 0$, which is a subalgebra of G_m, $G_h \subseteq G_m$.

Theorem 2.6.2 *For any Noether point symmetry (2.53) of the Lagrangian (2.54) there corresponds a function* $\phi\left(t, x^k, \dot{x}^k\right)$

$$\phi = \xi\left(\dot{x}^i \frac{\partial L}{\partial \dot{x}^i} - L\right) - \eta^i \frac{\partial L}{\partial x^i} + f \tag{2.56}$$

which is a first integral i.e. $\frac{d\phi}{dt} = 0$ *of the equations of motion. The function (2.56) is called a **Noether integral** (first integral) of (2.51).*

Since a Noether symmetry leaves the DEs (2.51) invariant, it is also a Lie symmetry of (2.51) and from (2.26) we can say that (2.56) satisfies the condition $X(\phi) = 0$, that is, Noether integrals are invariant functions of the Noether symmetry vector X.

The existence and the number of Noether symmetries characterize a dynamical system. If a dynamical system (2.51) on n degrees of freedom admits (at least) n linear independent first integrals[8] ϕ_J, $J = 1...n$, which are in involution, i.e.

$$\{\phi_J, \phi_K\} = 0 \tag{2.57}$$

where $\{\ ,\ \}$ is the Poisson bracket, then the solution of the dynamical system can be obtained by quadratures.

We calculate the Noether symmetries of the free particle and of the harmonic oscillator.

Example 2.6.3 *Find the Noether symmetries of the Lagrangian* $L = \frac{1}{2}\dot{x}^2$.

Solution: The Lagrangian corresponds to the equation of motion of the free particle $\ddot{x} = 0$. *In example 2.3.2, the Lie symmetries of the free particle were calculated. The generic Lie symmetry vector is applied to condition (2.55) and it is found that the generic Noether symmetry of the free particle is*

$$X = \left(a_1 + 2a_4 t + a_5 t^2\right)\partial_t + \left(a_2 + a_3 t + a_4 x + a_5 t x\right)\partial_x. \tag{2.58}$$

From the generic vector (2.58) and (2.56), it is found that the corresponding Noether integral is

$$\phi = a_1 \dot{x}^2 + a_2 \dot{x} + a_3\left(t\dot{x} - x\right) + a_4\left(t\dot{x}^2 - x\dot{x}\right) + a_5\left(t^2\dot{x}^2 - 2tx\dot{x} + x^2\right)$$

Example 2.6.4 *Find the Noether symmetries of the one dimensional harmonic oscillator with Lagrangian*

$$L = \frac{1}{2}\dot{x}^2 - \frac{1}{2}x^2. \tag{2.59}$$

Solution: To derive the Noether symmetries of (2.59) we apply theorem 2.6.1. Let $X = \xi(t,x)\partial_t + \eta(t,x)\partial_x$ *be the infinitesimal generator. The first prolongation* $X^{[1]}$ *is*

$$X^{[1]} = \xi\partial_t + \eta\partial_x + \eta_{[1]}\partial_{\dot{x}}.$$

[8] Not necessarilly Noether point integrals.

where $\eta_{[1]}=\left[\eta_{,t}+\dot{x}\left(\eta_{,x}-\xi_{,t}\right)-\dot{x}^{2}\xi_{,x}\right]$. *For the terms of (2.55) we have*

$$\frac{df}{dt}=f_{,t}+\dot{x}f_{,x}.$$

$$X^{[1]}L=\dot{x}\eta_{,t}+\dot{x}^{2}\left(\eta_{,x}-\xi_{,t}\right)-\dot{x}^{3}\xi_{,x}-x\eta$$

$$L\frac{d\xi}{dt}=\frac{1}{2}\dot{x}^{2}\xi_{,t}-\frac{1}{2}x^{2}\xi_{,t}+\frac{1}{2}\dot{x}^{3}\xi_{,x}-\frac{1}{2}x^{2}\dot{x}\xi_{,x}.$$

Replacing in (2.55) we find:

$$\begin{aligned}0 &= -\left[x\eta+\frac{1}{2}x^{2}\xi_{,t}+f_{,t}\right]+\dot{x}\left[\eta_{,t}-\frac{1}{2}x^{2}\xi_{,x}-f_{,x}\right]\\ &\quad+\dot{x}^{2}\left[\eta_{,x}-\frac{1}{2}\xi_{,t}\right]+\dot{x}^{3}\left[\xi_{,x}\right]\end{aligned}$$

The determining equations are

$$\xi_{,x}=0\quad,\quad\eta_{,x}-\frac{1}{2}\xi_{,t}=0 \tag{2.60}$$

$$\eta_{,t}-\frac{1}{2}x^{2}\xi_{,x}-f_{,x}=0\quad,\quad x\eta+\frac{1}{2}x^{2}\xi_{,t}+f_{,t}=0. \tag{2.61}$$

The solution of the system of equations (2.60)-(2.61) gives the generic Noether symmetry [19]

$$\begin{aligned}X &= \left(a_{1}+a_{4}\cos 2t+a_{5}\sin 2t\right)\partial_{t}+\\ &\quad+\left(a_{2}\sin t+a_{3}\cos t-a_{4}x\sin 2t+a_{5}x\cos 2t\right)\partial_{x}\end{aligned} \tag{2.62}$$

with the corresponding gauge function

$$f\left(t,x\right)=a_{2}x\cos t-a_{3}\sin t-a_{3}x^{2}\cos 2t-a_{5}x^{2}\sin 2t.$$

It should be underlined that the free particle and the harmonic oscillator have the same number of Noether symmetries. It can be easily noticed that the Lie algebras (2.58) and (2.62) are the same Lie algebra in different representations.

2.6.2 Noether point symmetries of PDEs

In case of PDEs

$$H\left(x^{i},u,u_{i},u_{ij}\right)=0 \tag{2.63}$$

which arise from a variational principle the following theorem holds.

Theorem 2.6.5 *The action of the transformation (2.41)-(2.42) on the Lagrangian*

$$L_{P}=L_{P}\left(x^{k},u,u_{k}\right) \tag{2.64}$$

leaves (2.63) invariant if there exists a vector field $F^{i}=F^{i}\left(x^{i},u\right)$ *such that*

$$X^{[1]}L_{P}+L_{P}D_{i}\xi^{i}=D_{i}F^{i}.$$

The generator of the point transformation (2.41)-(2.42) is called ***Noether symmetry.*** *The corresponding* ***Noether flow*** *is*

$$\Phi^i = \xi^k \left(u_k \frac{\partial L}{\partial u_i} - L \right) - \eta \frac{\partial L}{\partial u_i} + F^i \tag{2.65}$$

and satisfies the condition $D_i \Phi^i$.

The implementation of Noether flows in PDEs differs from that in the ODEs. Conservation flow (2.65) is used to reduce the order of (2.63) by defining a new dependent variable $v\left(x^i\right)$. It has been proved that the solution of the system

$$v_{,i}\left(x^k\right) = \Phi_i\left(x^k, u, u_k\right) \tag{2.66}$$

is also a solution of (2.63). Furthermore, it is feasible new symmetries to arise from the system (2.66) which are not symmetries of (2.63). These new symmetries have been called **potential symmetries** [20].

2.7 Collineations of Riemannian manifolds

Previously, we studied the case when a function is invariant under a point transformation. In the following subsections we consider the cases in which a function is invariant under a point transformation. In order to do this we shall need the concept of the Lie derivative. In particular, the geometric objects which will be considered the metric tensor g_{ij} and the connection coefficients Γ^i_{jk} in a Riemannian space.

Definition 2.7.1 *Consider an n dimensional space* A_n *of class* C^p. *An object is called a* ***geometric object*** *of class r* $(r \leq p)$ *if it has the following properties .*

i) In each coordinate system $\left\{x^i\right\}$, *it has a well determined set of components* $\Omega^a\left(x^k\right)$.

ii) Under a coordinate transformation $x^{i'} = J^i\left(x^k\right)$ *the new components* $\Omega^{a'}$ *of the object in the new coordinates* $\left\{x^{i'}\right\}$ *are represented as well determined functions of class* $r' = p - r$ *of the old components* Ω^a *in the old coordinates* $\left\{x^i\right\}$, *of the functions* J^i *and of their sth derivatives* $(1 \leq s \leq p)$, *that is, the new components* $\Omega^{i'}$ *of the object can be represented by equations of the form*

$$\Omega^{a'} = \Phi^a\left(\Omega^k, x^k, x^{k'}\right). \tag{2.67}$$

iii) The functions Φ^a *have the group properties, that is, they satisfy the following relations*

$$\Phi^a\left(\Phi\left(\Omega, x^k, x^{k'}\right), x^k, x^{k'}\right) = \Phi^a\left(\Omega, x^k, x^{k'}\right) \tag{2.68}$$

$$\Phi^a\left(F\left(\Omega, x^k, x^{k'}\right), x^{k'}, x^k\right) = \Omega^a \tag{2.69}$$

The coordinate transformation law $\Phi\left(\Omega, x^k, x^{k'}\right)$ characterizes the geometric object. In case that the function Φ contains only Ω and the partial derivatives of J^k with respect to x^k, the geometric object is said to be a **differential geometric object**.

Furthermore, we say that a geometric object is linear if for the transformation law $\Phi\left(\Omega, x^k, x^{k'}\right)$ it holds

$$\Phi^a\left(\Omega, x^k, x^{k'}\right) = J_b^a\left(x^k, x^{k'}\right)\Omega^b + C\left(x^k, x^{k'}\right). \tag{2.70}$$

When the transformation law is

$$\Phi^a\left(\Omega, x^k, x^{k'}\right) = J_b^a\left(x^k, x^{k'}\right)\Omega^b \tag{2.71}$$

we say that the geometric object is a **linear homogeneous geometric object**. One important calss of linear homogeneous geometric objects are the **tensors.**

2.7.1 Lie Derivative

Let Ω be a geometric object in A_n with transformation law (2.67) and consider the infinitesimal transformation

$$\bar{x}^i = x^i + \varepsilon\xi^i\left(x^k\right). \tag{2.72}$$

where $\xi = \xi^i\left(x^k\right)$ are the components of the infinitesimal generator. From the transformation law (2.67), the geometric object in the coordinate system $x^{i'} = \bar{x}^i$ is $\Phi\left(\Omega^k, x^k, x^{k'}\right)$. The Lie derivative of the geometric object Ω with respect to ξ is defined as follows

$$L_X\Omega = \lim_{\varepsilon\to 0}\frac{1}{\varepsilon}\left[\Phi\left(\Omega^k, x^k, x^{k'}\right) - \Omega\right]. \tag{2.73}$$

In order for the Lie derivative of a geometric object to be again a geometric object (not necessarily of the same type) it is necessary and sufficient that the geometric object be linear [21].

By definition, the Lie derivative of a geometric object depends on the transformation law. The transformation law (2.67) for functions is $f'\left(x^{i'}\right) = f\left(x^{i'}\right)$, hence, under the point transformation (2.72) we have

$$f'\left(x^{i'}\right) = f\left(x^i + \varepsilon\xi^i\right) = f\left(x^i\right) + \varepsilon f_{,i}\xi^i + O\left(\varepsilon^2\right).$$

From (2.73) the Lie derivative of functions f along the vector field ξ is computed to be

$$L_\xi f = f_{,i}\xi^i = \xi\left(f\right).$$

The transformation law for a tensor field T of rank (r,s) is

$$T^{i'_1\ldots i'_r}_{j'_i\ldots j'_s} = J^{i'_1}_{i_1}\ldots J^{i'_r}_{i_r}J^{j_1}_{j'_1}\ldots J^{j'_s}_{j_s}T^{i_1\ldots i_r}_{j_i\ldots j_s}$$

where J is the Jacobian of the transformation. Thus, from (2.73), the Lie derivative of T with respect to ξ is

$$\begin{aligned} L_\xi T^{i_1\ldots i_r}_{j_i\ldots j_s} &= \xi^k T^{i_1\ldots i_r}_{j_i\ldots j_{s,k}} - T^{m\ldots i_r}_{j_i\ldots j_s}\xi^{i_1}_{,m} - T^{i_1 m\ldots i_r}_{j_i\ldots j_s}\xi^{i_2}_m + \ldots \\ &\ldots + T^{i_1\ldots i_r}_{m\ldots j_s}\xi^m_{,j_1} + T^{i_1\ldots i_r}_{j_i m\ldots j_s}\xi^m_{j_2} + \ldots. \end{aligned} \tag{2.74}$$

In case of vector fields X and 1-forms ω expression (2.74) gives

$$L_{\xi}X = \xi^{k}X^{i}_{,k} - X^{k}\xi^{i}_{,k} = [\xi, X] \tag{2.75}$$

$$L_{\xi}\omega = \xi^{k}\omega_{i,k} - \omega_{k}\xi^{k}_{,i}. \tag{2.76}$$

For a second order tensor T_{ij}, expression (2.74) becomes

$$L_{\xi}T_{ij} = T_{ij,k}\xi^{k} + T_{ik}\xi^{k}_{,j} + T_{kj}\xi^{k}_{,i}. \tag{2.77}$$

The connection coefficients Γ^{i}_{jk} are linear differential geometric objects with transformation law

$$\Gamma^{i'}_{j'k'} = J^{i'}_{i}J^{j}_{j'}J^{k}_{k'}\Gamma^{i}_{jk} + \frac{\partial x^{i'}}{\partial x^{r}}\frac{\partial^{2}x^{r}}{\partial x^{j'}\partial x^{k'}}. \tag{2.78}$$

Connection coefficients have different transformation law from tensor fields, that is, the Lie derivative $L_{\xi}\Gamma^{i}_{jk}$ will be different from that of tensors. Applying (2.73),we find that the Lie derivative $L_{\xi}\Gamma^{i}_{jk}$ is expressed as follows

$$L_{\xi}\Gamma^{i}_{jk} = \xi^{i}_{,jk} + \Gamma^{i}_{jk,r}\xi^{r} - \xi^{i}_{,r}\Gamma^{r}_{jk} + \xi^{s}_{,j}\Gamma^{i}_{sk} + \xi^{s}_{,k}\Gamma^{i}_{js}. \tag{2.79}$$

In case of symmetric connection $\Gamma^{i}_{jk} = \Gamma^{i}_{kj}$, condition (2.79) can be written in the equivalent form

$$L_{\xi}\Gamma^{i}_{jk} = \xi^{i}_{;jk} - R^{i}_{jkl}\xi^{l} \tag{2.80}$$

where R^{i}_{jkl} is the curvature tensor and the semicolon ";" means covariant derivative.

Collineations

In section 2.2.2 we gave the conditions under which a function is invariant under a one parameter point transformation. Similarly for linear (homogeneous) differential geometric objects there is the following definition.

Definition 2.7.2 *A linear differential geometric object $\Omega\left(x^{i}\right)$ is invariant under a one parameter point transformation*

$$\bar{x}^{i} = \bar{x}^{i}\left(x^{k}, \varepsilon\right) \tag{2.81}$$

if and only if $\bar{\Omega}\left(\bar{x}^{i}\right) = \Omega\left(x^{i}\right)$ at all points where the one parameter point transformation acts. Equivalently, the Lie derivative of the geometric object Ω with respect to the infinitesimal generator of (2.81) vanishes, that is, $L_{\xi}\Omega = 0$.

A direct result which arises from the definition of the Lie derivative and the transformation law of linear differential geometric objects is that if a linear differential geometric object Ω is invariant under the transformation (2.81), then there exist a coordinate system with respect to which the components of Ω are independent of one of the coordinates.

One can generalize the concept of symmetry in the sense that one does not require the Lie derivative to be equal to zero, but to another tensor. That is, the Lie derivative of the linear differential geometric object Ω with respect to the infinitesimal generator ξ is

$$L_\xi \Omega = \Psi$$

where Ψ has the same numbers of components and symmetries of the indices with Ω. In this case the infinitesimal generator ξ is said to be a **collineation** of Ω, the type of collineations being defined by Ψ. Collineations are a powerful tool in the study of the geometric properties of Riemannian manifolds.

In Riemannian geometry, the geometric object Ω can be the metric tensor g_{ij} or any other geometric object defined from it (e.g. the connection coefficients).

Definition 2.7.3 *All collineations involving geometric objects Ω derived from the metric g_{ij} of a Riemannian space shall be called **geometric collineations**. In particular, the collineation defined by the metric $L_\xi g_{ij}$ is called the **generic collineation** because any other geometric collineation can be written in terms of it. Furthermore, the geometric collineations can be written in terms of the irreducible parts ψ, H_{ij} as follows*

$$L_\xi g_{ij} = 2\psi g_{ij} + 2H_{ij} \tag{2.82}$$

*where the function ψ is called the **conformal factor** and H_{ij} is a **symmetric traceless tensor**.*

The role of the quantities $\psi,\ H_{ij}$ is important because they can be used as the variables in terms of which one can study any geometric collineation. To do that, one has to express the Lie derivative of any metric tensor in terms of the generic symmetry variables $\psi,\ H_{ij}$ and their derivatives.

In the following we shall be interested in geometric collineations, particularly in the collineations of the metric and of the connection coefficients of a Riemannian space.

2.7.2 Motions of Riemannian spaces

Consider an n dimensional Riemannian space V^n with line element

$$ds^2 = g_{ij} dx^i dx^j. \tag{2.83}$$

where g_{ij} is the metric tensor.

Definition 2.7.4 *The point transformation (2.81) is called a motion of V^n, if and only if the line element is invariant under the action of (2.81). Equivalently, the Lie derivative of the metric tensor g_{ij} with respect to the infinitesimal generator ξ of (2.81) vanishes, i.e.*

$$L_\xi g_{ij} = 0. \tag{2.84}$$

The point transformations (2.81) which are motions of V^n form a group named the **group of motions**. Since g_{ij} is a metric, condition (2.84) can be written in the equivalent form

$$L_\xi g_{ij} = 2\xi_{(i;j)} = 0. \tag{2.85}$$

This equation is known as **Killing equation** and ξ is called an **isometry** or **Killing Vector** (KV). The KVs of a metric form a Lie algebra, which is called the **Killing algebra**.

Motions are important in physics. For instance, the Euclidian space admits as motions the group of translations and the group of rotations $T(3) \otimes so(3)$ and this implies the conservation of linear and angular momentum respectively. As a second example in Cosmology the assumption that the universe is isotropic and homogeneous about all points leads to the Friedmann Robertson Walker (FRW) spacetime.

The maximum dimension of the Killing algebra that V^n can admit is given in the following theorem.

Theorem 2.7.5 *If an n dimensional Riemannian space V^n admits a Killing algebra G_{KV}, then, $0 \leq \dim G_{KV} \leq \frac{1}{2}n(n+1)$.*

A Riemannian space which admits a Killing algebra of dimension $\frac{1}{2}n(n+1)$ is called a **maximally symmetric space**. For example, the Euclidian space E^3 and the Minkowski spacetime M^4 are maximally symmetric spaces.

A special class of KVs are the gradient KVs. A KV ξ is called **gradient** iff $\xi_{i;j} = 0$, that is,

$$\xi_{(i;j)} = 0 \text{ and } \xi_{[i;j]} = 0.$$

For every gradient KV ξ, there exists a function S so $S_{,k}g^{ik} = \xi^i$ and $S_{;ij} = 0$.

Theorem 2.7.6 *If V^n admits p gradient KVs (where $p \leq n$) then, V^n is called a p **decomposable space** and in this case there exists a coordinate system in which the line element (2.83) can be written in the form*

$$ds^2 = M_{\alpha\beta}dz^\alpha dz^\beta + h^{AB}\left(y^A\right)dx_A dx_B$$

where $\alpha, \beta = 1, 2.., p$, $A, B = p+1, ...n$ and $M_{a\beta} = diag\left(c_1, c_2, ..., c_p\right)$, $c_1, c_2, ..., c_p$ are constants.

Example 2.7.7 *Compute the KVs of the Euclidian sphere with line element*

$$ds^2 = d\phi^2 + \sin\phi^2 d\theta^2. \tag{2.86}$$

Solution: In order to find the KVs we solve the Killing equation (2.85). This leads to the system of equations

$$\xi^\theta_{,\theta} = 0$$

$$2\xi^\phi_{,\phi} + 2\xi^\theta \sin\theta\cos\theta = 0$$

$$\xi^\theta_{,\phi} + \xi^\phi_{,\theta} - 2\xi^\phi \cot\theta = 0.$$

whose solution are the elements of the $so(3)$ Lie algebra (see example 2.2.8).

The 2D Euclidian sphere (2.86) admits a three dimensional Killing algebra, hence, it is a maximally symmetric space. Moreover, all spaces of constant curvature are maximally symmetric spaces.

Conformal motions

When the point transformation (2.81) does not change the angle between two directions at a point, the transformation (2.81) is called a **conformal motion**. Technically we have the following definition.

Definition 2.7.8 *The infinitesimal generator ξ of the point transformation (2.81) is called* ***Conformal Killing Vector*** *(CKV) if the Lie derivative of the metric g_{ij} with respect to ξ is a multiple of g_{ij}. That is if,*

$$L_{\xi}g_{ij} = 2\psi g_{ij} \tag{2.87}$$

where $\psi = \frac{1}{n}\xi^{k}_{;k}$. In the case where $\psi_{;ij} = 0$, ξ is a special CKV (sp.CKV)[9] *and if ψ =constant, ξ is a Homothetic Killing Vector (HV).*

The CKVs of a metric form a Lie algebra, which is called the c**onformal algebra**, G_{CV}. Obviously the KVs and the homothetic vector are elements of the conformal algebra G_{CV}. If G_{HV} is the algebra of HVs (including the algebra G_{KV} of KVs), then the following theorem applies.

Theorem 2.7.9 *Let V^{n} be an n dimensional Riemannian space, $n \geq 2$, which admits a conformal algebra G_{CK}, a homothetic algebra G_{HV} and a Killing algebra G_{KV}, then*

i) $G_{KV} \subseteq G_{HV} \subseteq G_{CV}$.

ii) for arbitrary n, $G_{H-K} = G_{HV} - G_{HV} \cap G_{KV}$, then $0 \leq \dim G_{H-K} \leq 1$; that is, V^{n} admits at most one HV.

iii) V^{2} admits an infinite dimentional conformal algebra G_{CV},

iv) for $n > 2$, $0 \leq \dim G_{CV} \leq \frac{1}{2}(n+1)(n+2)$.

In the following by the term *proper conformal algebra* we mean the algebra of CKVs which are not HV/KVs. A generalization of theorem 2.7.6 for spaces which admit a gradient CKV is the following.

Theorem 2.7.10 *If V^{n} admits a gradient CKV then there exists a coordinate system in which the line element can be written as follows*

$$ds^{2} = f^{2}(x^{n})\left[dx^{n} + h^{AB}\left(x^{A}\right)dx^{A}dx^{B}\right]$$

where $A, B = 1, 2, ...n-1$. In these coordinates the CKV is $X_{C} = \partial_{x^{n}}$ with conformal factor $\psi = \frac{f_{,x^{n}}}{f}$. In the case in which $f(x^{n}) = \exp(x^{n})$, then $\psi = 1$, hence, X_{C} becomes HV and if $f_{,x^{n}} = 0$, X_{C} becomes KV.

Two metrics g_{ij}, $\bar{g}_{ij}$ are **conformally related** if $\bar{g}_{ij} = N^{2}g_{ij}$ where the function N^{2} is the **conformal factor**. If ξ is a CKV of the metric $\bar{g}_{ij}$ so that $L_{\xi}\bar{g}_{ij} = 2\bar{\psi}\bar{g}_{ij}$, then ξ is also a CKV of the metric g_{ij}, that is $L_{\xi}g_{ij} = 2\psi g_{ij}$ with conformal factor $\psi\left(x^{k}\right)$

$$\psi = \bar{\psi}N^{2} - NN_{,i}\xi^{i}. \tag{2.88}$$

[9] For the conformal factor of a sp.CKV holds $\psi_{;ij} = 0$, that is, $\psi_{,i}$ is a gradient KV. A Riemannian space admits a sp.CKV if and only if admits a gradient KV and a gradient HV [22].

The last relation implies that two conformally related metrics have the same conformal algebra, but with different subalgebras; that is, a KV for one may be proper CKV for the other. This is an important observation which shall be useful later. A special class of conformally related metrics are the conformally flat metrics. A space V^n is conformally flat if the metric g_{ij} of V^n satisfies the relation $g_{ij} = N^2 s_{ij}$ where s_{ij} is the metric of a flat space which has the same signature with g_{ij}.

For conformally flat spaces the following proposition aplies.

Proposition 2.7.11 *Let V^n be an n-dimensional Riemannian space, $n \geq 2$,*

i) If V^n, $n > 2$ is conformally flat then V^n admits a conformal algebra of dimension $\frac{1}{2}(n+1)(n+2)$.

ii) A three dimensional space is conformally flat if the Cotton-York tensor

$$C^{ij} = 2\varepsilon^{ikr}\left(R^j_k - \frac{1}{4}\delta^j_k\right)_{;r}$$

vanishes.

iii) V^n, $n \geq 4$ is conformally flat if the Weyl tensor

$$R_{ijkr} = C_{ijkr} + \frac{2}{n-2}\left(g_{i[k}R_{r]j} - g_{j[k}R_{r]i}\right) - \frac{2}{(n-1)(n-2)} R g_{i[k}g_{r]j}$$

vanishes.

iv) If V^n, $n > 3$ is a maximally symmetric space then V^n is conformally flat.

v) All two dimensional spaces are conformally flat .

A result which will be used in subsequent sections is the following.

Example 2.7.12 (The conformal algebra of the flat space.) *Consider a flat space of dimension $n > 2$ with metric*

$$ds^2 = \varepsilon dt^2 + \delta_{AB}dy^A dy^B \ , \ \varepsilon = \pm 1.$$

The conformal algebra of the space consists of the following vectors

$n-$ gradient KVs,

$$K^1_G = \partial_t \ , \ K^A_G = \partial_A$$

$\frac{n(n-1)}{2}$ non gradient KVs (rotations).

$$X^{1A}_R = y^A\partial_t - \varepsilon t\partial_A$$

$$X^{AB}_R = y^B\partial_A - y^A\partial_B$$

one gradient HV

$$H = t\partial_t + \sum_A y^A\partial_A$$

n special CKVs

$$X_C^1 = \frac{1}{2}\left(t^2 - \varepsilon \sum_A \left(y^A\right)^2\right)\partial_t + t\sum_A y^A \partial_A$$

$$X_C^A = ty^A\partial_t + \frac{1}{2}\left(\varepsilon t^2 + \left(y^A\right)^2 - \sum_{B\neq A}\left(y^B\right)^2\right)\partial_A + y^A \sum_{B\neq A} y^B \partial_B$$

where $y^A = 1...n-1$ *with conformal factor* $\psi_C^1 = t$ *and* $\psi_C^A = y^A$.

For $n > 2$ *the flat space does not admit proper (non special) CKVs.*

For $n = 2$ *the vector field*

$$X = \left[f_1\left(t + i\sqrt{\varepsilon}x\right) - f_2\left(t - i\sqrt{\varepsilon}x\right) + c_0\right]\partial_t + i\sqrt{\varepsilon}\left[f_1\left(t + i\sqrt{\varepsilon}x\right) + f_2\left(t - i\sqrt{\varepsilon}x\right)\right]\partial_x$$

is the generic CKV, that is, it includes the KVs, the HV, and the CKVs.

2.7.3 Symmetries of the connection

Let ξ be the generator of an infinitesimal transformation of (2.81). In a Riemannian space with metric g_{ij} we have the identity

$$L_\xi \Gamma^i_{jk} = g^{ir}\left[\left(L_\xi g_{rk}\right)_{;j} + \left(L_\xi g_{rj}\right)_{;k} - \left(L_\xi g_{jk}\right)_{;r}\right]. \tag{2.89}$$

If ξ is a HV or KV then from (2.89) follows that $L_\xi \Gamma^i_{jk}$ vanishes, which implies that the Γ^i_{jk} are invariant under the action of transformation (2.81).

Definition 2.7.13 *The infinitesimal generator* ξ *of the point transformation (2.81) caries a geodesic into a geodesic and also preserves the affine parameter iff the Lie derivative of connection coefficients* Γ^i_{jk} *with respect to* ξ *vanishes, that is iff*

$$L_\xi \Gamma^i_{jk} = 0 \tag{2.90}$$

The infinitesimal generator ξ *is called an* ***Affine Killing*** *vector or* ***Affine collineation*** *(AC).*

ACs of V^n form a Lie algebra, which is called **Affine algebra**, G_{AC}. Obviously the homothetic algebra[10] G_{HV}, is a subalgebra of G_{AC}, $G_{HV} \subseteq G_{AC}$. We shall say that a spacetime admits proper ACs when $\dim G_{HV} \prec \dim G_{AC}$.

For instance, in the case of flat space condition (2.90) becomes $\xi^i_{,jk} = 0$, therefore, the general solution is $\xi^i = A^i_j x^j + B^i$ where A^i_j, B^i are $n\left(n+1\right)$ constants. Therefore the flat space admits a $n\left(n+1\right)$ dimensional Affine algebra (including the KVs and the HV). We have the inverse result.

Theorem 2.7.14 *If an* n *dimensional Riemannian space* V^n *admits an Affine algebra* G_{AC} *and* $\dim G_{AC} = n\left(n+1\right)$, *then* V^n *is a flat space.*

[10] Note that the proper CKVs do not satisfy condition (2.90).

A generalization of affine symmetry which is of interest is the Projective collineation.

Definition 2.7.15 *The infinitesimal generator ξ of the point transformation (2.81) is called a* ***Projective Collineation*** *(PC) if there exists a function*[11] $\phi\left(x^k\right)$ *such that*

$$L_\xi \Gamma^i_{jk} = \phi_{,j}\delta^i_k + \phi_{,k}\delta^i_j \tag{2.91}$$

or equivalently

$$\xi_{(i;j);k} = 2g_{ij}\phi_{,k} + 2g_{k(i}\phi_{,j)}.$$

The function ϕ is called the ***projective function****. If the projective function satisfies the condition $\phi_{;ij} = 0$, then we say that ξ is a* ***special PC*** *(sp.PC). Projective transformations transform the system of geodesics (auto parallel curves) of V^n into the same system but they do not preserve the affine parameter.*

The PCs of V^n form a Lie algebra which is called **Projective algebra**, G_{PC}. The affine algebra G_{AC} is a subalgebra of G_{PC}, $G_{AC} \subseteq G_{PC}$.

In flat space condition (2.91) gives the generic projective collineation

$$\xi_i = A_{ij}x^j + \left(B_j x^j\right)x_i + C_i$$

where A_{ij}, B_j,C_i are $n\,(n+2)$ constants.

Theorem 2.7.16 *If an n dimensional Riemannian space V^n admits a projective algebra G_{PC}, then* $\dim G_{PC} \leq n(n+2)$*. In case* $\dim G_{PC} = n\,(n+2)$ *then V^n is a maximally symmetric space [23].*

Some useful propositions for the existence of sp. PCs are the following [23, 24, 25].

Proposition 2.7.17 *Let V^n be an n dimensional Riemannian space, then*

i) If V^n admits a $p \leq n$ dimensional Lie algebra of sp.PCs then also admits p gradient KVs and a gradient HV and if $p = n$ the space is flat (the reverse also holds true).

ii) A maximally symmetric space which admits a proper AC or a sp.PC is a flat space.

A Riemannian space is possible to admit more general collineations, e.g. Curvature collineations. A full classification of the collineations of a Riemannian space (with definite or indefinite metric) can be found in [26]. A summary of the above definitiosn is given in Table 2.1.

We note that the Lie symmetries $(sl\,(3,R))$ of the free particle form the projective algebra of the two dimensional Euclidian space. Therefore, the natural question which arises is the following:

Is there any connection between the Lie symmetries of differential equations of second order and collineations?

In the following chapters this will be confirmed and will be used to apply the Lie symmetries of DEs in various areas of Geometry and Physics.

[11] In general, ξ is PC if there exists a one form ω_i such that $L_\xi\Gamma^i_{jk} = 2\omega_{(j}\delta^i_{k)}$. In a Riemannian space, the one form ω_i is necessarily closed, that is, there exist a function ϕ such as $\omega_i = \phi_{,i}$.

Table 2.1: Collineations of a Riemannian space

Collineation $\mathcal{L}_{\xi}\mathbf{A}=\mathbf{B}$	**A**	**B**
Killing Vector (KV)	g_{ij}	0
Homothetic vector (HV)	g_{ij}	$2\psi g_{ij},\ \psi_{,i}=0$
Conformal Killing vector (CKV)	g_{ij}	$2\psi g_{ij},\ \psi_{,i}\neq 0$
Affine Collineation (AC)	Γ^{i}_{jk}	0
Proj. Collineations (PC)	Γ^{i}_{jk}	$2\phi_{(,j}\delta^{i}_{k)},\ \phi_{,i}\neq 0$
Sp. Proj. collineation (sp.PC)	Γ^{i}_{jk}	$2\phi_{(,j}\delta^{i}_{k)},\ \phi_{;jk}=0$

Part II

Symmetries of ODEs

Chapter 3

Lie symmetries of geodesic equations

3.1 Introduction

In a Riemannian space the affinely parameterized geodesics are determined uniquely by the metric. Therefore one should expect a close relation between the geodesics as a set of homogeneous ordinary differential equations (ODEs) linear in the highest order term and quadratic non-linear in first order terms, and the metric as a second order symmetric tensor. A system of such ODEs is characterized (perhaps not fully) by its Lie symmetries and a metric by its collineations. Therefore it is reasonable to expect that the Lie symmetries of the system of geodesics of a metric will be closely related with the collineations of the metric. That such a relation exists it is easy to see by the following simple example. Consider on the Euclidian plane a family of straight lines parallel to the $x-$axis. These curves can be considered either as the integral curves of the ODE $\frac{d^2y}{dx^2}=0$ or as the geodesics of the Euclidian metric dx^2+dy^2. Subsequently consider a symmetry operation defined by a reshuffling of these lines without preserving necessarily their parametrization. According to the first interpretation this symmetry operation is a Lie symmetry of the ODE $\ddot{y}=0$ and according to the second interpretation it is a (special) projective symmetry of the Euclidian two dimensional metric.

What has been said for a Riemannian space can be generalized to a space in which there is only a linear connection. In this case the geodesics are called autoparallels (or paths) and they comprise again a system of ODEs linear in the highest order term and quadratic non-linear in the first order terms. In this case one is looking for relations between the Lie symmetries of the autoparallels and the projective or affine collineations of the connection.

The above matters have been discussed extensively in a series of interesting papers. Classic is the work of Katzin and Levin [27, 28, 29]. Important contributions have also been done by Aminova [30, 31, 32, 33, 34], Prince and Crampin [35] and many others. More recent is the work of Feroze et al [36]. In [36] they have considered the KVs of the metric and their relation to the Lie symmetries of the system of affinely parameterized geodesics of maximally symmetric spaces of low dimension. In the same paper a conjecture is made, which essentially says that the maximally symmetric spaces of non-vanishing curvature do not admit further Lie symmetries.

In the following we consider the set of autoparallels - not necessarily affinely parameterized - of a symmetric connection. We find that the major symmetry condition relates the Lie symmetries with the special projective symmetries of the connection. A similar result has been obtained by Prince and Crampin in [35] using the bundle formulation of second order ODEs.

Furthermore, because the geodesic equations follow from the variation of the geodesic Lagrangian defined by the metric and due to the fact that the Noether symmetries are a subgroup of the Lie group of Lie symmetries of these equations, one should expect a relation / identification of the Noether symmetries of this Lagrangian with the projective collineations of the metric or with its degenerates. Recent work in this direction has been by Bokhari et all [37, 38] in which the relation of the Noether symmetries with the KVs of some special spacetimes is discussed.

In section 3.2 we derive the Lie symmetry conditions for a general system of second order ODE polynomial in the first derivatives. In section 3.4 we apply these conditions in the special case of Riemannian spaces and in Theorem 3.4.1 we give the Lie symmetry vectors in terms of the special projective collineations of the metric and their degenerates. In section 3.4.1 we give the second result of this work, that is Theorem 3.4.2, which relates the Noether symmetries of the geodesic Lagrangian defined by the metric with the homothetic algebra of the metric and comment on the results obtained so far in the literature. Finally in section 3.5 we apply the results to various cases and eventually we give the Lie symmetries, the Noether symmetries and the associated conserved quantities of Einstein spaces, the Gödel spacetime, the Taub spacetime and the Friedman Robertson Walker spacetimes.

3.2 The Lie symmetry conditions in an affine space

We consider the system of ODEs:

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k + \sum_{m=0}^{n} P^i_{j_1\ldots j_m}\dot{x}^{j_1}\ldots\dot{x}^{j_m} = 0 \tag{3.1}$$

where Γ^i_{jk} are the connection coefficients of the space and $P^i_{j_1\ldots j_m}(t,x^i)$ are smooth functions completely symmetric in the lower indices and derive the Lie point symmetry conditions in geometric form using the standard approach. Equation (3.1) is quite general and covers most of the standard cases autonomous and non autonomous equations. For instance for all $\mathbf{P}=0$ equation (3.1) becomes

$$\ddot{x} + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = 0 \tag{3.2}$$

which are the geodesic equations. In case $P^i{=}F^i$ and $P^i_{j_1\ldots j_m}=0$ equation (3.1) becomes

$$\ddot{x} + \Gamma^i_{jk}\dot{x}^j\dot{x}^k + F^i\left(t,x^i\right) = 0 \tag{3.3}$$

which are the equations of motions of a particle in a curved space under the action of the force **F**. Furthermore because the Γ^i_{jk}'s are not assumed to be symmetric, the results are valid in a space with torsion. Obviously they hold in a Riemannian space provided the connection coefficients are given in terms of the Christofell symbols.

Following, the standard procedure (see e.g. [1, 3]) we find that the Lie symmetry conditions for the values of $m \leq 4$ are[1]

$$L_\eta P^i + 2\xi_{,t} P^i + \xi P^i{}_{,t} + \eta^i{}_{,tt} + \eta^j{}_{,t} P^i_{.j} = 0 \tag{3.4}$$

$$L_\eta P^i_j + \xi_{,t} P^i_j + \xi P^i_{j,t} + \left(\xi_{,k}\,\delta^i_j + 2\xi_{,j}\,\delta^i_k\right) P^k + 2\eta^i{}_{,t|j} - \xi_{,tt}\,\delta^i_k + 2\eta^k{}_{,t} P^i_{.jk} = 0 \tag{3.5}$$

$$L_\eta P^i_{jk} + L_\eta \Gamma^i_{jk} + \left(\xi_{,d}\,\delta^i_{(k} + \xi_{,(k}\,\delta^i_{|d|}\right) P^d_{.j)} + \xi P^i_{.kj,t} - 2\xi_{,t(j}\,\delta^i_{k)} + 3\eta^d{}_{,t} P^i_{.dkj} = 0 \tag{3.6}$$

$$L_\eta P^i_{.jkd} - \xi_{,t} P^i_{.jkd} + \xi_{,e}\,\delta^i_{(k} P^e_{.dj)} + \xi P^i_{.jkd,t} + 4\eta^e{}_{,t} P^i_{.jkde} - \xi_{(,j|k}\delta^i_{d)} = 0 \tag{3.7}$$

and for $(\dot{x})^l, l \;\geq 4$

$$\begin{aligned} &L_\eta P^i_{j_1 \ldots j_m} + P^i_{j_1\ldots j_m}{}_{,t}\xi + (2-m)\,\xi_{,t} P^i_{j_1\ldots j_m} + \\ &+ \xi_{,r}\left(2-(m-1)\right) P^i_{j_1\ldots j_{m-1}}\delta^r_{j_m} + (m+1)\, P^i_{j_1\ldots j_{m+1}}\eta^{j_{m+1}}_{,t} + \xi_{,j} P^j_{j_1\ldots j_{m-1}}\delta^i_{j_m} = 0. \end{aligned} \tag{3.8}$$

From the above general relations it is possible to extract the Lie symmetry conditions for the various values of functions **P**. For example the Lie symmetry conditions for the geodesic equations (3.2) are as follows.

$$\eta^i{}_{,tt} = 0 \tag{3.9}$$

$$2\eta^i{}_{,t|j} - \xi_{,tt}\,\delta^i_k = 0 \tag{3.10}$$

$$L_\eta \Gamma^i_{jk} - 2\xi_{,t(j}\,\delta^i_{k)} = 0 \tag{3.11}$$

$$\xi_{(,j|k}\delta^i_{d)} = 0. \tag{3.12}$$

Moreover, the symmetry conditions for the system (3.3) are

$$L_\eta F^i + 2\xi_{,t} F^i + \xi F^i{}_{,t} + \eta^i{}_{,tt} = 0 \tag{3.13}$$

$$\left(\xi_{,k}\,\delta^i_j + 2\xi_{,j}\,\delta^i_k\right) F^k + 2\eta^i{}_{,t|j} - \xi_{,tt}\,\delta^i_k = 0 \tag{3.14}$$

$$L_\eta \Gamma^i_{jk} - 2\xi_{,t(j}\,\delta^i_{k)} = 0 \tag{3.15}$$

$$\xi_{(,j|k}\delta^i_{d)} = 0. \tag{3.16}$$

In the same manner we work for more terms of **P**. We note the appearance of the term $L_\eta \Gamma^i_{jk}$ in these expressions.

[1] The detailed calculations are given in Appendix 3.A.

3.3 Lie symmetries of autoparallel equation

Consider a C^p manifold M of dimension n, endowed with a Γ^i_{jk} symmetric[2] connection. In a local coordinate system $\{x^i : i = 1, \ldots, n\}$ the connection $\Gamma^i_{jk}\partial_i = \nabla_j \partial_k$ and the **autoparallels** of the connection are defined by the requirement

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k + \phi(t)\,\dot{x}^i = 0 \tag{3.17}$$

where t is a parameter along the paths. When $\phi(t)$ vanishes, we say that the autoparallels are **affinely parameterized** and in this case t is called an **affine parameter**, that is one has (3.2). Consider the infinitesimal transformation

$$\bar{t} = t + \varepsilon\xi\left(t, x^k\right)\ ,\quad \bar{x}^i = x^i + \varepsilon\eta^i\left(t, x^k\right) \tag{3.18}$$

with infinitesimal generator

$$X = \xi\left(t, x^k\right)\partial_t + \eta^i\left(t, x^k\right)\partial_t. \tag{3.19}$$

The autoparallels (3.17) are invariant under transformation (3.18) if

$$X^{[2]}\left(\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k + \phi(t)\,\dot{x}^i\right) = 0 \tag{3.20}$$

where $X^{[2]}$ is the second prolongation of (3.19). For $P^i_{j_1\cdots j_m} = 0$ for $m \neq 0$ and $P^i_{j_1} = \phi(t)$, $\Gamma^i_{jk}\left(x^k\right) + P^i_{jk}\left(t, x^k\right) = \Gamma^i_{jk}\left(t, x^k\right)$ and from conditions (3.4)-(3.8) we have the Lie symmetry conditions for the autoparallel equations (3.17)

$$\eta^i_{,tt} + \eta^i_{,t}\phi = 0 \tag{3.21}$$

$$\xi_{,tt}\delta^i_j - \xi\phi_{,t}\delta^i_j - 2\left[\eta^i_{,tj} + \eta^k_{,t}\Gamma^i_{(kj)}\right] - \phi\xi_{,t}\delta^i_j = 0 \tag{3.22}$$

$$\mathcal{L}_{\boldsymbol{\eta}}\Gamma^i_{(jk)} = -2\phi\xi_{(,j}\delta^i_{k)} + \xi\Gamma^i_{(kj),t} + 2\xi_{,t(j}\delta^i_{k)} \tag{3.23}$$

$$\xi_{(,j|k}\delta^i_{d)} = 0\ . \tag{3.24}$$

Define the quantity

$$\Phi = \xi_{,t} - \phi\xi. \tag{3.25}$$

Then condition (3.23) is written (note that $\phi_{,i} = 0$):

$$L_\eta\Gamma^i_{(jk)} = 2\Phi_{(,j}\delta^i_{k)} - \xi\Gamma^i_{(kj),t}. \tag{3.26}$$

If we consider the vector ξ=$\xi\partial_t$ (which does not have components along ∂_i) we compute

$$L_{\boldsymbol{\xi}}\Gamma^i_{(jk)} = \xi\Gamma^i_{(kj),t}$$

[2] The coefficients Γ^i_{jk} in general are not symmetric in the lower indices. In the autoparallel equation (3.17) the antisymmetric part of $\Gamma^i_{[jk]}$ (the torsion) does not play a role.

hence (3.26) is written

$$L_X\Gamma^i_{(jk)} = 2\Phi_{(,j}\delta^i_{k)}. \tag{3.27}$$

We observe that this condition is the condition for a projective collineation of the connection $\Gamma^i_{(jk)}$ along the symmetry vector X and with projecting function Φ. Concerning the other conditions we note that (3.22) can be written in covariant form as follows:

$$\Phi_{,t}\delta^i_j - 2\eta^i_{,t|j} = 0 \tag{3.28}$$

where $\eta^i_{,t|j} = \eta^i_{,tj} + \eta^k_{,t}\Gamma^i_{(kj)}$ is the covariant derivative with respect to $\Gamma^i_{(kj)}$ of the vector $\eta^i_{,t}$. Condition (3.24) implies that $\xi_{,i}$ is a gradient KV of the metric of the space x^i. Condition (3.21) is obviously in covariant form with respect to the index i.

We arrive at the following conclusion.

a) The conditions for the Lie symmetries of the autoparallel equations (3.17) are covariant equations because if we consider the connection in the augmented $n+1$ space $\{x^i, t\}$, all components of Γ which contain an index along the direction of t vanish, therefore the partial derivatives with respect to t can be replaced with covariant derivatives with respect to t.

b) Equation (3.21) gives the functional dependence of η^i on t and the non-affine parametrization function $\phi(t)$.

c) Equation (3.24) gives that the vector $\xi_{,i}$ is a gradient Killing vector of the $n-$dimensional space $\{x^i\}$.

d) Equation (3.22) relates the functional dependence of η^i and ξ in terms of t.

e) Equation (3.23) is the most important equation for our purpose, because it states that *the symmetry vector (3.18) is an affine collineation in the jet space* $\{t, x^i\}$ because it preserves both the geodesics and their affine parameter. In the space $\{x^i\}$ the vectors $\eta^i(t,x)\partial_{x^i}$ *are projective collineations* because they preserve the geodesics but not necessarily their parametrization.

In the following we restrict our considerations to the case of Riemannian connections that is the Γ^i_{jk} are symmetric and the covariant derivative of the metric vanishes.

3.4 Lie and Noether symmetries of geodesic equations

We compute the Lie symmetry vectors of geodesics equations (3.2) with affine parametrization; that is, we assume $\phi = 0$ and $\Gamma^i_{jk,t} = 0$. The later is a reasonable assumption because the $\Gamma^{i\prime}_{jk}$'s are computed in terms of the metric which does not depend on the parameter t. Under these assumptions the symmetry conditions for the geodesics equations (3.2) are (3.9)-(3.12). We proceed with the solution of this system of equations[3].

Equation (3.9) implies

$$\eta^i(t,x) = A^i(x)t + B^i(x) \tag{3.29}$$

[3] See also [35] Table II.

where $A^i(x), B^i(x)$ are arbitrary differentiable vector fields.

The solution of equation (3.12) is

$$\xi(t,x) = C_J(t)S^J(x) + D(t) \tag{3.30}$$

where $C_J(t), D(t)$ are arbitrary functions of the affine parameter t and $S^J(x)$ is a function whose gradient is a gradient KV, i.e.

$$S^J(x)_{|(i,j)} = 0. \tag{3.31}$$

The index J runs through the number of gradient KVs of the metric. Condition (3.10) gives

$$2A(x)^i_{|j} = \left[C_J(t)_{,tt}\, S^J(x) + D(t)_{,tt}\right]\delta^i_j. \tag{3.32}$$

Because the left hand side is a function of x only, we must have

$$D(t)_{,tt} = M \Rightarrow D(t) = \frac{1}{2}Mt^2 + Kt + L \ \text{ where } M, K, L \text{ constants} \tag{3.33}$$

$$C_J(t)_{,tt} = G_J = \text{constant } \Rightarrow C_J(t) = \frac{1}{2}G_J t^2 + E_J t + F_J \text{ where } G_J, E_J, F_J \text{ constants.} \tag{3.34}$$

Replacing in (3.32) we find

$$2A(x)^i_{;j} = \left(G_J S^J(x) + M\right)\delta^i_j \Rightarrow A(x)_{i;j} = \frac{1}{2}\left(G_J S^J(x) + M\right) g_{ij} \tag{3.35}$$

where we have lowered the index because the connection is metric (i.e. $g_{ij|k} = 0$). The last equation implies that the vector $A(x)^i$ is a conformal Killing vector with conformal factor $\psi = \frac{1}{2}(G_J S^J(x) + M)$. Because $A(x)_{[i;j]} = 0$ this vector is a gradient CKV.

We continue with condition (3.11) and replace $\eta^i(t,x)$ from (3.29)

$$L_A\Gamma^i_{jk}t + L_B\Gamma^i_{jk} = 2\xi_{,t\,(,j}\delta^i_{k)} = 2\left[\left(G_J t + E_J\right)S^J(x) + Mt + K\right]_{|(j}\delta^i_{k)} = 2\left(G_J t + E_J\right)S^J(x)_{|(j}\delta^i_{k)} \Rightarrow$$

$$L_A\Gamma^i_{jk} = 2G_J S^J(x)_{,(j}\delta^i_{k)} \tag{3.36}$$

$$L_B\Gamma^i_{jk} = 2E_J S^J(x)_{,(j}\delta^i_{k)}. \tag{3.37}$$

The last two equations imply that the vectors $A^i(x)$, $B^i(x)$ are *special projective collineations* or *affine collineations* of the metric - or one of their specializations - with projective functions $G_J S^J(x)$ and $E_J S^J(x)$ or zero respectively. Note that relations (3.36), (3.37) remain true if we add a KV to the vectors $A^i(x)$, $B^i(x)$, therefore these vectors are determined up to a KV[4].

It is well known that in a Riemannian space a CKV K^i with conformal factor $\psi(x^i)$ satisfies the identity:

$$L_K\Gamma^i_{(jk)} = g^{is}\left[\psi_{,j}g_{ks} + \psi_{,k}g_{js} - \psi_{,s}g_{jk}\right]. \tag{3.38}$$

[4]Because A^i is a projective collineation and a CKV it must be a HV.

Applying this identity to the CKV A^i we find:

$$G_J S^J(x)_{,k} = 0 \Rightarrow G_J S^J(x) = 2\rho = \text{constant}. \tag{3.39}$$

This implies that A^i *is a gradient HV (not necessarily proper) with homothetic factor* $\rho + \frac{1}{2}M$. Furthermore (3.35) implies:

$$\begin{aligned} 2A^i &= (2\rho + M)x^i + 2L^i \Rightarrow \\ A^i &= (\rho + \frac{1}{2}M)x^i + L^i \end{aligned} \tag{3.40}$$

where L^i is a non-gradient KV.

We continue with the special projective collineation vector B^i. For this vector we use the property that for a symmetric connection the following identity, holds

$$\mathcal{L}_{\mathbf{B}}\Gamma^i_{(jk)} = B^i_{;jk} - R^i_{jkl}B^l.$$

Replacing the left hand side from (3.37) we find

$$B^i_{;jk} - R^i_{jkl}B^l = 2E_J S^J(x)_{,(j}\delta^i_{k)}. \tag{3.41}$$

Contracting the indices i, j we find

$$\left(B^i_{;i} - (n+1)E_J S^J(x)\right)_{;k} = 0 \tag{3.42}$$

which implies

$$B^i_{;i} = (n+1)E_J S^J(x) + 2b \tag{3.43}$$

where b =constant. In case this vector is an affine collineation then $B^i_{;i} = 2b$. Using the above results we find for $\xi(t,x)$

$$\begin{aligned} \xi(t,x) &= C_J(t)S^J + D(t) \\ &= \left(\frac{1}{2}G_J t^2 + E_J t + F_J\right) S^J + \frac{1}{2}Mt^2 + Kt + L \\ &= \frac{1}{2}(G_J S^J + M)t^2 + (E_J S^J + K)t + F_J S^J + L \end{aligned}$$

We summarize the above results in the following Theorem.

Theorem 3.4.1 *The Lie symmetry vector*

$$X = \xi(t,x)\partial_t + \eta^i(t,x)\partial_{x^i}$$

of the equation of geodesics (3.2) in a Riemannian space is generated from the elements of the special projective algebra as follows.

Case A. The metric admits gradient KVs. Then

a. The function

$$\xi(t,x)=\frac{1}{2}\left(G_J S^J+M\right)t^2+\left[E_J S^J+K\right]t+F_J S^J+L, \tag{3.44}$$

where G_J, M, b, K, F_J and L are constants and the index J runs along the number of gradient KVs

b. The vector

$$\eta^i(t,x)=A^i(x)t+B^i(x)+D^i(x) \tag{3.45}$$

where the vector $A^i(x)$ is a gradient HV with conformal factor $\psi=\frac{1}{2}\left(G_J S^J+M\right)$ (if it exists), $D^i(x)$ is a non-gradient KV of the metric and $B^i(x)$ is either a special projective collineation with projection function $E_J S^J(x)$ or an AC and $E_J=0$ in (3.44).

Case B. The metric does not admit gradient KVs. Then

a. The function

$$\xi(t,x)=\frac{1}{2}Mt^2+Kt+L \tag{3.46}$$

b. The vector

$$\eta^i(t,x)=A^i(x)t+B^i(x)+D^i(x), \tag{3.47}$$

where $A^i(x)$ is a gradient HV with conformal factor $\psi=\frac{1}{2}M$, $D^i(x)$ is a non-gradient KV of the metric and $B^i(x)$ is an AC. If in addition the metric does not admit a gradient HV, then

$$\xi(t)=Kt+L \tag{3.48}$$

$$\eta^i(x)=B^i(x)+D^i(x). \tag{3.49}$$

3.4.1 Noether symmetries and conservation laws

In a Riemannian space the equations of geodesics (3.2) are produced from the geodesic Lagrangian:

$$L=\frac{1}{2}g_{ij}\dot{x}^i\dot{x}^j\ . \tag{3.50}$$

The infinitesimal generator

$$X=\xi\left(t,x^k\right)\partial_t+\eta^i\left(t,x^k\right)\partial_{x^i} \tag{3.51}$$

is a Noether symmetry of Lagrangian (3.50) if there exists a smooth function $f(t,x^i)$ such that

$$X^{[1]}L+\frac{d\xi}{dt}L=\frac{df}{dt} \tag{3.52}$$

where

$$X^{[1]}=\xi\left(t,x^k\right)\partial_t+\eta^i\left(t,x^k\right)\partial_{x^i}+\left(\eta^i_{[1]}\right)\partial_{\dot{x}^i}$$

is the first prolongation of **X**. We compute

$$X^{[1]}L=\frac{1}{2}\left(\eta^k g_{ij,k}\dot{x}^i\dot{x}^j+2\frac{d\eta^k}{dt}g_{ik}\dot{x}^i-2\dot{x}^i\dot{x}^j\frac{d\xi}{dt}g_{ij}\right).$$

Replacing the total derivatives in the rhs

$$\frac{d\xi}{dt} = \xi_{,t} + \dot{x}^k \xi_{,k}$$
$$\frac{d\eta^i}{dt} = \eta^i_{,t} + \dot{x}^k \eta^i_{,k}$$

we find that

$$X^{[1]}L = \frac{1}{2}\left(\begin{array}{c} \eta^k g_{ij,k}\dot{x}^i\dot{x}^j + 2\eta^i_{,t}g_{ij}\dot{x}^j + \eta^i_{,r}g_{ik}\dot{x}^k\dot{x}^r + \\ +\eta^i_{,r}g_{kj}\dot{x}^k\dot{x}^r - 2\xi_{,t}g_{ij}\dot{x}^i\dot{x}^j - 2\xi_{,k}g_{ij}\dot{x}^i\dot{x}^j\dot{x}^k \end{array}\right).$$

The term

$$\frac{d\xi}{dt}L = \frac{1}{2}\left(\xi_{,t} + \dot{x}^k\xi_{,k}\right) g_{ij}\dot{x}^i\dot{x}^j.$$

Finally the Noether symmetry condition (3.52) is

$$-2f_{,t} + \left[2\eta^i_{,t}g_{ij} - 2f_{,i}\right]\dot{x}^j - \xi_{,k}g_{ij}\dot{x}^i\dot{x}^j\dot{x}^k + \left[\eta^k g_{ij,k} + \eta^k_{,i}g_{ik} + \eta^k_{,i}g_{kj} - g_{ij}\xi_{,t}\right]\dot{x}^i\dot{x}^j = 0.$$

This relation is an identity hence the coefficient of each power of $\dot{x}^j$ must vanish. This results in the equations:

$$\xi_{,k} = 0 \tag{3.53}$$

$$L_\eta g_{ij} = 2\left(\frac{1}{2}\xi_{,t}\right) g_{ij} \tag{3.54}$$

$$\eta^i_{,t} g_{ij} = f_{,i} \tag{3.55}$$

$$f_{,t} = 0 \tag{3.56}$$

Condition (3.53) gives $\xi_{,k} = 0 \Rightarrow \xi = \xi\,(t)$.

Condition (3.56) implies $f\left(x^k\right)$ and then condition (3.53) gives that η^i is of the form:

$$\eta_i = f_{,i}t + K_i(x^j). \tag{3.57}$$

Then from (3.54) follows that $\xi_{,t}$ must be at most linear in t. Hence $\xi(t)$ must be at most a function of t^2. Furthermore from (3.54) follows that η^i is at most a CKV with conformal factor $\psi_H = \frac{1}{2}(At + B)$, where A, B are constants. We consider various cases.

Case 1: Suppose ξ =constant=C_1. Then η^i is a KV of the metric which is independent of t. This implies that either $f_{,i} = 0$ and f =constant $= A = 0$ or that $f_{,i}$ is a gradient KV. In this case the Noether symmetry vector is:

$$X^i = C_1\partial_t + g^{ij}\left(f_{,j}t + K_j(x^r)\right),$$

where K^i is a non-gradient KV of g_{ij}.

Case 2: Suppose $\xi = 2t$. Then η^i is a HV of the metric g_{ij} with homothetic factor 1. Then $\eta_i = H_i(x^j)$, $f_{,i} = 0 \Rightarrow f$ =constant= 0 where H^i is a HV of g_{ij} with homothetic factor ψ, not necessarily a gradient HV. In this case the Noether symmetry vector is:

$$X^i = 2\psi t\partial_t + H^i(x^r).$$

Case 3: $\xi(t) = t^2$. Then η^i is a HV of the metric g_{ij} (the variable t cancels) with homothetic factor 1. Again $f_{,i}$ is a gradient HV with homothetic factor ψ and the Noether symmetry vector is

$$X^i = \psi t^2\partial_t + g^{ij}f_{,j}t.$$

Therefore we have the result.

Theorem 3.4.2 *The Noether Symmetries of the geodesic Lagrangian follow from the KVs and the HV of the metric g_{ij} as follows:*

$$\begin{aligned} X &= \left(C_3\psi t^2 + 2C_2\psi t + C_1\right)\partial_t + \\ &+ \left[C_J S^{J,i} + C_I KV^{Ii} + C_{IJ}tS^{J,i} + C_2 H^i + C_3 t(GHV)^i\right]\partial_i \end{aligned} \tag{3.58}$$

with corresponding gauge function

$$f(x^k) = C_1 + C_2 + C_I + C_J + \left[C_{IJ}S^J\right] + C_3\left[GHV\right], \tag{3.59}$$

where $S^{J,i}$ are the C_J gradient KVs, KV^{Ii} are the C_I non-gradient KVs, H^i is a HV not necessarily gradient and $(GHV)^i$ is the gradient HV (if it exists) of the metric g_{ij}.

The importance of Theorems 3.4.1 and 3.4.2 are that one is able to compute the Lie symmetries and the Noether symmetries of the geodesic equations in a Riemannian space by computing the corresponding collineation vectors avoiding the cumbersome formulation of the Lie symmetry method. It is also possible to use the inverse approach and prove that a space does not admit KVs, HVs, ACs and special PCs by using the calculational approach of the Lie symmetry method (assisted with algebraic manipulation programmes) and avoid the hard approach of Differential Geometric methods. In Section 3.5 we demonstrate the use of the above results.

Noether Integrals of geodesic equations

We know that, if the infinitesimal generator (3.51) is a Noether symmetry with Noether function f, then the quantity:

$$\phi = \xi\left(\dot{x}^i\frac{\partial L}{\partial \dot{x}^i} - L\right) - \eta^i\frac{\partial L}{\partial \dot{x}^i} + f \tag{3.60}$$

is a First Integral of L which satisfies $X\phi = 0$, $\frac{d\phi}{dt} = 0$. For the Lagrangian defined by the metric g_{ij}, i.e. $L = \frac{1}{2}g_{ij}\dot{x}^i\dot{x}^j$, we compute:

$$\phi = \frac{1}{2}\xi g_{ij}\dot{x}^i\dot{x}^j - g_{ij}\eta^i\dot{x}^j + f. \tag{3.61}$$

In (3.58) we have computed the generic form of the Noether symmetry and the associated Noether function for this Lagrangian. Substituting into (3.61) we find the following expression for the generic First Integral of the geodesic equations:

$$\begin{aligned}\phi &= \frac{1}{2}\left[C_3\psi t^2 + 2C_2\psi t + C_1\right] g_{ij}\dot{x}^i\dot{x}^j \\ &+ \left[C_J S^{J,i} + C_I KV^{Ii} + C_{IJ} t S^{J,i} + C_2 H^i(x^r) + C_3 t (GHV)^{,i}\right] g_{ij}\dot{x}^j \\ &+ C_1 + C_2 + C_I + C_J + \left[C_{IJ} S^J\right] + C_3\left[GHV\right]. \end{aligned} \tag{3.62}$$

From the generic expression we obtain the following first integrals[5]

$$C_I \neq 0 : \phi_{C_I} = KV_i^I \dot{x}^i - C_I \tag{3.63}$$

$$C_J \neq 0 : g_{ij} S^{J,i}\dot{x}^j - C_J \tag{3.64}$$

$$C_{IJ} \neq 0 : t g_{ij} S^{J,i}\dot{x}^j - S^J \tag{3.65}$$

$$C_1 \neq 0 : \phi_{C_1} = \frac{1}{2} g_{ij}\dot{x}^i\dot{x}^j \tag{3.66}$$

$$C_2 \neq 0 : \phi_{C_2} = t\psi g_{ij}\dot{x}^i\dot{x}^j - g_{ij} H^i \dot{x}^j + C_2 \tag{3.67}$$

$$C_3 \neq 0 : \phi_{C_3} = \frac{1}{2} t^2 \psi g_{ij}\dot{x}^i\dot{x}^j - t(GHV)_{,i}\dot{x}^i + [GHV]. \tag{3.68}$$

We conclude that the first Integrals of the Noether symmetry vectors of the geodesic equations are:

a) linear, the $\phi_I, \phi_J, \phi_{IJ}$

b) quadratic, the $\phi_{c1}, \phi_{c2}, \phi_3$.

These results are compatible with the corresponding results of Katzin and Levine [29].

In a number of recent papers [39, 40, 41], the authors study the relation between the Noether symmetries of the geodesic Lagrangian. They also make a conjecture concerning the relation between the Noether symmetries and the conformal algebra of spacetime and concentrate especially on conformally flat spacetimes. In [41] it is also claimed that the author has found new conserved quantities for spaces of different curvatures, which seem to be of non Noetherian character. Obviously due to the above results the conjecture/results in these papers should be revised and the word ‘conformal’ should be replaced with the word ‘homothetic’.

3.5 Applications

We apply the general Theorems 3.4.1 and 3.4.2 in various curved spaces where we determine the Lie and the Noether symmetries of the corresponding geodesic equations.

[5] GHV stands for gradient HV

3.5.1 The geodesic symmetries of Einstein spaces

Suppose Y is a projective collineation with projection function $\phi(x^k)$, such that

$$L_Y\Gamma^i_{jk} = \phi_{,j}\delta^i_k + \phi_{,k}\delta^i_j.$$

For a proper Einstein space ($R \neq 0$) we have $R_{ab} = \frac{R}{n}g_{ab}$ from which follows [42]

$$L_Y g_{ab} = \frac{n(1-n)}{R}\phi_{;ab} - L_Y(\ln R)g_{ab}. \tag{3.69}$$

Using the contracted Bianchi identity

$$\left[R^{ij} - \frac{1}{2}Rg^{ij}\right]_{;j} = 0$$

it follows that in an Einstein space of dimension[6] $n > 2$ the curvature scalar R =constant and (3.69) reduces to

$$L_Y g_{ab} = \frac{n(1-n)}{R}\phi_{;ab}.$$

It follows that if Y^i generates either an affine or a special projective collineation, then $\phi_{;ab} = 0$. Hence X^a reduces to a KV. This means that proper Einstein spaces do not admit HV, ACs, special PCs and gradient KVs ([21, 23])

The above results and Theorem 3.4.1 lead to the following result.

Proposition 3.5.1 *The Lie symmetries of the geodesic equations in a proper Einstein space with curvature scalar $R \neq 0$ are given by the vectors*

$$X = (Kt + L)\,\partial_t + D^i(x)\,\partial_i$$

where $D^i(x)$ is a nongradient KV and K, L are constants

For the Noether symmetries of Einstein spaces we have the following

Proposition 3.5.2 *The Noether symmetries of the geodesic equations in a proper Einstein space with curvature scalar $R \neq 0$ are given by the vectors*

$$X = L\partial_t + D^i(x)\,\partial_i \quad , \quad f = \text{ constant}$$

Proposition 3.5.1 extends and amends the conjecture of [36] to the more general case of Einstein spaces.

We apply the results to the maximally symmetric space of Euclidian 2d sphere.

[6] Recall that all two dimensional spaces are Einstein spaces.

Euclidian 2d sphere

The geodesic Lagrangian of the Euclidian 2d sphere is

$$L\left(\phi,\dot{\phi},\theta,\dot{\theta}\right)=\frac{1}{2}\dot{\phi}^2+\sin^2\phi\ \dot{\theta}^2$$

and the geodesic equations are

$$\ddot{\phi}-\frac{1}{2}\sin 2\phi\ \dot{\theta}^2=0$$
$$\ddot{\theta}+\cot\phi\ \dot{\phi}\dot{\theta}=0$$

The Euclidian 2d sphere is an Einstein space with curvature scalar $R=2$, therefore propositions (3.5.1) and (3.5.2) apply. The KVs of the Euclidian 2d sphere are the elements of the $so\,(3)$ Lie algebra (See example 2.7.7)

$$X_1=\sin\theta\partial_\phi+\cos\theta\cot\phi\partial_\theta\quad,\ X_2=\cos\theta\partial_\phi-\sin\theta\cot\phi\partial_\theta\quad,\quad X_3=\partial_\theta.$$

Consequently the Lie symmetries of geodesic equations are the elements of the $so\,(3)$ plus the vectors $\partial_t,\ t\partial_t$. Likewise the Noether symmetries are the elements of $so\,(3)$ plus the vector ∂_t. The corresponding Noether integrals are

$$\phi_1=\dot{\theta}\sin^2\phi$$
$$\phi_2=\dot{\phi}\sin\theta+\frac{1}{2}\dot{\theta}\sin 2\phi\cos\theta$$
$$\phi_3=\dot{\phi}\cos\theta-\frac{1}{2}\dot{\theta}\sin 2\phi\sin\theta$$

and the Hamiltonian constant.

3.5.2 The geodesic symmetries of Gödel spacetime

The Gödel metric in Cartesian coordinates is

$$ds^2=-dt^2-2e^{ax}dtdy+dx^2-\frac{1}{2}e^{2ax}dy^2+dz^2.$$

The geodesic Lagrangian is

$$L=\frac{1}{2}\left(-t'^2-2e^{ax}t'y'+dx^2-\frac{1}{2}e^{2ax}y'^2+z'^2\right)\tag{3.70}$$

where $'$ means $\frac{d}{ds}$ where "s" is an affine parameter. The geodesic equations are

$$t''+2at'x'+ae^{ax}x'y'=0$$
$$x''+ae^{ax}t'y'+\frac{1}{2}ae^{2ax}y'^2=0$$
$$y''-2ae^{-ax}t'x'=0$$
$$z''=0.$$

Table 3.1: Lie algebra of the Gödel geodesic symmetries

$[X_I, X_J]$	X_1	X_2	X_3	X_4	X_5	X_6	X_7	X_8	X_9	X_{10}
X_1	0	X_1	0	0	0	0	0	0	X_8	0
X_2	$-X_1$	0	$-X_3$	0	0	0	0	0	$-X_9$	0
X_3	0	X_3	0	0	0	0	0	$-X_1$	$X_{10}-X_2$	$-X_3$
X_4	0	0	0	0	0	0	0	0	0	0
X_5	0	0	0	0	0	X_5	$-X_7$	0	0	0
X_6	0	0	0	0	$-X_5$	0	aX_7	0	0	0
X_7	0	0	0	0	X_7	$-aX_7$	0	0	0	0
X_8	0	0	X_1	0	0	0	0	0	0	X_8
X_9	$-X_8$	X_9	X_2-X_{10}	0	0	0	0	0	0	X_9
X_{10}	0	0	X_3	0	0	0	0	$-X_8$	$-X_{19}$	0

The special projective algebra of the Gödel metric has as follows:

$$Y^1 = \partial_z \ , \ Y^3 = \partial_x - ay\partial_y \ , \ Y^4 = \partial_t \ , \ Y^5 = \partial_y \ , \ Y^6 = z\partial_z$$

$$Y^2 = \left(-\frac{2}{a}e^{-ax}\right)\partial_t + y\partial_x + \left(\frac{2e^{-2ar} - a^2y^2}{2a}\right)\partial_y$$

where Y^1 is a gradient KV ($S_1 = z$), Y^{2-5} are non gradient KVs and Y^6 is a proper AC. The Gödel spacetime does not admit proper sp.PC [24].

Applying theorem (3.4.1) we find that, the Gödel spacetime admits ten Lie point symmetries as follows

$$X_1 = \partial_s \quad , \quad X_2 = s\partial_s \quad , \quad X_3 = z\partial_s \quad , \quad X_4 = Y^4$$

$$X_5 = Y^2 \quad , \quad X_6 = Y^3 \quad , \quad X_7 = Y^5$$

$$X_8 = Y^1 \ , \ X_9 = sY^1 \quad , \quad X_{10} = Y^6$$

whose Lie algebra is given in Table 3.1.

There are two Lie subalgebras. One spanned by the vectors $\{X_1, X_4, X_5, X_6, X_7, X_8, X_9\}$ and a second spanned by the vectors $\{X_2, X_3, X_{10}\}$. It can be shown that the first subalgebra consists of the Noether symmetries of the Lagrangian (3.70). The corresponding Noether integrals are

$$
\begin{aligned}
\phi_8 &= z' \\
\phi_9 &= sz' - z \\
\phi_4 &= t' + y'e^{ax} \\
\phi_6 &= x' + ay\phi_7 \\
\phi_7 &= e^{\alpha x}\left(t' + \frac{1}{2}y'e^{ax}\right) \\
\phi_5 &= \frac{2}{a}e^{-ax}t' + \frac{3}{a}y' + 2yx' + ay^2\phi_7.
\end{aligned}
$$

The Noether constant corresponding to the Noether symmetry $X_1 = \partial_s$ is the total energy i.e. the Hamiltonian.

3.5.3 The geodesic symmetries of Taub spacetime

Consider the Taub spacetime with line element

$$ds^2 = x^{-\frac{1}{2}}\left(-dt^2 + dx^2\right) + x\left(dy^2 + dz^2\right). \tag{3.71}$$

The geodesic Lagrangian is

$$L = \frac{1}{2}\left(x^{-\frac{1}{2}}\left(-t'^2 + x'^2\right) + x\left(y'^2 + z'^2\right)\right) \tag{3.72}$$

and the geodesic equations are

$$
\begin{aligned}
t'' - \frac{1}{2x}t'x' &= 0 \\
y'' + \frac{1}{x}y'x' &= 0 \\
z'' + \frac{1}{x}z'x' &= 0 \\
\ddot{x} - \frac{1}{4x}\left(t'^2 + x'^2\right) - \frac{x^{\frac{1}{2}}}{2}\left(y'^2 + z'^2\right) &= 0.
\end{aligned}
$$

In order to find the Lie symmetries of the geodesic equations for the Taub spacetime (3.71) we need to have the special projective algebra of (3.71). The spacetime (3.71) admits a five dimensional special projective algebra[7] which consists from four non gradient KVs Y^{1-4} and a non gradient HV Y^5 [43].

$$
\begin{aligned}
Y_1 &= \partial_t \ , \ Y_2 = z\partial_y - y\partial_z \\
Y_3 &= \partial_y \ , \ Y_4 = \partial_z
\end{aligned}
$$

$$Y_5 = \frac{4}{3}t\partial_t + \frac{4}{3}x\partial_x + \frac{1}{3}y\partial_y + \frac{1}{3}z\partial_z \ , \ \psi = 1.$$

[7] The spacetime (3.71) does not admit proper AC or sp.PC.

Table 3.2: Lie algebra of the Taub geodesic symmetries

$[X_I, X_J]$	X_1	X_2	X_3	X_4	X_5	X_6	X_7
X_1	0	X_1	0	0	0	0	0
X_2	$-X_1$	0	0	0	0	0	0
X_3	0	0	0	0	0	0	$\frac{4}{3}X_3$
X_4	0	0	0	0	X_6	$-X_5$	0
X_5	0	0	0	$-X_6$	0	0	$\frac{1}{3}X_5$
X_6	0	0	0	X_5	0	0	$\frac{1}{3}X_6$
X_7	0	0	$-\frac{4}{3}X_7$	0	$-\ \frac{1}{3}X_5$	$-\frac{1}{3}X_6$	0

Applying theorem (3.4.1) we find that the geodesic equations of (3.71) admit seven Lie symmetries

$$\begin{aligned} X_1 &= \partial_s \ \ , \ X_2 = s\partial_s \ , \ X_3 = \partial_t \ \ , \ X_4 = z\partial_y - y\partial_z \\ X_5 &= \partial_y \ \ , \ \ X_6 = \partial_z \ \ , \ X_7 = \frac{4}{3}t\partial_t + \frac{4}{3}x\partial_x + \frac{1}{3}y\partial_y + \frac{1}{3}z\partial_z \end{aligned}$$

with Lie algebra is given in Table 3.1.

Similarly from Theorem 3.4.2 we have that the geodesic Lagrangian (3.72) admits a six dimensional Noether algebra, with elements

$$X_1, \ X_3, \ X_4 \ , \ X_5 \ , \ X_6 \ , \ X_8 = 2X_2 + X_7$$

and correspoding Noether integrals

$$\begin{aligned} \phi_3 &= x^{-\frac{1}{2}}t' \ , \ \phi_5 = xy' \\ \phi_6 &= xz' \ , \ \phi_4 = x\left(zy' - yz'\right) \\ \phi_1 &= x^{-\frac{1}{2}}\left(-t'^2 + x'^2\right) + x\left(y'^2 + z'^2\right) \\ \phi_8 &= s\phi_1 - \frac{1}{3}x^{-\frac{1}{2}}\left[4xx' + x^{\frac{3}{2}}\left(yy' + zz'\right) - 4tt'\right]. \end{aligned}$$

3.5.4 The geodesic symmetries of a 1+3 decomposable spacetime metric

We consider next the metric which is a 1+3 decomposable, that is it has the form:

$$ds_4 = -d\tau^2 + U^2\delta_{\alpha\beta}dx^\alpha dx^\beta \tag{3.73}$$

Table 3.3: The conformal algebra of the 1+3 metric

K	CKVs of ds_3^2	ψ_3	#	CKVs of ds_{1+3}^2	ψ_{1+3}
1	$H = x^a\partial_a$	$\psi_+(H) = U\left(1-\frac{1}{4}x^\alpha x_\alpha\right)$	1	$H_1^+ = -\psi_+(H)\cos\tau\partial_\tau + H\sin\tau$	$\psi_+(H)\sin\tau$
1	$H = x^a\partial_a$	$\psi_+(H) = U\left(1-\frac{1}{4}x^\alpha x_\alpha\right)$	1	$H_2^+ = \psi_+(H)\sin\tau\partial_\tau + H\cos\tau$	$\psi_+(H)\cos\tau$
1	$C_\mu = \left(\delta_\mu^a - \frac{1}{2}Ux_\mu x^a\right)\partial_\alpha$	$\psi_+(C_\mu) = -Ux^\mu$	3	$Q_\mu^+ = -\psi_+(C_\mu)\cos\tau\partial_\tau + C_\mu\sin\tau$	$\psi_+(C_\mu)\sin\tau$
1	$C_\mu = \left(\delta_\mu^\alpha - \frac{1}{2}Ux_\mu x^a\right)\partial_\alpha$	$\psi_+(C_\mu) = -Ux^\mu$	3	$Q_{\mu+3}^+ = \psi_+(C_\mu)\sin\tau\partial_\tau + C_\mu\cos\tau$	$\psi_+(C_\mu)\cos\tau$
-1	$H = x^\alpha\partial_\alpha$	$\psi_-(H) = U\left(1+\frac{1}{4}x^\alpha x_\alpha\right)$	1	$H_1^- = \psi_-(H)\cosh\tau\partial_\tau + H\sinh\tau$	$\psi_-(H)\sinh\tau$
-1	$H = x^\alpha\partial_\alpha$	$\psi_-(H) = U\left(1+\frac{1}{4}x^\alpha x_\alpha\right)$	1	$H_2^- = \psi_-(H)\sinh\tau\partial_\tau + H\cosh\tau$	$\psi_-(H)\cosh\tau$
-1	$C_\mu = \left(\delta_\mu^\alpha + \frac{1}{2}Ux_\mu x^a\right)\partial_\alpha$	$\psi_-(C_\mu) = Ux^\mu$	3	$Q_\mu^- = \psi_-(C_\mu)\cosh\tau\partial_\tau + C_\mu\sinh\tau$	$\psi_-(C_\mu)\sinh\tau$
-1	$C_\mu = \left(\delta_\mu^\alpha + \frac{1}{2}Ux_\mu x^a\right)\partial_\alpha$	$\psi_-(C_\mu) = Ux^\mu$	3	$Q_{\mu+3}^- = \psi_-(C_\mu)\sinh\tau\partial_\tau + C_\mu\cosh\tau$	$\psi_-(C_\mu)\cosh\tau$

where Greek indices take the values $1, 2, 3$. It is well known [44] that this metric admits 15 CKVs. Seven of these vectors are KVs (the six nongradient KVs of the 3-metric $\mathbf{r}_{\mu\nu}, \mathbf{I}_\mu$ plus the gradient KV ∂_τ) and nine proper CKVs. The vectors of this conformal algebra are shown in Table 3.3.

According to Theorem 3.4.2 this metric admits the following Noether symmetries

$$\partial_s \ , \ s\partial_\tau \ , \ \mathbf{r}_{\mu\nu} \ , \ \mathbf{I}_\mu \ , \ \partial_\tau$$

with first integrals

$$\phi_s = \frac{1}{2}g_{ij}x'^i x'^j$$
$$\phi_\tau = \tau' \ , \ \phi_{\tau+1} = s\tau' - \tau$$
$$\phi_I = \mathbf{I}_i^I x'^i \ , \ \phi_r = \mathbf{r}_{(AB)_i} x'^j.$$

The Lie symmetries of the geodesic equations of (3.73) are the Noether symmetries plus the vectors $s\partial_s$, $\tau\partial_s$.

3.5.5 The geodesic symmetries of the FRW metrics

In a recent paper Bokhari and Kara [38] studied the Lie symmetries of the conformally flat Friedman Robertson Walker (FRW) metric with the view to understand how Noether symmetries compare with conformal Killing vectors. More specifically they considered the conformally flat FRW metric[8]

[8] The second metric $ds^2 = -t^{-\frac{4}{3}}dt^2 + dx^2 + dy^2 + dz^2$ they consider is the Minkowski metric whose Lie and Noether symmetries are well known.

Table 3.4: The conformal algebra of a flat 3d metric

CKV	**Components**	**#**	$\psi(\boldsymbol{\xi})$	**Comment**
P_I	∂_I	3	0	gradient KV
r_{AB}	$2\delta^d_{[A}x_{B]}\partial_d$	3	0	nongradient KV
H	$x^a\partial_a$	1	1	gradient HV
K_I	$\left[2x_Ix^d - \delta^d_I(x_ax^a)\right]\partial_d$	3	$2x_I$	nongradient SCKV

$$ds^2 = -dt^2 + t^{\frac{4}{3}}\left(dx^2 + dy^2 + dz^2\right)$$

and found that the Noether symmetries are the seven vectors

$$\partial_s,\ S^J\ ,\ r_{AB}$$

where S_J are the gradient KVs $\partial_x\partial_y, \partial_z$ and r_{AB} are the three nongradient KVs (generating $so\,(3)$) whereas the vector ∂_s counts for the gauge freedom in the affine parametrization of the geodesics. Therefore they confirm our Theorem 3.4.2 that the Noether vectors coincide with the KVs and the HV of the metric. Furthermore their claim that '*...the conformally transformed Friedman model admits additional conservation laws not given by the Killing or conformal Killing vectors*' is not correct.

In the following lines, we compute all the Noether point symmetries of the FRW spacetimes. To do that we have to have the homothetic algebra of these models [45]. There are two cases to consider, the conformally flat models ($K = 0$) and the non conformally flat models ($K \neq 0$).

In the following we need the conformal algebra of the flat metric, which in Cartesian coordinates (See example 2.7.12) is given in Table 3.4.

Case A**:** $K \neq 0$

The metric is

$$ds = R^2\left(\tau\right)\left[-d\tau^2 + \frac{1}{\left(1+\frac{1}{4}Kx^ix_i\right)^2}\left(dx^2 + dy^2 + dz^2\right)\right]. \tag{3.74}$$

For a general $R\left(\tau\right)$ this metric admits the nongradient KVs $\mathbf{P}_I,\ \mathbf{r}_{\mu\nu}$ (see Table 4) and does not admit a HV. Therefore the Noether symmetries of the geodesic Lagrangian

$$L = -\frac{1}{2}R^2\left(\tau\right)t'^2 + \frac{1}{2}\frac{R^2\left(\tau\right)}{\left(1+\frac{1}{4}Kx^ix_i\right)^2}\left(x'^2 + y'^2 + z'^2\right)$$

of the FRW metric (3.74) are:

$$\partial_s\ ,\ \mathbf{P}_I\ ,\ \mathbf{r}_{\mu\nu}$$

Table 3.5: The special forms of the scale factor for |K|=1

K	Proper CKV	#	Conformal Factor	$R(\tau)$ for KVs	$R(\tau)$ for HV
±1	$\mathbf{P}_\tau=\partial_\tau$	1	$(\ln R(\tau))_{,\tau}$	c	$\exp(\tau)$
1	$\mathbf{H}_1^+$	1	$-\frac{\psi_+(\mathbf{H})}{R(\tau)}\left(R(\tau)\cos\tau\right)_{,\tau}$	$\frac{c}{\cos\tau}$	$\nexists$
1	$\mathbf{H}_2^+$	1	$\frac{\psi_+(\mathbf{H})}{R(\tau)}\left(R(\tau)\sin\tau\right)_{,\tau}$	$\frac{c}{\sin\tau}$	$\nexists$
1	$\mathbf{Q}_\mu^+$	3	$-\frac{\psi_+(\mathbf{C}_\mu)}{R(\tau)}\left(R(\tau)\cos\tau\right)_{,\tau}$	$\frac{c}{\cos\tau}$	$\nexists$
1	$\mathbf{Q}_{\mu+3}^+$	3	$\frac{\psi_+(\mathbf{C}_\mu)}{R(\tau)}\left(R(\tau)\sin\tau\right)_{,\tau}$	$\frac{c}{\sin\tau}$	$\nexists$
-1	$\mathbf{H}_1^-$	1	$\frac{\psi_-(\mathbf{H})}{R(\tau)}\left(R(\tau)\cosh\tau\right)_{,\tau}$	$\frac{c}{\cosh\tau}$	$\nexists$
-1	$\mathbf{H}_2^-$	1	$\frac{\psi_-(\mathbf{H})}{R(\tau)}\left(R(\tau)\sinh\tau\right)_{,\tau}$	$\frac{c}{\sinh\tau}$	$\nexists$
-1	$\mathbf{Q}_\mu^-$	3	$\frac{\psi_-(\mathbf{C}_\mu)}{R(\tau)}\left(R(\tau)\cosh\tau\right)_{,\tau}$	$\frac{c}{\cosh\tau}$	$\nexists$
-1	$\mathbf{Q}_{\mu+3}^-$	3	$\frac{\psi_-(\mathbf{C}_\mu)}{R(\tau)}\left(R(\tau)\sinh\tau\right)_{,\tau}$	$\frac{c}{\sinh\tau}$	$\nexists$

with Noether integrals

$$\phi_s=\frac{1}{2}g_{ij}x'^i x'^j\ ,\ \phi_I=\mathbf{P}_i^I x'^i\ ,\ \phi_r=\ \mathbf{r}_{(AB)_i}x'^j. \tag{3.75}$$

Concerning the Lie symmetries we note that the FRW spacetimes do not admit ACs [46] and furthermore does not admit gradient KVs. Therefore they do not admit special PCs. The Lie symmetries of these spacetimes are then

$$\partial_s,\ s\partial_s,\ \mathbf{P}_I\ ,\ \mathbf{r}_{\mu\nu}.$$

For special functions $R(\tau)$ it is possible to have more KVs and HV. In Table 3.5 we give the special forms of the scale factor $R(t)$ and the corresponding extra KVs and HV for $K=\pm1$.

From Table 3.5 we infer the following additional Noether symmetries of the FRW-like Lagrangian for special forms of the scale factor

Case A(1): $R(t)=c$ =constant, the space is the 1+3 decomposable.

Case A(2) $K=1,\ R(t)=\exp(\tau)$. In this case the space is flat and admits as Lie point symmetries the $sl(4+2,R)$.

Case A(3a) $K=1,\ R(\tau)=\frac{c}{\cos\tau}$. In this case we have the additional non-gradient KVs $H_1^+,\ Q_\mu^+$. Therefore the Noether symmetries are:

$$\partial_s\ ,\ \mathbf{P}_I\ ,\ \mathbf{r}_{\mu\nu}\ ,H_1^+,\ Q_\mu^+$$

with Noether Integrals

$$\phi_s\ ,\ \phi_I\ ,\ \phi_r\ ,\ \phi_{H_1^+} = \left(H_1^+\right)_i x'^i \ \text{ and } \ \phi_{Q_\mu^+} = \left(Q_\mu^+\right)_i x'^i.$$

The Lie symmetries are

$$\partial_s\ ,\ s\partial_s\ ,\ \mathbf{P}_I\ ,\ \mathbf{r}_{\mu\nu}\ , H_1^+,\ Q_\mu^+$$

Case A(3b) $K=1,\ R(\tau)=\frac{c}{\sin\tau}$. In this case we have the two nongradient KVs $H_2^+,\ Q_{\mu+3}^+$. The Noether Symmetries are

$$\partial_s\ ,\ \mathbf{P}_I\ ,\ \mathbf{r}_{\mu\nu}\ , H_2^+,\ Q_{\mu+3}^+ \quad : \ f=\text{constant}$$

with Noether Integrals

$$\phi_s\ ,\ \phi_I\ ,\ \phi_r\ ,\ \phi_{H_2^+} = \left(H_2^+\right)_i x'^i \ \text{ and } \ \phi_{Q_{\mu+3}^+} = \left(Q_{\mu+3}^+\right)_i x'^i.$$

The Lie symmetries are

$$\partial_s\ ,\ s\partial_s\ ,\ \mathbf{P}_I\ ,\ \mathbf{r}_{\mu\nu}\ , H_2^+,\ Q_{\mu+3}^+.$$

Case A(4a) $K=-1,\ R(\tau)=\frac{c}{\cosh\tau}$. In this case we have the two additional nongradient KVs $H_1^-,\ Q_{\bar{\mu}}$. The Noether Symmetries are

$$\partial_s\ ,\ \mathbf{P}_I\ ,\ \mathbf{r}_{\mu\nu}\ , H_1^-,\ Q_{\bar{\mu}}$$

with Noether Integrals

$$\phi_s\ ,\ \phi_I\ ,\ \phi_r\ ,\ \phi_{H_1^-} = \left(H_1^-\right)_i x'^i \ \text{ and } \ \phi_{Q_\mu^-} = \left(Q_\mu^-\right)_i x'^i.$$

The Lie symmetries are

$$\partial_s\ ,\ s\partial_s\ ,\ \mathbf{P}_I\ ,\ \mathbf{r}_{\mu\nu}\ , H_1^-,\ Q_{\bar{\mu}}.$$

Case A(4b) $K=-1,\ R(\tau)=\frac{c}{\sin\tau}$, we have the nongradient KV $H_2^-,\ Q_{\mu+3}^-$. The Noether Symmetries are

$$\partial_s\ ,\ \mathbf{P}_I\ ,\ \mathbf{r}_{\mu\nu}\ , H_2^-,\ Q_{\mu+3}^-$$

with Noether Integrals

$$\phi_s\ ,\ \phi_I\ ,\ \phi_r\ ,\ \phi_{H_1^-} = \left(H_2^-\right)_i x'^i \text{ and } \ \phi_{Q_{\mu+3}^-} = \left(Q_{\mu+3}^-\right)_i x'^i.$$

The Lie symmetries are

$$\partial_s\ ,\ s\partial_s\ ,\ \mathbf{P}_I\ ,\ \mathbf{r}_{\mu\nu}\ , H_2^-,\ Q_{\mu+3}^-.$$

Case B: $K=0$

In this case the metric is

$$ds = R^2(t)\left(-dt^2+dx^2+dy^2+dz^2\right)$$

and admits three nongradient KVs $\mathbf{P}_I$ and three nongradient KVs $\mathbf{r}_{AB}$. Therefore the Noether symmetries are

$$\partial_s\ , \mathbf{P}_I\ ,\ \mathbf{r}_{AB}\ :\ f=\text{constant}$$

Table 3.6: The special forms of the scale factor for K=0

#	Proper CKV	Conformal Factor ψ	$R(\tau)$ for KVs	$R(\tau)$ for HV
1	$\mathbf{P}_{\tau}=\partial_{\tau}$	$(\ln R(\tau))_{,\tau}$	c	$\exp(\tau)$
3	$\mathbf{M}_{\tau\alpha}=x_{\alpha}\partial_{\tau}+\tau\partial_{\alpha}$	$x_{\alpha}(\ln R(\tau))_{,\tau}$	c	$\nexists$
1	$\mathbf{H}=\mathbf{P}_{\tau}+x^{a}\partial_{a}$	$\boldsymbol{\tau}(\ln R(\tau))+1$	c/τ	$\nexists$
1	$\mathbf{K}_{\tau}=2\tau\mathbf{H}+\left(x^{c}x_{c}-\tau^{2}\right)\partial_{\tau}$	$-(\ln R(\tau))_{,\tau}\left(-\tau^{2}+r^{2}\right)+2\epsilon\tau$	$\nexists$	$\nexists$
3	$\mathbf{K}_{\mu}=2x_{\mu}\mathbf{H}-\left(x^{c}x_{c}-\tau^{2}\right)\partial_{\mu}$	$2x_{\mu}\left[\tau(\ln R(\tau))_{,\tau}+1\right]$	c/τ	$\nexists$

with Noether Integrals

$$\phi_{s}=\frac{1}{2}g_{ij}x^{\prime i}x^{\prime j}\ ,\ \phi_{P_{I}}=\mathbf{P}_{i}^{I}x^{\prime i}\text{ and }\phi_{F}=\ \left(\mathbf{r}_{AB}\right)_{i}x^{\prime i}.$$

The Lie symmetries are

$$\partial_{s}\ ,\ s\partial_{s}\ ,\ \mathbf{P}_{I}\ ,\ \mathbf{r}_{AB}.$$

Again for special forms of the scale factor one obtains extra KVs and HV as shown in Table 3.6.

From Table 3.6 we have the following special cases.

Case B(1): $R(t)=c=$constant. Then the space is the Minkowski space and admits as Lie symmetries the $sl(4+2,R)$.

Case B(2): $R(t)=\exp(\tau)$. Then $\mathbf{P}_{\tau}$ becomes a gradient HV $\left(\psi=1,\text{ gradient function }\frac{1}{2}\exp(2\tau)\right)$. Hence the Noether symmetries are

$$\partial_{s}\ ,\mathbf{P}_{I}\ ,\ \mathbf{r}_{AB}\ ,\ 2s\partial_{s}+\mathbf{P}_{\tau}\ ,\ \ s^{2}\partial_{s}+s\mathbf{P}_{\tau}$$

with Noether Integrals

$$\phi_{s}\ ,\ \phi_{P_{I}}\ ,\ \phi_{F}\ ,\ \phi_{\mathbf{P}_{\tau}}=sg_{ij}x^{\prime i}x^{\prime j}-g_{ij}\left(\mathbf{P}_{\tau}\right)^{i}x^{\prime j}\text{ and }\ \phi_{\mathbf{Y+1}}=\frac{1}{2}s^{2}g_{ij}\dot{x}^{i}\dot{x}^{j}-s\left(\mathbf{P}_{\tau}\right)_{i}x^{\prime i}+\mathbf{P}_{\tau}.$$

The Lie symmetries are

$$\partial_{s}\ ,\ s\partial_{s}\ ,\ \ \mathbf{P}_{I}\ ,\ \mathbf{r}_{AB}\ ,\ \mathbf{P}_{\tau}\ ,s^{2}\partial_{s}+s\mathbf{P}_{\tau}.$$

Case B(3)**:** $R(t)=\tau^{-1}$. Then we have four additional nongradient KVs, the $\mathbf{H}$, and $\mathbf{K}_{\mu}$, and the Noether symmetries are:

$$\partial_{s}\ ,\ \mathbf{P}_{I},\ \mathbf{r}_{AB}\ ,\ \mathbf{H}\ ,\ \mathbf{K}_{\mu}$$

with Noether Integrals

$$\phi_{s}\ ,\ \phi_{P_{I}}\ ,\ \phi_{F}\ ,\ \phi_{\ \mathbf{H}}=(\mathbf{H})_{i}x^{\prime i}\ \text{ and }\ \phi_{\mathbf{K}_{\mu}}=\left(\mathbf{K}_{\mu}\right)_{i}x^{\prime i}.$$

The Lie symmetries are:

$$\partial_{s}\ ,\ s\partial_{s}\ ,\ \mathbf{P}_{I},\ \mathbf{r}_{AB}\ ,\ \mathbf{H}\ ,\ \mathbf{K}_{\mu}.$$

3.6 Conclusion

We derived the symmetry conditions for the admittance of a Lie point symmetry by the equations of autoparallels (paths) in an affine space. The important conclusion is that the Lie symmetry vector is an Affine Collineation in the jet space $\{t, x^i\}$ (it preserves both the autoparallels and their parametrization) while in the space $\{x^i\}$ the vectors $\eta^i(t, x)\partial_{x^i}$ are projective collineations (they preserve the autoparallels but not necessarily their parametrization).

The symmetry conditions are applied to the geodesics of a Riemannian space were they are solved and the generic Lie symmetry vector is obtained in terms of the special projective algebra (and its degeneracies KVs, HKV, ACs) of the metric. Furthermore we derived the Noether symmetries of the geodesic Lagrangian and it was proved that Noether symmetries are generateted from the homothetic algebra of the metric. We applied the results to the case of Einstein spaces and obtained the Lie symmetry vectors in terms of the KVs of the metric, in agreement with the conjecture made in [36].

Finally, the Lie and the Noether symmetries of the geodesic equations were computed in the Gödel spacetime, the Taub spacetime and the Friedman Robertson Walker spacetimes. In each case the Noether symmetries were computed explicitly together with the corresponding first integrals.

3.A The determining equations

Below we calculate the determining equations for the system (3.1). Let $X = \xi\left(t, x^k\right)\partial_t + \eta^i\left(t, x^k\right)\partial_t$ be the infinitesimal generator of a one parameter point transformation. X is a Lie symmetry of (3.1) if the following condition, holds

$$\eta^i_{[2]} = -X^{[1]}\left(\Gamma^i_{jk}\dot{x}^j\dot{x}^k + \sum_{m=0}^{n} P^i_{j_1\ldots j_m}\dot{x}^{j_1}\ldots\dot{x}^{j_m}\right) \tag{3.76}$$

where

$$X^{[2]} = X + \eta^i_{[1]}\partial_{\dot{x}^i} + \eta^i_{[2]}\partial_{\dot{x}^i}$$

is the second prolongation of X and $\eta^i_{[1]}, \eta^i_{[2]}$ are the prolongation functions

$$\eta^i_{[1]} = \left(\eta^i_{,t}\right) + \left(\eta^i_{,k} - \xi_{,t}\delta^i_k\right)\dot{x}^k - \left(\xi_{,j}\right)\dot{x}^i\dot{x}^j$$

$$\begin{aligned}\eta^i_{[2]} &= \left(\eta^i_{,tt}\right) + \left(2\eta^i_{,tj} - \xi_{,tt}\delta^i_j\right)\dot{x}^j + \left(\eta^i_{,jk} - 2\xi_{,tj}\delta^i_k\right)\dot{x}^j\dot{x}^k + \\ &\quad - \left(\xi_{,jk}\right)\dot{x}^i\dot{x}^j\dot{x}^k + \left(\eta^i_{,j} - \xi_{,j}\dot{x}^i\right)\ddot{x}^j - 2\ddot{x}^i\left(\xi_{,j}\dot{x}^j + \xi_{,t}\right).\end{aligned}$$

Replacing $\ddot{x}^i$ from (3.1) we find eventually

$$\begin{aligned}\eta^i_{[2]} &= \left(\eta^i_{,tt} - \eta^i_{,j}P^j + 2\xi_{,t}P^i\right) + \\ &\quad + \left(2\eta^i_{,tr} - \xi_{,tt} - \eta^i_{,j}P^j_r + \xi_{,j}P^j\delta^i_r + 2\xi_{,r}P^i + 2\xi_{,t}P^i_r\right)\dot{x}^r + \\ &\quad + \begin{pmatrix} \eta^i_{,rk} - 2\xi_{,tr}\delta^i_k - \eta^i_{,j}P^j_{rk} - \eta^i_{,j}\Gamma^j_{rk} + \\ +\xi_{,j}P^j_r\delta^i_k + 2\xi_{,r}P^i_k + 2\xi_{,t}\Gamma^i_{rk} + 2\xi_{,t}P^i_{rk} \end{pmatrix}\dot{x}^r\dot{x}^k \\ &\quad + \begin{pmatrix} -\xi_{,rk}\delta^i_s - \eta^i_{,j}P^j_{rks} + \xi_{,j}P^j_{rk}\delta^i_s + \\ +3\xi_{,r}\Gamma^i_{ks} + 2\xi_{,t}P^i_{rks} + 2\xi_{,r}P^i_{j_1j_2} \end{pmatrix}\dot{x}^s\dot{x}^r\dot{x}^k \\ &\quad + \left[-\left(\eta^i_{,j}\sum_{m=4}^{n} P^j_{j_1\ldots j_m}\dot{x}^{j_1}\ldots\dot{x}^{j_m}\right) + \xi_{,j}\dot{x}^i\sum_{m=3}^{n} P^j_{j_1\ldots j_m}\dot{x}^{j_1}\ldots\dot{x}^{j_m}\right] + \\ &\quad + \left[2\xi_{,r}\dot{x}^r\sum_{m=3}^{n} P^j_{j_1\ldots j_m}\dot{x}^{j_1}\ldots\dot{x}^{j_m} + \left(2\xi_{,t}\sum_{m=4}^{n} P^j_{j_1\ldots j_m}\dot{x}^{j_1}\ldots\dot{x}^{j_m}\right)\right]\end{aligned}$$

We have computed the lhs of (3.76). It remains to compute the rhs

$$X^{[1]}\left(\Gamma^i_{jk}\dot{x}^j\dot{x}^k + \sum_{m=0}^{n} P^i_{j_1\ldots j_m}\dot{x}^{j_1}\ldots\dot{x}^{j_m}\right) = X^{[1]}\left(\Gamma^i_{jk}\dot{x}^j\dot{x}^k\right) + X^{[1]}\left(\sum_{m=0}^{n} P^i_{j_1\ldots j_m}\dot{x}^{j_1}\ldots\dot{x}^{j_m}\right).$$

Analysis of the term $X^{[1]}\left(\Gamma^i_{jk}\dot{x}^j\dot{x}^k\right)$:

$$\begin{aligned}X^{[1]}\left(\Gamma^i_{jk}\dot{x}^j\dot{x}^k\right) &= X\left(\Gamma^i_{jk}\right)\dot{x}^j\dot{x}^k + 2\Gamma^i_{jk}\dot{x}^j\delta\dot{x}^k \\ &= \left(\xi\Gamma^i_{jk,t} + \eta^l\Gamma^i_{jk,l}\right)\dot{x}^j\dot{x}^k + 2\Gamma^i_{jk}\dot{x}^j\eta^k_{[1]}.\end{aligned}$$

hence

$$X^{[1]}\left(\Gamma^i_{jk}\dot{x}^j\dot{x}^k\right) = 2\Gamma^i_{rs}\eta^s_{,t}\dot{x}^r - 2\Gamma^i_{rs}\xi_{,r}\dot{x}^s\dot{x}^r\dot{x}^r + \\ + \left(\xi\Gamma^i_{rk,t} + \eta^l\Gamma^i_{rk,l} + 2\Gamma^i_{rs}\eta^s_{,k} - 2\Gamma^i_{rs}\xi_{,t}\delta^s_k\right)\dot{x}^r\dot{x}^k.$$

Analysis of the term $X^{[1]}\left(\sum_{m=0}^{n} P^i_{j_1\dots j_m}\dot{x}^{j_1}\dots\dot{x}^{j_m}\right)$:

$$X^{[1]}\left(\sum_{m=0}^{n} P^i_{j_1\dots j_m}\dot{x}^{j_1}\dots\dot{x}^{j_m}\right) = \sum_{m=0}^{n}\left(XP^i_{j_1\dots j_m}\right)\dot{x}^{j_1}\dots\dot{x}^{j_m} + \sum_{m=0}^{n} mP^i_{j_1\dots j_m}\dot{x}^{j_1}\dots\left(\eta^{j_m}_{[1]}\right)$$

The first term becomes

$$\sum_{m=0}^{n}\left(XP^i_{j_1\dots j_m}\right)\dot{x}^{j_1}\dots\dot{x}^{j_m} = \sum_{m=0}^{n}\left(P^i_{j_1\dots j_m\ ,t}\xi + P^i_{j_1\dots j_m\ ,r}\eta^r\right)\dot{x}^{j_1}\dots\dot{x}^{j_m} \\ = \left(P^i_{,t}\xi + P^i_{,r}\eta^r\right) + \left(P^i_{k,t}\xi + P^i_{k,r}\eta^r\right)\dot{x}^k + \\ + \left(P^i_{ks,t}\xi + P^i_{ks,r}\eta^r\right)\dot{x}^k\dot{x}^s + \left(P^i_{ksl,t}\xi + P^i_{ksl,r}\eta^r\right)\dot{x}^k\dot{x}^s\dot{x}^l + \\ + \sum_{m=4}^{n}\left(P^i_{j_1\dots j_m\ ,t}\xi + P^i_{j_1\dots j_m\ ,r}\eta^r\right)\dot{x}^{j_1}\dots\dot{x}^{j_m}.$$

The second term

$$\sum_{m=0}^{n} mP^i_{j_1\dots j_m}\left(\delta\dot{x}^{j_1}\right)\dots\delta\dot{x}^{j_m} = \sum_{m=0}^{n} mP^i_{j_1\dots j_m}\eta^{j_1}_{,t}\dots\dot{x}^{j_{m-1}} + \\ + \sum_{m=0}^{n} mP^i_{j_1\dots j_m}\left(\eta^{j_1}_{,s} - \delta^{j_1}_s\xi_{,t}\right)\dots\dot{x}^{j_m}\ \dot{x}^s \\ - \sum_{m=0}^{n} mP^i_{j_1\dots j_m}\dot{x}^{j_1}\dots\dot{x}^{j_m}\dot{x}^s\xi_{,s}.$$

The term $\sum_{m=0}^{n} mP^i_{j_1\dots j_m}\dot{x}^{j_1}\dots\dot{x}^{j_{m-1}}\eta^{j_m}_{,t}$ can be written as

$$\sum_{m=0}^{n} mP^i_{j_1\dots j_m}\dot{x}^{j_1}\dots\dot{x}^{j_{m-1}}\eta^{j_m}_{,t} = P^i_k\eta^k_{,t} + 2P^i_{ks}\eta^k_{,t}\dot{x}^s + 3P^i_{ksr}\eta^k_{,t}\dot{x}^s\dot{x}^r + \\ +4P^i_{rksl}\eta^r_{,t}\dot{x}^s\dot{x}^k\dot{x}^l + \sum_{m=5}^{n} mP^j_{j_1\dots j_m}\dot{x}^{j_1}\dots\dot{x}^{j_m}.$$

that is

$$\sum_{m=0}^{n} mP^i_{j_1\dots j_m}\left(\eta^{j_1}_{,s} - \delta^{j_1}_s\xi_{,t}\right)\dots\dot{x}^{j_m}\ \dot{x}^s = P^i_k\left(\eta^k_{,s} - \delta^k_s\xi_{,t}\right)\dot{x}^s + 2P^i_{kr}\left(\eta^k_{,s} - \delta^k_s\xi_{,t}\right)\dot{x}^s\dot{x}^r + \\ +3P^i_{krl}\left(\eta^r_{,s} - \delta^r_s\xi_{,t}\right)\dot{x}^s\dot{x}^r\dot{x}^l + \\ + \sum_{m=4}^{n} mP^i_{j1\dots j_m}\left(\eta^{j_1}_{,s} - \delta^{j_1}_s\xi_{,t}\right)\dot{x}^s\dots\dot{x}^{j_m}$$

$$
\begin{aligned}
-\sum_{m=0}^{n} mP^i_{j_1\ldots j_m}\dot{x}^{j_1}\ldots\dot{x}^{j_m}\dot{x}^s\xi_{,s} &= -\left(P^i_k\xi_{,s}\right)\dot{x}^k\dot{x}^s + \\
&\quad -2\left(P^i_{kr}\xi_{,s}\right)\dot{x}^k\dot{x}^r\dot{x}^s - \left(\sum_{m=3}^{n} mP^i_{j_1\ldots j_m}\dot{x}^{j_1}\ldots\dot{x}^{j_m}\dot{x}^s\xi_{,s}\right).
\end{aligned}
$$

Finally we have

$$
\begin{aligned}
\delta\left(\sum_{m=0}^{n} P^i_{j_1\ldots j_m}\dot{x}^{j_1}\ldots\dot{x}^{j_m}\right) &= \left(\left(P^i_{,t}\xi + P^i_{,r}\eta^r\right) + P^i_k\eta^k_{,t}\right) + \begin{pmatrix} P^i_{s,t}\xi + P^i_{s,r}\eta^r + \\ +2P^i_{ks}\eta^k_{,t} + P^i_k\left(\eta^k_{,s} - \delta^k_s\xi_{,t}\right)\end{pmatrix}\dot{x}^s + \\
&\quad + \begin{pmatrix} P^i_{ks,t}\xi + P^i_{ks,r}\eta^r + 3P^i_{ksr}\eta^r_{,t} + \\ +2P^i_{kr}\left(\eta^r_{,s} - \delta^r_s\xi_{,t}\right) - P^i_k\xi_{,s}\end{pmatrix}\dot{x}^k\dot{x}^s + \\
&\quad + \begin{pmatrix} P^i_{ksl,t}\xi + P^i_{ksl,r}\eta^r + 4P^i_{rksl}\eta^r_{,t} + \\ +3P^i_{krl}\left(\eta^r_{,s} - \delta^r_s\xi_{,t}\right) - 2P^i_{kr}\xi_{,s}\end{pmatrix}\dot{x}^k\dot{x}^s\dot{x}^l + \\
&\quad + \begin{pmatrix} \sum_{m=4}^{n}\left(P^i_{j_1\ldots j_m\ ,t}\xi + P^i_{j_1\ldots j_m\ ,r}\eta^r\right)\dot{x}^{j_1}\ldots\dot{x}^{j_m} + \\ + \sum_{m=5}^{n} mP^i_{j_1\ldots j_m}\eta^{j_1}_{,t}\ldots\dot{x}^{j_m} + \\ + \sum_{m=4}^{n} mP^i_{j_1\ldots j_m}\left(\eta^{j_1}_{,s} - \delta^{j_1}_s\xi_{,t}\right)\dot{x}^s\ldots\dot{x}^{j_m} + \\ -\left(\sum_{m=3}^{n} mP^i_{j_1\ldots j_m}\dot{x}^{j_1}\ldots\dot{x}^{j_m}\dot{x}^s\xi_{,s}\right)\end{pmatrix}.
\end{aligned}
$$

Collecting terms and setting the coefficient of each product of $\dot{x}^{j_1}$ equal to zero we obtain determining equations (3.4)-(3.8).

$\left(\dot{x}^i\right)^0$:

$$
\begin{aligned}
0 &= \left(\eta^i_{,tt} - \eta^i_{,j}P^j \ + 2\xi_{,t}P^i\right) + \left(\left(P^i_{,t}\xi + P^i_{,r}\eta^r\right) + P^i_k\eta^k_{,t}\right) \\
&= L_\eta P^i + 2\xi_{,t}P^i + P^i_{,t}\xi + P^i_k\eta^k_{,t} + \eta^i_{,tt}.
\end{aligned}
$$

$\left(\dot{x}^i\right)^1$:

$$
\begin{aligned}
0 &= \left[2\eta^i_{,tr} + 2\Gamma^i_{rs}\eta^s_{,t} - \xi_{,tt}\delta^i_r\right] - \eta^i_{,j}P^j_r + \left[\xi_{,j}P^j\delta^i_r + 2\xi_{,r}P^i + 2\xi_{,t}P^i_r\right] + \\
&\quad + P^i_{s,t}\xi + P^i_{s,r}\eta^r + 2P^i_{ks}\eta^k_{,t} + P^i_k\left(\eta^k_{,s} - \delta^k_s\xi_{,t}\right) \\
&= \left(2\eta^i_{,t|r} - \xi_{,tt}\delta^i_r\right) + L_\eta P^i_r + 2P^i_{ks}\eta^k_{,t} + \left(\xi_{,r}P^r\delta^i_r + 2\xi_{,r}P^i + P^i_{r,t}\xi + \xi_{,t}P^i_r\right).
\end{aligned}
$$

$\left(\dot{x}^i\right)^2$:

$$\begin{aligned} 0 &= L_\eta \Gamma^i_{jk} + \xi \Gamma^i_{rk,t} + \left[-2\xi_{,tr}\delta^i_k\right] + \left[-\eta^i_{,j}P^j_{rk}\right] + \left[\xi_{,j}P^j_r\delta^i_k + 2\xi_{,r}P^i_k\right] + \\ &\quad +2\xi_{,t}P^i_{rk} + P^i_{ks,t}\xi + P^i_{ks,r}\eta^r + 3P^i_{ksr}\eta^r_{,t} + 2P^i_{kr}\left(\eta^r_{,s} - \delta^r_s\xi_{,t}\right) - P^i_k\xi_{,r} \\ &= \left(L_X\Gamma^i_{jk} - 2\xi_{,tr}\delta^i_k\right) + L_\eta P^i_{rk} + P^i_{ks,t}\xi + 3P^i_{ksr}\eta^r_{,t} + \left(\xi_{,j}P^j_r\delta^i_k + \xi_{,r}P^i_k\right). \end{aligned}$$

$\left(\dot{x}^i\right)^3$:

$$\begin{aligned} 0 &= \left[\xi_{,rk}\delta^i_s + \xi_{,r}\Gamma^i_{ks}\right] - \eta^i_{,j}P^j_{rks} + \xi_{,j}P^j_{rk}\delta^i_s + 2\xi_{,t}P^i_{rks} + P^i_{ksl,t}\xi + \\ &\quad +P^i_{ksl,r}\eta^r + 4P^i_{rksl}\eta^r_{,t} + 3P^i_{krl}\left(\eta^r_{,s} - \delta^r_s\xi_{,t}\right) - 2P^i_{kr}\xi_{,s} + 2\xi_{,s}P^i_{kr} \\ &= \left[-\xi_{,r|k}\delta^i_s\right] + L_\eta P^i_{krs} - P^i_{krs}\xi_{,t} + \xi_{,j}P^j_{rk}\delta^i_s + P^i_{ksl,t}\xi + 4P^i_{rksl}\eta^r_{,t}. \end{aligned}$$

the rest terms gives

$$\begin{aligned} 0 &= \sum_{m=4}^{n}\left(L_\eta P^j_{j_1\ldots j_m}\dot{x}^{j_1}\ldots\dot{x}^{j_m}\right) + \sum_{m=4}^{n}\left(P^i_{j_1\ldots j_m\ ,t}\xi\right)\dot{x}^{j_1}\ldots\dot{x}^{j_m} + \\ &\quad + \left(2\xi_{,t}\sum_{m=4}^{n}P^i_{j_1\ldots j_m}\dot{x}^{j_1}\ldots\dot{x}^{j_m} - \sum_{m=4}^{n} mP^i_{j_1\ldots j_m}\dot{x}^{j_1}\ldots\dot{x}^{j_m}\right) + \\ &\quad +\xi_{,r}\dot{x}^r\left(2\sum_{K=m-1}^{n}P^i_{j_1\ldots j_K}\dot{x}^{j_1}\ldots\dot{x}^{j_K} - \sum_{K=m-1}^{n}KP^i_{j_1\ldots j_m}\dot{x}^{j_1}\ldots\dot{x}^{j_K}\right) + \\ &\quad + \left(\sum_{C=m+2}^{n}CP^i_{j_1\ldots j_c}\eta^{j_1}_{,t}\ldots\dot{x}^{j_c}\right) + \xi_{,j}\dot{x}^i\left(\sum_{K=m-1}^{n}P^j_{j_1\ldots j_K}\dot{x}^{j_1}\ldots\dot{x}^{j_K}\right). \end{aligned}$$

Chapter 4

Motion on a curved space

4.1 Introduction

The study of Lie point symmetries of a given system of ODEs consists of two steps: (a) the determination of the conditions, which the components of the Lie symmetry vectors must satisfy and (b) the solution of the system of these conditions. These conditions can be quite involved, but today it is possible to use algebraic computing programs to derive them (for a review see [47]). Therefore the essential part of the work is the second step. For a small number of equations (say up to three) one can possibly employ again computer algebra to look for a solution of the system. However for a large number of equations such an attempt is prohibitive and one has to go back to traditional methods to determine the solution.

In Chapter 3 the Lie and the Noether point symmetries of the geodesic equations were calculated in terms of the special projective algebra of the space. The purpose of the present chapter is to extend the previous results and to provide an alternative way to solve the system of Lie symmetry conditions for the second order equations of the form

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k = F^i. \tag{4.1}$$

Here $\Gamma^i_{jk}(x^r)$ are general functions, a dot over a symbol indicates derivation with respect to the parameter t along the solution curves and $F^i(x^j)$ is a C^p vector field. This type of equations is important, because it contains the equations of motion of a dynamical system in a Riemannian space, in which the functions $\Gamma^i_{jk}(x^r)$ are the connection coefficients of the metric and t being an affine parameter along the trajectory. In the following we assume this identification of Γ^i_{jk}'s[1].

The key idea, which is proposed here, is to express the system of Lie symmetry conditions of (4.1) in a Riemannian space in terms of collineation (i.e. symmetry) conditions of the metric. If this is achieved, then the

[1] Of course it is possible to look for a metric for which a given set of Γ^i_{jk} are the connection coefficients, or, even avoid the metric altogether. However we shall not attempt this in the present work. For such an attempt see [9].

Lie point symmetries of (4.1) will be related to the collineations of the metric, hence their determination will be transferred to the geometric problem of determining the generators of a specific type of collineations of the metric. One then can use of existing results of Differential Geometry on collineations to produce the solution of the Lie symmetry problem.

The natural question to ask is: If the Lie symmetries of the dynamical systems moving in a given Riemannian space are from the same set of collineations of the space, how will one select the Lie symmetries of a specific dynamical system? The answer is as follows. The left hand side of Equation (4.1) contains the metric and its derivatives and it is *common to all* dynamical systems moving in the same Riemannian space. Therefore geometry (i.e. collineations) enters in the left hand side of (4.1) only. A dynamical system is defined by the force field F^i, which enters into the right hand side of (4.1) only. We conclude that there must exist constraint conditions, which will involve the components of the collineation vectors and the force field F^i, which will select the appropriate Lie symmetries for a specific dynamical system.

Indeed Theorem 4.2.2 (see section 4.2) relates the Lie symmetry generators of (4.1) with the elements of the special projective Lie algebra of the space where motion occurs, and provides these necessary constraint conditions.

What has been said for the Lie point symmetries of (4.1) applies also to Noether symmetries. The Noether symmetries are Lie point symmetries which satisfy the constraint

$$X^{[1]}L + L\frac{d\xi}{dt} = \frac{df}{dt}. \tag{4.2}$$

Theorem 4.3.2 (see section 4.3) relates the generators of Noether symmetries of (4.1) with the homothetic algebra of the metric and provides the required constraint conditions.

In the following sections we apply Theorem 4.2.2 and Theorem 4.3.2 to determine all two dimensional (section 4.4) and all three dimensional (section 4.5) Newtonian dynamical systems moving under the action of a general force F^i, which admit Lie and Noether point symmetries. In section 4.6 we apply the results to determine the conservative dynamical systems which move in a two-dimensional space of constant non-vanishing curvature and admit Noether point symmetries. The case of a conservative force has been addressed previously for the two dimensional case by Sen [48] and more recently by Damianou *et al* [49] and for the three dimensional by [50]. As it will be shown both treatments are incomplete. We demonstrate the use of the results in two cases. The non-conservative Kepler - Ermakov system [51, 52, 53] and the case of the Hènon Heiles type potentials [54, 55]. In both cases we recover and complete the existing results.

4.2 Lie symmetries of a dynamical system in a Riemannian space

I section 3.2 the Lie symmetry conditions were calculated for a general system of ODEs polynomial in the velocities, therefore the Lie symmetry conditions (determining equations) for equation (4.1) with $F^i = F^i\left(t, x^k\right)$

are

$$L_{\eta}F^{i}+2\xi_{,t}F^{i}+\xi F_{,t}^{i}+\eta^{i}{}_{,tt}=0 \tag{4.3}$$

$$\left(\xi_{,k}\delta_{j}^{i}+2\xi_{,j}\delta_{k}^{i}\right)F^{k}+2\eta^{i}{}_{,t|j}-\xi_{,tt}\delta_{j}^{i}=0 \tag{4.4}$$

$$L_{\eta}\Gamma_{(jk)}^{i}=2\xi_{,t(j}\delta_{k)}^{i} \tag{4.5}$$

$$\xi_{(,i|j}\delta_{r)}^{k}=0. \tag{4.6}$$

Equation (4.5) means that η^{i} is a projective collineation of the metric with projective function $\xi_{,t}$. Furthermore, equation (4.6) means that $\xi_{,j}$ is a gradient KV of g_{ij}; that is, η^{i} is a special projective collineation of the metric. Equation (4.3) gives[2]

$$\left(L_{\eta}g^{ij}\right)F_{j}+g^{ij}L_{\eta}F_{j}+2\xi_{,t}g^{ij}F_{j}+\xi g^{ij}F_{j,t}+\eta_{,tt}^{i}=0. \tag{4.7}$$

This equation restricts η^{i} further because it relates it directly to the metric symmetries. Finally equation (4.4) gives

$$-\delta_{j}^{i}\xi_{,tt}+\left(\xi_{,j}\delta_{k}^{i}+2\delta_{j}^{i}\xi_{,k}\right)F^{k}+2\eta_{,tj}^{i}+2\Gamma_{jk}^{i}\eta_{,t}^{k}=0. \tag{4.8}$$

Equations (4.7),(4.8) are the constraint conditions which relate the components ξ, η^{i} of the Lie point symmetry vector with the vector F^{i}.

Proposition 4.2.1 *The Lie point symmetries of the dynamical system (4.1) where $F^{i}=F^{i}\left(t,x^{k}\right)$, are generated from the special projective algebra of the space where the motion occurs.*

In the case where the dynamical system (4.1) is autonomous and conservative, that is, $F^{i}=g^{ij}V_{,j}\left(x^{k}\right)$ and $V_{,j}$ is not a gradient KV of the metric, the solution of the determining equations is given by the following theorem (for a proof see Appendix 4.A).

Theorem 4.2.2 *The Lie point symmetries of the equations of motion of an autonomous conservative system*

$$\ddot{x}^{i}+\Gamma_{jk}^{i}\dot{x}^{j}\dot{x}^{k}=g^{ij}V_{,i} \tag{4.9}$$

in a general Riemannian space with metric g_{ij}, are given in terms of the generators Y^{i} of the special projective Lie algebra of the metric g_{ij} as follows.

Case I *Lie symmetries due to the affine algebra. The resulting Lie symmetries are*

$$\mathbf{X}=\left(\frac{1}{2}d_{1}a_{1}t+d_{2}\right)\partial_{t}+a_{1}Y^{i}\partial_{i} \tag{4.10}$$

where a_{1} and d_{1} are constants, provided the potential satisfies the condition

$$L_{Y}V^{,i}+d_{1}V^{,i}=0. \tag{4.11}$$

[2] Recall that $L_{\eta}F_{j}=F_{,jk}\eta^{k}+\eta^{k}{}_{,j}F_{,k}$.

Case IIa *The Lie symmetries are generated by the gradient homothetic algebra and* $Y^i \neq V^{,i}$. *The Lie symmetries are*

$$\mathbf{X} = 2\psi \int T(t)\, dt\partial_t + T(t)\, Y^i\partial_i \tag{4.12}$$

where the function $T(t)$ *is the solution of the equation* $T_{,tt} = a_1 T$ *provided the potential* $V(x^i)$ *satisfies the condition*

$$L_{\mathbf{Y}}V^{,i} + 4\psi V^{,i} + a_1 Y^i = 0. \tag{4.13}$$

Case IIb *The Lie symmetries are generated by the gradient HV* $Y^i = \kappa V^{,i}$, *where* κ *is a constant. In this case the potential is the gradient HV of the metric and the Lie symmetry vectors are*

$$\mathbf{X} = \left(-c_1\sqrt{\psi k}\cos\left(2\sqrt{\frac{\psi}{k}}t\right) + c_2\sqrt{\psi k}\sin\left(2\sqrt{\frac{\psi}{k}}t\right)\right)\partial_t + \left(c_1\sin\left(2\sqrt{\frac{\psi}{k}}t\right) + c_2\cos\left(2\sqrt{\frac{\psi}{k}}t\right)\right)H^{,i}\partial_i. \tag{4.14}$$

Case IIIa *The Lie symmetries due to the proper special projective algebra. In this case the Lie symmetry vectors are (the index* J *counts the gradient KVs)*

$$\mathbf{X}_J = (C(t)\, S_J + D(t))\, \partial_t + T(t)\, Y^i\partial_i \tag{4.15}$$

where the functions $C(t), T(t), D(t)$ *are solutions of the system of simultaneous equations*

$$D_{,t} = \frac{1}{2}d_1 T \quad , \quad T_{,tt} = a_1 T \quad , \quad T_{,t} = c_2 C \quad , \quad D_{,tt} = d_c C \quad , \quad C_{,t} = a_0 T \tag{4.16}$$

and in addition the potential satisfies the conditions

$$L_Y V^{,i} + 2a_0 S V^{,i} + d_1 V^{,i} + a_1 Y^i = 0 \tag{4.17}$$

$$\left(S_{,k}\delta^i_j + 2S_{,j}\,\delta^i_k\right)V^{,k} + \left(2Y^i{}_{;\,j} - a_0 S\delta^i_j\right)c_2 - d_c\delta^i_j = 0. \tag{4.18}$$

Case IIIb *Lie symmetries due to the proper special projective algebra and* $Y^i_J = \lambda S_J V^{,i}$, *where* $V^{,i}$ *is a gradient HV and* $S^{;i}_J$ *is a gradient KV. The Lie symmetry vectors are*

$$X_J = (C(t)\, S_J + d_1)\, \partial_t + T(t)\lambda S_J V^{,i}\partial_i \tag{4.19}$$

where the functions $C(t)$ *and* $T(t)$ *are computed from the relations*

$$T_{,tt} + 2C_{,t} = \lambda_1 T \quad , \quad T_{,t} = \lambda_2 C \quad , \quad C_{,t} = a_0 T \tag{4.20}$$

and the potential satisfies the conditions

$$L_{\mathbf{Y}_J}V^{,i} + \lambda_1 S_J V^{,i} = 0 \tag{4.21}$$

$$C\left(\lambda_1 S_J\delta^i_j + 2S_{J,j}\,V^{,i}\right) + \lambda_2\left(2\lambda S_{J,j}V^{.i} + (2\lambda S_J - a_0 S_J)\,\delta^i_j\right) = 0. \tag{4.22}$$

An immediate important application of Theorem 4.2.2 concerns the important case of spaces of constant curvature. As we have seen already (see also [23]) the special projective algebra of a space of constant curvature consists of non-gradient KVs only. Therefore we have the following corollary.

Corollary 4.2.3 *The Lie point symmetries of the equations of motion of an autonomous conservative system (4.9) in a space of constant curvature are elements of the non-gradient KVs algebra.*

This implies that in spaces of constant curvature it is enough to consider Case I only.

If the system (4.1) is autonomous but not conservative moving under the action of the external force F^i the previous results remain valid except the cases IIb, IIIb which are not applicable. We emphasize that (with appropriate adjustments) the results apply to affine spaces in which there exists a connection Γ^i_{jk} but not necessarily a metric.

4.3 Noether symmetries of a dynamical system in a Riemannian space

Consider a particle moving in a Riemannian space with metric g_{ij} under the influence of the potential $V\left(t,x^k\right)$. The Lagrangian describing the motion of the particle is

$$L=\frac{1}{2}g_{ij}\dot{x}^i\dot{x}^j-V\left(t,x^k\right). \tag{4.23}$$

A Lie symmetry vector $\mathbf{X}=\xi\left(t,x^k\right)\partial_t+\eta^i\left(t,x^k\right)\partial_{x^i}$ is a Noether symmetry of the Lagrangian if it satisfies the condition

$$\mathbf{X}^{[1]}L+\frac{d\xi}{dt}L=\frac{df}{dt} \tag{4.24}$$

where $\mathbf{X}^{[1]}$ is the first prolongation of $\mathbf{X}$. It can be shown that condition (4.24) is equivalent to the system of equations:

$$V_{,k}\eta^k+V\xi_{,t}+\xi V_{,t}=-f_{,t} \tag{4.25}$$

$$\eta^i_{,t}g_{ij}-\xi_{,j}V=f_{,j} \tag{4.26}$$

$$L_\eta g_{ij}=2\left(\frac{1}{2}\xi_{,t}\right)g_{ij} \tag{4.27}$$

$$\xi_{,k}=0. \tag{4.28}$$

Equation (4.28) implies $\xi=\xi\left(t\right)$, and then from (4.27) follows that η^i is a HV. Therefore we have the following result

Proposition 4.3.1 *The Noether symmetries of the Lagrangian (4.23) are generated, from the homothetic algebra of the metric g_{ij} of the space where motion occurs.*

In the case the potential is autonomous, that is $V\left(t,x^{k}\right)=V\left(x^{k}\right)$, the solution of (4.25)-(4.28) relates the Noether symmetries of (4.23) with the elements of the homothetic algebra of the metric g_{ij} as follows.

Theorem 4.3.2 *The Noether symmetries of an autonomous conservative dynamical system moving in a Riemannian space with metric g_{ij} described by the Lagrangian (4.23) are generated from the homothetic Lie algebra of the metric g_{ij} as follows.*

Case I. *The KVs and the HV satisfy the condition:*

$$V_{,k}Y^{k}+2\psi_{Y}V+c_{1}=0. \tag{4.29}$$

The Noether symmetry vector is

$$\mathbf{X}=2\psi_{Y}t\partial_{t}+Y^{i}\partial_{i},\quad f=c_{1}t, \tag{4.30}$$

where $T\left(t\right)=a_{0}\neq0$.

Case II. *The metric admits the gradient KVs S_{J}, the gradient HV $H^{,i}$ and the potential satisfies the condition*

$$V_{,k}Y^{,k}+2\psi_{Y}V=c_{2}Y+d. \tag{4.31}$$

In this case the Noether symmetry vector and the Noether function are

$$\mathbf{X}=2\psi_{Y}\int T\left(t\right)dt\partial_{t}+T\left(t\right)S_{J}^{;i}\partial_{i}\quad,\quad f\left(t,x^{k}\right)=T_{,t}S_{J}\left(x^{k}\right)\ +d\int Tdt. \tag{4.32}$$

and the functions $T(t)$ and $K\left(t\right)$ ($T_{,t}\neq0$) are computed from the relations

$$T_{,tt}=c_{2}T\ ,\ K_{,t}=d\int Tdt+constant \tag{4.33}$$

where c_{2} is a constant.

In addition to the above there is also the standard Noether symmetry ∂_{t}.

The first integrals for the Noether symmetry vectors have as follows.

Proposition 4.3.3 *For the Noether vector ∂_{t} the Noether integral is the Hamiltonian* E*. For the Noether vectors of Case I and Case II the Noether integrals are respectively:*

$$\phi_{I}=2\psi_{Y}t\mathrm{E}-g_{ij}Y^{i}\dot{x}^{j}+c_{1}t \tag{4.34}$$

$$\phi_{II}=2\psi_{Y}\mathrm{E}\int Tdt\ -Tg_{ij}H^{i}\dot{x}^{j}+T_{,t}H+d\int Tdt. \tag{4.35}$$

For the case of motion in spaces of constant curvature we have the following result.

Proposition 4.3.4 *The Noether symmetry vectors of the Lagrangian (4.23) of an autonomous conservative dynamical system moving in a space of constant curvature, are generated by the non-gradient KVs of the space. Hence only Case I survives.*

In the following sections, the autonomous non linear Newtonian systems which admit Lie and Noether point symmetries are calculated with the use of Theorems 4.2.2 and 4.3.2.

Table 4.1: Special projective algebra of the 2d Euclidian space

Collineation	**Gradient**	**Non-gradient**
Killing vectors	∂_x , ∂_y	$y\partial_x - x\partial_y$
Homothetic vector	$x\partial_x + y\partial_y$	
Affine Collineation	$x\partial_x$, $y\partial_y$	$y\partial_x$, $x\partial_y$
Sp. Projective collineation		$x^2\partial_x + xy\partial_y$, $xy\partial_x + y^2\partial_y$

4.4 2D autonomous systems which admit Lie/Noether point symmetries

In this section we apply Theorems 4.2.2 and 4.3.2 to determine *all* Newtonian dynamical systems with two degrees of freedom which admit at least one Lie/Noether symmetry. The reason for considering this problem is that a Lie/Noether symmetry lead to invariants/first integrals, which can be used in many ways in order to study a given system of differential equations e.g. to simplify, to determine the integrability of the system etc. Because the Newtonian systems move in E^2 we need to consider the generators of the special projective algebra of E^2 and then use the constraint conditions for each case to determine the functional form of the force field F^i.

We consider Cartesian coordinates so that the metric of the space is

$$ds^2 = dx^2 + dy^2.$$

In Table 4.1 we give the elements of the projective Lie algebra of E^2 in Cartesian coordinates. We note that the special projective algebra of the two dimensional Lorentz space

$$ds^2 = -dx^2 + dy^2$$

is the same with that of the space E^2, with the difference that the non gradient Killing vector is replaced with $y\partial_x + x\partial_y$. We shall use this observation in later chapters where we study the Lie and the Noether point symmetries in scalar field cosmology.

We examine the cases where the force F^i (a) is non-conservative and (b) is conservative. In certain cases the results are common to both cases, however for clarity it is better to consider the two cases separately. Finally for economy of space, easy reference and convenience we present the results in the form of Tables.

In order to indicate how the results of the Tables are obtained we consider Case I & II of theorem 4.2.2.

Table 4.2: Two dimensional Newtonian systems admit Lie symmetries (1/4)

Lie $\downarrow$ $F^i \rightarrow$	$\mathbf{F}^x(x,y)/\mathbf{F}^\theta(r,\theta)$	$\mathbf{F}^y(x,y)/\mathbf{F}^\theta(r,\theta)$
$\frac{d}{2}t\partial_t + \partial_x$	$e^{-dx}f(y)$	$e^{-dx}g(y)$
$\frac{d}{2}t\partial_t + \partial_y$	$e^{-dy}f(x)$	$e^{-dy}g(x)$
$\frac{d}{2}t\partial_t + (y\partial_x - x\partial_y)$	$f(r)\,e^{-d\theta}$	$g(r)\,e^{-d\theta}$
$\frac{d}{2}t\partial_t + x\partial_x + y\partial_y$	$x^{(1-d)}f\left(\frac{y}{x}\right)$	$x^{(1-d)}g\left(\frac{y}{x}\right)$
$\frac{d}{2}t\partial_t + x\partial_x$	$x^{(1-d)}f(y)$	$x^{-d}g(y)$
$\frac{d}{2}t\partial_t + y\partial_y$	$y^{-d}f(x)$	$y^{(1-d)}g(x)$
$\frac{d}{2}t\partial_t + y\partial_x$	$\left(\frac{x}{y}g(y) + f(y)\right)e^{-d\frac{x}{y}}$	$g(y)\,e^{-d\frac{x}{y}}$
$\frac{d}{2}t\partial_t + x\partial_y$	$f(x)\,e^{-d\frac{y}{x}}$	$\left(\frac{y}{x}f(x) + g(x)\right)e^{-d\frac{y}{x}}$

The Lie point symmetry vectors for Case I are given by (4.10) i.e.

$$\mathbf{X} = \left(\frac{1}{2}d_1a_1t + d_2\right)\partial_t + a_1Y^i\partial_i, \tag{4.36}$$

where a_1 and d_1 are constants and Y^i is a vector of the affine algebra of E^2. The force field must satisfy condition (4.11) i.e.

$$L_Y\mathbf{F} + d_1\mathbf{F} = 0.$$

Writing

$$\mathbf{F} = F^x\partial_x + F^y\partial_y \text{ and } \mathbf{Y} = Y^x\partial_x + Y^y\partial_y$$

we obtain a system of two differential equations involving the unknown quantities F^x, F^y and the known quantities Y^x, Y^y. For each vector $\mathbf{Y}$ we replace Y^x, Y^y from Table 4.1 and solve the system to compute F^x, F^y. For example for the gradient KV ∂_x we have $Y^x = 1, Y^y = 0$ and find the solution $F^x(x,y) = e^{-dx}f(y)$, $F^y(x,y) = e^{-dx}g(y)$ where d is a constant and $f(y), g(y)$ are arbitrary functions of their argument. Working similarly we determine the form of the force field for all cases of Theorem 4.2.2. The results are given in Tables 4.2 and 4.3.

Case III: Y^i is a special PC. There is only one dynamical system in this case, which is the forced oscillator, acted upon the external force $F^i = (\omega x + a)\partial_x + (\omega y + b)\partial_y$, that is the system is conservative. As it can be seen from Table 4.1 the Lie symmetry algebra of the forced oscillator is the $sl(4,R)$. This result agrees with that of [6].

Table 4.3: Two dimensional Newtonian systems admit Lie symmetries (2/4)

Lie $\downarrow$ $V\rightarrow$	$\mathbf{F}^{x}(x,y)$	$\mathbf{F}^{y}(x,y)$
$T(t)\partial_{x}$	$-mx+f(y)$	$g(y)$
$T(t)\partial_{y}$	$f(x)$	$-my+g(x)$
$2\int T(t)\,dt\ \partial_{t}+T(t)(x\partial_{x}+y\partial_{y})$	$-\frac{m}{4}x+x^{-3}f\left(\frac{y}{x}\right)$	$-\frac{m}{4}y+y^{-3}g\left(\frac{y}{x}\right)$

Except the above three cases we have to consider the Lie point symmetries generated from linear combinations of the vectors Y^{i}. It is found that the only new cases are the ones given given in Tables 4.14 and 4.15

We assume now F^{i} to be conservative with potential function $V(x,y)$. In this case the results of the previous Tables differentiate. The results of the calculations are given in Tables 4.4, 4.16 and 4.17 .

As it was stated in section 4.1 the determination of all two dimensional potentials which admit a Lie point symmetry has been addressed previously in [48] and [49]. Our results contain the results of both these papers and additionally some cases missing, mainly in the linear combinations of the HV with the KVs.

4.4.1 2D autonomous Newtonian systems which admit Noether symmetries

Noether symmetries are associated with a Lagrangian. Therefore we consider only the case in which the force F^{i} is conservative. Furthermore Noether symmetries are special Lie symmetries, hence we look into the two dimensional potentials which admit a Lie point symmetry. These potentials were determined in the previous section. We apply Theorem 4.3.2 to these potentials and select the potentials which admit a Noether symmetry. The calculations are similar to the ones for the Lie point symmetries and are omitted. The results are listed in Tables 4.5 and 4.6. In the next section we apply the same method to determine the three dimensional autonomous Newtonian systems which admit Lie and Noether symmetries.

Table 4.4: Two dimensional conservative Newtonian systems admit Lie symmetries (1/3)

Lie $\downarrow$ $V \rightarrow$	$\mathbf{d=0}$	$\mathbf{d\neq 0}$	$\mathbf{d=2}$
$\frac{d}{2}t\partial_t + \partial_x$	$c_1 x + f(y)$	$f(y)\,e^{-dx}$	$f(y)\,e^{-2x}$
$\frac{d}{2}t\partial_t + \partial_y$	$c_1 y + f(x)$	$f(x)\,e^{-dy}$	$f(x)\,e^{-2y}$
$\frac{d}{2}t\partial_t + (y\partial_x - x\partial_y)$	$\theta + f(r)$	$f(r)\,e^{-d\theta}$	$f(r)\,e^{-2\theta}$
$\frac{d}{2}t\partial_t + (x\partial_x + y\partial_y)$	$x^2 f\left(\frac{y}{x}\right)$	$x^{2-d} f\left(\frac{y}{x}\right)$	$c_1 \ln x + f\left(\frac{y}{x}\right)$
$\frac{d}{2}t\partial_t + x\partial_x$	$c_1 x^2 + f(y)$	$\nexists$	$\nexists$
$\frac{d}{2}t\partial_t + y\partial_y$	$c_1 y^2 + f(x)$	$\nexists$	$\nexists$
$\frac{d}{2}t\partial_t + y\partial_x$	$x^2 + y^2 + c_1 x$	$\nexists$	$\nexists$
$\frac{d}{2}t\partial_t + x\partial_y$	$x^2 + y^2 + c_1 y$	$\nexists$	$\nexists$

Lie $\downarrow$ $V \rightarrow$	$\mathbf{T}_{,tt} = \mathbf{mT}$.
$T(t)\,\partial_x$	$-\frac{mx^2}{2} + c_1 x + f(y)$
$T(t)\,\partial_y$	$-\frac{my^2}{2} + c_1 y + f(x)$
$2\int T(t)\,dt\ \partial_t + T(t)\,(x\partial_x + y\partial_y)$	$-\frac{m}{8}\left(x^2+y^2\right) + \frac{1}{x^2} f\left(\frac{y}{x}\right)$

Table 4.5: Two dimensional autonomous Newtonian systems admiting Noether symmetries (1/2)

Noether Symmetry	$\mathbf{V}(x,y)$	**Noether Symmetries**	$\mathbf{V}(x,y)$
∂_x	$cx+f(y)$	$\partial_x+b\partial_y$	$f(y-bx)-cx$
∂_y	$cy+f(x)$	$(a+y)\partial_x+(b-x)\partial_y$	$f\left(\frac{1}{2}\left(x^2+y^2\right)+ay-bx\right)$
$y\partial_x-x\partial_y$	$c\theta+f(r)$	$2t\partial_t+(x+ay)\partial_x+(y-ax)\partial_y$	$r^{-2}\ f(\theta-a\ln r)$
$2t\partial_t+x\partial_x+y\partial_y$	$x^{-2}f\left(\frac{y}{x}\right)$	$2t\partial_t+(a+x)\partial_x+(b+y)\partial_y$	$f\left(\frac{b+x}{a+x}\right)(a+x)^{-2}-c\,(a+x)^{-2}\left(\frac{1}{2}x^2+ax\right)$

Table 4.6: Two dimensional autonomous Newtonian systems admiting Noether symmetries (2/2)

Noether $\downarrow V\rightarrow$	$\mathbf{T}_{,tt}=\mathbf{mT}$
$T(t)\,\partial_x$	$f(y)-cx-\frac{m}{2}x^2$
$T(t)\,\partial_y$	$f(x)-cy-\frac{m}{2}y^2$
$2\int T(t)\,dt\ \partial_t+T(t)\,(x\partial_x+y\partial_y)$	$x^{-2}f\left(\frac{y}{x}\right)-\frac{m}{8}\left(x^2+y^2\right)$

Noether $\downarrow$ $V\rightarrow$	$\mathbf{T}_{,tt}=\mathbf{mT}$
$T(t)\,\partial_x+bT(t)\,\partial_y$	$-\frac{m}{2}\left(x^2+y^2\right)-\frac{m}{2}(y-bx)^2+f(y-bx)-cx$
$2\int T(t)\,dt\ \partial_t+T(t)\,((a+x)\,\partial_x+(b+y)\,\partial_y)$	$f\left(\frac{b+x}{a+x}\right)(a+x)^{-2}-\frac{c}{2}(a+x)^{-2}x\,(x+2a)$ $-\frac{xm(x+2a)}{8(a+x)^4}\left\{\begin{array}{c}\left((a+x)^2+a^2\right)y\,(y+2b)+\\ +x\,(x+2a)\,(b+(a+x))\,(-b+(a+x))\end{array}\right\}$

4.5 3D autonomous Newtonian systems admit Lie/Noether point symmetries

In this section we determine all Newtonian systems with *three* degrees of freedom which admit at least one Lie/Noether symmetry (except the obvious symmetry ∂_t). In order to use Theorem 4.2.2 we need the special projective algebra of the Euclidian 3D metric

Table 4.7: Three dimensional autonomous Newtonian systems admit Lie symmetries (1/2)

Lie symmetry	$\mathbf{F}_{\mu}\left(x_{\mu},x_{\nu},x_{\sigma}\right)$	$\mathbf{F}_{\nu}\left(x_{\mu},x_{\nu},x_{\sigma}\right)$	$\mathbf{F}_{\sigma}\left(x_{\mu},x_{\nu},x_{\sigma}\right)$
$\frac{d}{2}t\partial_t+\partial_\mu$	$e^{-dx_\mu}f\left(x_\nu,x_\sigma\right)$	$e^{-dx_\mu}g\left(x_\nu,x_\sigma\right)$	$e^{-dx_\mu}h\left(x_\nu,x_\sigma\right)$
$\frac{d}{2}t\partial_t+\partial_{\theta_{(\mu\nu)}}$	$e^{-d\theta_{(\mu\nu)}}f\left(r_{(\mu\nu)},x_\sigma\right)$	$e^{-d\theta_{(\mu\nu)}}g\left(r_{(\mu\nu)},x_\sigma\right)$	$e^{-d\theta_{(\mu\nu)}}h\left(r_{(\mu\nu)},x_\sigma\right)$
$\frac{d}{2}t\partial_t+R\partial_R$	$x_\mu^{1-d}f\left(\frac{x_\nu}{x_\mu},\frac{x_\sigma}{x_\mu}\right)$	$x_\mu^{1-d}g\left(\frac{x_\nu}{x_\mu},\frac{x_\sigma}{x_\mu}\right)$	$x_\mu^{1-d}h\left(\frac{x_\nu}{x_\mu},\frac{x_\sigma}{x_\mu}\right)$
$\frac{d}{2}t\partial_t+x_\mu\partial_\mu$	$x_\mu^{1-d}f\left(x_\nu,x_\sigma\right)$	$x_\mu^{1-d}g\left(x_\nu,x_\sigma\right)$	$x_\mu^{1-d}h\left(x_\nu,x_\sigma\right)$
$\frac{d}{2}t\partial_t+x_\nu\partial_\mu$	$e^{-d\frac{x_\mu}{x_\nu}}\left[\frac{x_\mu}{x_\nu}g\left(x_\nu,x_\sigma\right)+f\left(x_\nu,x_\sigma\right)\right]$	$e^{-d\frac{x_\mu}{x_\nu}}g\left(x_\nu,x_\sigma\right)$	$e^{-d\frac{x_\mu}{x_\nu}}h\left(x_\nu,x_\sigma\right)$

$$ds_E^2=dx^2+dy^2+dz^2. \tag{4.37}$$

This algebra consists of 15 vectors[3] as follows: Six KVs ∂_μ , $x_\nu\partial_\mu - x_\mu\partial_\nu$ one HV $R\partial_R$, nine ACs $x_\mu\partial_\mu$, $x_\nu\partial_\mu$ and three SPCs $x_\mu^2\partial_\mu + x_\mu x_\nu\partial_\nu + x_\mu x_\sigma\partial_\sigma$, where[4] $\mu\neq\nu\neq\sigma$, $r_{(\mu\nu)}^2 = x_\mu^2+x_\nu^2$, $\theta_{(\mu\nu)}=\arctan\left(\frac{x_\nu}{x_\mu}\right)$ and R,θ,ϕ are spherical coordinates.

In the computation of Lie symmetries we consider only the linearly independent vectors of the special projective group. We do not consider their linear combinations because the resulting Lie symmetries are too many; on the other hand they can be computed in the standard way.

4.5.1 3D autonomous Newtonian systems admit Lie point symmetries

In Tables 4.7 and 4.8 we list the Lie point symmetries and the functional dependence of the components of the force for Case I and II of Theorem 4.2.2.

For the remaining Case III of Theorem 4.2.2, we have that the force F^μ admits Lie symmetries generated from the proper sp. PCs if the force is the isotropic oscillator, that is, $F^\mu = (\omega x^\mu + c^\mu)\,\partial_\mu$ where $\omega,\ c^\mu$ are constants. From Tables 4.7 and 4.8 we infer that the isotropic oscillator admits 24 Lie point symmetries generating the $Sl\,(5,R)$, as many as the free particle [6].

In order to demonstrate the use of the above Tables let us require the equations of motion of a Newtonian dynamical system which is invariant under the $sl(2,R)$ algebra. We know [56] that $sl(2,R)$ is generated by the following Lie symmetries

$$\partial_t\ ,\ 2t\partial_t+R\partial_R\ ,t^2\partial_t+tR\partial_R.$$

[3] These vectors are not all linearly independent i.e. the HV and the rotations are linear combinations of the ACs
[4] If $x_\mu = x,$ then $\{x_\nu = y\ ,\ x_\sigma = z\}$ or $\{x_\nu = z\ ,\ x_\sigma = y\}$

Table 4.8: Three dimensional autonomous Newtonian systems admit Lie symmetries (1/2)

Lie symmetry	$\mathbf{F}_\mu(x_\mu, x_\nu, x_\sigma)$	$\mathbf{F}_\nu(x_\mu, x_\nu, x_\sigma)$	$\mathbf{F}_\sigma(x_\mu, x_\nu, x_\sigma)$
$t\partial_\mu$	$f(x_\nu, x_\sigma)$	$g(x_\nu, x_\sigma)$	$h(x_\nu, x_\sigma)$
$t^2\partial_t + tR\partial_R$	$\frac{1}{x_\mu^3} f\left(\frac{x_\nu}{x_\mu}, \frac{x_\sigma}{x_\mu}\right)$	$\frac{1}{x_\mu^3} g\left(\frac{x_\nu}{x_\mu}, \frac{x_\sigma}{x_\mu}\right)$	$\frac{1}{x_\mu^3} h\left(\frac{x_\nu}{x_\mu}, \frac{x_\sigma}{x_\mu}\right)$
$e^{\pm t\sqrt{m}}\partial_\mu$	$-mx_\mu + f(x_\nu, x_\sigma)$	$g(x_\nu, x_\sigma)$	$h(x_\nu, x_\sigma)$
$\frac{1}{\sqrt{m}} e^{\pm t\sqrt{m}}\partial_t \pm e^{\pm t\sqrt{m}} R\partial_R$	$-\frac{m}{4}x_\mu + \frac{1}{x_\mu^3} f\left(\frac{x_\nu}{x_\mu}, \frac{x_\sigma}{x_\mu}\right)$	$-\frac{m}{4}x_\nu + \frac{1}{x_\mu^3} g\left(\frac{x_\nu}{x_\mu}, \frac{x_\sigma}{x_\mu}\right)$	$-\frac{m}{4}x_\sigma + \frac{1}{x_\mu^3} h\left(\frac{x_\nu}{x_\mu}, \frac{x_\sigma}{x_\mu}\right)$

From Table 4.7 and from Table 4.8 we have that the force must be of the form

$$F = \left(\frac{1}{x_\mu^3} f\left(\frac{x_\nu}{x_\mu}, \frac{x_\sigma}{x_\mu}\right), \frac{1}{x_\mu^3} g\left(\frac{x_\nu}{x_\mu}, \frac{x_\sigma}{x_\mu}\right), \frac{1}{x_\mu^3} h\left(\frac{x_\nu}{x_\mu}, \frac{x_\sigma}{x_\mu}\right)\right) \tag{4.38}$$

therefore, the equations of motion of this system in Cartesian coordinates are:

$$(\ddot{x}, \ddot{y}, \ddot{z}) = \left(\frac{1}{x_\mu^3} f\left(\frac{x_\nu}{x_\mu}, \frac{x_\sigma}{x_\mu}\right), \frac{1}{x_\mu^3} g\left(\frac{x_\nu}{x_\mu}, \frac{x_\sigma}{x_\mu}\right), \frac{1}{x_\mu^3} h\left(\frac{x_\nu}{x_\mu}, \frac{x_\sigma}{x_\mu}\right)\right). \tag{4.39}$$

Immediately we recognize that this dynamical system is the well known and important generalized Kepler Ermakov system (see [56]). A different representation of $sl(2, R)$ consists of the vectors

$$\partial_t\ ,\ \frac{1}{\sqrt{m}} e^{\pm t\sqrt{m}}\partial_t \pm e^{\pm t\sqrt{m}} R\partial_R$$

For this representation from Table 4.8 we have

$$F' = -\frac{m}{4}(x_\mu, x_\nu, x_\sigma) + \left(\frac{1}{x_\mu^3} f\left(\frac{x_\nu}{x_\mu}, \frac{x_\sigma}{x_\mu}\right), \frac{1}{x_\mu^3} g\left(\frac{x_\nu}{x_\mu}, \frac{x_\sigma}{x_\mu}\right), \frac{1}{x_\mu^3} h\left(\frac{x_\nu}{x_\mu}, \frac{x_\sigma}{x_\mu}\right)\right) \tag{4.40}$$

which leads again to the *autonomous Kepler Ermakov system*. In a subsequent chapter, we shall apply the results obtained here to study the integrability of the 3D Hamiltonian Kepler–Ermakov system and generalize it in a Riemannian space.

In case the force is given by the potential $V = V(x^\mu)$, that is, that the system is conservative, we obtain the results in Table 4.9.

Table 4.9: Three dimensional conservative Newtonian systems admiting Lie symmetries

Lie /V(x,y,z)	$\mathbf{d=0}$	$\mathbf{d=2}$	$\mathbf{d\neq 0,2}$
$\frac{d}{2}t\partial_t+\partial_\mu$	$c_1x_\mu+f(x_\nu,x_\sigma)$	$e^{-2x_\mu}f(x_\nu,x_\sigma)$	$e^{-dx_\mu}f(x_\nu,x_\sigma)$
$\frac{d}{2}t\partial_t+\partial_{\theta_{(\mu\nu)}}$	$c_1\theta_{(\mu\nu)}+f\left(r_{(\mu\nu)},x_\sigma\right)$	$e^{-2\theta_{(\mu\nu)}}f\left(r_{(\mu\nu)},x_\sigma\right)$	$e^{-d\theta_{(\mu\nu)}}f\left(r_{(\mu\nu)},x_\sigma\right)$
$\frac{d}{2}t\partial_t+R\partial_R$	$x^2f\left(\frac{x_\nu}{x_\mu},\frac{x_\sigma}{x_\mu}\right)$	$c_1\ln(x_\mu)+f\left(\frac{x_\nu}{x_\mu},\frac{x_\sigma}{x_\mu}\right)$	$x^{2-d}f\left(\frac{x_\nu}{x_\mu},\frac{x_\sigma}{x_\mu}\right)$
$\frac{d}{2}t\partial_t+x_\mu\partial_\mu$	$c_1x_\mu^2+f(x_\nu,x_\sigma)$	$\nexists$	$\nexists$
$\frac{d}{2}t\partial_t+x_\nu\partial_\mu$	$c_1x_\mu+c_2\left(x_\mu^2+x_\nu^2\right)+f(x_\sigma)$	$\nexists$	$\nexists$

Lie	$\mathbf{V}(x,y,z)$	**Lie**	$\mathbf{V}(x,y,z)$
$t\partial_\mu$	$c_1x_\mu+f(x_\nu,x_\sigma)$	$e^{\pm t\sqrt{m}}\partial_\mu$	$-\frac{m}{2}x_\mu^2+c_1x_\mu+f(x_\nu,x_\sigma)$
$t^2\partial_t+tR\partial_R$	$\frac{1}{x_\mu^2}f\left(\frac{x_\nu}{x_\mu},\frac{x_\sigma}{x_\mu}\right)$	$\frac{1}{\sqrt{m}}e^{\pm t\sqrt{m}}\partial_t+e^{\pm t\sqrt{m}}R\partial_R$	$-\frac{m}{8}\left(x_\mu^2+x_\nu^2+x_\sigma^2\right)+\frac{1}{x_\mu^2}f\left(\frac{x_\nu}{x_\mu},\frac{x_\sigma}{x_\mu}\right)$

4.5.2 3D autonomous Newtonian systems which admit Noether point symmetries

In this section using theorem 4.3.2 we determine all autonomous Newtonian Hamiltonian systems with Lagrangian

$$L=\frac{1}{2}\left(\dot{x}^2+\dot{y}^2+\dot{z}^2\right)-V(x,y,z) \tag{4.41}$$

which admit a non-trivial Noether point symmetry. This problem has been considered previously in [49, 50], however as we shall show the results in these works are not complete. We note that the Lie symmetries of a conservative system are not necessarily Noether symmetries. The inverse is of course true.

Before we continue we note that the homothetic algebra of the Euclidian 3d space E^3 has dimension seven and consists of three gradient KVs ∂_μ with gradient function x_μ, three non-gradient KVs $x_\nu\partial_\mu-x_\mu\partial_\nu$ generating the rotational algebra $so(3)$, and a gradient HV $H^i=R\partial_R$ with gradient function $H=\frac{1}{2}R^2$, where $R^2=x^\mu x_\mu$.

The Noether point symmetries generated from the homothetic algebra i.e. the non-gradient $so(3)$ elements included, are shown in Table 4.10. Moreover, the Noether symmetries generated from the gradient homothetic algebra are listed in Table 4.11.

The corresponding Noether integrals are computed easily from proposition 4.3.3. In Tables 4.18, 4.19 and 4.20 (see Appendix 4.B) we give a complete list of the potentials resulting form the linear combinations of the

Table 4.10: Three dimensional conservative Newtonian systems admit Noether symmetries (1/5)

Noether Symmetry	$\mathbf{V(x,y,z)}$
∂_μ	$-px_\mu + f\left(x^\nu, x^\sigma\right)$
$x_\nu\partial_\mu - x_\mu\partial_\nu$	$-p\theta_{(\mu\nu)} + f\left(r_{(\mu\nu)}, x^\sigma\right)$
$2t\partial_t + R\partial_R$	$\frac{1}{R^2}f\left(\theta,\phi\right)$ or $\frac{1}{x_\mu^2}f\left(\frac{x_\nu}{x_\mu},\frac{x_\sigma}{x_\mu}\right)$

Table 4.11: Three dimensional conservative Newtonian systems admit Noether symmetries (2/5)

Noether Symmetry	$\mathbf{V(x,y,z)}$ / $\mathbf{T}_{,tt}\mathbf{= mT}$
$T\left(t\right)\partial_\mu$	$-\frac{m}{2}x_\mu^2 - px_\mu + f\left(x_\nu, x_\sigma\right)$
$\left(2\int T\left(t\right)dt\right)\partial_t + T\left(t\right)R\partial_R$	$-\frac{m}{8}R^2 + \frac{1}{R^2}f\left(\theta,\phi\right)$ or $-\frac{m}{8}R^2 + \frac{1}{x_\mu^2}f\left(\frac{x_\nu}{x_\mu},\frac{x_\sigma}{x_\mu}\right)$

elements of the homothetic algebra. From the Tables we infer that the isotropic linear forced oscillator admits 12 Noether point symmetries, as many as the free particle.

As it has been remarked above, the determination of the Noether point symmetries admitted by an autonomous Newtonian Hamiltonian system has been considered previously in [50]. Our results extend the results of [50] and coincide with them if we set the constant $p = 0$. For example in page 12 case 1 and page 15 case 6 of [50] the terms $-\frac{p}{a}x_\mu$ and $p\arctan\left(l\left(\theta,\phi\right)\right)$ are missing respectively. Furthermore the potential given in page/line 12/1, 13/2, 13/3 of [50] admits Noether symmetries only when $\lambda = 0$ and $b_{1,2}\left(t\right) = const$. This is due to the fact that the vectors given in [50] are KVs and in order to have $b_{,t} \neq 0$ they must be given by Case II of theorem 4.3.2 above, that is, the KVs must be gradient. However the KVs used are linear combinations of translations and rotations which are non-gradient.

It is possible that there exist integrable Newtonian dynamical systems for potentials not included in these Tables, for example systems which admit only dynamical symmetries [57, 58] with integrals quadratic in momenta [59]. However these systems are not integrable via Noether point symmetries.

We remark that from the above results we are also able to give, without any further calculations, the Lie and the Noether point symmetries of a dynamical system 'moving' in a three dimensional flat space whose metric has Lorentzian signature simply by taking one of the coordinates to be complex, for example by setting $x^1 = ix^1$.

4.6 Motion on the two dimensional sphere

A first application of the results of section 4.5.2 is the determination of Lie and Noether point symmetries admitted by the equations of motion of a Newtonian particle moving in a two dimensional space of constant non-vanishing curvature.

Before we continue it is useful to recall some facts concerning spaces of constant curvature. Consider a $n+1$ dimensional flat space with fundamental form

$$ds^2 = \sum_a c_a (dz^a)^2 \quad a = 1, 2..., n+1$$

where c_a are real constants. The hypersurfaces defined by

$$\sum_a c_a (dz^a)^2 = eR_0^2$$

where R_0 is an arbitrary constant and $e = \pm 1$ are called **fundamental hyperquadrics** of the space. When all coefficients c_a are positive the space is Euclidian and $e = +1$. In this case there is one family of hyperquadrics which is the hyperspheres. In all other cases (excluding the case when all c_a 's are negative) there are two families of hyperquadrics corresponding to the values $e = +1$ and $e = -1$. It has been shown that in all cases the hyperquadrics are spaces of constant curvature (see [60] p202).

Consider an autonomous dynamical system moving in the two dimensional sphere (Euclidian ($\varepsilon = 1$) or Hyperbolic ($\varepsilon = -1$)) with Lagrangian[5] [61]

$$L\left(\phi, \theta, \dot{\phi}, \dot{\theta}\right) = \frac{1}{2}\left(\dot{\phi}^2 + \text{Sinn}^2\phi\, \dot{\theta}^2\right) - V(\theta, \phi) \tag{4.42}$$

where

$$\text{Sinn}\phi = \begin{cases} \sin\phi & \varepsilon = 1 \\ \sinh\phi & \varepsilon = -1 \end{cases}, \quad \text{Cosn}\phi = \begin{cases} \cos\phi & \varepsilon = 1 \\ \cosh\phi & \varepsilon = -1. \end{cases}$$

The equations of motion are

$$\ddot{\phi} - \text{Sinn}\phi\, \text{Cosn}\phi\, \dot{\theta}^2 + V_{,\phi} = 0 \tag{4.43}$$

$$\ddot{\theta} + 2\frac{\text{Cosn}\phi}{\text{Sinn}\phi}\, \dot{\theta}\dot{\phi} + \frac{1}{\text{Sinn}^2\phi} V_{,\theta} = 0. \tag{4.44}$$

For the Lagrangian (4.42) proposition 4.3.4 applies and we use it to find the potentials $V(\theta, \phi)$ for which additional Noether point symmetries, hence Noether integrals are admitted.

The homothetic algebra of a metric of spaces of constant curvature consists only of non-gradient KVs (hence $\psi = 0$) as follows

[5] We use spherical coordinates which are natural in the case of spaces of constant curvature.

(a) $\varepsilon = 1$ (Euclidian case)

$$CK_e^1 = \sin\theta\partial_\phi + \cos\theta\cot\phi\partial_\theta,\ CK_e^2 = \cos\theta\partial_\phi - \sin\theta\cot\phi\partial_\theta,\ CK_e^3 = \partial_\theta \tag{4.45}$$

(b) $\varepsilon = -1$ (Hyperbolic case)

$$CK_h^1 = \sin\theta\partial_\phi + \cos\theta\ \mathrm{coth}\phi\partial_\theta,\ CK^2 = \cos\theta\partial_\phi - \sin\theta\ \mathrm{coth}\phi\partial_\theta,\ CK^3 = \partial_\theta. \tag{4.46}$$

Therefore the Noether vectors and the Noether function are

$$\mathbf{X} = CK_{e,h}^i\partial_i, \quad f = pt \tag{4.47}$$

provided the potential satisfies the condition

$$\mathcal{L}_{CK}V + p = 0. \tag{4.48}$$

The first integrals given by proposition 4.3.3 are

$$\phi_{II} = -g_{ij}^i CK_{e,h}^i \dot{x}^j + pt \tag{4.49}$$

and are time dependent if $p \neq 0$.

4.6.1 Noether Symmetries

We consider two cases, the case $V(\theta,\phi)$ =constant which concerns the geodesics of the space, and the case $V(\theta,\phi) \neq$constant.

For the case of geodesics it has been shown (see section 3.5) that the Noether point symmetries are the three elements of $so(3)$ with corresponding Noether integrals

$$I_{CK_{e,h}^1} = \dot{\phi}\sin\theta + \dot{\theta}\cos\theta\ \mathrm{Sinn}\phi\ \mathrm{Cosn}\phi \tag{4.50}$$

$$I_{CK_{e,h}^2} = \dot{\phi}\cos\theta - \dot{\theta}\sin\theta\ \mathrm{Sinn}\phi\ \mathrm{Cosn}\phi \tag{4.51}$$

$$I_{CK_{e,h}^3} = \dot{\theta}\ \mathrm{Sinn}^2\phi. \tag{4.52}$$

These integrals are in involution with the Hamiltonian hence the system is Liouville integrable.

In the case $V(\theta,\phi) \neq$constant we find the results of Table 4.12

The first integrals which correspond to each potential of Table 4.12 are in involution with the Hamiltonian and independent. Hence the corresponding systems are integrable. From Table 4.12 we infer the following result.

Proposition 4.6.1 *A dynamical system with Lagrangian (4.42) has one, two or four Noether point symmetries hence Noether integrals.*

Table 4.12: Noether symmetries/Integrals and potentials for the Lagrangian of the 2D sphere

Noether Symmetry	$\mathbf{V}(\theta,\phi)$	**Noether Integral**
$CK^1_{e,h}$	$F(\cos\theta\ \text{Sinn}\phi)$	$I_{CK^1_{e,h}}$
$CK^2_{e,h}$	$F(\sin\theta\ \text{Sinn}\phi)$	$I_{CK^2_{e,h}}$
$CK^3_{e,h}$	$F(\phi)$	$I_{CK^3_{e,h}}$
$aCK^1_{e,h}+bCK^2_{e,h}$	$F\left(\frac{1+\tan^2\theta}{\text{Sinn}^2\phi\ (a-b\tan\theta)^2}\right)$	$aI_{CK^1_{e,h}}+bI_{CK^2_{e,h}}$
$aCK^1_{e,h}+bCK^3_{e,h}$	$F(a\cos\theta\text{Sinn}\phi-\varepsilon\ b\ \text{Cosn}\phi)$	$aI_{CK^1_{e,h}}+bI_{CK^3_{e,h}}$
$aCK^2_{e,h}+bCK^3_{e,h}$	$F(a\sin\theta\text{Sinn}\phi-\varepsilon\ b\ \text{Cosn}\phi)$	$aI_{CK^2_{e,h}}+bI_{CK^3_{e,h}}$
$aCK^1_{e,h}+bCK^2_{e,h}+cCK^3_{e,h}$	$F((a\cos\theta-b\sin\theta)\ \text{Sinn}\phi-\varepsilon\ c\ \text{Cosn}\phi)$	$aI_{CK^1_{e,h}}+bI_{CK^2_{e,h}}+cI_{CK^3_{e,h}}$

Proof. For the case of the free particle we have the maximum number of four Noether symmetries (the rotation group $so(3)$ plus the ∂_t). In the case the potential is not constant the Noether symmetries are produced by the non-gradient KVs with Lie algebra

$$[X_A, X_B] = C^C_{AB} X_C$$

where $C^3_{12} = C^2_{31} = C^1_{23} = 1$ for $\varepsilon = 1$ and $\bar{C}^3_{21} = \bar{C}^1_{23} = \bar{C}^2_{31} = 1$ for $\varepsilon = -1$. Because the Noether point symmetries form a Lie algebra and the Lie algebra of the KVs is semisimple the system will admit either none, one or three Noether point symmetries generated by the KVs. The case of three is when $V(\theta,\phi) = V_0$ that is the case of geodesics, therefore the Noether point symmetries will be (including ∂_t) either one, two or four. ■

We note that the two important potentials of Celestial Mechanics, that is $V_1 = -\frac{\text{Cosn}\phi}{\text{Sinn}\phi}$, $V_2 = \frac{1}{2}\frac{\text{Sinn}^2\phi}{\text{Cosn}^2\phi}$ which according to Bertrand 's Theorem [61, 62, 63] produce closed orbits on the sphere are included in Table 4.12. Hence the dynamical systems they define are Liouville integrable via Noether point symmetries $CK^3_{e,h}$. The potential V_1 corresponds to the Newtonian Kepler potential and V_2 is the analogue of the harmonic oscillator. We also note that our results contain those of [61, 64]

We emphasize that the potentials listed in Table 4.12 concern dynamical systems with Lagrangian (4.42) which are integrable via Noether point symmetries.

4.7 Applications

In this section we demonstrate the application of the results of section 4.4 in two cases. The first case is the Kepler-Ermakov system, which (in general) is not a conservative dynamical system and the second is the Hènon - Heiles type potential.

4.7.1 Lie point symmetries of the Kepler-Ermakov system.

The Ermakov systems are time dependent dynamical systems, which contain an arbitrary function of time (the frequency function) and two arbitrary homogeneous functions of dynamical variables. A central feature of Ermakov systems is their property of always having a first integral. The Kepler-Ermakov system is an autonomous Ermakov system defined by the equations [65]

$$\ddot{x}+\frac{x}{r^{3}}H\left(x,y\right)-\frac{1}{x^{3}}f\left(\frac{y}{x}\right) = 0 \tag{4.53}$$

$$\ddot{y}+\frac{y}{r^{3}}H\left(x,y\right)-\frac{1}{y^{3}}g\left(\frac{y}{x}\right) = 0 \tag{4.54}$$

where H,f,g are arbitrary functions. In [51] it has been shown that this system admits Lie point symmetries for certain forms of the function $H\left(x,y\right)$. Furthermore it has been shown that for special classes of these equations there exists a Lagrangian (see also [52]).

In the following we demonstrate the use of our results by finding the Lie point symmetries simply by reading the entries of the proper Tables. Looking at the Tables we find that equations (4.53), (4.54) admit a Lie point symmetry for the following two cases.

Case 1. When $H\left(x,y\right)=\frac{h\left(\frac{y}{x}\right)}{x}$. Then from Tables 4.2 and 4.3 ($m=0$) we have that the Lie point symmetries are

$$X=\left(c_{1}+c_{2}2t+c_{3}t^{2}\right)\partial_{t}+\left(c_{2}x+c_{3}tx\right)\partial_{x}+\left(c_{2}y+c_{3}ty\right)\partial_{y}. \tag{4.55}$$

Case 2. When $H\left(x,y\right)=\omega^{2}r^{3}+\frac{h\left(\frac{y}{x}\right)}{x}$ where $m=-4\omega^{2}$ and $m\neq0$. In this case Table 4.3 for $m\neq0$ applies and the Lie point symmetry generator is

$$X=\left(c_{1}-\frac{c_{2}}{\omega}\cos\left(2\omega t\right)+\frac{c_{3}}{\omega}\sin\left(2\omega t\right)\right)\partial_{t}+\left(c_{2}\sin\left(2\omega t\right)+c_{3}\cos\left(2\omega t\right)\right)x\partial_{x}+\left(c_{2}\sin\left(2\omega t\right)+c_{3}\cos\left(2\omega t\right)\right)y\partial_{y}.$$

These symmetries coincide with the ones found in [51]. We note that in both cases the Lie symmetry vectors come from the HV $x\partial_{x}+y\partial_{y}$ of the Euclidean metric.

In a subsequent publication [52] it was shown that the Lagrangian considered in [51] was incorrect and that the correct Lagrangian is:

$$L=\frac{1}{2}\left(\dot{r}^{2}+r^{2}\dot{\theta}^{2}\right)-\frac{1}{2}\omega^{2}r^{2}-\frac{\mu}{2r^{2}}-\frac{C\left(\theta\right)}{2r^{2}} \tag{4.56}$$

where $C(\theta)=\sec^{2}\theta f(\tan\theta)+\csc^{2}\theta g(\tan\theta)$ and the functions f,g satisfy two compatibility conditions (see equation (5.2) of [52]).

We observe that the Lie symmetries are also Noether symmetries and that the Noether Integrals (in addition to the Hamiltonian E) corresponding the these Noether symmetries are

$$I_{1} = 2tE-r\dot{r} \tag{4.57}$$

$$I_{2} = t^{2}E-tr\dot{r}+\frac{1}{2}r^{2} \tag{4.58}$$

for $\omega=0$. When $\omega\neq0$ the Noether integrals are

$$I_1' = -\frac{1}{\omega}\cos(2\omega t)\,E-\sin(2\omega t)\,r\dot{r}+\omega\cos(2\omega t)\,r^2 \tag{4.59}$$

$$I_2' = \frac{1}{\omega}\sin(2\omega t)\,E-\cos(2\omega t)\,r\dot{r}-\omega\sin(2\omega t)\,r^2. \tag{4.60}$$

In total we have three Noether integrals. Since we do not look for generalized symmetries, we do not expect to find the Ermakov - Lewis invariant [53].

4.7.2 Point symmetries of the Hènon - Heiles potential

The Hènon - Heiles potential

$$V(x,y)=\frac{1}{2}\left(x^2+y^2\right)+x^2y-\frac{1}{3}y^2$$

has been used as a model for the galactic cluster. Computer analysis has suggested that for sufficiently small values of the energy, there exists a first integral independent of energy. In [54] it is proposed to study if there exists a Lie point symmetry of the potential which could justify such a first integral. Working in a slightly more general scenario, in [54] considered potentials of the form

$$V(x,y)=\frac{1}{2}\left(x^2+y^2\right)+Ax^3+Bx^2y+Cxy^2+Dy^3 \tag{4.61}$$

where A,B,C,D are real parameters. The Hènon - Heiles potential is the special case for $A=C=0,B=1,D=-\frac{1}{3}$.

Using standard Lie analysis in [54] it is shown that only the potentials $V_1(x,y)=\frac{1}{2}\left(x^2+y^2\right)+x^3$, $V_2(x,y)=\frac{1}{2}\left(x^2+y^2\right)+y^3$, $V_3(x,y)=\frac{1}{2}\left(x^2+y^2\right)\pm(ay\pm x)^3$, $V_4(x,y)=\frac{1}{2}\left(x^2+y^2\right)\pm(ay\mp x)^3$ admit Lie point symmetries, hence the Hènon - Heiles potential does not admit a Lie point symmetry and the existence of a first integral it is not justified. We apply the results of sections 4.4, 4.4.1 to give the Lie point symmetries and the Noether quantities of these potentials, simply by reading the relevant Tables.

The potential $V_1(x,y)$ is of the form $cy^2+f(x)$. Hence from Table 4.4 the Lie point symmetries admitted by this potential are:

$$X=c_0\partial_t+c_1\sin t\partial_y+c_2\cos t\partial_y+c_3y\partial_y.$$

We note that the Lie symmetry $y\partial_y$, which is due to the Affine collineation, has not been found in [54].

The potential $V_2(x,y)$ is obtained by $V_1(x,y)$ with x,y interchanged. Therefore the Lie point symmetries admitted by the potential $V_2(x,y)$ are

$$X=c_0\partial_t+c_1\sin t\partial_x+c_2\cos t\partial_x+c_3x\partial_x$$

and again in [54] the Lie point symmetry $y\partial_y$ is missing.

The potential $V_3(x,y)$ is of the form $\frac{1}{2}\left(x^2+y^2\right)+f(x-ay)$. Hence from Tables 4.16 and 4.17 the admitted Lie symmetries are

$$X=c_0\partial_t+(c_1\cos t+c_2\sin t)(a\partial_x\pm\partial_y)+c_3(ax+y)(a\partial_x+\partial_y).$$

Table 4.13: Noether symmetries of Hènon - Heiles potential

$\mathbf{V}(x,y)$	**Noether Symmetry**	**Noether Integral**
$\frac{1}{2}\left(x^2+y^2\right)+x^3$	$\sin t\partial_y$	$\dot{y}\sin t - y\cos t$
	$\cos t\partial_y$	$\dot{y}\cos t + y\sin t$
$\frac{1}{2}\left(x^2+y^2\right)+y^3$	$\sin t\partial_x$	$\dot{x}\sin t - x\cos t$
	$\cos t\partial_x$	$\dot{x}\cos t + x\sin t$
$\frac{1}{2}\left(x^2+y^2\right)\pm(ay\pm x)^3$	$\sin t\left(\mp a\partial_x+\partial_y\right)$	$\left(\mp a\dot{x}+\dot{y}\right)\sin t-\left(\mp ax+y\right)\cos t$
	$\cos t\left(\mp a\partial_x+\partial_y\right)$	$\left(\mp a\dot{x}+\dot{y}\right)\cos t+\left(\mp ax+y\right)\sin t$
$\frac{1}{2}\left(x^2+y^2\right)\pm(ay\mp x)^3$	$\sin t\left(\pm a\partial_x+\partial_y\right)$	$\left(\pm a\dot{x}+\dot{y}\right)\sin t-\left(\pm ax+y\right)\cos t$
	$\cos t\left(\pm a\partial_x+\partial_y\right)$	$\left(\pm a\dot{x}+\dot{y}\right)\cos t+\left(\pm ax+y\right)\sin t$

The potential $V_4(x,y)$ is of the same form as $V_3(x,y)$ with x,y interchanged. Therefore the Lie point symmetries are:

$$X = c_0\partial_t + a(c_1\cos t + c_2\sin t)\left(a\partial_x \mp \partial_y\right) + c_3\left(ax+y\right)\left(a\partial_x+\partial_y\right).$$

We observe that in all four cases the Lie point symmetries depend on four free parameters (the c_0, c_1, c_2, c_3). The parameter c_0 determines the vector $c_0\partial_t$ and the rest c_1, c_2, c_3 the $x-y$ part of the symmetry generators.

The Lie point symmetries which are possibly Noether symmetries are the ones generated by the KVs. We check that the Lie point symmetries which are due to the gradient KVs are Noether Symmetries of the potentials (plus the ∂_t whose Noether integral is the Hamiltonian). The Noether integrals and the Noether functions corresponding to each of these symmetries are given in Table 4.13. The results coincide with those of [54, 55].

4.8 Conclusion

We have shown in Theorem 4.2.2 and Theorem 4.3.2 that the Lie and the Noether point symmetries for the general class of equations of motion (4.1) are generated from the special projective Lie algebra and the homothetic Lie algebra respectively of the metric of the space where motion takes place. The specific subalgebra is determined by a set of differential conditions which involve the potential defining the dynamical system. The results apply to both conservative and non conservative dynamical systems. They also apply to affine spaces and they are independent of the signature of the metric and the dimension of the space.

The essence of the above is that they reduce the problem of finding the Lie and the Noether point symmetries of second order systems of equations of the form (4.1) to the geometric requirement of finding the special projective algebra of the a metric (or more general of an affine) space. Because there is a plethora of results in existing studies on the projective algebra of Riemannian spaces, it is possible that the problem of finding the Lie and the Noether point symmetries of an autonomous conservative dynamical is already solved! As it has been shown, one such case is the case of spaces of constant curvature. The power of the geometric approach is that it *gives all* the Lie and the Noether point symmetries without the use of computer programs.

An additional point, which could be of interest, is one to reverse the argument and use the computational approach of the Lie symmetries of the geodesic equations of a space to compute the projective group, which can be a formidable task in Differential Geometry. Aminova [32, 33, 34] has shown that if one chooses the Cartan parametrization of the geodesic equations then the Lie symmetries generate the projective algebra of the underline metric. Because up to now there do not seem to exist either a general method or general theorems which allow the computation of the projective algebra of a metric, this approach could be valuable.

We have applied these theorems to classify all two and three dimensional Newtonian dynamical systems which admit at least one Lie symmetry, and in the case of conservative forces, all two and three dimensional potentials $V\left(x^k\right)$ which admit a Lie symmetry and a Noether point symmetry. These results complete previous results [48, 49, 50] concerning the Noether point symmetries of the two and three dimensional Newtonian dynamical systems. We note that, due to the geometric derivation and the tabular presentation, the results can be extended easily to higher dimensional flat spaces; however at the cost of convenience because the linear combinations of the symmetry vectors increase dramatically.

We have demonstrated the application of the results in various important cases. We considered the Kepler-Ermakov system, which is an autonomous, but in general not conservative dynamical system and we determined the classes of this type of systems which admit Lie and Noether point symmetries; we also considered the case of the Hènon Heiles type potentials and determined their Lie point symmetries and their Noether symmetries. These results are compatible and complete previous results in the literature.

In the following chapter, we apply the results obtained here to study the Liouville integrability of the three dimensional Hamiltonian Kepler-Ermakov system and generalize it in a Riemannian space.

4.A Proof of main Theorem

Below, we give the proof of Theorem 4.2.2.

Equation (4.6) gives:

$$\xi\left(t,x^{i}\right)=C\left(t\right)S\left(x^{i}\right)+D\left(t\right) \tag{4.62}$$

where $S^{,i}$ is a gradient KV. Replacing this in (4.5) we find

$$L_{\eta}\Gamma^{i}_{(jk)}=2C_{,t}S_{,(j}\delta^{i}_{k)}. \tag{4.63}$$

Because $\Gamma^{i}_{(jk)}$ is a function of x^{i} only and $\eta^{i}(t,x^{i})$ we must have

$$\eta^{i}\left(t,x^{j}\right)=T\left(t\right)Y^{i}\left(x^{j}\right) \tag{4.64}$$

hence (4.63) becomes:

$$T\left(t\right)L_{Y}\Gamma^{i}_{jk}=2C_{,t}S_{,(j}\delta^{i}_{k)} \tag{4.65}$$

from which follows

$$C_{,t}=a_{0}T \tag{4.66}$$

$$L_{Y}\Gamma^{i}_{jk}=2\alpha_{0}S_{,(j}\delta^{i}_{k)} \tag{4.67}$$

where $a_{0}=0$, when Y^{i} is a KV,HKV,AC and $a_{0}\neq0$ if Y^{i} is a special PC.

The remaining equations (4.3)-(4.4) are written $(F^{i}=g_{ij}V^{,i}(x^{i}))$

$$TL_{Y}V^{,i}+2\left(C_{,t}S+D_{,t}\right)V^{,i}+T_{,tt}Y^{i}=0 \tag{4.68}$$

$$C\left(S_{,k}\delta^{i}_{j}+2S_{,j}\,\delta^{i}_{k}\right)V^{,k}+2T_{,t}Y^{i}{}_{;\,j}-\left(C_{,tt}S+D_{,tt}\right)\delta^{i}_{j}=0 \tag{4.69}$$

Equation (4.69) is written as

$$C(t)\left(S_{,k}\delta^{i}_{j}+2S_{,j}\,\delta^{i}_{k}\right)V^{,k}+2T_{,t}Y^{i}{}_{;\,j}+\left(-C_{,tt}\right)S\delta^{i}_{j}+\left(-D_{,tt}\right)\delta^{i}_{j}=0$$

which due to (4.66) is simplified as follows

$$C(t)\left(S_{,k}\delta^{i}_{j}+2S_{,j}\,\delta^{i}_{k}\right)V^{,k}+\left(2Y^{i}{}_{;\,j}-a_{0}S\delta^{i}_{j}\right)T_{,t}+\left(-D_{,tt}\right)\delta^{i}_{j}=0. \tag{4.70}$$

Collecting the results we have the system of equations

$$C_{,t}=a_{0}T \tag{4.71}$$

$$L_{Y}\Gamma^{i}_{jk}=2S_{,(j}\delta^{i}_{k)} \tag{4.72}$$

$$TL_{Y}V^{,i}+2\left(C_{,t}S+D_{,t}\right)V^{,i}+T_{,tt}Y^{i} = 0 \tag{4.73}$$

$$C\left(S_{,k}\delta^{i}_{j}+2S_{,j}\,\delta^{i}_{k}\right)V^{,k}+\left(2Y^{i}{}_{;\,j}-a_{0}S\delta^{i}_{j}\right)T_{,t}+\left(-D_{,tt}\right)\delta^{i}_{j} = 0 \tag{4.74}$$

where $a_0 = 0$, when Y^i is a KV,HKV,AC and $a_0 \neq 0$ if Y^i is a sp. PC. We consider various cases.

Case I. $T(t) = 0$. In this case (4.71) implies $C_{,t} = 0$ and (4.73),(4.74) give

$$D_{,t} V^{,i} = 0 \tag{4.75}$$

$$C\left(S_{,k}\delta^i_j + 2S_{,j}\,\delta^i_k\right) V^{,k} + \left(-D_{,tt}\right)\delta^i_j = 0 \tag{4.76}$$

From (4.75) follows $D_{,t} = 0$ and consequently (4.76) implies

$$C\left(S_{,k}\delta^i_j + 2S_{,j}\,\delta^i_k\right) V^{,k} = 0. \tag{4.77}$$

Because $\left(S_{,k}\delta^i_j + 2S_{,j}\,\delta^i_k\right) V^{,k} \neq 0$ it follows that $C(t) = 0$. Therefore in this case we have the Lie Symmetry

$$X = d_1 \partial_t \tag{4.78}$$

Case II. $T(t) = a_0 \neq 0$. Equation (4.71) implies $C_{,t} = a_0 a_1$. Then (4.73),(4.74) give:

$$a_1 L_Y V^{,i} + 2\left(C_{,t}S + D_{,t}\right) V^{,i} = 0 \tag{4.79}$$

$$C\left(S_{,k}\delta^i_j + 2S_{,j}\,\delta^i_k\right) V^{,k} + \left(-D_{,tt}\right)\delta^i_j = 0 \tag{4.80}$$

It follows $2a_0a_1 = c_1$, $\frac{2D_{,t}}{a_1} = d_1$ where

$$L_Y V^{,i} + \left(c_1 S + d_1\right) V^{,i} = 0 \tag{4.81}$$

Then (4.80) is written

$$C\left(S_{,k}\delta^i_j + 2S_{,j}\,\delta^i_k\right) V^{,k} = 0 \tag{4.82}$$

from which we infer that $C(t) = 0$. This means $a_0 = 0$ hence Y^i can only be KV, HV or AC and furthermore $c_1 = 0$. We conclude that provided the potential satisfies the condition

$$L_Y V^{,i} + d_1 V^{,i} = 0 \tag{4.83}$$

the symmetry vector is

$$X = \left(\frac{1}{2} d_1 a_1 t + d_2\right)\partial_t + a_1 Y^i \partial_i \tag{4.84}$$

where Y^i is a KV, HV or AC.

Case III. $T_{,t} \neq 0$. In this case we have the system of simultaneous equations

$$C_{,t} = a_0 T \tag{4.85}$$

$$T L_Y V^{,i} + 2\left(C_{,t}S + D_{,t}\right) V^{,i} + T_{,tt} Y^i = 0 \tag{4.86}$$

$$C\left(S_{,k}\delta^i_j + 2S_{,j}\,\delta^i_k\right) V^{,k} + \left(2Y^i{}_{;\,j} - a_0 S\delta^i_j\right) T_{,t} + \left(-D_{,tt}\right)\delta^i_j = 0 \tag{4.87}$$

Suppose that Y^i is a non-gradient KV or non-gradient HV or AC. Then S =constant so that $S^{,i}=0$. Then (4.87) becomes:

$$2T_{,t}Y_{i;\ j}+\left(-a_0T_{,t}S-D_{,tt}\right)g_{ij}=0 \tag{4.88}$$

and follows (by taking the antisymmetric part in the indices i,j) that $T_{,t}=0$ contrary to our assumption. Therefore Y^i must be a gradient KV, gradient HV or sp.PC. We consider various subcases.

Case III. a. Y^i is a gradient KV/HKV and $Y^i\neq V^{,i}$. Then we have $a_0=0$, $C(t)=c_1$ =constant and equations (4.86),(4.87) are written as follows:

$$L_YV^{,i}+2\frac{D_{,t}}{T}V^{,i}+\frac{T_{,tt}}{T}Y^i = 0 \tag{4.89}$$

$$c_1\left(S_{,k}\delta^i_j+2S_{,j}\,\delta^i_k\right)V^{,k}+\left(2\psi T_{,t}-D_{,tt}\right)\delta^i_j = 0. \tag{4.90}$$

From (4.89) we infer

$$2\frac{D_{,t}}{T}=d_1\ \ ,\ \ \frac{T_{,tt}}{T}=a_1 \tag{4.91}$$

and

$$L_YV^{,i}+d_1V^{,i}+a_1Y^i=0. \tag{4.92}$$

From (4.90) we find

$$2\psi T_{,t}-D_{,tt}=m \tag{4.93}$$

that is:

$$c_1\left(S_{,k}\delta^i_j+2S_{,j}\,\delta^i_k\right)V^{,k}+m\delta^i_j=0. \tag{4.94}$$

The last relation is satisfied only for $c_1=0$, $m=0$. Then (4.91),(4.93) give:

$$\frac{T_{,tt}}{T}=a_1\ \ ,\ D(t)=\frac{1}{2}d_1\int T(t)\,dt\ ,\ d_1=4\psi. \tag{4.95}$$

We conclude that provided the potential satisfies equation (4.92) and $d_1=4\psi$ where $\psi=0$ for a KV and $\psi=1$ for a HV, we have the Lie symmetry vector

$$X=\ \frac{1}{2}d_1\int T(t)\,dt\partial_t+T(t)\,Y^i\partial_i. \tag{4.96}$$

Case III. b. Y^i is a gradient HV and $Y^i=\kappa V^{,i}$. In this case we have $a_0=0$, $C(t)=c_1$ =constant and the system of equations (4.86),(4.87) becomes:

$$L_YV^{,i}+\left(2\frac{D_{,t}}{T}+\kappa\frac{T_{,tt}}{T}\right)V^{,i} = 0 \tag{4.97}$$

$$c_1\left(S_{,k}\delta^i_j+2S_{,j}\,\delta^i_k\right)V^{,k}+\left(2\psi T_{,t}-D_{,tt}\right)\delta^i_j = 0. \tag{4.98}$$

From (4.98) follows $c_1=0$ which implies the equation

$$2\psi T_{,t}-D_{,tt}=0. \tag{4.99}$$

Because $Y^i = V^{,i}$ the $L_Y V^{,i} = 0$ and we have the second condition

$$2D_{,t} + \kappa T_{,tt} = 0. \tag{4.100}$$

We conclude that in this case the Lie symmetry vector is

$$X = D(t)\,\partial_t + T(t)\,V^{,i}\partial_i \tag{4.101}$$

where the functions $T(t)$, $D(t)$ are solutions of the system of equations (4.98) and (4.98).

Case III. c. Y^i is a special PC. In this case the system of symmetry conditions reads

$$C_{,t} = a_0 T \tag{4.102}$$

$$L_Y V^{,i} + 2\left(\frac{C_{,t}}{T}S + \frac{D_{,t}}{T}\right)V^{,i} + \frac{T_{,tt}}{T}Y^i = 0 \tag{4.103}$$

$$\left(S_{,k}\delta^i_j + 2S_{,j}\,\delta^i_k\right)V^{,k} + \left(2Y^i{}_{;\,j} - a_0 S\delta^i_j\right)\frac{T_{,t}}{C} + \left(-\frac{D_{,tt}}{C}\right)\delta^i_j = 0. \tag{4.104}$$

Using (4.102) we write (4.103) as

$$L_Y V^{,i} + 2a_0 S V^{,i} + 2\frac{D_{,t}}{T}V^{,i} + \frac{T_{,tt}}{T}Y^i = 0 \tag{4.105}$$

from which follows

$$\frac{D_{,t}}{T} = \frac{1}{2}d_1 \;\;,\;\; \frac{T_{,tt}}{T} = a_1 \tag{4.106}$$

where

$$L_Y V^{,i} + 2a_0 S V^{,i} + d_1 V^{,i} + a_1 Y^i = 0 \tag{4.107}$$

Then relation (4.104) implies the conditions

$$\frac{T_{,t}}{C} = c_2 \;,\; \frac{D_{,tt}}{C} = d_c \tag{4.108}$$

and

$$\left(S_{,k}\delta^i_j + 2S_{,j}\,\delta^i_k\right)V^{,k} + \left(2Y^i{}_{;\,j} - a_0 S\delta^i_j\right)c_2 - d_c\delta^i_j = 0. \tag{4.109}$$

We conclude that in this case provided the potential function satisfies (4.107), we have the Lie symmetry vector

$$X = \left(C(t)\,S + D(t)\right)\partial_t + T(t)\,Y^i\partial_i \tag{4.110}$$

where the functions $C(t), D(t), T(t)$ are computed form the equations (4.102), (4.106), (4.108).

Case III. d. Y^i is a special PC of the form $Y^i = \lambda S V^{,i}$, λ =constant. and $S^{,i}$ is a gradient KV of the metric.

This case is possible only when the potential is such that the vector $V^{,i}$ is the gradient HV of the metric (if the metric admits one). Then it is easy to show that due to (4.85) equation (4.86) becomes:

$$L_Y V^{,i} + 2\frac{D_{,t}}{T}V^{,i} + \left(2\frac{C_{,t}}{T} + \lambda\frac{T_{,tt}}{T}\right) SV^{,i} = 0.$$

We compute

$$L_Y V^{,i} = \left[\lambda SV^{,r}, V^{,i}\right]\partial_r = -\lambda S_{,j}V^{,j}V^{,i} \tag{4.111}$$

therefore

$$-\lambda S_{,j}V^{,j}V^{,i} + 2\frac{D_{,t}}{T}V^{,i} + \left(2\frac{C_{,t}}{T} + \lambda\frac{T_{,tt}}{T}\right) SV^{,i} = 0. \tag{4.112}$$

It follows

$$D_{,t} = 0 \tag{4.113}$$

$$2\frac{C_{,t}}{T} + \lambda\frac{T_{,tt}}{T} = \lambda_1 \tag{4.114}$$

and the condition

$$-\lambda S_{,j}V^{,j} + \lambda_1 S = 0 \Rightarrow \lambda S_{,j}V^{,i} = \lambda_1 S. \tag{4.115}$$

Condition (4.87) now reads

$$C\left(S_{,k}\delta^i_j + 2S_{,j}\,\delta^i_k\right)V^{,k} + \left(2Y^i{}_{;\,j} - a_0 S\delta^i_j\right)T_{,t} = 0 \Rightarrow$$
$$C\left(\lambda_1 S\delta^i_j + 2S_{,j}\,V^{,i}\right) + \left(2\lambda S_{,j}V^{\cdot i} + (2\lambda S - a_0 S)\,\delta^i_j\right)T_{,t} = 0$$

from which follows $\frac{T_{,t}}{C} = \lambda_2$; that is,

$$C\left(\lambda_1 S_J\delta^i_j + 2S_{J,j}\,V^{,i}\right) + \lambda_2\left(2\lambda S_{,j}V^{\cdot i} + (2\lambda S - a_0 S)\,\delta^i_j\right) = 0.$$

We conclude that in this case we have the Lie symmetry vector

$$X = C(t)\,S\partial_i + T(t)\,SV^{,i}\partial_i$$

where the functions $C(t), T(t)$ are computed form the solution of the system of simultaneous equations

$$C_{,t} = a_0 T\ ,\ T_{,t} = \lambda_2 C$$

$$2C_{,t} + \lambda T_{,tt} = \lambda_1 T.$$

4.B Tables of Newtonian systems admit Lie and Noether symmetries

Table 4.14: Two dimensional Newtonian systems admiting Lie symmetries (3/4)

Lie $\downarrow$ $F^i \rightarrow$	$\mathbf{F}^x(x,y)/\mathbf{F}^r(r,\theta)$	$\mathbf{F}^y(x,y)/\mathbf{F}^\theta(r,\theta)$
$\frac{d}{2}t\partial_t+\partial_x+b\partial_y$	$f(y-bx)e^{-dx}$	$g(y-bx)e^{-dx}$
$\frac{d}{2}t\partial_t+(a+x)\partial_x+(b+y)\partial_y$	$f\left(\frac{b+y}{a+x}\right)(a+x)^{(1-d)}$	$g\left(\frac{b+y}{a+x}\right)(a+x)^{(1-d)}$
$\frac{d}{2}t\partial_t+(a+x)\partial_x+(b+hy)\partial_y$	$f\left(\left(\frac{b}{h}+y\right)(a+bx)^{-\frac{h}{b}}\right)(a+bx)^{1-\frac{d}{b}}$	$g\left(\left(\frac{b}{h}+y\right)(a+bx)^{-\frac{h}{b}}\right)(a+bx)^{\frac{h-d}{b}}$
$\frac{d}{2}t\partial_t+(x+y)\partial_x+(x+y)\partial_y$	$\begin{pmatrix} f(y-x)x+ \\ +g(y-x)\end{pmatrix}(y+x)^{-\frac{d}{2}}$	$\begin{pmatrix} f(y-x)y+ \\ -g(y-x)\end{pmatrix}(y+x)^{-\frac{d}{2}}$
$\frac{d}{2}t\partial_t+\left(a^2x+ay\right)\partial_x+$ $+(ax+y)\partial_y$	$a(ax+y)^{-\frac{d}{1+a^2}}\times$ $\times\begin{pmatrix} xa^2f\left(y-\frac{x}{a}\right)+ \\ +g\left(y-\frac{x}{a}\right)\end{pmatrix}$	$a^2(ax+y)^{-\frac{d}{1+a^2}}\times$ $\times\begin{pmatrix} af\left(y-\frac{x}{a}\right)+ \\ -g\left(y-\frac{x}{a}\right)\end{pmatrix}$
$\frac{d}{2}t\partial_t+(-ay+x)\partial_x+(ax+y)\partial_y$	$f(\theta-a\ln r)r^{1-d}$	$g(\theta-a\ln r)r^{1-d}$

Table 4.15: Two dimensional Newtonian systems admiting Lie symmetries (4/4)

Lie $\downarrow$ $F^i \rightarrow$	$\mathbf{F}^x(x,y)$	$\mathbf{F}^y(x,y)$
$T(t)(\partial_x+b\partial_y)$	$-mx+f(y-bx)$	$-mbx+g(y-bx)$
$2\int T(t)\,dt\ \partial_t+T(t)[(a+x)\partial_x+(b+y)\partial_y]$	$-\frac{m}{4}(a+x)+(a+x)^{-3}f\left(\frac{b+y}{a+x}\right)$	$-\frac{m}{4}(b+y)+g\left(\frac{b+y}{a+x}\right)(a+x)^{-3}$

Table 4.16: Two dimensional conservative Newtonian systems admiting Lie symmetries (2/3)

Lie $\downarrow$ $V \rightarrow$	$\mathbf{T}_{,tt}=\mathbf{mT}$
$T(t)(a\partial_x+b\partial_y)$	$-\frac{m}{2}(x^2+y^2)+c_1x+f(ay-bx)$
$2\int T(t)\,dt\ \partial_t+T(t)[(a+x)\partial_x+(b+y)\partial_y]$	$-\frac{m}{8}\left(x^2+y^2+2ax+2by\right)+(a+x)^{-2}f\left(\frac{b+y}{a+x}\right)$

Table 4.17: Two dimensional conservative Newtonian systems admiting Lie symmetries (3/3)

Lie $\downarrow$ $V \rightarrow$	$\mathbf{d=0}$	$\mathbf{d\neq 0}$
$\frac{d}{2}t\partial_t + a\partial_x + b\partial_y$	$f(ay-bx)$	$[c_1 + f(ay-bx)]\, e^{-d\frac{x}{a}}$
$\frac{d}{2}t\partial_t + (a+x)\partial_x + (b+y)\partial_y$	$f\left(\frac{b+y}{a+x}\right)(a+x)^2$	$f\left(\frac{b+y}{a+x}\right)(a+x)^{(2-d)}$
$\frac{d}{2}t\partial_t + (x+y)\partial_x + (x+y)\partial_y$	$f(y-x) + c_1(x+y)^2$	$(x+y)^{\left(2-\frac{d}{2}\right)}$
$\frac{d}{2}t\partial_t + \left(a^2x+ay\right)\partial_x + (ax+y)\partial_y$	$c_1\left(x^2+y^2\right) + f(ay-x)$	$c_1(ax+y)^{\left(2-\frac{d}{1+a^2}\right)}$
$\frac{d}{2}t\partial_t + (-ay+x)\partial_x + (ax+y)\partial_y$	$f(\theta - a\ln r)\, r^2$	$f(\theta - a\ln r)\, r^{2-d}$

Lie $\downarrow$ $V \rightarrow$	$\mathbf{d=2}$	$\mathbf{d=1}$
$\frac{d}{2}t\partial_t + \partial_x + b\partial_y$	$[c_1 + f(y-bx)]\, e^{-2\frac{x}{a}}$	$[c_1 + f(y-bx)]\, e^{-\frac{x}{a}}$
$\frac{d}{2}t\partial_t + (a+x)\partial_x + (b+y)\partial_y$	$f\left(\frac{b+y}{a+x}\right) + c_1\ln(a+x)$	$f\left(\frac{b+y}{a+x}\right)(a+x)$
$\frac{d}{2}t\partial_t + (x+y)\partial_x + (x+y)\partial_y$	$(x+y)$	$(x+y)^{\frac{3}{2}}$
$\frac{d}{2}t\partial_t + \left(a^2x+ay\right)\partial_x + (ax+y)\partial_y$	$\ln(ax+y) \quad \left(d = 2\left(1+a^2\right)\right)$	$(ax+y)^{\left(\frac{1+2a^2}{1+a^2}\right)}$
$\frac{d}{2}t\partial_t + (-ay+x)\partial_x + (ax+y)\partial_y$	$c_1\ln r + f(\theta - a\ln r)$	$c_1 r + f(\theta - a\ln r)\, r$

Table 4.18: Three dimensional conservative Newtonian systems admitingt Noether symmetries (3/5)

Noether Symmetry	$\mathbf{V(x,y,z)}$
$a\partial_\mu + b\partial_\nu$	$-\frac{p}{a}x_\mu + f\left(x^\nu - \frac{b}{a}x^\mu, x^\sigma\right)$
$a\partial_\mu + b\left(x_\nu\partial_\mu - x_\mu\partial_\nu\right)$	$-\frac{p}{\lvert b\rvert}\arctan\left(\frac{\lvert b\rvert x_\mu}{\lvert(a+bx_\nu)\rvert}\right) + f\left(\frac{1}{2}r_{(\mu\nu)} + \frac{a}{b}x^\nu, x^\sigma\right)$
$a\partial_\mu + b\left(x_\sigma\partial_\nu - x_\nu\partial_\sigma\right)$	$-\frac{p}{\lvert b\rvert}\theta_{(\nu\sigma)} + f\left(r_{(\nu\sigma)}, x^\mu - \frac{a}{b}\theta_{(\nu\sigma)}\right)$
$a\left(x_\nu\partial_\mu - x_\mu\partial_\nu\right) +$	$\frac{p}{a}\arctan\left(\frac{ax_\nu + bx_\sigma}{x_\mu\sqrt{a^2+b^2}}\right) +$
$+\, b\left(x_\sigma\partial_\mu - x_\mu\partial_\sigma\right)$	$+\frac{1}{a}f\left(x_\sigma - \frac{a}{b}x_\nu, x_\nu^2\left(1 - \left(\frac{a}{b}\right)^2 + \frac{2b}{a}\frac{x_\sigma}{x_\nu}\right) + x_\mu^2\right)$
$2bt\partial_t + a\partial_\mu + bR\partial_R$	$-p\frac{x_\mu(2a+bx_\mu)}{2\left(a+bx_\mu^2\right)} + \frac{1}{\left(a+bx_\mu^2\right)}f\left(\frac{x_\nu}{a+bx_\mu}, \frac{x_\sigma}{a+bx_\mu}\right)$
$2bt\partial_t + a\theta_{(\mu\nu)}\partial_{\theta_{(\mu\nu)}} + bR\partial_R$	$\frac{1}{r_{(\mu\nu)}^2}f\left(\theta_{(\mu\nu)} - \frac{a}{b}\ln r_{(\mu\nu)}, \frac{x_\sigma}{r_{(\mu\nu)}}\right)$

Table 4.19: Three dimensional conservative Newtonian systems admiting Noether symmetries (4/5)

Noether Symmetry	$\mathbf{V(x,y,z)}$
$a\partial_\mu + b\partial_\nu + c\partial_\sigma$	$-\frac{p}{a}x_\mu + f\left(x^\nu - \frac{b}{a}x^\mu, x^\sigma - \frac{c}{a}x^\mu\right)$
$a\partial_\mu + b\partial_\nu + c\left(x_\nu\partial_\mu - x_\mu\partial_\nu\right)$	$\frac{p}{\lvert c\rvert}\arctan\left(\frac{(b-cx_\mu)}{\lvert(a+cx_\nu)\rvert}\right)$
	$+f\left(\frac{c}{2}r_{(\mu\nu)} - bx_\mu + ax_\nu, x_\sigma\right)$
$a\partial_\mu + b\partial_\nu + c\left(x_\sigma\partial_\mu - x_\mu\partial_\sigma\right)$	$-\frac{p}{\lvert c\rvert}\arctan\left(\frac{\lvert c\rvert x_\mu}{\lvert a+cx_\sigma\rvert}\right)$
	$+\, f\left(x_\nu - \frac{1}{\lvert c\rvert}\arctan\left(\frac{\lvert c\rvert x_\mu}{\lvert a+cx_\sigma\rvert}\right), \frac{1}{2}r_{(\mu\sigma)} - \frac{a}{c}x_\sigma\right)$
$a\partial_\mu + b\left(x_\nu\partial_\mu - x_\mu\partial_\nu\right) +$	$\frac{p}{\sqrt{b^2+c^2}}\arctan\left(\frac{\left(ab+b^2x_\nu+bcx_\sigma\right)}{\lvert bx_\mu\rvert\sqrt{b^2+c^2}}\right) +$
$+\, c\left(x_\sigma\partial_\mu - x_\mu\partial_\sigma\right)$	$+f\left(x_\mu^2 + x_\nu^2\left(1 - \frac{c^2}{b^2}\right) + \left(\frac{2a}{b} + \frac{2c}{b}x_\sigma\right)x_\nu, x_\sigma - \frac{c}{b}x_\nu\right)$
$so(3)$ linear combination	$p\arctan\left(\lambda\left(\theta,\phi\right)\right) +$
	$+\, F\left(R, b\tan\theta\sin\phi + c\cos\phi - aM_1\right)$
$2ct\partial_t + a\partial_\mu + b\theta_{(\nu\sigma)}\partial_{\theta_{(\nu\sigma)}} + cR\partial_R$	$\frac{1}{r_{(\nu\sigma)}^2}f\left(\theta_{(\nu\sigma)} - \frac{b}{c}\ln r_{(\nu\sigma)}, \frac{a+cx_\mu}{cr_{(\nu\sigma)}}\right)$
$2lt\partial_t + \left(a\partial_\mu + b\partial_\nu + c\partial_\sigma + lR\partial_R\right)$	$-\frac{px(2a+cx_\mu)}{2(a+cx_\mu)^2} + \frac{1}{(a+lx_\mu)^2}f\left(\frac{b+lx_\nu}{l(a+lx_\mu)}, \frac{c+lx_\sigma}{l(a+lx_\mu)}\right)$

Table 4.20: Three dimensional conservative Newtonian systems admiting Noether symmetries (5/5)

Noether Symmetry	$\mathbf{V(x,y,z)}$ / $\mathbf{T}_{,tt}=\mathbf{mT}$
$T(t)(a\partial_\mu + b\partial_\nu + c\partial_\sigma)$	$-\frac{m}{2a}R^2 + f\left(x^\nu - \frac{b}{a}x^\mu, x^\sigma - \frac{c}{a}x^\mu\right)$
$\left(2l\int T(t)\,dt\right)\partial_t +$	$\frac{1}{(a+lx_\mu)^2} f\left(\frac{b+lx_\nu}{l(a+lx_\mu)}, \frac{c+lx_\sigma}{l(a+lx_\mu)}\right) +$
$+T(t)(a\partial_\mu + b\partial_\nu + c\partial_\sigma + lR\partial_R)$	$-\frac{m}{8}\left(R^2 + \frac{2a}{l}x_\mu + \frac{2c}{l}x_\nu + \frac{2b}{l}x_\sigma\right)$

Where $\lambda(\phi,\theta) = \left(\left(a^2+b^2\right)\cos\phi - bc\tan\theta\sin\phi + cM_1\right) \times$

$$\times \left\{M_2\left[-b^2M_1^2 - 2b\tan\theta\sin\phi M_1 - a^2\sin^2\phi\tan^2\theta\right]\right\}^{-\frac{1}{2}}$$

$M_1 = \frac{1}{\cos\theta}\sqrt{\sin^2\phi\left(2\cos^2\theta - 1\right)}$, $M_2 = \sqrt{a^2+b^2+c^2}$

Chapter 5

The autonomous Kepler Ermakov system in a Riemannian space

5.1 Introduction

The Ermakov system has its roots in the study of the one dimensional time dependent harmonic oscillator

$$\ddot{x} + \omega^2(t)x = 0. \tag{5.1}$$

Ermakov [66] obtained a first integral J of this equation by introducing the auxiliary equation

$$\ddot{\rho} + \omega^2(t)\rho = \rho^{-3} \tag{5.2}$$

eliminating the $\omega^2(t)$ term and multiplying with the integrating factor $\rho\dot{x} - \dot{\rho}x$

$$J = \frac{1}{2}\left[(\rho\dot{x} - \dot{\rho}x)^2 + (x/\rho)^2\right]. \tag{5.3}$$

The Ermakov system was rediscovered nearly a century after its introduction [67] and subsequently was generalized beyond the harmonic oscillator to a two dimensional dynamical system which admits a first integral [68]. In a series of papers the Lie, the Noether and the dynamical symmetries of this generalized system have been studied. A short review of these studies and a detailed list of relevant references can be found in [69]. Earlier reviews of the Ermakov system and its numerous applications in divertive areas of Physics can be found in [70, 71].

The general Ermakov system does not admit Lie point symmetries. The form of the most general Ermakov system which admits Lie point symmetries has been determined in [72] and it is called the Kepler Ermakov system [65, 56]. It is well known that these Lie point symmetries are a representation of the $sl(2,R)$ algebra.

In an attempt to generalize the Kepler Ermakov system to higher dimensions, Leach [56] used a transformation to remove the time dependent frequency term and then demanded that the autonomous 'generalized' Kepler Ermakov system will posses two properties: (a) a first integral, the Ermakov invariant and (b) $sl(2,R)$ invariance wrt to Lie symmetries. It has been shown, that the invariance group of the Ermakov invariant is reacher than $sl(2,R)$ [73]. The purpose of the present work is to use Leach's proposal and generalize the autonomous Kepler Ermakov system in two directions: (a) to higher dimensions using the $sl(2,R)$ invariance with respect to Noether symmetries (provided the system is Hamiltonian) and (b) in a Riemannian space which admits a gradient homothetic vector (HV).

The generalization of the autonomous Kepler Ermakov system to three dimensions using Lie symmetries has been done in [56]. In the following sections, we use the results of Chapter 4 to generalize the subset of autonomous Hamiltonian Kepler Ermakov systems to three dimensions via Noether symmetries. We show, that there is a family of three dimensional autonomous Hamiltonian Kepler Ermakov systems parametrized by an arbitrary function f which admits the elements of $sl(2,R)$ as Noether point symmetries. Each member of this family admits two first integrals, the Hamiltonian and the Ermakov invariant.

We use this result in order to determine all three dimensional Hamiltonian Kepler Ermakov systems which are Liouville integrable via Noether point symmetries. To do this we need to determine all members of the family, that is, those functions f for which the corresponding system admits an additional Noether symmetry.

The results of Chapter 4 indicate that there are two cases to be considered, i.e. Noether point symmetries resulting from linear combinations of (a) translations and (b) rotations (elements of the $so(3)$ algebra). In each case we determine the functions f and the required extra time independent first integral.

The above scenario can be generalized to an n dimensional Euclidian space as Leach indicates in [56], however at the cost of major complexity and number of cases to be considered. Indeed as it can be seen by the results of Chapter 4, the situation is complex enough even for the three dimensional case.

We continue with the generalization of the Kepler Ermakov system in a different and more drastic direction. We note that the Ermakov systems considered so far are based on the Euclidian space, therefore we may call them Euclidian Ermakov systems. Furthermore the $sl(2,R)$ symmetry algebra of the autonomous Kepler Ermakov system is generated by the trivial symmetry ∂_t and the gradient HV of the Euclidian two dimensional space E^2. Using this observation we generalize the autonomous Kepler Ermakov system (not necessarily Hamiltonian) in an n dimensional Riemannian space which admits a gradient HV using either Lie or Noether point symmetries. The new dynamical system we call the Riemannian Kepler Ermakov system. This generalization makes possible the application of the autonomous Kepler Ermakov system in General Relativity and in particular in Cosmology.

Concerning General Relativity, we determine the four dimensional autonomous Riemannian Kepler Ermakov system and the associated Riemannian Ermakov invariant in the spatially flat Freedman - Robertson - Walker (FRW) spacetime and we use previous results to calculate the extra Noether point symmetries. The applications to cosmology concern two models for dark energy on a locally rotational symmetric (LRS) space time. The first model involves a scalar field with an exponential potential minimally interacting with a perfect fluid with a stiff equation of state. The second cosmological model is the $f(R)$ modified gravity model of $\Lambda_{bc}CDM$. It is shown, that, in both models the gravitational field equations define an autonomous Riemannian Kepler Ermakov system which is integrable via Noether integrals.

In section 5.2, we review the main features of the two dimensional autonomous Euclidian Kepler Ermakov system. In section 5.3, we discuss the general scheme of generalization of the two dimensional autonomous Euclidian Kepler Ermakov system to higher dimensions and to a Riemannian space which admits a gradient HV. In section 5.4, we consider the generalization to the 3D autonomous Euclidian Hamiltonian Kepler Ermakov system by Noether point symmetries and determine all such systems which are Liouville integrable. In section 5.6, we define the autonomous Riemannian Kepler Ermakov system by the requirements that it will admit (a) a first integral (the Ermakov invariant) and (b) posses $sl(2,R)$ invariance. In section 5.6.1, we consider the non-conservative autonomous Riemannian Kepler Ermakov system and derive the Riemannian Ermakov invariant and in section 5.6.2 we repeat the same for the autonomous Hamiltonian Riemannian Kepler Ermakov system. In the remaining sections we discuss the applications of the autonomous Hamiltonian Riemannian Kepler Ermakov system in General Relativity and in Cosmology.

5.2 The two dimensional autonomous Kepler Ermakov system

In [72] Hass and Goedert considered the most general 2d Newtonian Ermakov system to be defined by the equations:

$$\ddot{x} + \omega^2(t, x, y, \dot{x}, \dot{y})x = \frac{1}{yx^2} f\left(\frac{y}{x}\right) \tag{5.4}$$

$$\ddot{y} + \omega^2(t, x, y, \dot{x}, \dot{y})y = \frac{1}{xy^2} g\left(\frac{y}{x}\right). \tag{5.5}$$

This system admits the Ermakov first integral

$$I = \frac{1}{2}(x\dot{y} - y\dot{x})^2 + \int^{y/x} f(\tau)\, d\tau + \int^{y/x} g(\tau)\, d\tau. \tag{5.6}$$

If one considers the transformation:

$$\Omega^2 = \omega^2 - \frac{1}{xy^3} g\left(\frac{y}{x}\right)$$

$$F\left(\frac{y}{x}\right) = f\left(\frac{y}{x}\right) - \frac{x^2}{y^2} g\left(\frac{y}{x}\right)$$

then equations (5.4)-(5.5) take the form

$$\ddot{x} + \Omega^2(x, y, \dot{x}, \dot{y})x = \frac{1}{x^2 y} F\left(\frac{y}{x}\right) \tag{5.7}$$

$$\ddot{y} + \Omega^2(x, y, \dot{x}, \dot{y})y = 0. \tag{5.8}$$

Due to the second equation, except for special cases, the new function Ω is independent of t; it depends only on the dynamical variables x, y and possibly on their derivative. The Ermakov first integral in the new variables is:

$$I=\frac{1}{2}(x\dot{y}-y\dot{x})^2+\int^{y/x}F(\lambda)\,d\lambda. \tag{5.9}$$

The system of equations (5.7)-(5.8) defines the most general 2D Ermakov system and produces all its known forms for special choices of the function Ω. For example, the weak Kepler Ermakov system [56] is defined by the equations [65]

$$\ddot{x}+\omega^2(t)x+\frac{x}{r^3}H(x,y)-\frac{1}{x^3}f\left(\frac{y}{x}\right) = 0 \tag{5.10}$$

$$\ddot{y}+\omega^2(t)y+\frac{y}{r^3}H(x,y)-\frac{1}{y^3}g\left(\frac{y}{x}\right) = 0 \tag{5.11}$$

where H, f, g are arbitrary functions of their argument, Ω is of the form

$$\Omega^2(x,y)=\omega^2(t)+H(x,y)/r^3 \tag{5.12}$$

and the Ermakov first integral becomes

$$I=\frac{1}{2}(x\dot{y}-y\dot{x})^2+\int^{\frac{y}{x}}\left[\lambda f(\lambda)-\lambda^{-3}g(\lambda)\right]d\lambda. \tag{5.13}$$

The weak Kepler Ermakov system does not admit Lie point symmetries. However, the property of having a first integral prevails. The system of equations (5.10), (5.11) admits the $sl(2,R)$ as Lie point symmetries [52] only for $H(x,y)=-\mu^2r^3+\frac{h\left(\frac{y}{x}\right)}{x}$ where μ is either a real or a pure imaginary number. This is the Kepler Ermakov system defined by the equations

$$\ddot{x}+\left(\omega^2(t)-\mu^2\right)x+\frac{1}{r^3}h\left(\frac{y}{x}\right)-\frac{1}{x^3}f\left(\frac{y}{x}\right) = 0 \tag{5.14}$$

$$\ddot{y}+\left(\omega^2(t)-\mu^2\right)y+\frac{1}{r^3}\frac{y}{x}h\left(\frac{y}{x}\right)-\frac{1}{y^3}g\left(\frac{y}{x}\right) = 0. \tag{5.15}$$

It is well known (see [52]) that the oscillator term $\omega^2(t)-\mu^2$ in (5.14)-(5.15) is removed if one considers new variables T, X, Y defined by the relations:

$$T=\int\rho^{-2}dt, X=\rho^{-1}x\ , Y=\rho^{-1}y \tag{5.16}$$

where ρ is any smooth solution of the time dependent oscillator equation

$$\ddot{\rho}+\left(\omega^2(t)-\mu^2\right)\rho=0. \tag{5.17}$$

In [52] it is commented that "the effect of μ^2 is to shift the time dependent frequency function". However this is true as long as $\omega(t)\neq 0$. When $\omega(t)=0$, one has the autonomous Kepler Ermakov system whose Lie symmetries span the $sl(2,R)$ algebra with different representations for $\mu=0$ and $\mu\neq 0$.

Before we justify the need for the consideration of the two cases $\mu = 0$ and $\mu \neq 0$, we note that by applying the transformation

$$s = \int v^{-2} dT \ , \ \bar{x} = v^{-1} X \ , \ \bar{y} = v^{-1} Y \tag{5.18}$$

where ν satisfies the Ermakov Pinney equation

$$\frac{d^2 v}{dT^2} + \frac{\mu^2}{v^3} = 0 \tag{5.19}$$

to the transformed equations

$$\frac{d^2 X}{dT^2} + \frac{1}{R^3} h\left(\frac{Y}{X}\right) - \frac{1}{X^3} f\left(\frac{Y}{X}\right) = 0 \tag{5.20}$$

$$\frac{d^2 Y}{dT^2} + \frac{1}{R^3}\frac{Y}{X} h\left(\frac{Y}{X}\right) - \frac{1}{Y^3} g\left(\frac{Y}{X}\right) = 0 \tag{5.21}$$

we retain the term μ^2 and obtain the autonomous Kepler Ermakov system of [53]

$$\ddot{x} - \mu^2 x + \frac{1}{r^3} h\left(\frac{y}{x}\right) - \frac{1}{x^3} f\left(\frac{y}{x}\right) = 0 \tag{5.22}$$

$$\ddot{y} - \mu^2 y + \frac{1}{r^3}\frac{y}{x} h\left(\frac{y}{x}\right) - \frac{1}{y^3} g\left(\frac{y}{x}\right) = 0. \tag{5.23}$$

The above transformations show that the consideration of the autonomous Kepler Ermakov system is not a real restriction.

We discuss now the need for the consideration of the cases $\mu = 0$ and $\mu \neq 0$. In Section 4.7, we have determined the Lie symmetries of the autonomous 2d Kepler Ermakov system and we have found two cases. Case I concerns the autonomous Kepler Ermakov system with $\mu = 0$ and has the Lie symmetry vectors

$$\mathbf{X} = \left(\bar{c}_1 + \bar{c}_2 2t + \bar{c}_3 t^2\right)\partial_t + \left(\bar{c}_2 + \bar{c}_3 t\right) r\partial_r \qquad (\ \mu = 0) \tag{5.24}$$

The second case, Case II, concerns the same system with $\mu \neq 0$ and has the Lie symmetry vectors

$$\mathbf{X} = \left(c_1 + c_2 \frac{1}{\mu} e^{2\mu t} - c_3 \frac{1}{\mu} e^{-2\mu t}\right)\partial_t + \left(c_2 e^{2\mu t} + c_3 e^{-2\mu t}\right)\ r\partial_r \qquad (\ \mu \neq 0) \tag{5.25}$$

where in both cases $r\partial_R = x\partial_x + y\partial_y$ is the gradient HV of the 2D Euclidian metric. Each set of vectors in (5.25)-(5.24) is a representation of the $sl(2, R)$ algebra and furthermore each set of vectors is constructed from the vector ∂_t and the gradient HV $r\partial_r$ of the Euclidian two dimensional space E^2.

The essence of the difference between the two representations is best seen in the corresponding first integrals. If a Kepler Ermakov system is Hamiltonian then the Lie point symmetries are also Noether point symmetries therefore in order to find these integrals we determine the Noether invariants. The Noether symmetries of the Kepler Ermakov system have been determined in Section 4.7. For the convenience of the reader we repeat the relevant material.

Equations (5.22), (5.23) follow from the Lagrangian [52]

$$L = \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) - \frac{\mu^2}{2} r^2 - \frac{C(\theta)}{2r^2} \tag{5.26}$$

where $C(\theta) = c + \sec^2\theta f(\tan\theta) + \csc^2\theta g(\tan\theta)$ provided the functions f, g satisfy the constraint:

$$\sin^2\theta f'(\tan\theta) + \cos^2\theta\, g'(\tan\theta) = 0. \tag{5.27}$$

The Ermakov invariant in this case is [52].

$$J = r^4\dot{\theta}^2 + 2C(\theta). \tag{5.28}$$

Because the system is autonomous the first Noether integral is the Hamiltonian

$$E = \frac{1}{2}\left(\dot{r}^2 + r^2\dot{\theta}^2\right) + \frac{1}{2}\mu^2 r^2 + \frac{1}{r^2}F(\theta) \tag{5.29}$$

In addition to the Hamiltonian, there exist two additional time dependent Noether integrals as follows:
$\mu = 0$

$$I_1 = 2tE - r\dot{r} \tag{5.30}$$

$$I_2 = t^2E - tr\dot{r} + \frac{1}{2}r^2 \tag{5.31}$$

$\mu \neq 0$

$$I_1' = \left(\frac{1}{\mu}E - r\dot{r} + \mu r^2\right)e^{2\mu t} \tag{5.32}$$

$$I_2' = \left(\frac{1}{\mu}E + r\dot{r} + \mu r^2\right)e^{-2\mu t}. \tag{5.33}$$

We note that the Noether integrals corresponding to the representation (5.24) are linear in t, whereas the ones corresponding to the representation (5.25) are exponential. Therefore the consideration of the cases $\mu = 0$ and $\mu \neq 0$ is not spurious otherwise we loose important information. This latter fact is best seen in the applications of Noether symmetries to field theories where the main core of the theory is the Lagrangian. In these cases the potential is given and, as it has been shown in Chapter 4, a given potential admits certain Noether symmetries only; therefore one has to consider all possible cases. We shall come to this situation in section 5.8 where it will be found that the potential selects the representation (5.25).

To complete this section, we mention that for a Hamiltonian Kepler Ermakov system the Ermakov invariant (5.28) is constructed [53] from the Hamiltonian and the Noether invariants (5.32),(5.33) as follows:

$$J = E^2 - I_1' I_2'.$$

Finally in [53], it is shown that the Ermakov invariant is generated by a dynamical Noether symmetry of the Lagrangian (5.26), a result which is also confirmed in [74].

5.3 Generalizing the autonomous Kepler Ermakov system

We consider the generalization of the two dimensional autonomous Kepler Ermakov system [56, 72, 75, 76, 77, 78] using a geometric point of view. From the results presented so far we have the following:

(i) Equations (5.4)-(5.5) which define the Ermakov system employ coordinates in the Euclidian two dimensional space, therefore the system is the *Euclidian* Ermakov system.

(ii) The autonomous 2D Euclidian Kepler Ermakov system is defined by equations (5.22) and (5.23)

(iii) The Lie symmetries of the Kepler Ermakov system span the $sl(2,R)$ algebra. These symmetries are constructed from the vector ∂_t and the gradient HV of the space E^2.

(iv) For the autonomous Hamiltonian Kepler Ermakov system the Lie symmetries reduce to Noether point symmetries and the Ermakov invariant follows from a combination of the resulting three Noether integrals, two of which are time dependent. Furthermore, the Ermakov invariant is the Noether integral of a dynamical Noether symmetry.

The above observations imply that we may generalize the Kepler Ermakov system in two directions:

a. Increase the number of dimensions by defining the n dimensional Euclidian Kepler Ermakov system and/or

b. Generalize the background Euclidian space to be a Riemannian space and obtain the Riemannian Kepler Ermakov system.

Concerning the defining characteristics of the Kepler Ermakov system we distinguish three different properties of reduced generality: The property of having a first integral; the property of admitting Lie/Noether point symmetries, the $sl(2,R)$ invariance and the property of being Hamiltonian and admitting $sl(2,R)$ invariance via Noether point symmetries.

Following Leach [56] we generalize the autonomous Kepler Ermakov system to higher dimensions by the requirement: The generalized autonomous (Euclidian) Kepler Ermakov system admits the $sl(2,R)$ algebra as a Lie symmetry algebra.

5.4 The three dimensional autonomous Euclidian Kepler Ermakov system

The generalization of the autonomous Euclidian Kepler Ermakov system using $sl(2,R)$ invariance of Lie symmetries has been done in [56, 77, 78]. In this section, using of the results of Chapter 4, we give the generalization of the autonomous Euclidian Hamiltonian Kepler Ermakov system to three dimensions by demanding $sl(2,R)$ invariance with respect to Noether point symmetries. The reason for attempting this generalization is that it leads to the potentials for which the corresponding extended systems are Liouville integrable. Furthermore indicates the path to the n dimensional Riemannian Kepler Ermakov system.

Depending on the value $\mu \neq 0$ or $\mu = 0$, we consider the three dimensional Hamiltonian Kepler Ermakov systems of type I and type II.

5.4.1 The 3D autonomous Hamiltonian Kepler Ermakov system of type I ($\mu \neq 0$)

For $\mu \neq 0$, the admitted Noether symmetries are required to be (see (5.25))

$$X^1 = \partial_t,\ X_{\pm} = \frac{1}{\mu} e^{\pm 2\mu t}\partial_t \pm e^{\pm 2\mu t} R\partial_R. \tag{5.34}$$

From Table 4.11 and $T(t) = \frac{1}{\mu}e^{\pm 2\mu t}$ of section 4.5.2, we find that for these vectors the potential is

$$V(R,\phi,\theta) = -\frac{\mu^2}{2}R^2 + \frac{1}{R^2} f(\theta,\phi)$$

hence, the Lagrangian is

$$L = \frac{1}{2}\left(\dot{R}^2 + R^2\dot{\phi}^2 + R^2\sin^2\phi\,\dot{\theta}^2\right) + \frac{\mu^2}{2}R^2 - \frac{1}{R^2}f(\theta,\phi). \tag{5.35}$$

The equations of motion, that is, the equations defining the generalized dynamical system are

$$\ddot{R} - R\dot{\phi}^2 - R\sin^2\phi\,\dot{\theta}^2 - \mu^2 R - \frac{2}{R^3}f = 0 \tag{5.36}$$

$$\ddot{\phi} + \frac{2}{R}\dot{R}\dot{\phi} - \sin\phi\cos\phi\,\dot{\theta}^2 + \frac{1}{R^4}f_{,\phi} = 0 \tag{5.37}$$

$$\ddot{\theta} + \frac{2}{R}\dot{R}\dot{\theta} + \cot\phi\,\dot{\theta}\dot{\phi} + \frac{1}{R^4\sin^2\phi}f_{,\theta} = 0. \tag{5.38}$$

The Noether integrals corresponding to the Noether vectors are

$$E = \frac{1}{2}\left(\dot{R}^2 + R^2\dot{\phi}^2 + R^2\sin^2\phi\,\dot{\theta}^2\right) - \frac{\mu^2}{2}R^2 + \frac{1}{R^2}f(\theta,\phi) \tag{5.39}$$

$$I_+ = \frac{1}{\mu}e^{2\mu t}E - e^{2\ \mu t}R\dot{R} + \mu e^{2\mu t}R^2 \tag{5.40}$$

$$I_- = \frac{1}{\mu}e^{-2\mu t}E + e^{-2\mu t}R\dot{R} + \mu e^{-2\mu t}R^2 \tag{5.41}$$

where E is the Hamiltonian. The Noether integrals $I_{\pm}$ are time dependent. Following [53] we define the time independent combined first integral

$$J = E^2 - \ I_+ I_- = R^4\dot{\phi}^2 + R^4\sin^2\phi\,\dot{\theta}^2 + 2f(\theta,\phi). \tag{5.42}$$

Using (5.42) the equation of motion (5.36) becomes

$$\ddot{R} - \mu^2 R = \frac{J}{R^3} \tag{5.43}$$

which is the autonomous Ermakov- Pinney equation [79]. Therefore J is the Ermakov invariant [56].

An alternative way to construct the Ermakov invariant (5.42) is to use dynamical Noether symmetries [58]. Indeed one can show that the Lagrangian (5.35) admits the dynamical Noether symmetry $X_D = K^i_j \dot{x}^j \partial_i$ where K_{ij} is a Killing tensor of the second rank whose non-vanishing components are $K_{\phi\phi} = R^4$, $K_{\theta\theta} = R^4\sin^2\phi$. The dynamical Noether symmetry vector is $X_D = R^2\left(\dot{\phi}\partial_\phi + \dot{\theta}\partial_\theta\right)$ with gauge function $2f(\theta,\phi)$.

5.4.2 The 3D autonomous Hamiltonian Kepler Ermakov system of type II ($\mu = 0$)

For $\mu = 0$, the Noether point symmetries are required to be (see (5.24)) [56]

$$X^1 = \partial_t \ , \ X^2 = 2t\partial_t \ + R\partial_R \ , \ X^3 = t^2\partial_t \ + tR\partial_R. \tag{5.44}$$

From Table 4.10 and from Table 4.11 for $T(t) = t$ of section 4.5.2, we find that the potential is

$$V(R, \phi, \theta) = \frac{1}{R^2} f(\theta, \phi)$$

hence, the Lagrangian is

$$L' = \frac{1}{2}\left(\dot{R}^2 + R^2\dot{\phi}^2 + R^2\sin^2\phi\dot{\theta}^2\right) - \frac{1}{R^2} f(\theta, \phi). \tag{5.45}$$

The equations of motion are (5.36) - (5.38) with $\mu = 0$. The Noether invariants of the Lagrangian (5.45) are

$$E = \frac{1}{2}\left(\dot{R}^2 + R^2\dot{\phi}^2 + R^2\sin^2\phi\dot{\theta}^2\right) + \frac{1}{R^2} f(\theta, \phi) \tag{5.46}$$

$$I_1 = 2tE' - R\dot{R} \tag{5.47}$$

$$I_2 = t^2E' - tR\dot{R} + \frac{1}{2}R^2. \tag{5.48}$$

We note that the time dependent first integrals $I_{1,2}$ are linear in t whereas the corresponding integrals $I_\pm$ of the case $\mu \neq 0$ are exponential. From $I_{1,2}$ we define the time independent first integral $J = 4I_2E' - I_1^2$ which is calculated to be

$$J = R^4\dot{\phi}^2 + R^4\sin^2\phi\ \dot{\theta}^2 + 2f(\theta, \phi). \tag{5.49}$$

Using (5.49) the equation of motion for $R(t)$ becomes $\ddot{R} - \frac{J'}{R^3} = 0$ which is the one dimensional Ermakov-Pinney equation, hence J' is the Ermakov invariant [56]. As it was the case with the three dimensional Hamiltonian Kepler Ermakov system of type I, the Lagrangian (5.45) admits the dynamical Noether symmetry $X_D = R^2\left(\dot{\phi}\partial_\phi + \dot{\theta}\partial_\theta\right)$ whose integral is the (5.49).

5.5 Integrability of 3D autonomous Euclidian Kepler Ermakov system

The 3d autonomous Hamiltonian Euclidian Kepler Ermakov systems need three independent first integrals in involution in order to be Liouville integrable. As we have shown each system has the two Noether integrals (E, J) , therefore, we need one more Noether symmetry. Such a symmetry exists only for special forms of the arbitrary function $f(\theta, \phi)$. From tables 4.10, 4.11, 4.18, 4.19 and 4.20 of Chapter 4 we find that extra Noether symmetries are possible only[1] for linear combinations of translations (i.e. vectors of the form $\sum_{A=1}^{3} a^A\partial_A$ where a^A are constants) and/or rotations (i.e. elements of $so(3)$).

[1] The linear combination of an element of $so(3)$ with a translation does not give a potential, hence, an additional Noether symmetry.

5.5.1 Noether symmetries generated from the translation group

We determine the functions $f(\theta,\phi)$ for which the 3D autonomous Hamiltonian Euclidian Kepler Ermakov system admits extra Noether point symmetries for linear combinations of the translation group.

The Lagrangian (5.35)

In Cartesian coordinates the Lagrangian (5.35) is

$$L\left(x^{j},\dot{x}^{j}\right)=\frac{1}{2}\left(\dot{x}^{2}+\dot{y}^{2}+\dot{z}^{2}\right)+\frac{\mu^{2}}{2}\left(x^{2}+y^{2}+z^{2}\right)-\frac{1}{x^{2}}f_{I}\left(\frac{y}{x},\frac{z}{x}\right) \tag{5.50}$$

where $f_I=\left(1+\frac{y^2}{x^2}+\frac{z^2}{x^2}\right)^{-1}$. From Table 4.20 with $m=-\mu^2$, $p=0$ we find that the Lagrangian (5.50) admits Noether symmetries, which are produced from a linear combination of translations, if the function $f_I\left(\frac{y}{x},\frac{z}{x}\right)$ has the form

$$f_{I}\left(\frac{y}{x},\frac{z}{x}\right)=\frac{1}{\left(1-\frac{a}{b}\frac{y}{x}\right)^{2}}F\left(\frac{b\frac{z}{x}-c\frac{y}{x}}{\left(1-\frac{a}{b}\frac{y}{x}\right)}\right). \tag{5.51}$$

In this case, the Lagrangian (5.50) admits at least the following two extra Noether symmetries

$$X_{\pm}=e^{\pm\mu t}\sum_{A=1}^{3}a^{A}\partial_{A} \tag{5.52}$$

with corresponding Noether integrals

$$I_{\pm}=e^{\pm\mu t}\left(\sum_{A=1}^{3}a^{A}\dot{x}_{A}\right)\mp\mu e^{\pm\mu t}\left(\sum_{A=1}^{3}a^{A}x_{A}\right). \tag{5.53}$$

We note that the first integrals $I_\pm$ are time dependent; however the first integral

$$J_{2}=I_{+}I_{-}=\left(a\dot{x}+b\dot{y}+c\dot{z}\right)^{2}+\mu^{2}\left(ax+by+cz\right)^{2} \tag{5.54}$$

is time independent. As it was the case with the Ermakov invariant (5.42) the integral J_2 is possible to be constructed directly from the dynamical Noether symmetry $X_D'=K^i_{(2).j}\dot{x}^i\partial_i$, where $K_{(2)ij}$ is a Killing tensor of the second rank [58, 59], with non-vanishing components

$$\begin{aligned}K_{11}&=a^{2}\ ,\ K_{22}=b^{2}\ ,\ K_{33}=c^{2}\\K_{(12)}&=2ab\ ,\ K_{(13)}=2ac\ ,\ K_{(23)}=2bc\end{aligned}$$

so that the dynamical symmetry vector is

$$X_{D}'=\left(a^{2}+ab+ac\right)\dot{x}\partial_{x}+\left(b^{2}+ab+bc\right)\dot{y}\partial_{y}+\left(c^{2}+ac+bc\right)\dot{z}\partial_{z}. \tag{5.55}$$

The Ermakov invariant J (see (5.42)) in Cartesian coordinates is

$$J=2E\left(x^{2}+y^{2}+z^{2}\right)-\left(x\dot{x}+y\dot{y}+z\dot{z}\right)^{2}. \tag{5.56}$$

The first integrals J, J_2 are not in involution. Using the Poisson brackets we construct new first integrals and at some stage one of them will be in involution. These new first integrals can also be constructed form corresponding dynamical Noether symmetries.

An example of a known Lagrangian of the form (5.50) is the three body Calogero-Moser Lagrangian [80, 81, 82]

$$L = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - \frac{\mu^2}{2}\left(x^2 + y^2 + z^2\right) - \frac{1}{(x-y)^2} - \frac{1}{(x-z)^2} - \frac{1}{(y-z)^2}. \tag{5.57}$$

The extra Noether symmetries of this Lagrangian are produced by the vector (5.52) for $a^A = (1, 1, 1)$.

The Lagrangian (5.45)

In Cartesian coordinates the Lagrangian (5.45) is

$$L\left(x^j, \dot{x}^j\right) = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - \frac{1}{x^2} f_{II}\left(\frac{y}{x}, \frac{z}{x}\right). \tag{5.58}$$

According to Tables 4.19 and 4.20 (with $m = 0$, $p = 0$), the Lagrangian (5.58) admits extra Noether point symmetries for a linear combination of translations if the function f is of the form (5.51). In this case the corresponding Noether integrals are

$$I_1' = \sum_{A=1}^{3} a^A \dot{x}_A \ , \ I_2' = t \sum_{A=1}^{3} a^A \dot{x}_A - \sum_{A=1}^{3} a^A x_A. \tag{5.59}$$

Example of such a Lagrangian is the Calogero-Moser Lagrangian [80] (without the oscillator term)

$$L = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - \frac{1}{(x-y)^2} - \frac{1}{(x-z)^2} - \frac{1}{(y-z)^2}. \tag{5.60}$$

For the Lagrangian (5.60), we have the first integrals E, J, I_1', I_2'. From the integrals J, I_1' we construct the integral $\Phi = \{I_1', \{J, I_1'\}\}$. It is easy to show that the integrals E, I_1', Φ are in involution hence the dynamical system is Liouville integrable. We remark, that, the first integrals E, J, I_1', I_2' can also be computed by making use of the Lax pair tensor [82].

5.5.2 Noether symmetries generated from $so(3)$

The elements of $so(3)$ in spherical coordinates are the three vectors $CK^{1,2,3}$

$$CK^1 = \sin\theta\partial_\phi + \cos\theta\cot\phi\partial_\theta,\ CK^2 = \cos\theta\partial_\phi - \sin\theta\cot\phi\partial_\theta,\ CK^3 = \partial_\theta \tag{5.61}$$

which are also KVs for the Euclidian sphere.

In this case, the symmetry condition becomes

$$L_{CK}\left[\frac{1}{R^2} f(\theta, \phi)\right] + p = 0 \tag{5.62}$$

or, equivalently

$$\frac{1}{R^2}\left(R^2 g_{ij} CK^i f^{,j}\right) + p = 0 \Rightarrow g_{ij} CK^i_{(1,2,3)} f^{,j} + p = 0 \tag{5.63}$$

where g_{ij} is the metric of the Euclidian sphere, that is

$$ds^2 = d\phi^2 + \sin^2\phi \, d\theta^2. \tag{5.64}$$

We infer that the problem of determining the extra Noether point symmetries of Lagrangian (5.35) generated from elements of the $so(3)$ is equivalent to the determination of the Noether point symmetries for motion on the 2D sphere.

It is easy to show, that, the integrals of Table 4.12 of section 4.6 are in involution with the Hamiltonian and the Ermakov invariant, therefore, the system is Liouville integrable via Noether point symmetries.

The above results are extended to the case in which the system moves on the hyperbolic sphere that is, it has Lagrangian

$$L = \frac{1}{2}\left(\dot{R}^2 + R^2\dot{\phi}^2 + R^2 \sinh^2\phi \, \dot{\theta}^2\right) + \frac{\mu^2}{2}R^2 - \frac{1}{R^2} g(\theta, \phi). \tag{5.65}$$

We reach at the following conclusion.

Proposition 5.5.1 *The three dimensional autonomous Hamiltonian Kepler Ermakov system with Lagrangian (5.35) is Liouville Integrable via Noether point symmetries, which are generated from a linear combination of the three elements of the $so(3)$ algebra, if and only if the equivalent dynamical system in the fundamental hyperquadrics of the three dimensional flat space is integrable.*

We note that it is possible a three dimensional autonomous Kepler Ermakov system to admit more Noether symmetries which are due to the rotation group and the translation group (but not to a linear combination of elements from the two groups). For example, the 3D Kepler Ermakov system with Lagrangian [50]

$$L = \frac{1}{2}\left(\dot{x}^2 + \dot{y}^2 + \dot{z}^2\right) - \frac{1}{x^2\left(1 - \frac{y}{x} - \frac{z}{x}\right)^2} \tag{5.66}$$

has the following extra Noether point symmetries (in addition to the elements of $sl(2, R)$)

$$\begin{aligned} Y^1 &= \partial_x + \partial_y, \; Y^2 = \partial_x + \partial_z \\ Y^3 &= t(\partial_x + \partial_y), \; Y^4 = t(\partial_x + \partial_z) \\ Y^5 &= (y - z)\partial_x - (x + z)\partial_y + (x + y)\partial_z. \end{aligned}$$

The vectors $Y^{1,2}$ and $Y^{3,4}$ follow from (5.52) for $a_1^A = (1, 1, 0)$ and $a_2^A = (1, 0, 1)$ respectively, whereas Y^5 is a linear combination of the three elements of $so(3)$. The Noether integrals of the Noether symmetries Y^{1-5} are respectively

$$I_{Y_1} = \dot{x} + \dot{y} \tag{5.67}$$

$$I_{Y_2} = \dot{x} + \dot{z} \tag{5.68}$$

$$I_{Y_3} = t\,(\dot{x} + \dot{y}) - (x + y) \tag{5.69}$$

$$I_{Y_4} = t\,(\dot{x} + \dot{z}\ ) - (x + z) \tag{5.70}$$

$$I_{Y_5} = (y - z)\,\dot{x} - (x + z)\,\dot{y} + (x + y)\,\dot{z}. \tag{5.71}$$

It is clear that in order to extend the Kepler Ermakov system to higher dimensions one needs to have the type of results of Chapter 4; therefore, the remark made in [56], that the 'notion is easily generalized to higher dimensions' has to be understood as referring to the general scenario and not to the actual work.

5.6 The autonomous Riemannian Kepler Ermakov system

As it has been noted in section 5.2, the Kepler Ermakov systems considered so far in the literature are Newtonian Kepler Ermakov systems. In this section we make a drastic step forward and introduce the autonomous Riemannian Kepler Ermakov systems of dimension n. The generalization we consider is based on the following definition

Definition 5.6.1 *The n dimensional autonomous Riemannian Kepler Ermakov system is an autonomous dynamical system which:*
a. It is defined on a Riemannian space which admits a gradient HV
b. Admits a first integral, which we name the Riemannian Ermakov first integral and it is characterized by the requirement that the corresponding equation of motion takes the form of the Ermakov Pinney equation.
c. It is invariant at least under the $sl(2,R)$ algebra, which is generated by the vector ∂_t and the gradient HV of the space.

There are two types of n dimensional autonomous Riemannian Kepler Ermakov systems. The ones which are not Hamiltonian and admit the $sl(2,R)$ algebra as Lie point symmetries and the ones which are conservative and admit the $sl(2,R)$ algebra as Noether point symmetries.

5.6.1 The non Hamiltonian autonomous Riemannian Kepler Ermakov system

Consider an n dimensional Riemannian space which admits a gradient HV. It is well known, that the metric of this space can always be written in the form [83, 84]

$$ds^2 = du^2 + u^2 h_{AB} dy^A dy^B \tag{5.72}$$

where the Latin capital indices $A,B,..$ take the values $1,\ldots,n-1$ and $h_{AB}=h_{AB}\left(y^{C}\right)$ is the generic $n-1$ metric. The gradient HV of the metric is the vector $H^{i}=u\partial_{u}$ $(\psi=1)$ generated from the function $H=\frac{1}{2}u^{2}$. We note the relation

$$h_{DA}\Gamma_{BC}^{A}=\frac{1}{2}h_{DB,C} \tag{5.73}$$

where Γ_{BC}^{A} are the connection coefficients of the $(n-1)$ metric h_{AB}. In that space, we consider a particle moving under the action of the force

$$F^{i}=F^{u}(u,y^{C})\frac{\partial}{\partial u}+F^{A}(u,y^{C})\frac{\partial}{\partial y^{A}}.$$

The equations of motion $\frac{Dx^{i}}{Dt}=F^{i}$ when projected along the direction of u and in the $(n-1)$ space give the equations

$$u''-uh_{AB}y'^{A}y'^{B} = F^{u} \tag{5.74}$$

$$y''^{A}+\frac{2}{u}u'y'^{A}+\Gamma_{BC}^{A}y'^{B}y'^{C} = F^{A} \tag{5.75}$$

where $u'=\frac{du}{ds}$ and s is an affine parameter.

Because the system is autonomous admits the Lie point symmetry ∂_{t}. Using the vector ∂_{t} and the gradient HV $H^{i}=u\partial_{u}$ we construct two representations of $sl(2,R)$ by means of the sets of vectors (see (5.34) and (5.44))

$$\partial_{s},\ 2s\partial_{s}+u\partial_{u},\ s^{2}\partial_{t}+su\partial_{u} \qquad \text{when } \mu=0 \tag{5.76}$$

$$\partial_{s},\ \frac{1}{\mu}e^{\pm2\mu s}\partial_{s}\pm e^{\pm2\mu s}u\partial_{u} \qquad \text{when } \mu\neq0 \tag{5.77}$$

and require that the vectors in each set will be Lie point symmetries of the system of equations (5.74),(5.75). In Appendix 5.A we show that the requirement of the invariance of the force under both representations (5.76), (5.77) of $sl(2,R)$ demands that the force be of the form

$$F^{i}=\left(\mu^{2}u+\frac{1}{u^{3}}G^{u}\left(y^{C}\right)\right)\partial_{u}+\frac{1}{u^{4}}G^{A}\left(y^{C}\right)\partial_{A}. \tag{5.78}$$

Replacing F^{i} in the system of equations (5.74),(5.75) we find

$$u''-uh_{AB}y'^{A}y'^{B} = \mu^{2}u+\frac{1}{u^{3}}G^{u} \tag{5.79}$$

$$y''^{A}+\frac{2}{u}u'y'^{A}+\Gamma_{BC}^{A}y'^{B}y'^{C} = \frac{1}{u^{4}}G^{A}. \tag{5.80}$$

Multiplying the second equation with $2u^{4}h_{DA}y'^{D}$ and using (5.73) we have

$$u^{4}\frac{d}{ds}\left(h_{DB}y'^{D}y'^{B}\right)+4u^{3}h_{DA}u'y'^{A}y'^{D}=2G_{D}y'^{D} \tag{5.81}$$

from which follows

$$\frac{d}{ds}\left(u^{4}h_{DB}y'^{D}y'^{B}\right)=2G_{D}y'^{D}. \tag{5.82}$$

The rhs is a perfect differential if $G_D = -\Sigma_{,D}$ where $\Sigma(y^A)$ is a differentiable function. If this is the case we find the first integral

$$J = u^4 h_{DB} y'^D y'^B + 2\Sigma\left(y^C\right). \tag{5.83}$$

We note that J involves the arbitrary metric h_{AB} and the function $\Sigma(y^A)$. Furthermore equations (5.79), (5.80) become

$$u'' - u h_{AB} y'^A y'^B = \mu^2 u + \frac{1}{u^3} G^u\left(y^C\right) \tag{5.84}$$

$$y''^A + \frac{2}{u} u' y'^A + \Gamma^A_{BC} y'^B y'^C = -\frac{1}{u^4} h^{AB} \Sigma\left(y^C\right)_{,B}. \tag{5.85}$$

These are the equations defining the n dimensional autonomous Riemannian Kepler Ermakov system.

Using the first integral (5.83), the equation of motion (5.84) is written as follows

$$u'' = \mu^2 u + \frac{\bar{G}\left(y^C\right)}{u^3} \tag{5.86}$$

where $\bar{G} = J + G^u\left(y^C\right) - 2\Sigma\left(y^C\right)$. This is the Ermakov-Pinney equation; hence, we identify (5.83) as the Riemannian Ermakov integral of the autonomous Riemannian Kepler Ermakov system.

5.6.2 The autonomous conservative Riemannian Kepler Ermakov system

In the following we assume that the force is derived from the potential $V\left(u, y^C\right)$, that is, the dynamical system is conservative so that the equations of motion follow from the Lagrangian

$$L = \frac{1}{2}\left(u'^2 + u^2 h_{AB} y'^A y'^B\right) - V\left(u, y^C\right). \tag{5.87}$$

The Hamiltonian is

$$E = \frac{1}{2}\left(u'^2 + u^2 h_{AB} y'^A y'^B\right) + V\left(u, y^C\right). \tag{5.88}$$

The equations of motion, i.e. the Euler-Lagrange equations, are

$$u'' - u h_{AB} y'^A y'^B + V_{,u} = 0 \tag{5.89}$$

$$y''^A + \frac{2}{u} u' y'^A + \Gamma^A_{BC} y'^B y'^C + \frac{1}{u^2} h^{AB} V_{,B} = 0. \tag{5.90}$$

The demand that Lagrangian (5.87) admits Noether point symmetries which are generated from the gradient HV leads to the following cases.

Case A: The Lagrangian (5.87) admits the Noether point symmetries (5.76) if the potential is of the form

$$V\left(u, y^C\right) = \frac{1}{u^2} V\left(y^C\right). \tag{5.91}$$

The Noether integrals of these Noether point symmetries are

$$E_A = \frac{1}{2}\left(u'^2 + u^2 h_{AB} y'^A y'^B\right) + \frac{1}{u^2} V\left(y^C\right) \tag{5.92}$$

$$I_1 = 2sE - uu' \tag{5.93}$$

$$I_2 = s^2 E - suu' + \frac{1}{2} u^2 \tag{5.94}$$

where E_A is the Hamiltonian.

Case B: The Lagrangian (5.87) admits the Noether point symmetries (5.77) if the potential is of the form

$$V\left(u,y^{c}\right)=-\frac{\mu^{2}}{2}u^{2}+\frac{1}{u^{2}}V'\left(y^{C}\right). \tag{5.95}$$

The Noether integrals of these Noether point symmetries are

$$E_{B} = \frac{1}{2}\left(u'^{2}+u^{2}h_{AB}y'^{A}y'^{B}\right)-\frac{\mu^{2}}{2}u^{2}+\frac{1}{u^{2}}V'\left(y^{C}\right) \tag{5.96}$$

$$I_{+} = \frac{1}{\mu}e^{2\mu s}E-e^{2\mu s}uu'+\mu e^{2\mu s}u^{2} \tag{5.97}$$

$$I_{-} = \frac{1}{\mu}e^{-2\mu s}E+e^{-2\mu s}uu'+\mu e^{-2\mu s}u^{2} \tag{5.98}$$

where E_B is the Hamiltonian.

Using the Noether integrals we construct the Riemannian Ermakov invariant J_G, which is common for both Case A and Case B, as follows

$$J_{G}=u^{4}h_{DB}y'^{D}y'^{C}+2V'\left(y^{C}\right). \tag{5.99}$$

This coincides with the invariant first integral defined in (5.83). We note that with the use of the first integral (5.99) the Hamiltonians (5.92) and (5.96) take the form

$$E=\frac{1}{2}u'^{2}-\frac{\mu^{2}}{2}u^{2}+\frac{J}{2u^{2}} \tag{5.100}$$

which is the Hamiltonian for the Ermakov Pinney equation.

As it was the case with the Euclidian case of section 5.2, it can be shown that the Riemannian Ermakov invariant (5.99) is due to a dynamical Noether symmetry[58]. We collect the results in the following proposition.

Proposition 5.6.2 *In a Riemannian space with metric g_{ij} which admits a gradient HV, the equations of motion of a Hamiltonian system moving under the action of the potential ($\mu\epsilon\mathbb{C}$)*

$$V\left(u,y^{c}\right)=-\frac{\mu^{2}}{2}u^{2}+\frac{1}{u^{2}}V'\left(y^{C}\right) \tag{5.101}$$

admit the $sl(2,R)$ invariance and also an invariant first integral, the Riemannian Ermakov invariant. This latter quantity is also possible to be identified as the Noether integral of a dynamical Noether symmetry.

Without going into details we state the following general result.

Proposition 5.6.3 *Consider an n dimensional Riemannian space with an r decomposable metric, which in the Cartesian coordinates $x_1,\ldots,x_r$, has the general form*

$$ds^{2}=p\eta_{\Sigma\Lambda}dz^{\Sigma}dz^{\Lambda}+h_{ij}dx^{i}dx^{j}\ \ ,\ i,j=r+1,\ldots,n\ ,\ \Sigma=1,\ldots,r \tag{5.102}$$

where $\eta_{\Sigma\Lambda}$ is a flat non degenerate metric (of arbitrary signature). If there exists a potential, so that the vectors $e^{\pm\mu s}\sum_M a^M \partial_M$ are Noether point symmetries, where a_M are constants, with Noether integrals

$$I_{\pm} = e^{\pm\mu s}\sum_M a^M z'_M \mp \mu e^{\pm\mu s}\sum_M a^M z_M \tag{5.103}$$

the combined first integral $I = I_+ I_-$ is time independent and it is the result of a dynamical Noether symmetry.

In the remaining sections we consider applications of the autonomous Riemannian Kepler Ermakov system in General Relativity and in Cosmology.

5.7 The autonomous Riemannian Kepler Ermakov system in General Relativity

Below, we study the integrability of the Riemannian Kepler Ermakov system via Noether point symmetries in a conformally flat spacetime which admits a homothetic Lie algebra with a gradient (proper) HV.

5.7.1 The Riemannian Kepler Ermakov system on a 4D FRW spacetime

Consider the spatially flat FRW spacetime with metric

$$ds^2 = du^2 - u^2\left(dx^2 + dy^2 + dz^2\right). \tag{5.104}$$

This metric admits the gradient HV $u\partial_u$ and six non gradient KVs [45, 44] which are the KVs of E^3.

We consider the autonomous Riemannian Kepler Ermakov system defined by the Lagrangian (see (5.101)) ($\mu \in \mathbb{C}$)

$$L = \frac{1}{2}\left(u'^2 - u^2\left(x'^2 + y'^2 + z'^2\right)\right) + \frac{\mu^2}{2}u^2 - \frac{1}{u^2}V(x,y,z). \tag{5.105}$$

The Euler Lagrange equations are

$$u'' + u\left(x'^2 + y'^2 + z'^2\right) - \mu^2 u - \frac{2V(x,y,z)}{u^3} = 0 \tag{5.106}$$

$$x^{\sigma\prime\prime} + \frac{2}{u}u' x^{\sigma\prime} - \frac{V^{,\sigma}(x,y,z)}{u^4} = 0 \tag{5.107}$$

where $\sigma = 1,2,3$. The Lagrangian (5.105) has the form of the Lagrangian (5.87) for the potential $V\left(u,y^C\right) = -\frac{\mu^2}{2}u^2 + \frac{1}{u^2}V(x,y,z)$ hence according to proposition 5.6.2 possesses $sl(2,R)$ invariance under Noether point symmetries for *both* representations (5.76) and (5.77). The two time independent invariants are the Hamiltonian and the Riemannian Ermakov invariant (proposition 5.6.2)

$$E = \frac{1}{2}\left(u'^2 - u^2\left(x'^2 + y'^2 + z'^2\right)\right) - \frac{\mu^2}{2}u^2 + \frac{1}{u^2}V(x,y,z) \tag{5.108}$$

$$J_{G_4} = u^4\left(x'^2 + y'^2 + z'^2\right) + 2V(x,y,z). \tag{5.109}$$

Table 5.1: Potentials for which the Kepler Ermakov in the 4D FRW space admit Noether symmetries

Noether Symmetry	$\mathbf{V(x,y,z)}$	**Noether Integral**
$a\partial_\mu + b\partial_\nu + c\partial_\sigma$	$-\frac{p}{a}x_\mu + f\left(x^\nu - \frac{b}{a}x^\mu, x^\sigma - \frac{c}{a}x^\mu\right)$	$I_{T3} = u^2\left(aI_\mu + bI_\nu + cI_\sigma\right)$
$a\partial_\mu + b\partial_\nu + c\left(x_\nu\partial_\mu - x_\mu\partial_\nu\right)$	$\frac{p}{\lvert c\rvert}\arctan\left(\frac{(b-cx_\mu)}{\lvert(a+cx_\nu)\rvert}\right)$	$I_{T2Rr} = u^2\left(aI_\mu + I_\nu + I_{R_{\mu\nu}}\right)$
	$+f\left(\frac{c}{2}r_{(\mu\nu)} - bx_\mu + ax_\nu, x_\sigma\right)$	
$a\partial_\mu + b\partial_\nu + c\left(x_\sigma\partial_\mu - x_\mu\partial_\sigma\right)$	$-\frac{p}{\lvert c\rvert}\arctan\left(\frac{\lvert c\rvert x_\mu}{\lvert a+cx_\sigma\rvert}\right)$	$I_{T2R2} = u^2\left(aI_\mu + I_\nu + I_{R\mu\sigma}\right)$
	$+f\left(x_\nu - \frac{1}{\lvert c\rvert}\arctan\left(\frac{\lvert c\rvert x_\mu}{\lvert a+cx_\sigma\rvert}\right), \frac{1}{2}r_{(\mu\sigma)} - \frac{a}{c}x_\sigma\right)$	
$a\partial_\mu + b\left(x_\nu\partial_\mu - x_\mu\partial_\nu\right) +$	$\frac{p}{\sqrt{b^2+c^2}}\arctan\left(\frac{\left(ab+b^2x_\nu+bcx_\sigma\right)}{\lvert bx_\mu\rvert\sqrt{b^2+c^2}}\right) +$	$I_{T1R3} = u^2\left(aI_\mu + I_{R_{\mu\nu}} + I_{R\mu\sigma}\right)$
$+c\left(x_\sigma\partial_\mu - x_\mu\partial_\sigma\right)$	$+f\left(x_\mu^2 + x_\nu^2\left(1-\frac{c^2}{b^2}\right) + \left(\frac{2a}{b} + \frac{2c}{b}x_\sigma\right)x_\nu, x_\sigma - \frac{c}{b}x_\nu\right)$	
$so(3)$ linear combination	$F\left(b\tan\theta\sin\phi + c\cos\phi - aM_1\right)$	$I_{R3} = u^2\left(I_{R_{\mu\nu}} + I_{R_{\mu\sigma}} + I_{R_{\nu\sigma}}\right)$

Note that had we considered the representation (5.76) only (that is we had set $\mu = 0$) then we would have lost all information concerning the system defined for $\mu \neq 0$! We emphasize that in applications to Physics the major datum is the Lagrangian and not the equations of motion, therefore one should not make mathematical assumptions which restrict the physical generality.

To assure Liouville integrability we need one more Noether symmetry whose Noether integral is in involution with E, J_{G_4}. This is possible for certain forms of the potential $V(x,y,z)$. Using the general results of section 4.5 where all 3D potentials are given which admit extra Noether symmetries we find the results of Table 5.1.

Proposition 5.7.1 *The Lagrangian (5.105) admits an extra Noether symmetry if and only if the potential $V(x,y,z)$ has a form given in Table 5.1*[2].

For example if $V(x,y,z) = \left(x^2+y^2+z^2\right)^n$ then the system admits three extra Noether symmetries which are the elements of $so(3)$. If $V(x,y,z) = V_0$ then the system admits six extra Noether symmetries (the KVs of the three dimensional Euclidian space).

5.7.2 The Riemannian Kepler Ermakov system on a 3D FRW spacetime

Consider the three dimensional Lorentzian metric

$$ds^2 = du^2 - u^2\left(dx^2 + dy^2\right) \tag{5.110}$$

[2] Where $I_\mu = \delta_{\mu\rho}\dot{x}^\rho$ and $I_{R\mu\nu} = \delta_{\rho[\mu}\delta_{\nu]\sigma}x^\sigma\dot{x}'^\rho$

Table 5.2: Potentials for which the Kepler Ermakov in the 3D FRW space admit Noether symmetries

Noether Symmetry	$\mathbf{V}(x,y)$	**Noether Integral**
∂_x	$f(y)$	$I_x = u^2 x'$
∂_y	$f(x)$	$I_y = u^2 y'$
$y\partial_x - x\partial_y$	$f\left(x^2+y^2\right)$	$I_{xy} = u^2\left(yx' - xy'\right)$
$\partial_x + b\partial_y$	$f(y-bx)$	$I_{xby} = u^2\left(x' + by'\right)$
$(a+y)\,\partial_x + (b-x)\,\partial_y$	$f\left(\frac{1}{2}\left(x^2+y^2\right) + ay - bx\right)$	$I_{abxy} = (a+y)\,u^2x' + (b-x)\,u^2y'$
$\partial_x,\ \partial_y\ ,\ y\partial_x - x\partial_y$	V_0	$I_x\ , I_y\ ,\ I_{xy}$

which admits the gradient HV $u\partial_u$ and the three KVs of the Euclidian metric E^2. In that space consider the Lagrangian

$$L' = \frac{1}{2}\left(u'^2 - u^2\left(x'^2 + y'^2\right)\right) + \frac{\mu^2}{2}u^2 - \frac{1}{u^2}V(x,y)\,. \tag{5.111}$$

According to proposition 5.6.2 this Lagrangian admits as Noether point symmetries the elements of $sl(2,R)$. Then from proposition 5.6.2 we have that the Noether invariants of these symmetries are

$$E = \frac{1}{2}\left(u'^2 - u^2\left(x'^2 + y'^2\right)\right) - \frac{\mu^2}{2}u^2 + \frac{1}{u^2}V(x,y) \tag{5.112}$$

$$J_{G_3} = u^4\left(x'^2 + y'^2\right) + 2V(x,y)\,. \tag{5.113}$$

The requirement that the Lagrangian admits an additional Noether symmetry leads to the condition $L_{KV}V(x,y) + p = 0$, therefore in that case we have a 2D potential and we can use the results of Chapter 4. If we demand the new Noether integral to be time independent $(p=0)$ then the potential $V(x,y)$ and the new Noether integrals are given in Table 5.2.

Lagrangians with kinetic term $T_K = \frac{1}{2}\left(u'^2 - u^2\left(x'^2 + y'^2\right)\right)$ appear in cosmological models. In the following section we discuss such applications.

5.8 The Riemannian Kepler Ermakov system in cosmology

Below, we consider two cosmological models for dark energy, a scalar field cosmology and an $f(R)$ cosmology in a locally rotational symmetric (LRS) spacetime.

5.8.1 The case of scalar field cosmology

Consider the Class A LRS spacetime

$$ds^2 = -N^2(t)\,dt^2 + a^2(t)\,e^{-2\beta(t)}dx + a^2(t)\,e^{\beta(t)}\left(dy^2+dz^2\right) \tag{5.114}$$

which is assumed to contain a scalar field with exponential potential $V(\phi) = V_0 e^{-c\phi}$, $c \neq \sqrt{6k}$ and a perfect fluid with a stiff equation of state $p = \rho$, where p is the pressure and ρ is the energy density of the fluid. The conservation equation for the matter density gives

$$\dot{\rho} + 6\rho\frac{\dot{a}}{a} = 0 \rightarrow \rho = \frac{\rho_0}{a^6}. \tag{5.115}$$

Einstein field equations for the comoving observers $u^a = \frac{1}{N(t)}\partial_t$, $u^a u_a = -1$ follow from the autonomous Lagrangian [85, 86]

$$L = -3\frac{a}{N}\dot{a}^2 + \frac{3}{4}\frac{a^3}{N}\dot{\beta} + \frac{k}{2}\frac{a^3}{N}\dot{\phi}^2 - Nka^3e^{-c\phi} - N\frac{\rho_0}{a^3}. \tag{5.116}$$

We set $N^2 = e^{c\phi}$ and the Lagrangian becomes

$$L = (-3\dot{a}^2 + \frac{3}{4}a^2\dot{\beta}^2 + \frac{k}{2}a^2\dot{\phi}^2)ae^{-\frac{c}{2}\phi} - ka^3V_0e^{-\frac{c}{2}\phi} - \frac{\rho_0}{a^3e^{\frac{c}{2}\phi}}. \tag{5.117}$$

The Hamiltonian is

$$E = (-3\dot{a}^2 + \frac{3}{4}a^2\dot{\beta}^2 + \frac{k}{2}a^2\dot{\phi}^2)ae^{-\frac{c}{2}\phi} + ka^3V_0e^{-\frac{c}{2}\phi} + \frac{\rho_0}{a^3e^{\frac{c}{2}\phi}} = 0 \tag{5.118}$$

If we consider the transformation

$$a^3 = e^{x+y} \ , \ \phi = \frac{1}{3}\sqrt{\frac{6}{k}}\,(x-y) \tag{5.119}$$

where

$$x = \frac{1}{1-\bar{c}}\ln\left(\frac{|1-\bar{c}|}{\sqrt{2}}ue^{z}\right) \ , \ y = \frac{1}{1+\bar{c}}\ln\left(\frac{1+\bar{c}}{\sqrt{2}}ue^{-z}\right) \ , \ \bar{c} = \frac{c}{\sqrt{6k}} \neq 1 \tag{5.120}$$

the Lagrangian (5.117) becomes

$$L = -\frac{2}{3}\dot{u}^2 + u^2\left(\frac{2}{3}\dot{z}^2 - \frac{3}{8kV_0}\dot{\beta}^2\right) - \frac{\mu^2}{2}u^2 + \frac{kV_0\rho_0}{\mu^2}\frac{1}{u^2} \ , \ \mu^2 = kV_0\left(1-\bar{c}^2\right) \neq 0. \tag{5.121}$$

We consider a 2D Riemannian space with metric defined by the kinematic terms of the Lagrangian, that is

$$ds^2 = \left(-6da^2 + \frac{3}{2}a^2d\beta^2 + ka^2d\phi^2\right)ae^{-\frac{c}{2}\phi} \tag{5.122}$$

We show easily that this metric admits the gradient HV $H^i = \frac{4}{6k-c^2}\left(ka\partial_a + c\partial_\phi\right)$ with gradient function $H = \frac{8k\varepsilon}{c^2-6k}ae^{-\frac{c}{2}\phi}$. Therefore the Lagrangian (5.117) defines an autonomous Hamiltonian Riemannian Kepler Ermakov system with potential ($\mu \neq 0$)

$$V(u, y^A) = -\frac{1}{2}\mu^2u^2 + \frac{kV_0\rho_0}{\mu^2}\frac{1}{u^2}. \tag{5.123}$$

Because $\mu \neq 0$ this Lagrangian admits $sl(2,R)$ invariance only for the representation (5.76) (an additional result which shows the necessity for the consideration of the cases $\mu = 0$ and $\mu \neq 0$!).

Using proposition 5.6.2, we write the Ermakov invariant

$$J = u^4 \left(\frac{2}{3}\dot{z}^2 + \frac{3}{8kV_0}\dot{\beta}^2 \right) + \frac{kV_0\rho_0}{\mu^2}\frac{1}{u^2}. \tag{5.124}$$

The second invariant is the Hamiltonian

$$E = -\frac{2}{3}\dot{u}^2 + u^2 \left(\frac{2}{3}\dot{z}^2 + \frac{3}{8kV_0}\dot{\beta}^2 \right) + \frac{\mu^2}{2}u^2 - \frac{kV_0\rho_0}{\mu^2}\frac{1}{u^2}. \tag{5.125}$$

We find that the Lagrangian admits three more Noether symmetries

$$\partial_\beta,\ \partial_z\ ,\ z\partial_\beta - \beta\partial_z \tag{5.126}$$

with corresponding integrals

$$I_1 = u^2\dot{\beta}\ ,\ I_2 = u^2\dot{z}\ ,\ I_3 = u^2 \left(\frac{3}{8kV_0}z\dot{\beta} - \frac{2}{3}\beta\dot{z} \right). \tag{5.127}$$

It is easy to show that three of the integrals are in involution, therefore the system is Liouville integrable.

5.8.2 The case of $f(R)$ Cosmology

Consider the modified Einstein-Hilbert action

$$S = \int d^4x\sqrt{-g}f(R) \tag{5.128}$$

where $f(R)$ is a smooth function of the curvature scalar R. The resulting field equations for this action in the metric variational approach are [87]

$$f'R_{ab} - \frac{1}{2}fg_{ab} + g_{ab}\Box f' - f'_{;ab} = 0 \tag{5.129}$$

where $f' = \frac{df(R)}{dR}$ and $f'' \neq 0$. In the LRS spacetime (5.114), with $N(t) = 1$, these equations for comoving observers are the Euler-Lagrange equations of the Lagrangian

$$L = \left(6af'\dot{a}^2 + 6a^2f''\dot{a}\dot{R} - \frac{3}{2}f'a^3\dot{\beta}^2 \right) + a^3\left(f'R - f\right). \tag{5.130}$$

The Hamiltonian is

$$E = \left(6af'\dot{a}^2 + 6a^2f''\dot{a}\dot{R} - \frac{3}{2}f'a^3\dot{\beta}^2 \right) - a^3\left(f'R - f\right) = 0. \tag{5.131}$$

Again we consider the 3d Riemannian space whose metric is defined by the kinematic part of the Lagrangian (5.130)

$$ds^2 = 12af'da^2 + 12a^2f''da\ dR - 3a^3f'd\beta^2. \tag{5.132}$$

This metric admits the gradient HV

$$H^i = \frac{1}{2}\left(a\partial_a + \frac{f'}{f''}\partial_R\right) \tag{5.133}$$

with gradient function $H = 3a^3 f'$.

In order to determine the function $f(R)$ we demand the geometric condition that Lagrangian (5.130) admits $s(2,R)$ invariance via Noether symmetries. Then for each representation (5.76), (5.77) we have a different function $f(R)$ hence a different physical theory.

The representation (5.76) in the present context is:

$$\partial_t\ ,\ 2t\partial_t + \frac{1}{2}\left(a\partial_a + \frac{f'}{f''}\partial_R\right)\ ,\ t^2\partial_t + \frac{t}{2}\left(a\partial_a + \frac{f'}{f''}\partial_R\right). \tag{5.134}$$

The Noether conditions become

$$-4a^3 f' R + \frac{7}{2}a^3 f + p = 0. \tag{5.135}$$

These vectors are Noether symmetries if $p = 0$ and

$$f\left(R\right) = R^{\frac{7}{8}}. \tag{5.136}$$

However power law $f\left(R\right)$ theories are not cosmologically viable [88].

The second representation (5.77) in the present context gives the vectors

$$\partial_t\ ,\ \frac{1}{\mu}e^{\pm 2\mu t}\partial_t \pm \frac{1}{2}e^{\pm 2\mu t}\left(a\partial_a + \frac{f'}{f''}\partial_R\right). \tag{5.137}$$

The Noether conditions give

$$-4a^3 f' R + \frac{7}{2}a^3 f + 3\mu^2 a^3 f' + p = 0. \tag{5.138}$$

These vectors are Noether symmetries if the constant $p = 0$ and the function

$$f\left(R\right) = \left(R - 2\Lambda\right)^{\frac{7}{8}} \tag{5.139}$$

where $2\Lambda = 3\mu^2$. This model is the viable Λ_{bc}CDM-like cosmological with $b = 1, c = \frac{7}{8}$. [89].

We note that if we had not considered the latter representation then we would loose this interesting result. The importance of the result is due to the fact that it follows from a geometric assumption which is beyond and above the physical considerations. Furthermore the assumption of Noether symmetries provides the Noether integrals which allow for an analytic solution of the model.

For the function (5.139) the Lagrangian (5.130) becomes for both cases (if $\Lambda = 0$ we have the power-law $f\left(R\right) = R^{\frac{7}{8}}$)

$$L = \frac{21}{4}a\left(R - 2\Lambda\right)^{-\frac{1}{8}}\dot{a}^2 - \frac{21}{16}a^2\left(R - 2\Lambda\right)^{-\frac{9}{8}}\dot{a}\dot{R} - \frac{21}{8}a^3\left(R - 2\Lambda\right)^{-\frac{1}{8}}\dot{\beta}^2 - \frac{a^3}{8}\frac{\left(R - 16\Lambda\right)}{\left(R - 2\Lambda\right)^{\frac{1}{8}}}. \tag{5.140}$$

Furthermore there exist a coordinate transformation for which the metric (5.132) is written in the form of (5.87).

We introduce new variables u, v, w with the relations

$$a = \left(\frac{21}{4}\right)^{-\frac{1}{3}} \sqrt{ue^{v}} \ , \ R = 2\Lambda + \frac{e^{12v}}{u^4}, \ \beta = \sqrt{2}w. \tag{5.141}$$

In the new variables the Lagrangian (5.140) takes the form

$$L = \frac{1}{2}\dot{u}^2 - \frac{1}{2}u^2\left(\dot{v}^2 + \dot{w}^2\right) + \frac{\mu^2}{2}u^2 - \frac{1}{42}\frac{e^{12v}}{u^2}. \tag{5.142}$$

The Hamiltonian (5.131) in the new coordinates is

$$E = \frac{1}{2}\dot{u}^2 - \frac{1}{2}u^2\left(\dot{v}^2 + \dot{w}^2\right) - \frac{\mu^2}{2}u^2 + \frac{1}{42}\frac{e^{12v}}{u^2}. \tag{5.143}$$

The Lagrangian (5.142) defines a Hamiltonian Riemannian Kepler Ermakov system with potential

$$V(u,v) = -\frac{\mu^2}{2}u^2 + \frac{1}{42}\frac{e^{12v}}{u^2}$$

from which follows the potential $V(v) = \frac{1}{42}e^{12v}$. In addition to the Hamiltonian the dynamical system admits the Riemannian Ermakov invariant

$$J_f = u^4\left(\dot{v}^2 + \dot{w}^2\right) + \frac{1}{21}e^{12v}. \tag{5.144}$$

The Lagrangian (5.142) admits the extra Noether point symmetry ∂_w with Noether integral $I_w = u^2\dot{w}$ (see Table 5.2). The three integrals E, I_w and J_f are in involution and independent, therefore the system is integrable.

5.9 Conclusion

In this Chapter we have considered the generalization of the autonomous Kepler Ermakov dynamical system in the spirit of Leach [56], that is using invariance with respect to the $sl(2,R)$ Lie and Noether algebra. We have generalized the autonomous Newtonian Hamiltonian Kepler Ermakov system to three dimensions using Noether rather than Lie point symmetries and have determined all such systems which are Liouville integrable via Noether point symmetries. We introduced the autonomous Riemannian Kepler Ermakov system in a Riemannian space which admits a gradient HV. This system is the generalization of the autonomous Euclidian Kepler Ermakov system and opens new fields of applications for the autonomous Kepler Ermakov system, especially in relativistic Physics. Indeed we have determined the autonomous Riemannian Kepler Ermakov system in a spatially flat FRW spacetime which admits a gradient HV. As a further application we have considered two types of cosmological models, which are described by the autonomous Riemannian Kepler Ermakov system, the scalar field cosmology with exponential potential and $f(R)$ gravity in an LRS spacetime.

5.A Appendix

We require that the force admits two Lie symmetries which are due to the gradient HV $H=u\partial_u$ (if we require the force to be invariant under three Lie symmetries which are due to the gradient HV then it is reduced to the isotropic oscillator). From Theorem 4.2.2 of Chapter 4 we have the following cases.

(I) Case $\mu=0$

In this case the Lie symmetries are

$$\partial_s,\ 2s\partial_s+u\partial_u,\ s^2\partial_s+su\partial_u.$$

The condition which the force must satisfy is

$$L_H F^i + dF^i = 0.$$

Replacing components we find the equations

$$\begin{aligned}\left(\frac{\partial}{\partial u}F^u\right)u+(d-1)F^u &= 0\\ \left(\frac{\partial}{\partial u}F^A\right)u+dF^A &= 0\end{aligned}$$

from which follows

$$F^u=\frac{1}{u^{(d-1)}}F^u\ ,\ F^A=\frac{1}{u^d}F^A.$$

Because the HV is gradient, Case II of Theorem 4.2.2 applies and gives the condition

$$L_H F^i + 4F^i + a_1 H^i = 0$$

from which follows $a_1=0$ and $d=4$. Therefore

$$F^u=\frac{1}{u^3}G^u\left(y^C\right)\ ,\ F^A=\frac{1}{u^4}G^A\left(y^C\right).$$

(II) Case $\mu\neq 0$

In this case the Lie symmetries are

$$\partial_s,\ \frac{1}{\mu}e^{\pm 2\mu s}\partial_s \pm e^{\pm 2\mu s}u\partial_u$$

The condition which the force must satisfy is

$$L_H F^i + 4F^i + a_1 H^i = 0$$

We demand $a_1\neq 0$ and obtain the system of equations:

$$\begin{aligned}\left(\frac{\partial}{\partial u}F^u\right)u+3F^u+a_1 u &= 0\\ \left(\frac{\partial}{\partial u}F^A\right)u+4F^A &= 0\end{aligned}$$

whose solution is

$$F^u=\mu^2 u+\frac{1}{u^3}G^u\ ,\ F^A=\frac{1}{u^4}G^A$$

where we have set $a_1=-4\mu^2$.

Part III

Symmetries of PDEs

Chapter 6

Lie symmetries of a general class of PDEs

6.1 Introduction

In the previous chapters we studied the relation between point symmetries (Lie and Noether symmetries) of second order ordinary differential equations. Particularly, we considered the case of geodesic equations and the equations of motion of a particle moving in a Riemannian space. We made clear that there exists a unique relation between the point symmetries and the special projective Lie algebra of the underlying space in which the motion occurs.

In subsequent sections, we will attempt to extend the relation between point symmetries and collineations of the space to the case of second order partial differential equations. Obviously, a global answer to this problem is not possible. However, it will be shown that for many interesting PDEs, the Lie point symmetries are indeed obtained from the collineations of the metric. Pioneering work in this direction is the work of Ibragimov [4]. Recently, Bozhkov et al. [90] have studied the Lie and the Noether point symmetries of the Poisson equation and showed that the Lie symmetries of the Poisson PDE are generated from the conformal algebra of the metric.

In this chapter we show that for a general class of second order PDEs, there is a close relation between the Lie symmetries and the conformal algebra of the underlying space. Subsequently, we apply these results to a number of interesting PDEs and regain existing results in a unified manner.

In Section 6.2 we examine the generic PDE of the form

$$A^{ij}u_{ij} - F(x^i, u, u_i) = 0 \tag{6.1}$$

and derive the Lie symmetry conditions. Furthermore, in case A is independent on u, i.e. $A_{,u} = 0$, then the Lie point symmetries of (6.1) are related to the Conformal vectors (CVs) of the linear homogeneous differential geometric object A. In Section 6.3 we consider $F(x,u,u_i)$ to be linear in u_i and determine the Lie point symmetry conditions in geometric form. In sections 6.4 and 6.5, we apply the results of section 6.3 in order to determine the Lie point symmetries of the Poisson equation, the Yamabe equation and the heat equation with flux in a n dimensional Riemannian space. It will be shown that the Lie symmetry vectors of the Poisson and the Yamabe equation are obtained from the conformal algebra of the geometric object A^{ij} [90] whereas the Lie symmetries of the heat equations are obtained from the homothetic algebra of the metric. Furthermore, we determine the Lie symmetries of Laplace equation, the Yamabe equation and the homogeneous heat equation in various Riemannian spaces.

6.2 The case of the second order PDEs

Attempting to establish a general relation between the Lie symmetries of a second order PDE of the form (6.1) and the collineations of a Riemannian space we derive the Lie symmetry conditions of (6.1) and relate them with the collineations of the coefficients $A^{ij}(x,u)$ which we consider to be the components of a metric. According to the standard approach [1, 3, 4, 91] the symmetry condition is

$$X^{[2]}(H) = \lambda H \quad , \quad \mathrm{mod}\, H = 0 \tag{6.2}$$

where $\lambda(x^i,u,u_i)$ is a function to be determined. $X^{[2]}$ is the second prolongation of the Lie symmetry vector

$$X = \xi^i\left(x^i,u\right)\frac{\partial}{\partial x^i} + \eta\left(x^i,u\right)\frac{\partial}{\partial u} \tag{6.3}$$

given by the expression

$$X^{[2]} = \xi^i\frac{\partial}{\partial x^i} + \eta\frac{\partial}{\partial u} + \eta_i^{[1]}\frac{\partial}{\partial u_i} + \eta_{ij}^{[1]}\frac{\partial}{\partial u_{ij}} \tag{6.4}$$

where[1]

$$\begin{aligned}
\eta_i^{[1]} &= \eta_{,i} + u_i\eta_u - \xi^j_{,i}u_j - u_iu_j\xi^j_{,u} \\
\eta_{ij}^{[2]} &= \eta_{ij} + (\eta_{ui}u_j + \eta_{uj}u_i) - \xi^k_{,ij}u_k + \eta_{uu}u_iu_j - (\xi^k_{.,ui}u_j + \xi^k_{.,uj}u_i)u_k \\
&\quad +\eta_u u_{ij} - (\xi^k_{.,i}u_{jk} + \xi^a_{.,j}u_{ik}) - (u_{ij}u_k + u_iu_{jk} + u_{ik}u_j)\,\xi^k_{.,u} - u_iu_ju_k\xi^k_{uu}.
\end{aligned}$$

The introduction of the function $\lambda(x^i,u,u_i)$ in (6.2) causes the variables x^i, u, u_i to be independent[2].

The symmetry condition (6.2) when applied to (6.1) gives:

$$A^{ij}\eta_{ij}^{[2]} + \left(XA^{ij}\right)u_{ij} - X^{[1]}(F) = \lambda(A^{ij}u_{ij} - F) \tag{6.5}$$

[1] See section 2.3.1.
[2] See Ibragimov [4] p. 115

from which follows

$$\begin{aligned}0 &= A^{ij}\eta_{ij} - \eta_{,i}g^{ij}F_{,u_j} - X(F) + \lambda F \\ &+ 2A^{ij}\eta_{ui}u_j - A^{ij}\xi^a_{,ij}u_a - u_i\eta_u g^{ij}F_{,u_j} + \xi^k_{,i}u_k g^{ij}F_{,u_j} \\ &+ A^{ij}\eta_{uu}u_iu_j - 2A^{ij}\xi^k_{.,uj}u_iu_k + u_iu_k\xi^k_{,u}g^{ij}F_{,u_j} \\ &+ A^{ij}\eta_u u_{ij} - 2A^{ij}\xi^k_{.,i}u_{jk} + (\xi^kA^{ij}_{,k} + \eta A^{ij}_{,u})u_{ij} - \lambda A^{ij}u_{ij} \\ &- A^{ij}\left(u_{ij}u_a + u_iu_{ja} + u_{ia}u_j\right)\xi^a_{.,u} - u_iu_ju_aA^{ij}\xi^a_{uu}.\end{aligned} \tag{6.6}$$

We note that we cannot deduce the symmetry conditions before we select a specific form for the function $F(x^i, u, u_i)$. However, we may determine the conditions which are due to the second derivative of u because in these terms no F terms are involved. This observation significantly reduces the complexity of the remaining symmetry conditions. Following this, we have the condition

$$\begin{aligned}0 &= A^{ij}\eta_u u_{ij} - A^{ij}(\xi^k_{.,i}u_{ja} + \xi^k_{.,j}u_{ik}) + (\xi^kA^{ij}_{,k} + \eta A^{ij}_{,u})u_{ij} - \lambda A^{ij}u_{ij} \\ &- A^{ij}\left(u_{ij}u_a + u_iu_{ja} + u_{ia}u_j\right)\xi^a_{.,u} - u_iu_ju_aA^{ij}\xi^a_{uu}\end{aligned}$$

from which the following system of equations results

$$\begin{aligned}A^{ij}\left(u_{ij}u_k + u_{jk}u_i + u_{ik}u_j\right)\xi^k_{.,u} &= 0 \\ A^{ij}\eta_u u_{ij} - A^{ij}(\xi^k_{.,i}u_{jk} + \xi^k_{.,j}u_{ik}) + (\xi^kA^{ij}_{,k} + \eta A^{ij}_{,u})u_{ij} - \lambda A^{ij}u_{ij} &= 0 \\ A^{ij}\xi^a_{,uu} &= 0.\end{aligned}$$

The first equation is

$$A^{ij}\xi^k_{.,u} + A^{kj}\xi^i_{.,u} + A^{ik}\xi^j_{.,u} = 0 \Leftrightarrow A^{(ij}\xi^{k)}_{.,u} = 0. \tag{6.7}$$

From the second equation we get

$$A^{ij}\eta_u + \eta A^{ij}_{,u} + \xi^kA^{ij}_{,k} - A^{kj}\xi^i_{.,k} - A^{ik}\xi^j_{.,k} - \lambda A^{ij} = 0. \tag{6.8}$$

and the last equation gives the constraint

$$A^{ij}\xi^k_{,uu} = 0. \tag{6.9}$$

It can be easily shown that condition (6.7) implies $\xi^k_{.,u} = 0,$ which is a well known result[3]. From the analysis so far, we obtain the first result

Proposition 6.2.1 *For the infinitesimal generator (6.3) for all second order PDEs of the form (6.1), holds* $\xi^i_{.,u} = 0,$ *that is,* $\xi^i = \xi^i(x^j)$. *Furthermore, condition (6.9) is identically satisfied.*

[3]We give a simple proof for $n = 2$ in Appendix 6.A. A detailed and more general proof can be found in [92].

There remains the third symmetry condition (6.8). We consider the following cases.

$i, j \neq 0$:

We write (6.8) in an alternative form by considering A^{ij} to be linear a homogeneous differential geometric object as follows:

$$L_\xi A^{ij} = \lambda A^{ij} - (\eta A^{ij})_{,u}. \tag{6.10}$$

Then it follows:

Proposition 6.2.2 *For the infinitesimal generator (6.3) for all second order PDEs of the form (6.1) for which* $A^{ij}{}_{,u} = 0$*, i.e.* $A^{ij} = A^{ij}(x^i)$*, the vector* ξ^i *is a CKV of the linear homogeneous differential geometric object* A^{ij} *with conformal factor* $(\lambda - \eta_u)(x)$*.*

Assuming[4] $A^{tt} = A^{ti} = 0$, we have

- for $i = j = 0$ nothing
- for $i, j \neq 0$ gives (6.10) and
- for $i = 0, j \neq 0$ (6.10) becomes

$$\begin{aligned} A^{tj}\eta_u + \eta A^{tj}_{,u} + \xi^k A^{tj}_{,k} - A^{kj}\xi^t_{.,k} - A^{tk}\xi^j_{.,k} - \lambda A^{tj} = 0 \Rightarrow \\ A^{kj}\xi^t_{.,k} = 0 \end{aligned} \tag{6.11}$$

which leads to the following general result.

Proposition 6.2.3 *For all second order PDEs of the form* $A^{ij}u_{ij} - F(x^i, u, u_i) = 0$*, for which* A^{ij} *is nondegenerate the* $\xi^t_{.,k} = 0$*, that is,* $\xi^t = \xi^t(t)$ *.*

By using that $\xi^i_{,u} = 0$, the symmetry condition (6.6) is simplified as follows

$$\begin{aligned} 0 = & A^{ij}\eta_{ij} - \eta_{,i}A^{ij}F_{,u_j} - X(F) + \lambda F \\ & + 2A^{ij}\eta_{ui}u_j - A^{ij}\xi^a_{,ij}u_a - u_i\eta_u A^{ij}F_{,u_j} + \xi^k_{,i}u_k A^{ij}F_{,u_j} \\ & + A^{ij}\eta_{uu}u_i u_j + A^{ij}\eta_u u_{ij} - 2A^{ij}\xi^k_{.,i}u_{jk} \\ & + (\xi^k A^{ij}_{,k} + \eta A^{ij}_{,u})u_{ij} - \lambda A^{ij}u_{ij} \end{aligned} \tag{6.12}$$

which together with the condition (6.10) are the complete set of symmetry conditions for all second order PDEs of the form $A^{ij}u_{ij} - F(x^i, u, u_i) = 0$. This class of PDEs is quite general. This fact makes the above result very useful..

In order to continue, we need to consider special forms for the function $F(x, u, u_i)$.

[4] The index t refers to the coordinate x^0 whenever it is involved.

6.3 The Lie symmetry conditions for a linear function $F(x,u,u_i)$

Consider the function $F(x,u,u_i)$ to be linear in u_i, that is, to be of the form

$$F(x,u,u_i) = B^k(x,u)u_k + f(x,u) \tag{6.13}$$

where $B^k(x,u)$ and $f(x,u)$, are arbitrary functions of their arguments. In this case, the PDE (6.1) is of the form

$$A^{ij}u_{ij} - B^k(x,u)u_k - f(x,u) = 0. \tag{6.14}$$

The Lie symmetries of this type of PDEs have been studied previously by Ibragimov[4]. Assuming that at least one of the components of $\mathbf{A}$ is $A_{ij} \neq 0$ the Lie symmetry conditions are (6.12) and (6.10).

Replacing $F(x,u,u_1)$ in (6.12) we find[5]

$$\begin{aligned} 0 = {} & A^{ij}\eta_{ij} - \eta_{,i}g^{ij}B_j - \xi^k f_{,k} - \eta f_{,u} + \lambda f \\ & + 2A^{ij}\eta_{ui}u_j - A^{ij}\xi^a_{,ij}u_a - u_i\eta_u g^{ij}B_j + \xi^k_{,i}u_k g^{ij}B_j + \lambda B^k u_k - \eta B^k_{,u}u_k - \xi^l B^k_{,l}u_k \\ & + A^{ik}\eta_{uu}u_i u_k \qquad (6.15) \\ & + A^{ij}\eta_u u_{ij} - 2A^{kj}\xi^i_{.,k}u_{ji} + (\xi^k A^{ij}_{,k} + \eta A^{ij}_{,u})u_{ij} - \lambda A^{ij}u_{ij} \qquad (6.16) \end{aligned}$$

from which the subsequent equations follow

$$A^{ij}\eta_{ij} - \eta_{,i}B^i - \xi^k f_{,k} - \eta f_{,u} + \lambda f = 0 \tag{6.17}$$

$$-2A^{ik}\eta_{ui} + A^{ij}\xi^k_{,ij} + \eta_u B^k - \xi^k_{,i}B^i + \xi^i B^k_{,i} - \lambda B^k + \eta B^k_{,u} = 0 \tag{6.18}$$

$$A^{ik}\eta_{uu} = 0. \tag{6.19}$$

Equation (6.19) gives (because at least one $A^{ik} \neq 0$)

$$\eta = a(x^i)u + b(x^i). \tag{6.20}$$

Equation (6.18) gives the constraint

$$-2A^{ik}a_{,i} + aB^k + auB^k_{,u} + A^{ij}\xi^k_{,ij} - \xi^k_{,i}B^i + \xi^i B^k_{,i} - \lambda B^k + bB^k_{,u} = 0.$$

We summarize the above results as follows.

Proposition 6.3.1 *The determining equations (the Lie symmetry conditions) for the second order PDEs (6.14), in which at least one of the $A_{ij} \neq 0$, are*

$$A^{ij}(a_{ij}u + b_{ij}) - (a_{,i}u + b_{,i})B^i - \xi^k f_{,k} - auf_{,u} - bf_{,u} + \lambda f = 0 \tag{6.21}$$

[5] We ignore the terms with u_{ij} because we have already used them to obtain condition (6.10). Indeed, it is clear that these terms give $A^{ij}\eta_u - 2A^{ij}\xi^k_{.,i} + \xi^k A^{ij}_{,k} + \eta A^{ij}_{,u} - \lambda A^{ij} = 0$, which is precisely condition (6.10).

$$A^{ij}\xi^k_{,ij} - 2A^{ik}a_{,i} + aB^k + auB^k_{,u} - \xi^k_{,i}B^i + \xi^i B^k_{,i} - \lambda B^k + bB^k_{,u} = 0 \tag{6.22}$$

$$L_{\xi^i\partial_i}A^{ij} = (\lambda - a)A^{ij} - \eta A^{ij}{}_{,u} \tag{6.23}$$

$$\eta = a(x^i)u + b(x^i) \tag{6.24}$$

$$\xi^k_{,u} = 0 \Leftrightarrow \xi^k(x^i). \tag{6.25}$$

Pay paricular attention to the fact that for all second order PDEs of the form (6.14), for which, $A^{ij}_{,u} = 0$, i.e $A^{ij} = A^{ij}(x^i)$, the $\xi^i(x^j)$ is a CKV of the metric A^{ij}. Also, in this case $\lambda = \lambda(x^i)$. This result establishes the relation between the Lie symmetries of this type of PDEs with the collineations of the metric defined by the coefficients A_{ij}.

Moreover, in case the coordinates are t, x^i (where $i = 1, ..., n$) $A^{tt} = A^{tx^i} = 0$ and A^{ij} is a nondegenerate metric we have that

$$\xi^t_{,i} = 0 \Leftrightarrow \xi^t(t). \tag{6.26}$$

These symmetry relations coincide with those given in [4]. Finally, note that equation (6.22) can be written as

$$A^{ij}\xi^k_{,ij} - 2A^{ik}a_{,i} + [\xi, B]^k + (a - \lambda)B^k + (au + b)B^k_{,u} = 0. \tag{6.27}$$

Having derived the Lie symmetry conditions for the type of PDEs (6.14) we continue with the computation of the Lie symmetries of some important PDEs of this form. Before we proceed, we state two Lemmas which will be used later (for details, see Appendix 6.B).

Lemma 6.3.2 *For the Lie derivative of the connection coefficients, the following properties hold.*

a. In flat space (in which $\Gamma^i_{jk} = 0$) the following identity holds:

$$L_\xi \Gamma^k_{ij} = \xi^k_{,ij}. \tag{6.28}$$

b. For a general metric g_{ij} satisfying the condition $L_{\xi^i\partial_i}g_{ij} = -(\lambda - a)g_{ij}$ the following relation holds:

$$g^{jk}L_\xi\Gamma^i_{.jk} = g^{jk}\xi^i_{,\ jk} + \Gamma^i{}_{,l}\xi^l - \xi^i{}_{,l}\Gamma^l + (a - \lambda)\Gamma^i. \tag{6.29}$$

Lemma 6.3.3 *Assume that the vector ξ^i is a CKV of the metric g_{ij} with conformal factor $-(\lambda - a)$ i.e. $L_{\xi^i\partial_i}g_{ij} = -(\lambda - a)g_{ij}$. Then, the following statement is true:*

$$g^{jk}L_\xi\Gamma^i_{.jk} = \frac{2-n}{2}(a - \lambda)^{,i} \tag{6.30}$$

where $n = g^{jk}g_{kj}$ is the dimension of the space.

In the following sections, we study the Lie symmetries of two second order PDEs which are important in physics. Particularly, we examine the relation between the Lie symmetries of the Poisson equation and the Heat conduction equation in a Riemannian manifold with the conformal group of the space. We will show that the Lie symmetries of the heat conduction equation relates to the homothetic group of the underlying space whereas the Lie symmetries of the Poisson equation are associated to the conformal group of the metric that defines the Laplace operator.

6.4 Symmetries of the Poisson equation in a Riemannian space

The Lie symmetries of the Poisson equation

$$\Delta u - f\left(x^i, u\right) = 0. \tag{6.31}$$

where $\Delta = \frac{1}{\sqrt{|g|}}\frac{\partial}{\partial x^i}\left(\sqrt{|g|}g^{ij}\frac{\partial}{\partial x^j}\right)$ is the Laplace operator of the metric g_{ij}, for $f = f(u)$ have been given in [4, 90]. Here we generalize this result [6] for $f = f\left(x^i, u\right)$.

Theorem 6.4.1 *The Lie symmetries of the Poisson equation (6.31) are generated from the CKVs of the metric g_{ij} defining the Laplace operator, as follows*

a) for $n > 2$, the Lie symmetry vector is

$$X = \xi^i\left(x^k\right)\partial_i + \left(\frac{2-n}{2}\psi\left(x^k\right)u + a_0 u + b\left(x^k\right)\right)\partial_u \tag{6.32}$$

where $\xi^i\left(x^k\right)$ is a CKV with conformal factor $\psi\left(x^k\right)$ and the following condition holds

$$\frac{2-n}{2}\Delta\psi u + g^{ij}b_{i;j} - \xi^k f_{,k} - \frac{2-n}{2}\psi u f_{,u} - \frac{2+n}{2}\psi f - b f_{,u} = 0. \tag{6.33}$$

b) for $n = 2$, the Lie symmetry vector is

$$X = \xi^i\left(x^k\right)\partial_i + \left(a_0 u + b\left(x^k\right)\right)\partial_u \tag{6.34}$$

where $\xi^i\left(x^k\right)$ is a CKV with conformal factor $\psi\left(x^k\right)$ and the following condition holds

$$g^{ij}b_{;ij} - \xi^k f_{,k} - a_0 u f_{,u} + \left(a_0 - 2\psi\right)f - b f_{,u} = 0. \tag{6.35}$$

In the following subsections, we apply Theorem 6.4.1 for special forms of the function $f\left(x^i, u\right)$.

6.4.1 Lie symmetries of Laplace equation

The Laplace equation

$$\Delta u = 0 \tag{6.36}$$

follows from the Poison equation (6.31) if we consider $f\left(x^i, u\right) = 0$. Therefore, Theorem 6.4.1 applies and we have the following result [90].

Theorem 6.4.2 *The Lie symmetries of Laplace equation (6.36) are generated from the CKVs of the metric g_{ij} defining the Laplace operator as follows*

a) for $n > 2$, the Lie symmetry vector is

$$X = \xi^i\left(x^k\right)\partial_i + \left(\frac{2-n}{2}\psi\left(x^k\right)u + a_0 u + b\left(x^k\right)\right)\partial_u \tag{6.37}$$

[6] The proof is given in Appendix 6.B.

where ξ^i is a CKV with conformal factor $\psi\left(x^k\right)$, $b\left(x^k\right)$ is a solution of (6.36) and the following condition is satisfied

$$\Delta\psi = 0. \tag{6.38}$$

b) for $n = 2$, the Lie symmetry vector is

$$X = \xi^i\left(x^k\right)\partial_i + \left(a_0 u + b\left(x^k\right)\right)\partial_u \tag{6.39}$$

where ξ^i is a CKV with conformal factor $\psi\left(x^k\right)$ and $b\left(x^k\right)$ is a solution of (6.36).

6.4.2 Symmetries of conformal Poisson equation in a Riemannian space

If in the Poisson equation (6.31) we replace $f\left(x^i, u\right)$ with

$$f = -M_0 R u + \bar{f}\left(x^i, u\right) \tag{6.40}$$

where R is the Ricci scalar of the metric which defines the Laplace operator Δ and

$$M_0 = \frac{2-n}{4\left(n-1\right)} \tag{6.41}$$

then, equation (6.31) becomes

$$\bar{L}_g u - \bar{f}\left(x^i, u\right) = 0. \tag{6.42}$$

where $\bar{L}_g$ is the conformal Laplace or Yamabe operator acting on functions on V^n defined by

$$\bar{L}_g = \Delta + \frac{n-2}{4\left(n-1\right)}R. \tag{6.43}$$

Equation (6.42) is called the conformal Poisson or Yamabe equation and plays a central role in the study of a conformal class of metrics by means of the Yamabe invariant (see, e.g. [93]). In order to investigate the Lie symmetries of (6.42), we make use of Theorem 6.4.1 and find the following result [7].

Theorem 6.4.3 *The Lie symmetries of the conformal Poisson equation (6.42) are generated from the CKVs of the metric g_{ij} defining the conformal Laplace operator, as follows*

$$X = \xi^i\left(x^k\right)\partial_i + \left(\frac{2-n}{2}\psi\left(x^k\right)u + a_0 u + b\left(x^k\right)\right)\partial_u \tag{6.44}$$

where $\xi^i\left(x^k\right)$ is a CKV with conformal factor $\psi\left(x^k\right)$ and the following condition holds

$$-\xi^k \bar{f}_{,k} - \frac{2-n}{2}\psi u \bar{f}_k - \frac{2+n}{2}\psi\bar{f} + g^{ij}b_{;ij} - b\left(-M_0 R + \bar{f}_u\right) = 0. \tag{6.45}$$

[7] The proof is given in Appendix 6.B.

6.4.3 Lie symmetries of the conformal Laplace equation

The conformal Laplace equation

$$\bar{L}_g u = 0. \tag{6.46}$$

is the conformal Poisson equation (6.42) for $\bar{f}\left(x^i, u\right) = 0$. Therefore, Theorem 6.4.3 applies and we have the following result.

Theorem 6.4.4 *The Lie symmetries of the conformal Laplace equation (6.46) are generated from the CKVs of the metric g_{ij} of the conformal Laplace operator as follows*

$$X = \xi^i\left(x^k\right)\partial_i + \left(\frac{2-n}{2}\psi\left(x^k\right)u + a_0 u + b\left(x^k\right)\right)\partial_u \tag{6.47}$$

where ξ^i is a CKV with conformal factor $\psi\left(x^k\right)$ and $b\left(x^k\right)$ is a solution of (6.46).

In order to compare the Lie symmetries of Laplace equation (6.36) and of the conformal Laplace equation (6.46), we apply the results of the Theorems 6.4.2 and 6.4.4 in the case of the FRW spacetime with the following line element

$$ds^2 = R^2\left(\tau\right)\left(-d\tau^2 + dx^2 + dy^2 + dz^2\right). \tag{6.48}$$

The Laplace operator for the space with Line element (6.48) is

$$\Delta = \frac{1}{R^2}\eta^{ij}\partial_i\partial_j - 2\frac{R_{,\tau}}{R^3}\delta^i_\tau\partial_i$$

where η_{ij} is the metric of the Minkowski spacetime.

According to Theorem 6.4.2, the Laplace equation

$$\frac{1}{R^2}\eta^{ij}u_{ij} - 2\frac{R_{,\tau}}{R^3}\delta^i_\tau u_i = 0 \tag{6.49}$$

admits eight Lie point symmetries; six Lie symmetries are the KVs[8] of (6.48) plus the two Lie symmetries $\left(a_0 u + b\left(x^k\right)\right)\partial_u$ because, for general $R\left(\tau\right)$, the conformal factors of the CKVs of (6.48) do not satisfy (6.49).

On the contrary, according to Theorem (6.4.4), the conformal Laplace equation

$$\frac{1}{R^2}\eta^{ij}u_{ij} - 2\frac{R_{,\tau}}{R^3}\delta^i_\tau u_i - \frac{R_{,\tau\tau}}{R^3}u = 0 \tag{6.50}$$

admits seventeen Lie symmetries; fifteen are the CKVs of the metric (6.48) plus the two Lie symmetries $\left(a_0 u + b\left(x^k\right)\right)\partial_u$.

For special functions $R\left(\tau\right)$, the Laplace equation (6.49) admits extra Lie symmetries; however, the conformal Laplace equation (6.50) does not admit extra Lie symmetries.

A direct result, which arises from Theorems 6.4.2 and 6.4.4, is that, if V^n is an n dimensional Riemannian space, $n > 2$, then, if the Laplace equation (6.36) in V^n is invariant under a Lie group G_L, then, G_L is a subgroup

[8] For the conformal algebra of the FRW spacetime see Chapter 3, [45].

of $\bar{G}_{L_C}$, i.e. $G_L \subseteq \bar{G}_{LC}$ where $\bar{G}_{LC}$ is a Lie group which leaves invariant the conformal Laplace equation (6.46). The Lie algebras $G_L, \bar{G}_{LC}$ are identical if the V^n does not admit proper CKVs or if all the conformal factors of the CKVs of V^n are solutions of the Laplace equation (6.36). Moreover, if V^n is a conformally flat spacetime then, the conformal Laplace equation (6.46) admits a Lie algebra of $\frac{1}{2}(n+1)(n+2)+2$ dimension. For instance, the Laplace equation in the three dimensional sphere[9] admits eight Lie point symmetries [94] while, on the contrary, the Yamabe equation admits twelve Lie symmetries.

6.5 The heat conduction equation with a flux in a Riemannian space

The heat equation with a flux in an n dimensional Riemannian space with metric g_{ij} is

$$H(u) = q\left(t, x^i, u\right) \tag{6.51}$$

where

$$H(u) = \Delta u - u_t.$$

and Δ is the Laplace operator.

The term $q\left(t, x^i, u\right)$ indicates that the system exchanges energy with the environment. In this case, the Lie symmetry vector is

$$\mathbf{X} = \xi^i\left(x^j, u\right)\partial_i + \eta\left(x^j, u\right)\partial_u$$

where $a = t, i$. For this equation, we have

$$A^{tt} = 0,\ A^{ti} = 0,\ A^{ij} = g^{ij}, B^i = \Gamma^i(t, x^i), B^t = 1, f(x, u) = q\left(t, x^k, u\right).$$

For this PDE, the symmetry conditions (6.21) - (6.26) become

$$\eta = a(t, x^i)u + b(t, x^i) \tag{6.52}$$

$$\xi^t = \xi^t(t) \tag{6.53}$$

$$g^{ij}(a_{ij}u + b_{ij}) - (a_{,i}u + b_{,i})\Gamma^i - (a_{,t}u + b_{,t}) + \lambda q = \xi^t q_{,t} + \xi^k q_{,k} + \eta q_{,u} \tag{6.54}$$

$$g^{ij}\xi^k_{,ij} - 2g^{ik}a_{,i} + a\Gamma^k - \xi^k_{,i}\Gamma^i + \xi^i\Gamma^k_{,i} - \lambda\Gamma^k = 0 \tag{6.55}$$

$$L_{\xi^i\partial_i} g_{ij} = (a - \lambda)g_{ij}. \tag{6.56}$$

The solution of the symmetry conditions is summarized in Theorem 6.5.1 (for an sketch proof see Appendix 6.B).

[9] The three dimensional sphere is a conformally flat space and it is maximall symmetric [23].

Theorem 6.5.1 *The Lie symmetries of the heat equation with flux i.e.*

$$g^{ij}u_{ij} - \Gamma^i u_i - u_t = q\left(t,x,u\right) \tag{6.57}$$

in a n dimensional Riemannian space with metric g_{ij} are constructed from the homothetic algebra of the metric as follows:

a. Y^i is a nongradient HV/KV.

The Lie symmetry is

$$X = \left(2c_2\psi t + c_1\right)\partial_t + c_2 Y^i\partial_i + \left(a\left(t\right)u + b\left(t,x\right)\right)\partial_u \tag{6.58}$$

where $a(t), b\left(t,x^k\right), q\left(t,x^k,u\right)$ must satisfy the constraint equation

$$-a_t u + H\left(b\right) - \left(au+b\right)q_{,u} + aq - \left(2\psi c_2 qt + c_1 q\right)_t - c_2 q_{,i}Y^i = 0. \tag{6.59}$$

b. $Y^i = S^{;i}$ is a gradient HV/KV.

The Lie symmetry is

$$X = \left(2\psi\int Tdt + c_1\right)\partial_t + TS^{;i}\partial_i + \left(\left(-\frac{1}{2}T_{,t}S + F\left(t\right)\right)u + b\left(t,x\right)\right)\partial_u \tag{6.60}$$

where $F(t), T(t), b\left(t,x^k\right), q\left(t,x^k,u\right)$ must satisfy the constraint equation

$$\begin{aligned} 0 = & \left(-\frac{1}{2}T_{,t}\psi + \frac{1}{2}T_{,tt}S - F_{,t}\right)u + H\left(b\right) + \\ & -\left(\left(-\frac{1}{2}T_{,t}S + F\right)u + b\right)q_{,u} + \left(-\frac{1}{2}T_{,t}S + F\right)q - \left(2\psi q\int Tdt + c_1 q\right)_t - Tq_{,i}S^{;i}. \end{aligned} \tag{6.61}$$

Below, we apply Theorem 6.5.1 for special forms of the function $q\left(t,x,u\right)$.

6.5.1 The homogeneous heat equation

In the case $q\left(t,x,u\right) = 0,$ i.e equation (6.51) is the homogeneous heat equation, we have the following result.

Theorem 6.5.2 *The Lie symmetries of the homogeneous heat equation in an $n-$dimensional Riemannian space*

$$g^{ij}u_{ij} - \Gamma^i u_i - u_t = 0 \tag{6.62}$$

are constructed from the homothetic algebra of the metric g_{ij} as follows

a. If Y^i is a nongradient HV/KV of the metric g_{ij}, the Lie symmetry is

$$X = \left(2\psi c_1 t + c_2\right)\partial_t + c_1 Y^i\partial_i + \left(a_0 u + b\left(t,x^i\right)\right)\partial_u \tag{6.63}$$

where $c_1, c_2, , a_0$ are constants and $b\left(t,x^i\right)$ is a solution of the homogeneous heat equation.

b. If $Y^i = S^{,i}$, that is, Y^i is a gradient HV/KV of the metric g_{ij}, the Lie symmetry is

$$X = (c_3\psi t^2 + c_4 t + c_5)\partial_t + (c_3 t + c_4)S^i\partial_i + \left(-\frac{c_3}{2}S - \frac{c_3}{2}n\psi t + c_5\right)u\partial_u + b\left(t,x^i\right)\partial_u \tag{6.64}$$

where c_3, c_4, c_5 are constants and $b\left(t,x^i\right)$ is a solution of the homogeneous heat equation.

In order to compare the above result with the existing results in the literature, we consider the heat equation in a Euclidian space of dimension n. Then, in Cartesian coordinates $g_{ij} = \delta_{ij}$ and $\Gamma^i = 0$, therefore, the homogeneous heat equation is

$$\delta^{ij} u_{ij} - u_t = 0. \tag{6.65}$$

The homothetic algebra of space consists of the n gradient KVs ∂_i with generating functions x^i, the $\frac{n(n-1)}{2}$ nongradient KVs X_{IJ}, which are the rotations and a gradient HV H^i with gradient function $H = R\partial_R$. According to Theorem 6.5.2, the Lie symmetries of the heat equation in the Euclidian n dimensional space are (we may take $\psi = 1$)

$$\begin{aligned} X &= \left[c_3\psi t^2 + (c_4 + 2\psi c_1)t + c_5 + c_2\right]\partial_t + \left[c_1 Y^i + (c_3 t + c_4)S^i\right]\partial_i + \\ &+ \left[\left(a_0 + \frac{c_3}{2}S + \frac{c_3}{2}n\psi t - c_5\right)u + b\left(t, x^i\right)\right]\partial_u. \end{aligned} \tag{6.66}$$

This result is consistent with the results of [1] pg. 158.

Next, we consider the de Sitter spacetime (a four dimensional space of constant curvature and Lorentzian character) whose metric is

$$ds^2 = \frac{\left(-d\tau^2 + dx^2 + dy^2 + dz^2\right)}{\left(1 + \frac{K}{4}\left(-\tau^2 + x^2 + y^2 + z^2\right)\right)^2} \tag{6.67}$$

It is known that the homothetic algebra of this space consists of the ten KVs

$$X_1 = (-x\tau)\partial_\tau + \left(\frac{\left(-\tau^2 - x^2 + y^2 + z^2\right)}{2} - \frac{2}{K}\right)\partial_x + (-yx)\partial_y + (-zx)\partial_x$$

$$X_2 = (y\tau)\partial_\tau + (yx)\partial_x + \left(\frac{\left(-x^2 - z^2 + y^2 + \tau^2\right)}{2} + \frac{2}{K}\right)\partial_y + (yz)\partial_x$$

$$X_3 = (z\tau)\partial_\tau + (zx)\partial_x + (zy)\partial_y + \left(\frac{\left(-x^2 - y^2 + z^2 + \tau^2\right)}{2} + \frac{2}{K}\right)\partial_x$$

$$X_4 = \left(\frac{\left(x^2 + y^2 + z^2 + \tau^2\right)}{2} - \frac{2}{K}\right)\partial_\tau + (\tau x)\partial_x + (\tau y)\partial_y + (\tau z)\partial_x$$

$$X_5 = x\partial_\tau + \tau\partial_x \ , \ X_6 = y\partial_\tau + \tau\partial_y \ , \ X_7 = z\partial_\tau + \tau\partial_z \ , \ X_8 = y\partial_x - x\partial_y$$

$$X_9 = z\partial_x - x\partial_z \ , \ X_{10} = z\partial_y - y\partial_z$$

all of which are nongradient. According to Theorem 6.5.2, the Lie symmetries of the heat equation in de Sitter space are

$$\partial_t + \sum_{A=1}^{10} c_A X_A + (a_0 u + b(x, u))\partial_u.$$

From Theorem 6.5.2 we have the following additional results.

Corollary 6.5.3 *The one dimensional homogenous heat equation admits a maximum number of seven Lie point symmetries (module a solution of the heat equation).*

Proof. The homothetic group of a one dimensional metric $ds^2 = g^2(x)\,dx^2$ consists of one gradient KV (the $\frac{1}{g(x)}\partial_x$) and one gradient HV ($\frac{1}{g(x)}\int g(x)\,dx\ \partial_x$). According to theorem 6.5.2, from the KV we have two Lie symmetries and from the gradient HV another two Lie point symmetries. To these we need to add the two Lie point symmetries $X = a_0 u\partial_u + b(t,x^i)\,\partial_u$ and the trivial Lie symmetry ∂_t where $b(t,x^i)$ is a solution of the heat equation. ■

Corollary 6.5.4 *The homogeneous heat equation in a space of constant curvature of dimension* n *has at most* $(n+3)+\frac{1}{2}n(n-1)$ *Lie symmetries (modulo a solution of the heat equation).*

Proof. A space of constant curvature of dimension n admits $n+\frac{1}{2}n(n-1)$ nongradient KVs. To these we need to add the Lie symmetries $X = c\partial_t + a_0 u\partial_u + b(t,x^i)\,\partial_u$. ■

Corollary 6.5.5 *The heat conduction equation in a space of dimension* n *admits at most* $\frac{1}{2}n(n+3)+5$ *Lie symmetries (modulo a solution of the heat equation) and if this is the case, the space is flat.*

Proof. The space with the maximum homothetic algebra is the flat space which admits n gradient KVs, $\frac{1}{2}n(n-1)$ nongradient KVs and one gradient HV. Therefore, from Case 1 of Theorem 6.5.2 we have $(n+1)+\frac{1}{2}n(n-1)$ Lie symmetries. From Case 2 of Theorem 6.5.2, we have $(n+1)$ Lie symmetries and to these we have to add the Lie symmetries $X = c_1\partial_t + a_0 u\partial_u + b(t,x^i)\,\partial_u$ where $b(t,x^i)$ is a solution of the heat equation. The set of all these symmetries is $1+2n+\frac{1}{2}n(n-1)+2+1+1 = \frac{1}{2}n(n+3)+5$ [4]. ■

6.5.2 Case $q(t,x,u) = q(u)$

Let $q(t,x,u) = q(u)$, then the heat conduction equation (6.51) becomes

$$g^{ij}u_{ij} - \Gamma^i u_i - u_t = q(u). \tag{6.68}$$

From Theorem 6.5.1, we have the following results.

Theorem 6.5.6 *The Lie symmetries of the heat equation (6.68) in an* $n-$*dimensional Riemannian space with metric* g_{ij} *are constructed form the homothetic algebra of the metric as follows.*

a. Y^i *is a HV/KV. The Lie symmetry is*

$$X = (2c\psi t + c_1)\,\partial_t + cY^i\partial_i + (a(t)\,u + b(t,x))\,\partial_u \tag{6.69}$$

where the functions $a(t)$, $b(t,x)$ *and* $q(u)$ *satisfy the condition*

$$-a_t u + H(b) - (au+b)\,q_{,u} + (a - 2\psi c)\,q = 0. \tag{6.70}$$

b. $Y^i = S^{,i}$ *is a gradient HV/KV. The Lie symmetry is*

$$X = \left(2\psi\int Tdt + c_1\right)\partial_t + TS^{,i}\partial_i + \left(\left(-\frac{1}{2}T_{,t}S + F(t)\right)u + b(t,x)\right)\partial_u \tag{6.71}$$

Table 6.1: Functions of $q(u)$ where the Heat equation admits Lie symmetries

Function $q(u)$	Lie Symmetry vector
$q(u) = q_0 u$	$\left(\psi T_0 t^2 + 2c\psi t + c_1\right)\partial_t + \left(cY^i + T_0 t S^{,i}\right)\partial_i +$
	$+\left(\left[-2\psi c q_0 t + a_0 + T_0\left(-\frac{1}{2}S - \psi q_0 t^2 - \frac{1}{2}t\right)\right]u + b(t,x)\right)\partial_u$ where $H(b) - bq_0 = 0$
$q(u) = q_0 u^n$	$(2c\psi t + c_1)\partial_t + cY^i\partial_i + \left(\frac{2\psi c}{1-n}u\right)\partial_u$
$q(u) = u\ln u$	$c_1\partial_t + \left(Y^i + T_0 e^{-t}K^{,i}\right)\partial_i + \left(a_0 e^{-t}u\right)\partial_u$, $K^{,i}$
$q(u) = e^u$	$(2c\psi t + c_1)\partial_t + cY^i\partial_i + (-2\psi c)\partial_u$

where $b(t,x)$ is a solution of the homogeneous heat equation, the functions $T(t)$, $F(t)$ and the flux $q(u)$ satisfy the equation:

$$\left(-\frac{1}{2}T_{,t}\psi + \frac{1}{2}T_{,tt}S - F_{,t}\right)u + H(b) - \left(\left(-\frac{1}{2}T_{,t}S + F\right)u + b\right)q_{,u} + \left(-\frac{1}{2}T_{,t}S + F\right)q - 2\psi qT = 0 \quad (6.72)$$

For various cases of $q(u)$, we obtain the results of Table[10] 6.1.

6.6 Conclusion

The main result of this chapter is Proposition 6.2.2, which states that the Lie symmetries of the PDEs of the form (6.2) are obtained from the conformal vectors of the metric defined by the coefficients A_{ij}, provided $A_{ij,u} = 0$. This result is quite general and covers many well known and important PDEs of Physics. The geometrization of Lie point symmetries and their association with the collineations of the metric dissociates their determination from the dimension of the space because the collineations of the metric depend (in general) on the type of the metric and not on the dimensions of the space where the metric resides. Furthermore, this association provides a wealth of results of Differential Geometry on collineations, which is possible to be used in the determination of Lie pont symmetries.

We have applied the above theoretical results to the Poisson equation, the conformal Poisson (Yamabe) equation and the heat equation. We proved that the Lie symmetries of the Poisson equations (Laplace/Yamabe) are generated from the elements of the conformal group of the metric that defines the Laplace/Yamabe operator.

For the heat conduction equation, we proved that the Lie symmetries are generated from the homothetic group of the underlying metric. Furthermore, we specialized the equation to the homogeneous heat conduction equation and regained the existing results for the Newtonian case.

[10] Where int Table 6.1 Y^i is a HV/KV, $S^{,i}$ is a gradient HV/KV and $K^{,i}$ is a gradient KV.

In the following chapter. we apply the theoretical results of this chapter to study the correlation of point symmetries between Classical and Quantum systems.

6.A Appendix A

We prove the statement for $n = 2$. The generalization to any n is straightforward. For a general proof, see [92]. We consider A as a matrix and assume that the inverse of this matrix exists. We denote the inverse matrix with B and we get from (6.7)

$$\begin{aligned} B_{ij}A^{ij}\xi^k_{.,u} + B_{ij}A^{kj}\xi^i_{.,u} + B_{ij}A^{ik}\xi^j_{.,u} = 0 \\ 2\xi^k_{.,u} + \delta^k_i\xi^i_{.,u} + \delta^k_j\xi^j_{.,u} = 0 \Rightarrow \\ \xi^k_{.,u} = 0. \end{aligned} \tag{6.73}$$

Now, assume that the tensor A does not have an inverse. Then, we consider $n = 2$ and write:

$$[A] = \begin{bmatrix} A_{11} & A_{12} \\ A_{12} & A_{22} \end{bmatrix} \Rightarrow \det A = A^{11}A^{22} - (A^{12})^2 = 0$$

where at least one of the $A^{ij} \neq 0$. Assume $A^{11} \neq 0$. Then, equation (6.7) for $i = j = k = 1$ gives

$$3A^{11}\xi^1_{.,u} = 0 \Rightarrow \xi^1_{.,u} = 0.$$

The same equation for $i = j = k = 2$ gives

$$3A^{22}\xi^2_{.,u} = 0$$

therefore, either $\xi^2_{.,u} = 0$ or $A^{22} = 0$. If $A^{22} = 0$, then, from the condition $\det A^{ij} = 0$, we have $A^{12} = 0$; hence, $A_{ij} = 0$, which we do not assume. Thus, $\xi^2_{.,u} = 0$.

We consider now equations $i = j \neq k$ and find

$$A^{ii}\xi^k_{.,u} + A^{ki}\xi^i_{.,u} + A^{ik}\xi^i_{.,u} = 0.$$

Because $i \neq k$, this gives $A^{ii}\xi^k_{.,u} = 0$ and because we have assumed $A^{11} \neq 0$ it follows $\xi^2_{.,u}$. Therefore, we find $\xi^k_{.,u} = 0$.

6.B Appendix B

Proof of Lemma 6.3.2. By using the formula

$$L_\xi\Gamma^i_{.jk} = \Gamma^i_{.jk,l}\xi^l + \xi^i_{,\ jk} - \xi^i_{\ ,l}\Gamma^l_{.jk} + \xi^s_{\ ,j}\Gamma^i_{.sk} + \xi^s_{\ ,k}\Gamma^i_{.sj}$$

we have

$$\begin{aligned} g^{jk}L_\xi\Gamma^i_{.jk} &= \Gamma^i_{,l}\xi^l + g^{jk}\xi^i_{\ ,,jk} - g^{jk}_{\quad ,l}\xi^l\Gamma^i_{.jk} - \xi^i_{\ ,l}\Gamma^l + 2g^{jk}\xi^s_{\ ,j}\Gamma^i_{.sk} \\ &= \Gamma^i_{,l}\xi^l + g^{jk}\xi^i_{\ ,\ jk} - \xi^i_{\ ,l}\Gamma^l \\ &- [g^{jl}\xi^k_{,l} + g^{kl}\xi^j_{,l} - (\lambda - a))g^{jk}]\Gamma^i_{.jk} + 2g^{jk}\xi^s_{\ ,j}\Gamma^i_{\ sk} \end{aligned}$$

that is,

$$g^{jk}L_\xi\Gamma^i_{.jk} == \Gamma^i_{,l}\xi^l + g^{jk}\xi^i_{,\ jk} - \xi^i{}_{,l}\Gamma^l + 2(g^{jl}\xi^k{}_{,j}\Gamma^i{}_{kl} - g^{lj}\xi^k{}_{,j}\Gamma^i{}_{kl}) - (\lambda - a)\Gamma^i$$

Therefore,

$$g^{jk}L_\xi\Gamma^i_{.jk} = g^{jk}\xi^i_{,\ jk} + \Gamma^i{}_{,l}\xi^l - \xi^i{}_{,l}\Gamma^l + (a-\lambda)\Gamma^i.$$

■

Proof of Lemma 6.3.3. By using the identity

$$L_\xi\Gamma^i_{.jk} = \frac{1}{2}g^{ir}\left[\nabla_k L_\xi g_{jr} + \nabla_j L_\xi g_{kr} - \nabla_r L_\xi g_{kj}\right] \tag{6.74}$$

and replacing $L_\xi g_{ij} = (a-\lambda)g_{ij}$ because ξ is a CKV, we find

$$\begin{aligned} L_\xi\Gamma^i_{.jk} &= \frac{1}{2}g^{ir}\left[(a-\lambda)_{,k}g_{jr} + (a-\lambda)_{,j}g_{kr} - (a-\lambda)_{,r}g_{kj}\right] \\ &= \frac{1}{2}\left[(a-\lambda)_{,k}\delta^i_j + (a-\lambda)_{,j}\delta^i_k - g^{ir}(a-\lambda)_{,r}g_{kj}\right]. \end{aligned}$$

By contracting with g^{jk} it follows

$$g^{jk}L_\xi\Gamma^i_{.jk} = \frac{2-n}{2}(a-\lambda)^{,i}.$$

■

Proof of Theorem 6.4.1. The Lie symmetry conditions for the Poisson equation (6.31) are

$$g^{ij}(a_{ij}u + b_{ij}) - (a_{,i}u + b_{,i})\Gamma^i - \xi^k f_{,k} - auf_{,u} - bf_{,u} + \lambda f = 0 \tag{6.75}$$

$$g^{ij}\xi^k_{,ij} - 2g^{ik}a_{,i} + a\Gamma^k - \xi^k_{,i}\Gamma^i + \xi^i\Gamma^k_{,i} - \lambda\Gamma^k = 0 \tag{6.76}$$

$$L_{\xi^i\partial_i}g_{ij} = (a-\lambda)g_{ij} \tag{6.77}$$

$$\eta = a(x^i)u + b(x^i)\ ,\ \xi^k_{,u} = 0 \tag{6.78}$$

Equation (6.76) becomes (see [90])

$$g^{jk}L_\xi\Gamma^i_{.jk} = 2g^{ik}a_{,i} \tag{6.79}$$

From (6.77), ξ^i is a CKV, then equations (6.79) give

$$\frac{2-n}{2}(a-\lambda)^{,i} = 2a^{,i} \to (a-\lambda)^i = \frac{4}{2-n}a^{,i}.$$

We define

$$\psi = \frac{2}{2-n}a + a_0 \tag{6.80}$$

where $\psi = \frac{1}{2}(a-\lambda)$ is the conformal factor of ξ^i i.e. $L_\xi g_{ij} = 2\psi g_{ij}$. Furthermore, we have

$$\begin{aligned} (2-n)\,\lambda^i &= (2-n)\,a^i - 4a^i \\ (2-n)\,\lambda^i &= -(n+2)\,a^i \end{aligned}$$

Finally, from (6.75), we have the constraint

$$g^{ij}a_{i;j}u + g^{ij}b_{;ij} - \xi^{k}f_{,k} - auf_{,u} + \lambda f - bf_{,u} = 0. \tag{6.81}$$

For $n = 2$, holds that $g^{jk}L_{\xi}\Gamma^{i}_{.jk} = 0$; this means that $a_{,i} = 0 \rightarrow a = a_0$. From (6.77), ξ^i is a CKV with conformal factor

$$2\psi = (a_0 - \lambda) \tag{6.82}$$

and $\lambda = a_0 - 2\psi$. Finally, from (6.75), we have the constraint

$$g^{ij}b_{;ij} - \xi^{k}f_{,k} - a_0 uf_{,u} + (a_0 - 2\psi)\, f - bf_{,u} = 0. \tag{6.83}$$

■

Proof of Theorem 6.4.3. By replacing $f\left(x^k, u\right) = -M_0 R + \bar{f}\left(x^i, u\right)$ in (6.33) where $M_0 = \frac{n-2}{4(n-1)}$, we have the symmetry condition

$$\frac{2-n}{2}\Delta\psi u + g^{ij}b_{i;j} + M_0\xi^k R_{,k}u - \xi^k \bar{f}_{,k} + 2M_0\psi uR - \frac{2-n}{2}\psi u\bar{f}_k - \frac{2+n}{2}\psi\bar{f} - b\left(-M_0R + \bar{f}_u\right) = 0 \tag{6.84}$$

But ξ^i is a CKV with conformal factor ψ. That implies [95]

$$\xi^k R_{,k} = -2\psi R - 2\,(n-1)\,\Delta\psi \tag{6.85}$$

and this implies for the terms in (6.84)

$$0 = \left(\frac{2-n}{2}\Delta\psi + M_0\xi^k R_{,k} + 2M_0\psi R\right)u.$$

Hence, condition become (6.84) finally becomes

$$-\xi^k \bar{f}_{,k} - \frac{2-n}{2}\psi u\bar{f}_k - \frac{2+n}{2}\psi\bar{f} + g^{ij}b_{i;j} - b\left(-M_0R + \bar{f}_u\right) = 0.$$

■

Proof of Theorem 6.5.1. Condition (6.56) means that ξ^i is a CKV of the metric g_{ij} with conformal factor $a(t,x^k) - \lambda(t,x^k)$. Condition (6.55) implies $\xi^k = T\,(t)\,Y^k\left(x^j\right)$, where Y^i is a HV with conformal factor ψ, that is, we have:

$$L_{Y^i}g_{ij} = 2\psi g_{ij} \quad , \ \psi\text{=constant.}$$

and

$$\xi^t_{,t} = a - \lambda$$

from which follows

$$\xi^t\,(t) = 2\psi\int T dt. \tag{6.86}$$

$$-2g^{ik}a_{,i} + T_{,t}Y^k = 0. \tag{6.87}$$

Condition (6.54) becomes

$$H(a)u + H(b) + (a - \xi^t_{,t})q = \xi^t q_{,t} + T(t)Y^k q_{,k} + \eta q_{,u} \Rightarrow$$

$$H(a)u + H(b) - (au + b)\, q_{,u} + aq - (\xi^t q)_{,t} - T(t)Y^k q_{,k} = 0$$

$$H\left(a\right)u + H\left(b\right) - \left(au + b\right) q_{,u} + aq - \left(2\psi q \int T dt\right)_{t} - Tq_{,i}Y^i = 0. \tag{6.88}$$

We consider the following cases:

Case 1: Y^k is a HV/KV. From (6.87), we have that $T_{,t} = 0 \rightarrow T\left(t\right) = c_2$ and $a_{,i} = 0 \rightarrow a\left(t, x^k\right) = a\left(t\right)$. Then, (6.88) becomes

$$-a_t u + H\left(b\right) - \left(au + b\right) q_{,u} + aq - \left(2\psi c_2 qt + c_1 q\right)_t - c_2 q_{,i} Y^i = 0 \tag{6.89}$$

Case 2: Y^k is a gradient HV/KV, that is $Y^k = S^{,k}$. From (6.87), we have

$$a\left(t, x^k\right) = -\frac{1}{2} T_{,t} S + F\left(t\right). \tag{6.90}$$

By replacing in (6.88), we find the constraint equation

$$\begin{aligned} 0 = &\left(-\frac{1}{2}T_{,t}\psi + \frac{1}{2}T_{,tt}S - F_{,t}\right)u + H\left(b\right) + \\ &- \left(\left(-\frac{1}{2}T_{,t}S + F\right)u + b\right)q_{,u} + \left(-\frac{1}{2}T_{,t}S + F\right)q - \left(2\psi q \int T dt + c_1 q\right)_t - Tq_{,i}S^{,i}. \end{aligned} \tag{6.91}$$

■

Chapter 7

Point symmetries of Schrödinger and the Klein Gordon equations

7.1 Introduction

The Schrödinger and the Klein Gordon equations are two important equations of Quantum Physics. Therefore, it is important that we determine their Lie symmetries and use them either in order to find invariant solutions using Lie symmetry methods [1]. In order to achieve this, we notice that the Schrödinger equation is a special case of the heat conduction equation and the Klein Gordon equation is a special form of the Poisson equation. The Lie symmetries of the heat equation and of the Poisson equation in a general Riemannian space were determined in Chapter 6. Thus, we apply these results to find the Lie symmetries of the Schrödinger and the Klein Gordon equation in a general Riemannian space.

An important element of the present study is the concept of conformally related Lagrangians, that is, Lagrangians that under under a conformal transformation of the metric and the potential lead to the same equations but for different dynamic variables. The condition for this is that the Hamiltonian vanishes. Because the dynamic variables of these Lagrangians are not the standard ones in general the Hamiltonian is not relevant to the energy of the system.

From each Lagrangian describing a dynamical system, we define a metric called the kinematic metric, characteristic to the dynamical system described by this Lagrangian. As it will be seen, the conformal symmetries of this metric are in close relation to the Noether symmetries of the equations of motion. Furthermore, the kinetic metric of the Lagrangian defines the Laplace operator; hence, consequently the Lie symmetries of the corresponding Poisson equation are expressed in terms of the conformal symmetries of the kinematic metric. We extend these results to the Yamabe operator and study the Lie symmetries of the conformal Klein Gordon equation.

In section 7.2, we consider the classical Lagrangian

$$L\left(x^{k},\dot{x}^{k}\right)=\frac{1}{2}g_{ij}\left(x^{k}\right)\dot{x}^{i}\dot{x}^{j}-V\left(x^{k}\right) \tag{7.1}$$

in a general Riemannian space and we show that the Noether point symmetries of two conformally related Lagrangians are generated from the conformal algebra of the metric g_{ij}.

In section 7.3, we study the Lie point symmetries of Schrödinger and the Klein Gordon equation by using the resutls of Chapter 6. Using the geometric character of the Noether symmetries for the Lagrangian (7.1) and that of the Lie symmetries of the Schrödinger and of the Klein Gordon equation we establish the connection between the two. More specifically, it will be shown that if an element of the homothetic group of the kinetic metric generates a Noether point symmetry for the classical Lagrangian, then it also generates a Lie point symmetry for the Schrödinger equation. Concerning the Klein Gordon we find that the Noether symmetry of the Lagrangian (7.1) must have a constant Noether gauge function in order to be admitted.

In section 7.4, we examine the case of Noether symmetries whose Noether gauge functions are not constant. We will consider the cases the kinematic metric admits a gradient Killing vector (KV) or a gradient homothetic vector (HV) which produces Noether point symmetries for the Lagrangian (7.1) and show that the Lie symmetry in both cases is indeed a non-local symmetry of the Klein Gordon equation. In section 7.5, we demonstrate the use of the previous general results to various interesting practical situations.

To complete our analysis, in section 7.7, we examine the WKB approximation. In that case, the solution of the Klein Gordon equation satisfies the null Hamilton Jacobi equation. We derive the symmetry condition of the null Hamilton Jacobi equation and we prove that its Lie point symmetries are generated from the CKVs of the underlying space. Furthermore, there exists a unique relation between the Lie point symmetries of the Hamilton Jacobi and the Lie symmetries of the Euler-Lagrange equations of a classical particle; in particularly, the Lie point symmetries of the Euler-Lagrange equations which are generated from the homothetic algebra of the Riemannian space are generating point symmetries for the Hamilton Jacobi equation; that is, the Lie symmetry algebra of the Hamilton Jacobi equation can be greater than the Noether algebra of the classical Lagrangian (7.1).

7.2 Noether symmetries of Conformal Lagrangians

Consider the Lagrangian of a particle moving under the action of a potential $V(x^k)$ in a Riemannian space with metric g_{ij}

$$L=\frac{1}{2}g_{ij}\dot{x}^{i}\dot{x}^{j}-V\left(x^{k}\right) \tag{7.2}$$

where $\dot{x}=\frac{dx}{dt}$. The equations of motion follow from the action

$$S=\int dt\left(L\left(x^{k},\dot{x}^{k}\right)\right)=\int dt\left(\frac{1}{2}g_{ij}\dot{x}^{i}\dot{x}^{j}-V\left(x^{k}\right)\right). \tag{7.3}$$

Consider the change of variable $t \to \tau$ defined by the requirement

$$d\tau = N^2\left(x^i\right) dt. \tag{7.4}$$

In the new coordinates (τ, x^i), the action becomes

$$S = \int \frac{d\tau}{N^2\left(x^k\right)} \left(\frac{1}{2} g_{ij} N^4\left(x^k\right) x'^i x'^j - V\left(x^k\right) \right) \tag{7.5}$$

where $x'^i = \frac{dx}{d\tau}$ and the Lagrangian is transformed to the new Lagrangian

$$\bar{L}\left(x^k, x'^k\right) = \frac{1}{2} N^2\left(x^k\right) g_{ij} x'^i x'^j - \frac{V\left(x^k\right)}{N^2\left(x^k\right)}. \tag{7.6}$$

If we consider a conformal transformation (not a coordinate transformation!) of the metric $\bar{g}_{ij} = N^2\left(x^k\right) g_{ij}$ and a new potential function $\bar{V}\left(x^k\right) = \frac{V\left(x^k\right)}{N^2\left(x^k\right)}$ then, the new Lagrangian $\bar{L}\left(x^k, x'^k\right)$ in the new coordinates τ, x^k, takes the form,

$$\bar{L}\left(x^k, x'^k\right) = \frac{1}{2} \bar{g}_{ij} x'^i x'^j - \bar{V}\left(x^k\right) \tag{7.7}$$

implying that equation (7.7) is of the same form as the Lagrangian L in equation (7.2). From now on, the Lagrangian $L\left(x^k, \dot{x}^k\right)$ of equation (7.2) and the Lagrangian $\bar{L}\left(x^k, x'^k\right)$ of equation (7.7) will be called conformal. In this framework, the action remains the same, i.e. it is invariant under the change of parameter and the equations of motion in the new variables (τ, x^i) will be the same with the equations of motion for the Lagrangian L in the original coordinates (t, x^k) .

In Chapter 4, it was shown that the Noether point symmetries of a Lagrangian of the form (7.2) follow from the homothetic algebra of the metric g_{ij} (see Theorem 4.3.2). The same applies to the Lagrangian $\bar{L}\left(x^k, x'^k\right)$ and the metric $\bar{g}_{ij}$. The conformal algebra of the metrics $g_{ij}, \bar{g}_{ij}$ (as a set) is the same; however, their closed subgroups of HVs and KVs are generally different. Hence, the following Corollary holds.

Corollary 7.2.1 *The Noether point symmetries of the conformally related Lagrangians (7.2), (7.7) are contained in the common conformal algebra of the metrics $g_{ij}, \bar{g}_{ij}$.*

Now, we formulate and prove the following Lemma;

Lemma 7.2.2 *The Euler-Lagrange equations for two conformal Lagrangians transform covariantly under the conformal transformation relating the Lagrangians if and only if the Hamiltonian vanishes.*

Proof. Consider the Lagrangian $L = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j - V\left(x^k\right)$ whose Euler-Lagrange equations are:

$$\ddot{x}^i + \Gamma^i_{jk} \dot{x}^i \dot{x}^j + V^{,i} = 0 \tag{7.8}$$

where Γ^i_{jk} are the Christofell symbols. The corresponding Hamiltonian is given by

$$E = \frac{1}{2} g_{ij} \dot{x}^i \dot{x}^j + V\left(x^k\right) \ . \tag{7.9}$$

For the conformally related Lagrangian $\bar{L}\left(x^k, x'^k\right) = \left(\frac{1}{2}N^2\left(x^k\right)g_{ij}x'^i x'^j - \frac{V\left(x^k\right)}{N^2\left(x^k\right)}\right)$ where $N_{,j} \neq 0$ the resulting Euler Lagrange equations are

$$x''^i + \hat{\Gamma}^i_{jk}x'^j x'^k + \frac{1}{N^4}V^{,i} - \frac{2V}{N^5}N^{,i} = 0 \tag{7.10}$$

where

$$\hat{\Gamma}^i_{jk} = \Gamma^i_{jk} + (\ln N)_{,k}\delta^i_j + (\ln N)_{,j}\delta^i_k - (\ln N)^{,i}g_{jk} \tag{7.11}$$

and the corresponding Hamiltonian is

$$\bar{E} = \frac{1}{2}N^2\left(x^k\right)g_{ij}\dot{x}^i\dot{x}^j + \frac{V\left(x^k\right)}{N^2\left(x^k\right)}. \tag{7.12}$$

In order to show that the two equations of motion are conformally related we start from equation(7.10) and apply the conformal transformation

$$x'^i = \frac{dx^i}{d\tau} = \frac{dx^i}{dt}\frac{dt}{d\tau} = \dot{x}^i\frac{1}{N^2}$$
$$x''^i = \ddot{x}^i\frac{1}{N^4} - 2\dot{x}^i\dot{x}^j\left(\ln N\right)_{,j}\frac{1}{N^4}.$$

By replacing in equation(7.10). we find:

$$\ddot{x}^i\frac{1}{N^4} - 2\dot{x}^i\dot{x}^j\left(\ln N\right)_{,j}\frac{1}{N^4} + \frac{1}{N^4}\hat{\Gamma}^i_{jk}\dot{x}^j\dot{x}^k + \frac{1}{N^4}V^{,i} - \frac{2V}{N^5}N^{,i} = 0$$

By replacing $\hat{\Gamma}^i_{jk}$ from equation(7.11), we have

$$\ddot{x}^i - 2\dot{x}^i\dot{x}^j\left(\ln N\right)_{,j} + \Gamma^i_{jk}\dot{x}^j\dot{x}^k + 2(\ln N)_{,j}\dot{x}^j\dot{x}^i$$
$$-(\ln N)^{,i}g_{jk}\dot{x}^j\dot{x}^k + V^{,i} - 2V(\ln N)^{,i} = 0$$

from which follows

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^j\dot{x}^k + V^{,i} - (\ln N)^{,i}\left(g_{jk}\dot{x}^j\dot{x}^k + 2V\right) = 0.$$

Obviously, the above Euler-Lagrange equations coincide with equations(7.8) if and only if $g_{jk}\dot{x}^j\dot{x}^k + 2V = 0$, which implies that the Hamiltonian of equation(7.9) vanishes. The steps are reversible; hence, the inverse is also true. ■

The physical meaning of this result is that systems with vanishing energy are conformally invariant at the level of the equations of motion.

7.3 Lie point symmetries of Schrödinger and the Klein Gordon equation

In this section we study the Lie point symmetries of Schrödinger and the Klein Gordon equation in a Riemannian manifold. To do this, we use the results of Chapter 6. Furthermore, we will study the relation between Noether pont symmetries of classical Lagrangians and Lie point symmetries of the Schrödinger and the Klein Gordon equation with the same "kinetic" metric and the same potential.

7.3.1 Symmetries of the Schrödinger equation

The Schrödinger equation[1]

$$g^{ij}u_{ij} - \Gamma^i u_i - u_t = V(x)\,u \tag{7.13}$$

is a special form of the heat conduction equation (6.51) with $q(t,x,u) = V(x)\,u$. Therefore, it is possible to study the Lie point symmetries of the Schrödinger equation using Theorem 6.5.1 which, in this case, takes the following form.

Theorem 7.3.1 *The Lie point symmetries of the Schrödinger equation (7.13) are generated from the elements of the homothetic algebra of the metric g_{ij} as follows.*

a. Y^i is a non-gradient HV/KV. The Lie symmetry is

$$X = (2c\psi t + c_1)\,\partial_t + cY^i\partial_i + (a_0 u + b(t,x))\,\partial_u \tag{7.14}$$

with constraint equations

$$H(b) - bV = 0\ ,\quad cL_Y V + 2\psi cV + a_0 = 0. \tag{7.15}$$

b $Y^i = H^{,i}$ is a gradient HV/KV. The Lie symmetry is

$$X = \left(2\psi\int T dt + c_1\right)\partial_t + TS^{,i}\partial_i + \left(\left(-\frac{1}{2}T_{,t}S + F(t)\right)u + b(t,x)\right)\partial_u$$

with constraint equations

$$H(b) - bV = 0 \tag{7.16}$$

$$L_H V + 2\psi V - \frac{1}{2}c^2 H + d = 0 \tag{7.17}$$

and the functions T, F are computed from the relations

$$T_{,tt} = c^2 T\ ,\ \frac{1}{2}T_{,t}\psi + F_{,t} = dT. \tag{7.18}$$

From the form of the symmetry vectors and the symmetry conditions for the Schrödinger and the Lagrangian of the classical particle (7.2) we have the following result.

Proposition 7.3.2 *If a KV/HV of the metric g_{ij} produces a Lie point symmetry for the Schrödinger equation (7.13), then generates a Noether point symmetry for the Lagrangian (7.2) in the space with metric g_{ij} and potential $V\left(x^k\right)$. The reverse is also true.*

[1] We have absorbed the constant $\hbar$ and the imaginary unit i, in the variables x^k, t respectively

7.3.2 Symmetries of the Klein Gordon equation

The Klein-Gordon equation

$$\Delta u - V\left(x^k\right)u = 0 \tag{7.19}$$

follows from the Poisson equation (6.31) if we take $f\left(x^i, u\right) = V\left(x^i\right)u$. Therefore, Theorem 6.4.1 applies and we have the following result

Theorem 7.3.3 *The Lie point symmetries of the Klein Gordon equation (7.19) are generated from the CKVs of the metric g_{ij} defining the Laplace operator, as follows*

a) for $n > 2$, the Lie symmetry vector is

$$X = \xi^i\left(x^k\right)\partial_i + \left(\frac{2-n}{2}\psi\left(x^k\right)u + a_0 u + b\left(x^k\right)\right)\partial_u \tag{7.20}$$

where ξ^i is a CKV with conformal factor $\psi\left(x^k\right)$, $b\left(x^k\right)$ is a solution of (7.19) and the following condition is satisfied

$$\xi^k V_{,k} + 2\psi V - \frac{2-n}{2}\Delta\psi = 0. \tag{7.21}$$

b) for $n = 2$, the Lie symmetry vector is

$$X = \xi^i\left(x^k\right)\partial_i + \left(a_0 u + b\left(x^k\right)\right)\partial_u \tag{7.22}$$

where ξ^i is a CKV with conformal factor $\psi\left(x^k\right)$, $b\left(x^k\right)$ is a solution of (7.19) and the following condition is satisfied

$$\xi^k V_{,k} + 2\psi V = 0. \tag{7.23}$$

By comparing the symmetry condition of the Klein Gordon equation (7.19) and the classical Lagrangian (7.2) and by taking into consideration that for a special CKV/HV/KV the conformal factor satisfies the condition $\psi_{;ij} = 0$, we deduce the following result.

Proposition 7.3.4 *For $n > 2$, the Lie point symmetries of the Klein Gordon equation for the metric g_{ij} which defines the Laplace operator are related to the Noether point symmetries of the classical Lagrangian for the same metric and the same potential as follows*

a) If a KV or HV of the metric g_{ij} generates a Lie point symmetry of the Klein Gordon equation (7.19), then it also produces a Noether point symmetry of the classical Lagrangian with gauge function a constant.

b) If a special CKV or a proper CKV satisfying the condition $\Delta\psi = 0$ of the metric g_{ij} generates a Lie point symmetry of the Klein Gordon equation (7.19), then it also generates a Noether point symmetry of the conformally related Lagrangian if there exists a conformal factor $N\left(x^k\right)$, such that the CKV becomes a KV or a HV.

For $n = 2$, the results are different and are given below.

7.3.3 Symmetries of the Conformal Klein Gordon equation

The conformal Klein Gordon equation (or Yamabe Klein Gordon)

$$\bar{L}_g u - V\left(x^k\right)u = 0. \tag{7.24}$$

is the conformal Poisson equation (6.42) for $\bar{f}\left(x^i,u\right) = V\left(x^k\right)u$. Therefore, Theorem 6.4.3 is valid and we have the result.

Theorem 7.3.5 *The Lie point symmetries of the conformal Klein Gordon equation (7.24) are as follows*

a) For $n>2$, they are generated from the CKVs of the metric g_{ij} of the conformal Laplace operator, as follows

$$X = \xi^i\left(x^k\right)\partial_i + \left(\frac{2-n}{2}\psi\left(x^k\right)u + a_0 u + b\left(x^k\right)\right)\partial_u \tag{7.25}$$

where ξ^i is a CKV with conformal factor $\psi\left(x^k\right)$, and the following conditions are satisfied

$$\xi^k V_{,k} + 2\psi V = 0 \tag{7.26}$$

$$\bar{L}_g b - Vb = 0. \tag{7.27}$$

b) For $n=2$, equation (7.24) is the Laplace Klein Gordon equation (7.19) and the results of theorem 7.3.3 apply.

Comparing the symmetry condition of the conformal Klein Gordon equation (7.24) and the classical Lagrangian (7.2) we have the following proposition.

Proposition 7.3.6 *a) If a CKV of the metric g_{ij} ($\dim g_{ij} \succeq 2$), which defines the conformal Laplace operator, produces a Lie point symmetry of the conformal Klein Gordon equation (7.24), then the same vector generates a Noether point symmetry of the conformally related Lagrangian provided there exists a conformal factor $N\left(x^k\right)$ such that the CKV becomes a KV/HV of g_{ij}.*

b) If a KV/HV of the metric g_{ij} generates a Lie point symmetry for the conformal Klein Gordon equation (7.24) then the same vector generates a Noether point symmetry for the classical Lagrangian with gauge function a constant.

7.4 $sl(2,R)$ and the Klein Gordon equation

In the previous considerations, we have showed that the Lie point symmetries of the Klein Gordon equation induce Noether point symmetries for the classical Lagrangian if the gauge function is a constant. In this section, we investigate the case when the induced Noether symmetry has a gauge function which is not a constant. As we shall show in this case, the induced Noether symmetry comes from a generalized Lie symmetry of the Klein Gordon equation.

It is clear that if a KV/HV produces a Noether point symmetry for the classical Lagrangian satisfying conditions (4.29) with $d \neq 0$ or (4.31) with $m \neq 0$ (or $d \neq 0$) of Theorem 4.3.2, then it does not produce a Lie symmetry for the Klein Gordon equation. However, a gradient KV/HV which generates a Noether point symmetry for the classical Lagrangian satisfies only condition (4.31) with $d = 0$ and $m \neq 0$ and leads to two well known dynamical systems, the oscillator and the Ermakov system. The Lie and the Noether point symmetries of these dynamical systems have been considered previously in Chapter 5; however, we briefly reproduce these results in the current framework for competences.

7.4.1 The oscillator

First, we consider the case in which the metric admits a gradient KV which generates Lie point symmetries of the classical Lagrangian, provided condition (4.31) is satisfied. It is well known that if a metric admits a gradient KV, then it is decomposable and can be written in the form

$$ds^2 = dx^2 + h_{AB}dy^A dy^B \tag{7.28}$$

where the gradient KV is $S^{,i} = \partial_x$ $(S = x)$ and $h_{AB} = h_{AB}\left(y^C\right)$ is the tensor projecting normal to the KV. In these coordinates the Lagrangian takes the form

$$L = \frac{1}{2}\left(\dot{x}^2 + h_{AB}\dot{y}^A\dot{y}^B\right) - V\left(x, y^C\right). \tag{7.29}$$

The Lie point symmetry condition for the gradient KV becomes

$$V_{,x} + \mu^2 x = 0$$

from which follows that the potential is

$$V\left(x, y^C\right) = -\frac{1}{2}\mu^2 x^2 + F\left(y^C\right). \tag{7.30}$$

The Noether point symmetries are the vectors $e^{\pm\mu t}\partial_x$ with respective gauge function $f\left(t, x, y^A\right) = \mu e^{\pm}x$. The corresponding Noether integrals are

$$I_{\pm} = e^{\pm\mu t}\dot{x} \mp \mu e^{\pm\mu t}x \tag{7.31}$$

It can be easily shown that the combined Noether integral $I_0 = I_+ I_-$ is time independent and equals

$$I_0 = \dot{x}^2 - \mu^2 x^2. \tag{7.32}$$

The Laplace Klein Gordon equation defined by the metric (7.28) and the potential (7.30) is

$$u_{xx} + h^{AB}u_A u_B - \Gamma^A u_A - \mu^2 x^2 u - F\left(y^C\right)u = 0. \tag{7.33}$$

This equation does not admit a Lie point symmetry for general h_{ab}, $F\left(y^C\right)$. However, it is separable with respect to x in the sense that the solution can be written in the form $u\left(x, y^A\right) = w\left(x\right)S\left(y^A\right)$. This implies that the operator $\hat{I} = D_x D_x - \mu^2 x^2 - I_0$ satisfies $\hat{I}u = 0$, which means that the Klein Gordon equation (7.33) possesses a Lie Bäcklund symmetry [96, 97] with generating vector $\bar{X} = \left(u_{xx} - \mu^2 x^2\right)\partial_u$.

Concerning the conformal Klein Gordon equation

$$u_{xx} + h^{AB}u_A u_B - \Gamma^A u_A + \frac{n-2}{4(n-1)}Ru - \mu^2 x^2 u - 2\bar{F}\left(y^C\right)u = 0 \tag{7.34}$$

because for a KV, say ξ^a, we have $L_\xi R = 0$ [95] hence $R = R\left(y^C\right)$, equation (7.34) is written in the form of the Laplace Klein Gordon equation with $F\left(y^C\right) = 2\bar{F}\left(y^C\right) - \frac{n-2}{4(n-1)}R\left(y^C\right)$ and the previous result applies.

7.4.2 The Kepler Ermakov potential with an oscillator term

We assume now that there exists a gradient HV which produces a Noether point symmetry for the classical Lagrangian under the constraint condition (4.31). It is well known [83, 98] that if a metric admits a gradient HV, then there exists a coordinate system in which the metric has the form

$$ds^2 = dr^2 + r^2 h_{AB} dy^A dy^B$$

where the HV is $H^{,i} = r\partial_r$, $\left(\psi = 1\ ,\ H = \frac{1}{2}r^2\right)$ and $h_{AB} = h_{AB}\left(y^C\right)$ is the tensor projecting normal to $H^{,i}$.s For these coordinates, the Lagrangian is

$$L = \frac{1}{2}\left(\dot{r}^2 + r^2 h_{AB}\dot{y}^A\dot{y}^B\right) - V\left(r, y^C\right) \tag{7.35}$$

and the gradient HV generates Lie point symmetries only for the Ermakov potential extended by the oscillator term, that is,

$$V\left(r, y^C\right) = -\frac{1}{2}\mu^2 r^2 + \frac{F\left(y^C\right)}{r^2}. \tag{7.36}$$

The admitted Noetheroint symmetries generated from the gradient HV $H^{,i}$ are the vectors $X_\pm = \frac{1}{\mu}e^{\pm 2\mu t}\partial_t \pm e^{\pm 2\mu t}r\partial_r$ with corresponding gauge functions $f\left(t, r, y^A\right) = \mu e^{\pm 2\mu t}r^2$ and corresponding Noether integrals (5.97) and (5.98). From the Noether integrals (5.97),(5.98) and the Hamiltonian h of (7.35) we construct the time independent first integral $\Phi_0 = h^2 - I_+ I_-$, which is

$$\Phi_0 = r^4 h_{DB}\dot{y}^A\dot{y}^B + 2F\left(y^C\right). \tag{7.37}$$

This is the well known Ermakov invariant [67, 53].

The Laplace Klein Gordon equation defined by the metric (7.35) and the potential (7.36) is

$$u_{rr} + \frac{1}{r^2}h^{AB}u_{AB} + \frac{n-1}{r}u_r - \frac{1}{r^2}\Gamma^A u_A + \mu^2 r^2 + \frac{2}{r^2}F\left(y^C\right) = 0. \tag{7.38}$$

This equation does not admit a Lie point symmetry. However it is separable in the sense that $u\left(r, y^C\right) = w(r)\,S\left(y^C\right)$. Then the operator

$$\hat{\Phi} = h^{AB}D_A D_B - \Gamma^A D_A + 2F\left(y^C\right) - \Phi_0$$

satisfies the equation $\hat{\Phi}u = 0$ which means that (7.38) admits the Bäcklund symmetry with generator $\bar{X} = \left(\hat{\Phi}u\right)\partial_u$ [96, 97].

Concerning the conformal Klein Gordon equation, the Ricci scalar of the metric (7.35) and the HV satisfy the condition $L_H R + 2R = 0$ [95], that is $R = \frac{1}{r^2}\bar{R}\left(y^C\right)$. Then, as in the case of the gradient KV we absorb the term $\bar{R}\left(y^C\right)$ into the potential and we obtain the same results with the Laplace Klein Gordon equation.

From the above, we conclude that, although in the two cases considered above the Lie symmetries do not transfer from the classical to the "quantum" level the generalized symmetries do transfer.

7.5 Applications

We apply the previous general results in two practical cases. The first case concerns the Newtonian central motion and the second the classification of potentials in two and three dimensional flat spaces for which the Schrödinger equation and the Klein Gordon equation admit Lie symmetries.

7.5.1 Euclidean central force

Consider the autonomous classical Lagrangian

$$L = \frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\theta}^2 - Cr^{-(2m+2)}. \tag{7.39}$$

It is well known that (7.39) admits as Noether point symmetries (a) the gradient KV ∂_t (autonomous) with Noether integral the Hamiltonian and (b) the KV $X_N = \partial_\theta$, with constant gauge function and Noether integral the angular momentum $p_\theta = r^2\dot{\theta} = I_0$. Lagrangian (7.39) admits extra Noether symmetries for the values $m = -1$ (the free particle) and $m = 0$ (the Ermakov potential) [79]. In the following we assume $m \neq -1, 0$; hence, we do not expect to find symmetries.

From the Lagrangian (7.39), we consider the kinematic metric and define the Schrödinger equation

$$u_{rr} + \frac{1}{r^2}u_{\theta\theta} + \frac{1}{r}u_r + Cr^{-(2m+2)}u - u_t = 0 \tag{7.40}$$

and the Klein Gordon equation (because the dimension of the kinematic metric is two, the Laplace and the Yamabe operators coincide)

$$u_{rr} + \frac{1}{r^2}u_{\theta\theta} + \frac{1}{r}u_r + Cr^{-(2m+2)}u = 0. \tag{7.41}$$

The application of the results of the previous sections give the following

a. The Schrödinger equation (7.40) admits as Lie point symmetries the vectors

$$\partial_t\ ,\ \partial_\theta\ ,\ u\partial_u$$

b. The Klein Gordon equation (7.41) admits as Lie point symmetries the vectors

$$\partial_\theta \ , \ u\partial_u \ , \ b(r,\theta)\,\partial_\theta$$
$$X_1 = r^{m+1}\cos(m\theta)\,\partial_r + r^m\sin(m\theta)\,\partial_\theta$$
$$X_2 = r^{m+1}\sin(m\theta)\,\partial_r - r^m\cos(m\theta)\,\partial_\theta$$

It can be easily observed that X_1, X_2 are proper CKVs of the two dimensional flat kinematic metric.

Concerning the Schrödinger equation, it has the same Lie point symmetries as the Lagrangian (7.39) hence there is nothing more to do. However, the Klein Gordon equation has the extra Lie symmetries X_1, X_2; hence it is possible to apply the results of proposition 7.3.6 in order to find a conformally related Lagrangian which will admit the pair of symmetries ∂_t, X_1 or ∂_t, X_2.

It is important to note that if we use the zero order invariants of the Lie symmetry ∂_t to reduce the Schrödinger equation (7.40), we find that the reduced equation is the Klein Gordon equation (7.41). Therefore, the symmetries X_1, X_2 are Type II hidden symmetries [99, 14, 100] for the Schrödinger equation (7.40).

Let us consider the vector X_1. It is easy to show, that the conformal metric

$$ds^2 = N^2(r,\theta)\left(dr^2 + r^2 d\theta^2\right)$$

where $N(r,\theta) = r^{-(m+1)} g\left(r^{-1}\sin^{\frac{1}{m}}(m\theta)\right)$ and g is an arbitrary function of $r^{-1}\sin^{\frac{1}{m}}(m\theta)$ admits X_1 as a KV. This leads to the family of conformal Lagrangians

$$\bar{L} = r^{-2(m+1)} g^2\left(r^{-1}\sin^{\frac{1}{m}}(m\theta)\right)\left(\frac{1}{2}r'^2 + \frac{1}{2}r^2\theta'^2\right) - \frac{C}{g^2\left(r^{-1}\sin^{\frac{1}{m}}(m\theta)\right)} \tag{7.42}$$

where "$'$" means derivative with respect to the conformal "time" τ and the coordinate transformation is $dt = N^{-2}(r,\theta)\,d\tau$. According to proposition 7.3.6, the vector field X_1 generates a Noether point symmetry for the Lagrangian (7.42).

Working similarly for the vector X_2, we find *another* family of conformally related Lagrangians which admit X_2 as a Noether point symmetry.

The conformally related Lagrangians are possible to admit additional Noether point symmetries than the sets ∂_t, X_1 or ∂_t, X_2. For example in the case of X_1 we consider the conformally related Lagrangian defined by the function $g = 1$, i.e.

$$\bar{L} = r^{-2(m+1)}\left(\frac{1}{2}r'^2 + \frac{1}{2}r^2\theta'^2\right) - C$$

which, by means of the coordinate transformation $r = R^{-\frac{1}{m}}$, becomes

$$\bar{L} = \frac{1}{2}R'^2 + \frac{1}{2}R^2\theta'^2 - C. \tag{7.43}$$

This is the Lagrangian of the free particle moving in the 2D flat space.

Table 7.1: The 2D Klein Gordon admitting Lie symmetries from the homothetic group

Lie Symmetry	$\mathbf{V}(x,y)$	**Lie Symmetry**	$\mathbf{V}(x,y)$
∂_x	$f(y)$	$\partial_x + b\partial_y$	$f(y-bx)$
∂_y	$f(x)$	$(a+y)\partial_x + (b-x)\partial_y$	$f\left(\frac{1}{2}\left(x^2+y^2\right)+ay-bx\right)$
$y\partial_x - x\partial_y$	$f(r)$	$(x+ay)\partial_x + (y-ax)\partial_y$	$r^{-2}\ f(\theta - a\ln r)$
$x\partial_x + y\partial_y$	$x^{-2}f\left(\frac{y}{x}\right)$	$(a+x)\partial_x + (b+y)\partial_y$	$f\left(\frac{b+x}{a+x}\right)(a+x)^{-2}$

7.5.2 Lie symmetry classification of Schrödinger and the Klein Gordon equations in Euclidian space

In Chapter 4, all two and three dimensional potentials for which the corresponding Newtonian dynamical systems admit Lie and/or Noether point were determined. Using these results, we determine all Schrödinger and Klein Gordon equations in Euclidian 2D and 3D space which admit Lie point symmetries.

The Schrödinger equation in Euclidian space is

$$\delta^{ij}u_{ij} + V\left(x^k\right)u = u_t. \tag{7.44}$$

From proposition 7.3.2, we have that the potentials for which the Schrödinger equation (7.44) admits Lie symmetries are the same with the ones admitted by the classical Lagrangian. Therefore, the results of Chapter 4 apply directly and give all potentials for which the Schrödinger equation (7.44) admits at least one Lie symmetry.

We consider the Klein Gordon equation in flat space;that is,

$$\delta^{ij}u_{ij} + V\left(x^k\right)u = 0. \tag{7.45}$$

In this case, the conditions are different and we find that equation (7.45) admits a Lie point symmetry due to a HV/KV for the following potentials taken from the corresponding Tables of Chapter 4. In Table 7.1 and Table 7.2 we provide the potentials where the 2D and 3D Klein Gordon equation (7.45) admits Lie point symmetries generated from the elements of the homothetic group of the Euclidian space.

As we have seen in section 7.3.2, the Lie point symmetries of the Klein Gordon equation are generated from the conformal group of the space; therefore, we have to consider the admitted CKVs in addition to the HV and the KVs.

Table 7.2: The 3D Klein Gordon admitting Lie symmetries from the homothetic group

Lie Symmetry	$\mathbf{V(x,y,z)}$
$a\partial_\mu + b\partial_\nu + c\partial_\sigma$	$f\left(x^\nu - \frac{b}{a}x^\mu, x^\sigma - \frac{c}{a}x^\mu\right)$
$a\partial_\mu + b\partial_\nu + c\left(x_\nu\partial_\mu - x_\mu\partial_\nu\right)$	$+f\left(\frac{c}{2}r_{(\mu\nu)} - bx_\mu + ax_\nu, x_\sigma\right)$
$a\partial_\mu + b\partial_\nu + c\left(x_\sigma\partial_\mu - x_\mu\partial_\sigma\right)$	$+f\left(x_\nu - \frac{1}{\lvert c\rvert}\arctan\left(\frac{\lvert c\rvert x_\mu}{\lvert a+cx_\sigma\rvert}\right), \frac{1}{2}r_{(\mu\sigma)} - \frac{a}{c}x_\sigma\right)$
$a\partial_\mu + b\left(x_\nu\partial_\mu - x_\mu\partial_\nu\right) +$ $+ c\left(x_\sigma\partial_\mu - x_\mu\partial_\sigma\right)$	$+f\left(x_\mu^2 + x_\nu^2\left(1 - \frac{c^2}{b^2}\right) + \left(\frac{2a}{b} + \frac{2c}{b}x_\sigma\right)x_\nu, x_\sigma - \frac{c}{b}x_\nu\right)$
$so\left(3\right)$ linear combination	$F\left(R, b\tan\theta\sin\phi + c\cos\phi - aM_1\right)$
$a\partial_\mu + b\theta_{(\nu\sigma)}\partial_{\theta_{(\nu\sigma)}} + cR\partial_R$	$\frac{1}{r_{(\nu\sigma)}^2}f\left(\theta_{(\nu\sigma)} - \frac{b}{c}\ln r_{(\nu\sigma)}, \frac{a+cx_\mu}{cr_{(\nu\sigma)}}\right)$
$\left(a\partial_\mu + b\partial_\nu + c\partial_\sigma + lR\partial_R\right)$	$\frac{1}{(a+lx_\mu)^2}f\left(\frac{b+lx_\nu}{l(a+lx_\mu)}, \frac{c+lx_\sigma}{l(a+lx_\mu)}\right)$

The two dimensional case

We recall that the conformal algebra of a two dimensional space is infinite dimensional [101] and in coordinates with line element $ds^2 = 2dwdz$ are given by the vectors $X = F\left(w\right)\partial_w + G\left(z\right)\partial_z$, with conformal factor $\psi = \frac{1}{2}\left(F_{,w} + G_{,z}\right)$. In the coordinates $\left(w,z\right)$, the 2D Klein Gordon Klein Gordon equation (7.45) is

$$u_{wz} + V\left(w,z\right)u = 0.$$

The Lie symmetry condition (7.21) becomes

$$\left(FV\right)_{,w} + \left(GV\right)_{,z} = 0 \tag{7.46}$$

from which follows that there are infinite many potentials for which the 2D Klein Gordon equation admits Lie point symmetries.

The three dimensional case

The 3D Euclidian space admits the three special CKVs

$$K_C^\mu = \frac{1}{2}\left(\left(x^\mu\right)^2 - \left(\left(x^\sigma\right)^2 + \left(x^\nu\right)^2\right)\right)\partial_i + x^\mu x^\nu\partial_x + x^\mu x^\nu\partial_z \ , \ \mu,\sigma,\nu = 1,2,3$$

with corresponding conformal factor $\psi_C = x^\mu$.

From the symmetry condition (7.21) of theorem 7.3.3, it follows that a special CKV generates the Lie point symmetry $X = K_C^{\mu} - \frac{1}{2}x^{\mu}u\partial_u$ for the 3D Klein Gordon (7.45) only for the potential

$$V(x,y,z) = \frac{1}{(x^{\sigma})^2}F\left(\frac{x^{\nu}}{x^{\sigma}}, \frac{\delta_{\kappa\lambda}x^{\kappa}x^{\lambda}}{x^{\sigma}}\right).$$

One is possible to continue with the linear combinations and deduce all cases that the 3D Klein Gordon equation admits a Lie symmetry. These results hold for both the Klein Gordon and the conformal Klein Gordon equation. It particular one can show that the results remain still valid for the conformal Klein Gordon equation provided the metric defining the conformal Laplacian is conformally flat.

7.6 The Klein Gordon equation in a spherically symmetric space-time

In this section, we consider the Lie point symmetries of the Klein Gordon equation in a non-flat space and in particular in the static spherically symmetric empty space-time; that is the exterior Schwarzschild solution given by the metric (τ is the radial coordinate)

$$ds^2 = -a^2(\tau)\,dt^2 + d\tau^2 + b^2(\tau)\left(d\theta^2 + \sin^2\theta d\phi^2\right). \tag{7.47}$$

The Lagrangian of Einstein field equations for this space-time is [102, 103]

$$L = 2ab'^2 + 4ba'b' + 2a \tag{7.48}$$

where "$'$" means derivative with respect to the radius τ. If we see the Lagrangian (7.48) as a dynamical system in the space of variables $\{a,b\}$, then this system is "autonomous"; hence, admits the Noether symmetry ∂_τ with corresponding Noether integral the "Hamiltonian" h =constant. (h *is not* the "energy" because the coordinate is the radial distance not the time)

$$h = 2ab'^2 + 4ba'b' - 2a.$$

It can be shown directly that $h = \frac{2}{a}G_1^1$, where G_1^1 is the Einstein tensor. Because the space is empty, from Einstein's equations follows that $h = 0$. The Euler-Lagrange equations are

$$a'' - \frac{a}{2b^2}b'^2 + \frac{1}{b}a'b' + \frac{1}{2}\frac{a}{b^2} = 0$$

$$b'' + \frac{1}{2b}b'^2 - \frac{1}{2b} = 0.$$

We end up with a system of three equations whose solution will give the functions $a(\tau), b(\tau)$. It is found that the solution of the system does not give these functions in the well known closed form. This is due to the Lagrangian we have considered; Indeed we shall show below that it is possible to find the solution in closed form by considering a Lagrangian conformally related to the Lagrangian (7.48).

Applying Theorem 4.3.2, we find that the Lagrangian (7.48) admits the Noether point symmetry

$$X_1 = 2\tau\partial_\tau + H$$

where $H = -2a\partial_a + 2b\partial_b$ is a non-gradient homothetic vector of the two dimensional kinetic metric

$$d\bar{s}^2 = 2adb^2 + 4bdadb \tag{7.49}$$

defined from the Lagrangian (7.48).

The Klein Gordon equation defined by the metric (7.49) with potential $V(a,b) = 2a$, is

$$-\frac{a}{4b^2}u_{aa} + \frac{1}{2b}u_{ab} - \frac{1}{4b^2}u_a - 2au = 0 \tag{7.50}$$

and admits as Lie symmetries the vectors [103]

$$u\partial_u \quad , \quad b(a,b)\,\partial_u$$

$$H = -2a\partial_a + 2b\partial_b \quad , \quad X_2 = \frac{1}{ab}\partial_a \quad , \quad X_3 = \frac{a}{2b}\partial_a - \partial_b$$

where the vectors X_2, X_3 are proper CKVs of the two dimensional metric (7.49).

It is possible to find solutions of the Klein Gordon equation (7.50) which are invariant with respect to one of the admitted Lie symmetries.

For example, let us consider the Lie point symmetry $H_u = H - 2cu\partial_u$. The zero order invariants of H_u are $w = ab\ ,\ u = a^c\Phi(w)$. Replacing in the PDE we find the solution [2]

$$u(a,b) = a^c\left[c_1 I_c^B\left(2\sqrt{2}ab\right) + c_2 K_c^B\left(2\sqrt{2}ab\right)\right]$$

where I^B, K^B are the Bessel modified functions [103]. Working similarly for the Lie point symmetry $H + eX_2 - cu\partial_u$ we find the solution

$$u(a,b) = \left(a^2 - eb^{-1}\right)^{\frac{c}{4}}\left[c_1 I_{-\frac{c}{2}}\left(2\sqrt{2b\left(a^2b-e\right)}\right) + c_2 K_{\frac{c}{2}}^B\left(2\sqrt{2b\left(a^2b-e\right)}\right)\right].$$

One can find more solutions using linear combinations of the Lie symmetries.

Following proposition 7.3.6, we look for a conformal metric for which one of the CKVs X_2, X_3 becomes a KV and write the corresponding conformally related Lagrangian which admits this CKV as a Noether point symmetry.

We consider the vector X_2 and the conformally related metric

$$ds^2 = N^2\left(4adb^2 + 8bda\ db\right) \tag{7.51}$$

[2] We have found this by making use of the library SADE [104] of MAPLE.

where $N(a,b) = g(b)\sqrt{a}$ and $g(b)$ is an arbitrary function of its argument. It is easy to show that the vector X_2 is a KV of this metric hence a Noether symmetry for the family of conformally related Lagrangians

$$\bar{L} = N^2 \left[2a \left(\frac{db}{dr} \right)^2 + 4b \left(\frac{da}{dr} \right) \left(\frac{db}{dr} \right) \right] + \frac{2a}{N^2} \tag{7.52}$$

where we have considered the coordinate transformation $d\tau = \frac{dr}{N^2(a,b)}$. The Noether function for this Noether symmetry is the "Hamiltonian" of the Lagrangian (7.52). This constant and the two Lagrange equations for the "generalized" coordinates a, b provide a system of differential equations which will give the functions $a(\tau), b(\tau)$.

We consider $g(b) = g_0$ =constant, that is from the family of Lagrangians (7.52) we take the Lagrangian

$$L = g_0^2 \left[2a^2 \left(\frac{db}{dr} \right)^2 + 4ab \left(\frac{da}{dr} \right) \left(\frac{db}{dr} \right) \right] + \frac{2}{g_0^2}. \tag{7.53}$$

For this Lagrangian we have the following system of equations:

a. The "Hamiltonian" of the Lagrangian (7.53)

$$a^2 \left(\frac{db}{dr} \right)^2 + 2b^2 \left(\frac{da}{dr} \right) \left(\frac{db}{dr} \right) - V_0 = 0 \tag{7.54}$$

b. The Euler Lagrange equations of (7.53) with respect to the variables a, b

$$\frac{d^2a}{dr^2} + \frac{1}{a^2} \left(\frac{da}{dr} \right)^2 + \frac{2}{b} \left(\frac{da}{dr} \right) \left(\frac{db}{dr} \right) = 0 \tag{7.55}$$

$$\frac{d^2b}{dr^2} = 0 \tag{7.56}$$

where we have set $V_0 = g_0^{-4}$. The solution of the system of equations (7.54)-(7.56) is

$$b(r) = b_1 r + b_2 \quad , \quad a^2(r) = \frac{V_0 r + 2a_1 b_1}{2b_1 (b_1 r + b_2)}.$$

Under the linear transformation $b_1 r = b_1 R - \frac{b_2}{b_1}$, and if we set $V_0 = 2(b_1)^2$, $a_1 = -2m + b_2$, we obtain the exterior Schwarzschild solution in the standard coordinates

$$ds^2 = -\left(1 - \frac{2m}{R}\right) dt^2 + \left(1 - \frac{2m}{R}\right)^{-1} dR^2 + R^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{7.57}$$

The choice of the function $g(b)$ is essentially a choice of the coordinate system. Obviously the final solution must always be the exterior Schwarzschild solution. In order to show this let us consider $g(b) = \sqrt{b}$ so that $d\tau = (a(r) b(r))^{-1} dr$. Then, we get the Lagrangian

$$\bar{L} = 2a^2 b \left(\frac{db}{dr} \right)^2 + 4ab^2 \left(\frac{da}{dr} \right) \left(\frac{db}{dr} \right) + \frac{2}{b}$$

and the system of equations

$$\frac{d^2b}{dr^2} + \frac{1}{b} \left(\frac{db}{dr} \right)^2 = 0$$

$$2a^2b\left(\frac{db}{dr}\right)^2+4ab^2\left(\frac{da}{dr}\right)\left(\frac{db}{dr}\right)-\frac{2}{b}=0$$
$$\frac{d^2a}{dr^2}+\frac{1}{a}\left(\frac{da}{dr}\right)+\frac{2}{b}\left(\frac{da}{dr}\right)\left(\frac{db}{dr}\right)-\frac{a}{2b^2}\left(\frac{db}{dr}\right)^2+\frac{1}{2ab^4}=0$$

The solution of the system is

$$b^2\left(r\right)=r\ ,\ a^2\left(r\right)=\left(\sqrt{r}\right)^{-1}\left(4\sqrt{r}+a_1\right)$$

from which follows: $d\tau^2=\left(4r+\sqrt{r}a_1\right)^{-1}dr^2$. Therefore, for this choice of Lagrangian, the metric is

$$ds^2=-\left(\frac{4\sqrt{r}+a_1}{\sqrt{r}}\right)dt^2+\frac{1}{4r+\sqrt{r}a_1}dr^2+r\left(d\theta+\sin^2\phi d\theta^2\right). \tag{7.58}$$

If we make the transformation $r=R^2$, $dt\rightarrow\frac{1}{2}dt$ and $a_1=-8m$, we retain the metric (7.58) in the standard form (7.51).

Working similarly, we find that X_3 becomes a KV for the conformal metric (7.51) if $N_3\left(a,b\right)=f\left(a^2b\right)\sqrt{a}$ and generates a Noether point symmetry for the conformal Lagrangian (7.52) (with N_3 in the place of N) and continue as above.

7.7 WKB approximation

In WKB approximation, we search for solutions of the Klein Gordon equation of the form $u=A_ne^{iS\left(x^k\right)}$, where $S\left(x^k\right)$ has to satisfy the null Hamilton Jacobi equation [105, 106].

$$g^{ij}S_{,i}S_{,j}+\bar{V}\left(x^k\right)=0 \tag{7.59}$$

where g^{ij} is the metric defining the Yamabe operator and $\bar{V}\left(x\right)=\frac{n-2}{4(n-1)}R-V\left(x^k\right)$. We study the symmetries of the PDE (7.59).

We search for Lie point symmetries of the form [1]

$$X=\xi^i\left(x^i,S\right)\partial_i+\eta\left(x^i,S\right)\partial_S.$$

The symmetry condition is

$$X^{[1]}\left(H_{null}\right)=\lambda\left(x^k\right)\left(H_{null}\right)\ ,\ mod\left[H_{null}\right]=0$$

where $X^{[1]}$ is the first prolongation and $\lambda=\lambda\left(x^i\right)$. Replacing the first prolongation $X^{[1]}$ in the symmetry condition and collecting terms of powers of $S_{,i}$ we find the following result.

Theorem 7.7.1 *The Lie point symmetries of the null Hamilton Jacobi equation (7.59) are the vectors*

$$X=\xi^i\left(x^k\right)\partial_i+\left(a_0S+b_0\right)\partial_S \tag{7.60}$$

where $\xi^i\left(x^k\right)$ is a CKV of the metric g_{ij}, a_0,b_0 are constants and the following condition holds

$$V_k\xi^k+2\psi V-a_0V=0. \tag{7.61}$$

Proof . The symmetry condition is

$$X^{[1]}\left(H_{null}\right)=\lambda\left(H_{null}\right)\ ,\quad mod\left[H_{null}\right]=0$$

where $X^{[1]}$ is the first prolongation and we set $\lambda=\lambda\left(x^{i}\right)$. Replacing $X^{[1]}$, we find

$$X^{[1]}H_{null}=\frac{1}{2}g_{,k}^{ij}\xi^{k}S_{i}S_{j}+g^{ij}S_{j}\eta_{i}^{[1]}+V_{k}\xi^{k}$$

where

$$g^{ij}u_{j}\eta_{i}^{[1]}=g^{ij}S_{j}\eta_{,i}+g^{ij}S_{j}S_{i}\eta_{s}-g^{rj}S_{i}S_{j}\xi_{,r}^{j}-g^{ir}S_{r}S_{i}S_{j}\xi_{,u}^{j}.$$

We can easily show that $\xi_{,u}^{i}=0$ so that the the symmetry conditions become:

$$g_{,k}^{ij}\xi^{k}-2g^{r(j}\xi_{,r}^{j)}+2g^{ij}\eta_{S}=\lambda g^{ij}\tag{7.62}$$

$$V_{k}\xi^{k}=\lambda V\tag{7.63}$$

$$\eta_{,i}=0.\tag{7.64}$$

From (7.62) and (7.64) we have that $\eta=a_{0}S+b_{0}$. Then, the conditions (7.62),(7.63) become

$$L_{\xi}g_{ij}=\left(a_{0}-\lambda\right)g_{ij}\tag{7.65}$$

$$V_{k}\xi^{k}=\lambda V.\tag{7.66}$$

Setting $a_{0}-\lambda=2\psi$ or $\lambda=a_{0}-2\psi$, we see that condition (7.65) implies that the Lie point symmetries of the null Hamilton Jacobi equation (7.59) are generated from the CKVs of the the kinematic metric g_{ij}. Finally, condition (7.66) becomes

$$V_{k}\xi^{k}+2\psi V-a_{0}V=0.$$

■

Comparing the symmetry condition (7.61) of the null Hamilton Jacobi equation and the symmetry condition (7.26) of the Yamabe Klein Gordon equation, we have

Proposition 7.7.2 *If a CKV of the metric which defines the Yamabe operator generates a Lie point symmetry for the Yamabe Klein Gordon equation (7.24), then the same vector generates a Lie point symmetry of the null Hamilton Jacobi equation (7.59). The reverse holds if the CKV generating Lie symmetry for the null Hamilton Jacobi equation (7.59) satisfies condition (7.61) with* $a_{0}=0$.

Proposition 7.7.2 relates the Lie point symmetries of the Hamilton Jacobi equation with the Lie point symmetries of the Yamabe Klein Gordon when the constant $a_{0}=0$. The question which arises is what happens to the Lie symmetries when $a_{0}\neq0$. The answer is given in the following proposition

Proposition 7.7.3 *If a CKV generates Lie point symmetry for the null Hamilton Jacobi equation (7.59) satisfying condition (7.61) with* $a_0 \neq 0$*, then this CKV produces a Lie point symmetry for the Euler-Lagrange equations for a conformally related Lagrangian if there exist a conformal factor* $N\left(x^k\right)$ *for which the CKV becomes KV or HV.*

Proof . The Lie point symmetries of the autonomous system

$$\ddot{x}^i + \Gamma^i_{jk}\dot{x}^i\dot{x}^j + V^{,i} = 0 \tag{7.67}$$

are produced from the special projective algebra of the metric provided the potential satisfies the condition

$$L_\eta V^{,i} + d_0 V^{,i} = 0 \tag{7.68}$$

where d_0 is a constant and L_η denotes Lie derivative with respect to η^i. Equation (7.67) remains the same for the conformally related Lagrangian, that is, we have

$$x''^i + \bar{\Gamma}^i_{jk}x'^i x'^j + V'^{,i} = 0.$$

Therefore, the symmetry group will be again the special projective group of the conformal metric. The special projective group is not preserved under a conformal transformation but the subgroups of KVs and the homothetic group are preserved. Then these two subgroups are common for both metrics. Subsequetly, if the CKV of the metric g_{ij} becomes a KV/HV of the conformal metric $\bar{g}_{ij} = N^2 g_{ij}$ then it will be a Lie point symmetry of the Euler Lagrange equations of the conformally related Lagrangian $\bar{L}$. This symmetry will not be a Noether symmetry of $\bar{L}$ except in the case that $d_0 = 2\bar{\psi}$ $\left(\bar{\psi} = 1 \text{ for HV and } \psi = 0 \text{ for KV}\right)$. ■

7.8 Conclusion

We have determined the Lie point symmetries of Schrödinger equation and the Klein Gordon equation in a general Riemannian space. It has been shown that these symmetries are related to the homothetic algebra and the conformal algebra of the metric. Furthermore, these symmetries have been related to the Noether point symmetries of the classical Lagrangian for which the metric g_{ij} is the kinematic metric. More precisely, for the Schrödinger equation (7.13) it has been shown that if a KV/HV of the metric g_{ij} produces a Lie point symmetry of the Schrödinger equation, then it produces a Noether point symmetry for the Classical Lagrangian in the space with metric g_{ij} and potential $V(x^k)$. For the Klein Gordon equation the situation is different; the Lie point symmetries of the Klein Gordon are generated by elements of the conformal group of the metric g_{ij}. The KVs and the HV of this group produce a Noether symmetry of the classical Lagrangian with a constant gauge function. However the proper CKVs produce a Noether point symmetry for the conformal Lagrangian if there exists a conformal factor $N\left(x^k\right)$ such that the CKV becomes a KV/HV of g_{ij}.

We have applied these general results to three cases of practical interest: the motion in a central potential, the classification of all potentials in Euclidian 2D and 3D space for which the Schrödinger equation and the

Klein Gordon equation admit a Lie point symmetry and finally we have considered the Lie symmetries of the Klein Gordon equations in the static, spherically symmetric empty spacetime. In the last case, we have demonstrated the role of Lie symmetries and that of the conformal Lagrangians in the determination of the closed form solution of Einstein equations.

Furthermore, we investigated the Lie symmetries of the null Hamilton Jacobi equation and we proved that if a CKV generates a point symmetry for the Klein Gordon equation, then it also generates a Lie point symmetry for the null Hamilton Jacobi equation.

The knowledge of the Lie symmetries of the Schrödinger equation and the Klein Gordon equation in a general Riemannian space makes possible the determination of solutions of these equations which are invariant under a given Lie symmetry. In addition, they can be used in Quantum Cosmology [107, 108, 109, 110, 105] to determine the form of solutions of the Wheeler–DeWitt equation [111] in a given Riemannian space.

Chapter 8

The geometric origin of Type II hidden symmetries

8.1 Introduction

Lie symmetries assist us in the simplification of differential equations (DEs) by means of reduction. As it was indicated in Chapter 2, the reduction is different for ordinary differential equations (ODEs) and partial differential equations (PDEs). In the case of ODEs, the use of a Lie symmetry reduces the order of ODE by one while in the case of PDEs, the reduction by a Lie symmetry reduces by one the number of independent and dependent variables, but not the order of the PDE. A common characteristic in the reduction of both cases is that the Lie symmetry which is used for the reduction is not admitted as such by the reduced DE, it is "lost".

It has been found that the reduced equation is possible to admit more Lie symmetries than the original equation. These new Lie symmetries have been termed Type II hidden symmetries. Also, if one works in the reverse way and either increases the order of an ODE or increases the number of independent and dependent variables of a PDE, then, it is possible that the new (the 'augmented') DE admits new point symmetries not admitted by the original DE. This type of Lie symmetries are called Type I hidden symmetries.

The Type I and Type II hidden symmetries have studied extensively in the recent years by various authors (see e.g. [112, 99, 14, 100, 113, 114, 115]). In the following sections, we consider mainly the Type II hidden symmetries as they are the ones which could be used to reduce further the reduced DE.

The origin of Type II hidden symmetries is different for the ODEs and the PDEs, although it has been shown recently that they are nearly the same [116]. For the case of ODEs, the inheritance or not of a Lie symmetry, the X_2 say, by the reduced ODE depends on the commutator of that symmetry with the symmetry used for the reduction, the X_1 say. For example, if only two Lie symmetries X_1, X_2 are admitted by the original equation and the commutator $[X_1, X_2] = cX_2$ where c may be zero, then reduction by X_1 results in X_2 being a nonlocal symmetry for the reduced ODE while reduction by X_2 results in X_1 being an inherited Lie symmetry of the reduced ODE. In the reduction by X_1, the symmetry X_2 is a Type I hidden symmetry of the original equation relative to the reduced equation. In the case of more than two Lie symmetries the situation is the same, if the Lie bracket gives a third Lie symmetry, the X_3 say. Then, the point like nature of a symmetry is preserved only if reduction is performed using the normal subgroup and X_3 has a certain expression [117].

The above scenario is transferred to PDEs as follows. The reduced PDE loses the symmetry used to reduce the number of variables and it may lose other Lie symmetries depending on the structure of the associated Lie algebra depending if the admitted subgroup is normal or not [117]. Similarly, if X_1, X_2 are Lie symmetries of the original PDE with commutator $[X_1, X_2] = cX_2$ where c may be zero, then reduction by X_2 results in X_1 being a symmetry of the reduced PDE while reduction by X_1 results in an expression which has no relevance for the PDE [117].

In addition to that scenario, B. Abraham - Shrauner and K.S. Govinder have proposed a new potential source for the Type II hidden symmetries [118] based on the observation that different PDEs with the same variables, which admit different Lie symmetry algebras, may be reduced to the same target PDE. Based on that observation, they propose that the target PDE inherits Lie symmetries from all reduced PDEs, which explains why some of the new symmetries are not admitted by the specific PDE used for the reduction. In this context arises the problem of identifying the set of all PDEs which lead to the same reduced PDE after reduction by a Lie symmetry. In a recent paper [116], it has been shown that this is also the case with the ODEs; that is, it is shown that different differential equations which can be reduced to the same equation provide point sources for each of the Lie symmetries of the reduced equation even though any particular of the higher order equations may not provide the full complement of Lie symmetries. Therefore, concerning the ODEs the Lie symmetries of the reduced equation can be viewed as having two sources. Firstly, the point and nonlocal symmetries of *a given* higher order equation and secondly, the point symmetries of *a variety* of higher order ODEs. Finally, in a newer paper [118], it has been shown by a counter example that Type II hidden symmetries for PDEs can have a nonpoint origin, i.e. they arise from contact symmetries or even nonlocal symmetries of the original equation. Other approaches may be found in [119].

In the present Chapter we will study the reduction and the consequent existence of Type II hidden symmetries of the homogeneous heat equation

$$\Delta u - u_t = 0 \tag{8.1}$$

and the Laplace equation

$$\Delta u = 0 \tag{8.2}$$

in certain classes of Riemannian spaces, where

$$\Delta = \frac{1}{\sqrt{|g|}} \frac{\partial}{\partial x^i} \left(\sqrt{|g|} g^{ij} \frac{\partial}{\partial x^j} \right)$$

is the Laplace operator. In a general Riemannian space, the homogeneous heat equation (8.1) admits three

Lie point symmetries and the Laplace equation (8.2) admits two Lie point symmetries, which are not useful for reduction. This implies that if we wish to find 'sound' reductions of equations (8.1) and (8.2), we have to consider Riemannian spaces which admit some type of symmetry(ies) of the metric (these symmetries are not Lie symmetries and are called collineations). Indeed, as it has been shown in Chapter 6, the Lie symmetries of the homogeneous heat and the Laplace equation in a Riemannian space are generated from the elements of the homothetic algebra and the conformal algebra of the space respectively. Thus, one expects that in spaces with a nonvoid conformal algebra there will be Lie symmetry vectors which will allow for the reduction of equations (8.1), (8.2) and the possibility of the existence of Type II hidden symmetries.

In section 8.3 we reduce the Laplace equation with the extra Lie symmetries existing in (a) a decomposable space - that is a Riemannian space which admits a gradient Killing vector (KV) - (b) in a space which admits a gradient Homothetic vector (HV) and (c) in a space which admits a special Conformal killing vector (sp.CKV). In section 8.4, we apply the results of the previous section and find the Type II symmetries of the Laplace equation in four and three dimensional Minkowski spacetimes. Also we fully recover previous results [100]. In order to study the reduction of Laplace equation by a non-gradient HV and a proper CKV we consider two further examples. In section 8.5, we consider the algebraically special vacuum solution of Einstein's equations known as Petrov type III [120] and we make the reduction using the Lie point symmetry generated by the nongradient HV. Moreover we do the same in a conformally flat spacetime where the proper CKVs generate Lie symmetries.

In section 8.6, we reduce the homogeneous heat equation (8.1) with the extra Lie symmetries existing (i) in a space which admits a gradient KV and (ii) in a space which admits a gradient HV. In section 8.7, we consider the special cases of the previous section, that is, a decomposable space whose nondecomposable part is a maximally symmetric space of non-vanishing curvature and the spatially flat Friedmann Robertson Walker (FRW) space time used in Cosmology. Finally in section 8.8 we consider the reduction of the homogeneous heat equation in the Petrov type III spacetime using the Lie symmetry which is generated by the HV.

We emphasize that all results are derived in a purely geometric manner without the use of a computer package. However, we have verified them with the libraries PDEtools and SADE [121, 104] of Maple[1].

8.2 Lie symmetries of Laplace equation in certain Riemannian spaces

In a general Riemannian space, Laplace equation (8.2) admits the Lie symmetries

$$X_u = u\partial_u \ , \ X_b = b\,(t,x)\,\partial_u$$

where $b\,(t,x)$ is a solution of Laplace equation. These symmetries are too general to provide useful reductions and lead to reduced PDEs which posses Type II hidden symmetries. However if we restrict our considerations

[1] www.maplesoft.com

to spaces which admit a conformal algebra (proper or not) then we will have new Lie symmetries, hence new reductions of Laplace equation eqn (8.2), which might lead to Type II hidden symmetries.

In the following sections, we consider spaces in which the metric g_{ij} can be written in a generic form. The spaces we shall consider are: a. Spaces which admit a gradient KV (decomposable spaces) b. Spaces which admit a gradient HV and c. Spaces which admit a sp.CKV.

The generic form of the metric for each type is as follows $(A, B, \ldots = 1, 2, \ldots, n)$:

a. If a $1+n-$dimensional Riemannian space admits a gradient KV, the $S^i = \partial_z \;\; (S = z)$ say, then the space is decomposable along ∂_z and the metric is written as (see e.g. [44])

$$ds^2 = dz^2 + h_{AB} y^A y^B \ , \ h_{AB} = h_{AB}\left(y^C\right)$$

b. If a $1+n-$dimensional Riemannian space admits a gradient HV, the $H^i = r\partial_r \;\left(H = \frac{1}{2}r^2\right), \ \psi_H = 1$ say, then the metric can be written in the generic form [83]

$$ds^2 = dr^2 + r^2 h_{AB} dy^A dy^B \ , \quad h_{AB} = h_{AB}\left(y^C\right)$$

c. If a $1+n-$dimensional Riemannian space admits a sp.CKV then admits a gradient KV and a gradient HV [83, 22] and the metric can be written in the generic form

$$ds^2 = -dz^2 + dR^2 + R^2 f_{AB}\left(y^C\right) dy^A dy^B$$

while the sp.CKV is $C_S = \frac{z^2+R^2}{2}\partial_z + zR\partial_R$ with conformal factor $\psi_{C_S} = z$.

The Riemannian spaces which admit non-gradient proper HV do not have a generic form for their metric. However, the spaces for which the HV acts simply transitively are a few and are given together with their homothetic algebra in [120]. A special class of these spaces are the algebraically special vacuum space-times known as Petrov type N, II , III, D. In section 8.5.1, we shall consider the reduction of Laplace equation in the Petrov type III spacetime whose metric is

$$ds^2 = 2d\rho dv + \frac{3}{2}x d\rho^2 + \frac{v^2}{x^3}\left(dx^2 + dy^2\right)$$

with the symmetry generated by the non-gradient HV $H = v\partial_v + \rho\partial_\rho \ , \ \psi_H = 1$. The reduction of Laplace equation in the rest of the Petrov types is similar both in the working method and results and there is no need to consider them explicitly.

Finally we shall consider the conformally flat space

$$ds^2 = e^{2t}\left[dt^2 - \delta_{AB} y^A y^B\right]$$

which admits a proper CKV which produces a Lie symmetry[2] and we reduce Laplace equation using this symmetry.

In what follows all spaces are assumed to be of dimension $n > 2$.

[2] According to theorem 6.4.2, the condition for this is that the conformal factor satisfies Laplace equation

8.3 Reduction of the Laplace equation in certain Riemannian spaces

As we have seen in Chapter 6, the Lie symmetries of Laplace equation (8.2) in a Riemannian space are generated from the CKVs (not necessarily proper) whose conformal factor satisfies Laplace equation. This condition is satisfied trivially by the KVs ($\psi = 0$), the HV ($\psi_{;i} = 0$) and the sp.CKVs ($\psi_{;ij} = 0$). Therefore these vectors (which span a subalgebra of the conformal group) are among the Lie symmetries of Laplace equation. Concerning the proper CKVs it is not necessary that their conformal factor satisfies the Laplace equation, therefore they may not produce Lie symmetries for Laplace equation.

8.3.1 Riemannian spaces admitting a gradient KV

Without loss of generality we assume the gradient KV to be the ∂_z, so that the metric has the generic form

$$ds^2 = dz^2 + h_{AB}dy^A dy^B \ , \ h_{AB} = h_{AB}\left(y^C\right) \tag{8.3}$$

where h_{AB} $A, B, C = 1, ..., n$ is the metric of the $n-$ dimensional space. For the metric (8.3) Laplace equation (8.2) takes the form

$$u_{zz} + h^{AB}u_{AB} - \Gamma^A u_B = 0. \tag{8.4}$$

and admits as *extra* Lie symmetry the gradient KV ∂_z.

We reduce (8.4) by using the zeroth order invariants y^A , $w = u$ of the extra Lie symmetry ∂_z. Taking these invariants as new coordinates, equation (8.4) reduces to

$${}_h\Delta w = 0 \tag{8.5}$$

which is Laplace equation in the n dimensional space with metric h_{AB}. Now we recall the result that the conformal algebra of the n metric h_{AB} and the $1+n$ metric (8.3) are related as follows [44]:

a. The KVs of the n metric are identical with those of the $n+1$ metric

b. The $1+n$ metric admits a HV if and only if the n metric admits one and if ${}_nH^A$ is the HV of the n metric then the HV of the $1+n$ metric is given by the expression

$${}_{1+n}H^{\mu} = z\delta^{\mu}_z +_n H^A\delta^{\mu}_A \ \mu = x, 1, ..., n. \tag{8.6}$$

d. The $1+n$ metric admits CKVs if and only if the n metric h_{AB} admits a gradient CKV (for details see [44]).

Therefore Type II hidden symmetries for (8.2) exist if the n metric h_{AB} admits more symmetries. Specifically, the sp.CKVs of the h_{AB} metric as well as the proper CKVs whose conformal function is a solution of Laplace equation (8.62) generate Type II hidden symmetries.

8.3.2 Riemannian spaces admitting a gradient HV.

In Riemannian spaces which admit a gradient HV, H say, there exists a coordinate system in which the metric is written in the form [83]

$$ds^2 = dr^2 + r^2 h_{AB}\left(y^C\right) dy^A dy^B \tag{8.7}$$

and the gradient HV is $H = r\partial_r$. In these coordinates the Laplacian (8.2) takes the form

$$u_{rr} + \frac{1}{r^2} h^{AB} u_{AB} + \frac{(n-1)}{r} u_r - \frac{1}{r^2}\Gamma^A u_A = 0 \tag{8.8}$$

and admits the extra Lie symmetry (see Theorem 6.4.2) H. We reduce (8.8) using H.

The zeroth order invariants of H are y^A , $w\left(y^A\right)$ and using them it follows easily that the reduced equation is

$$_h\Delta u = 0 \tag{8.9}$$

that is, the Laplacian defined by the metric h_{AB}.

It is easy to establish the following results concerning the conformal algebras of the metrics (8.7) and h_{AB}.

1. The KVs of h_{AB} are also KVs of (8.7).
2. The HV of (8.7) is independent from that of h_{AB}.
3. The metric (8.7) admits proper CKVs if and only if the n metric h_{AB} admits gradient CKVs. This is because (8.7) is conformally related with the decomposable metric

$$ds^{2\prime} = dr^2 + h_{AB}\left(y^C\right) dy^A dy^B. \tag{8.10}$$

The above imply, that Type II hidden symmetries we shall have from the HV of the metric h_{AB}, the sp.CKVs and finally from the proper CKVs of h_{AB} whose conformal factor is a solution of Laplace equation (8.9).

8.3.3 Riemannian spaces admitting a sp.CKV

It is known [22], that if an $n = m + 1$ dimensional ($n > 2$) Riemannian space admits sp.CKVs then also admits a gradient HV and as many gradient KVs as the number of sp.CKVs. In these spaces there exists always a coordinate system in which the metric is written in the form [83]

$$ds^2 = -dz^2 + dR^2 + R^2 f_{AB}\left(y^C\right) dy^A dy^B \tag{8.11}$$

where ∂_z is the gradient KV and $z\partial_z + R\partial_R$ is the gradient HV. $f_{AB}\left(y^C\right)$, $A, B, C, .. = 1, 2, , ...m - 1$ is an $m-1$ dimensional metric. For a general $m-1$ dimensional metric f_{AB} the n dimensional metric (8.11) admits the following special Conformal group

$$\begin{aligned} K_G &= \partial_z \ , \ H = z\partial_z + R\partial_R \\ C_S &= \frac{z^2 + R^2}{2}\partial_z + zR\partial_R \end{aligned}$$

where K_G is a gradient KV, H is a gradient HV and C_S is a sp.CKV with conformal factor $\psi_{C_S} = z$. In these coordinates Laplace equation (8.2) takes the form

$$-u_{zz} + u_{RR} + \frac{1}{R^2} h^{AB} u_{AB} + \frac{(m-1)}{R} u_R - \frac{1}{R^2} \Gamma^A u_A = 0. \tag{8.12}$$

From Theorem 6.4.2, we have that the extra Lie symmetries of (8.12) are the vectors

$$\begin{aligned} X^1 &= K_G \ , \ X^2 = H \\ X^3 &= C_S + 2pzu\partial_u \end{aligned}$$

where $2p = \frac{1-m}{2}$ and the non zero commutators are

$$\left[X^1, X^2\right] = X^1 \ , \ \left[X^2, X^3\right] = X^3$$

$$\left[X^1, X^3\right] = X^2 + 2pX_u.$$

We consider the reduction of (8.12) with each of the extra Lie symmetries.

Reduction with the gradient KV X^1.

The first order invariants of X^1 are R, y^C, $w\left(R, y^C\right)$ and by using them we reduce the Laplacian (8.12) to (8.8) which admits the Lie symmetry X^2 generated by the HV. This result is expected because $\left[X^1, X^2\right] = X^1$ [100] hence the Lie symmetry X^2 is inherited. Therefore, in this reduction the Type II symmetries are generated from the CKVs of the metric (8.7). It is possible to continue the reduction by the gradient HV H and then we find the results of section 8.3.2.

Reduction with the gradient HV X^2.

The reduction with a gradient HV has been studied in section 8.3.2. To apply the results of section 8.3.2 in the present case we have to bring the metric (8.11) to the form (8.7). For this we consider the transformation

$$z = r\sinh\theta \ , \ R = r\cosh\theta$$

which brings (8.11) to

$$ds^2 = dr^2 + r^2\left(-d\theta^2 + \cosh^2\theta f_{AB} y^A y^B\right) \tag{8.13}$$

so that the metric h_{AB} of (8.7) is

$$ds_h^2 = \left(-d\theta^2 + \cosh^2\theta f_{AB} y^A y^B\right). \tag{8.14}$$

The reduced equation of (8.12) under the Lie symmetry generated by the gradient HV is Laplace equation in the space (8.14). For this reduction we do not have inherited symmetries and there exist Type II hidden symmetries as stated in section 8.3.2.

Reduction with the sp.CKV X^3.

Before we reduce (8.12) with the symmetry generated by the sp CKV X^3, it is best to write the metric (8.11) in new coordinates. We introduce the new variable x via the relation

$$z=\sqrt{\frac{R\left(xR-1\right)}{x}}. \tag{8.15}$$

In the new variables the metric (8.11) becomes

$$ds^2=-\frac{R}{4x^3\left(xR-1\right)}dx^2-\frac{2xR-1}{2x^2\left(xR-1\right)}dxdR-\frac{1}{4xR\left(xR-1\right)}dR^2+R^2f_{AB}\left(y^C\right)dy^Ady^B \tag{8.16}$$

the Laplacian (8.12):

$$\begin{aligned}0 \;=\; & \frac{x^2}{R^2}u_{xx}-2\frac{x}{R}\left(2xR-1\right)u_{xR}+u_{RR}+\frac{1}{R^2}f^{AB}u_{AB}+ \\ & +\frac{(m-1)}{R}u_R-\frac{x}{R^2}\left(m-1\right)\left(2xR-1\right)u_x-\frac{1}{R^2}\Gamma^Au_A\end{aligned} \tag{8.17}$$

and the Lie symmetry X^3

$$X^3=\sqrt{\frac{R\left(xR-1\right)}{x}}R\partial_R+2p\sqrt{\frac{R\left(xR-1\right)}{x}}u\partial_u.$$

The zeroth order invariants of X^3 are x,y^A , $w=uR^{-2p}$. We choose x,y^A to be the independent variables and $w=w\left(x,y^A\right)$ the dependent one. By replacing in (8.17) we find the reduced equation

$$x^2w_{xx}+f^{AB}w_{AB}-\Gamma^Aw_A-2p\left(2p+1\right)w=0 \tag{8.18}$$

We consider cases.

The case $m\succeq 4$.

If $2p+1\neq 0,\ m\succeq 4$ then (8.18) becomes

$${}_{(m\succeq 4)}\bar{\Delta}w-2p\left(2p+1\right)V\left(x\right)w=0 \tag{8.19}$$

where $V\left(x\right)=x^{\frac{2}{2-m}}$ and ${}_{(m\succeq 4)}\bar{\Delta}$ is the Laplace operator for the metric

$$d\bar{s}^2_{(m\succeq 4)}=\frac{1}{V\left(x\right)}\left(\frac{1}{x^2}dx^2+f_{AB}dy^Ady^B\right). \tag{8.20}$$

Equation (8.19) is the Klein Gordon equation in a space with potential $V\left(x\right)=x^{\frac{2}{2-m}}$ and metric (8.20). Considering the new transformation $\phi=\int\sqrt{\frac{1}{xV}}dx$ or $x=\left(m-2\right)^{2-m}\phi^{m-2}$ the metric (8.20) is written

$$d\bar{s}^2_{(m\succeq 4)}=d\phi^2+\phi^2\bar{f}_{AB}dy^Ady^B \tag{8.21}$$

where $\bar{f}_{AB}=\left(m-2\right)^{-2}f_{AB}$ whereas the potential $V\left(\phi\right)=\frac{(2-m)^2}{\phi^2}$ which is the well known Ermakov potential [69].

This means that the gradient HV $\phi\partial_{\phi}$, $\psi_{\phi}=1$, is a Lie symmetry of (8.19) which is the Lie symmetry X^{2}. Therefore, if the metric $\bar{f}_{AB}$ admits proper CKVs which satisfy the conditions of Theorem 7.3.3, then these vectors generate Type II hidden symmetries of (8.12).

The case $m=3$.

If $2p+1=0,$ then $m=3$ and f_{AB} is a two dimensional metric. In this case, equation (8.18) becomes

$$x^{2}w_{xx}+f^{AB}w_{AB}-\Gamma^{A}w_{A}=0 \tag{8.22}$$

or, by multiplying with x^{2}

$$_{(m=3)}\bar{\Delta}w=0 \tag{8.23}$$

which is the Laplacian in the three dimensional space with metric

$$d\bar{s}_{(m=3)}^{2}=\frac{1}{x^{4}}dx^{2}+\frac{1}{x^{2}}f_{AB}dy^{A}dy^{B}. \tag{8.24}$$

By making the new transformation $x=\frac{1}{r},$ (8.23) is of the form (8.7) and admits the gradient HV $r\partial_{r}$ which gives an inherited symmetry. We conclude that Type II hidden symmetries of (8.24) will be generated from the proper CKVs of the metric (8.24) which satisfy the condition of Theorem 6.4.2.

The case $m=2$.

For $m=2,$ f_{AB} is a one dimensional metric and (8.11) is

$$ds^{2}=-dz^{2}+dR^{2}+R^{2}d\theta^{2} \tag{8.25}$$

which is a flat metric[3]. In this space, Laplace equation (8.12) admits ten Lie point symmetries, as many as the elements of the Conformal algebra of the flat 3D space. Six of these vectors are KVs, one vector is a gradient HV and three are sp.CKVs (see example 2.7.12). We reduce the Laplace equation with the symmetry X^{3} and the reduced equation is (8.18) which for $f_{AB}=\delta_{\theta\theta}$ becomes

$$x^{2}w_{xx}+w_{\theta\theta}+\frac{1}{4}w=0. \tag{8.26}$$

Equation (8.26) is in the form of (6.13) (see Chapter 6) with $A^{ij}=diag\left(x^{2},1\right)$ and $B^{i}=0$. By replacing in the symmetry conditions (6.21)-(6.25) we find the Lie symmetries

$$X=\xi^{i}\partial_{i}+\left(a_{0}w+b\right)\partial_{w}$$

where ξ^{i} are the CKVs of the two dimensional space with metric A^{ij}. In this case, all proper CKVs of the two dimensional space A^{ij} generate Type II Lie symmetries. Recall that the conformal algebra of a two dimensional space is infinite dimensional.

[3] The only three dimensional space who admits sp.CKV is the flat space, because in that case we also have a gradient HV and a gradient KV.

8.4 Type II hidden symmetries of the 3D and the 4D wave equation

In this section we apply the general results of the previous section to specific spaces where the metric is known.

8.4.1 Laplacian in M^4

The Laplace equation in the four dimensional Minkowski spacetime M^4

$$ds^2 = -dt^2 + dx^2 + dy^2 + dz^2 \tag{8.27}$$

is the wave equation [100] in E^3

$$u_{tt} - u_{xx} - u_{yy} - u_{zz} = 0. \tag{8.28}$$

The conformal algebra of the metric (8.27) is generated by 15 vectors (see example 2.7.12). From theorem 6.4.2 we have that the extra Lie point symmetries of (8.28) are the vectors

$$K_G^1\ ,\ K_G^A\ ,\ X_R^{1A}\ ,\ X_R^{AB}\ ,\ H\ ,\ X_C^1 - tu\partial_u\ ,\ X_C^A - y^A u\partial_u \tag{8.29}$$

where $y^A = (x, y, z)$.

The nonzero commutators are

$$\begin{aligned}
\left[K_G^I, X_R^{IJ}\right] &= -K_G^J\ ,\ \left[K_G^I,\ H\right] = K_G^I \\
\left[K_G^I, X_C^I\right] &= H - X_u\ ,\ \left[K_G^I, X_C^J\right] = X_R^{IJ} \\
\left[H, X_C^I\right] &= X_C^I\ ,\ \left[X_R^{IJ}, X_C^I\right] = X_C^J\ .
\end{aligned}$$

Reduction with a gradient KV

We choose to make reduction of (8.28) with the gradient KV $K_G^z = \partial_z$. The reduced equation is

$$w_{tt} - w_{xx} - w_{yy} = 0 \tag{8.30}$$

which is Laplace equation in the space M^3. The extra Lie symmetries of (8.30) are

$$K_G^1\ ,\ K_G^x\ ,\ K_G^y\ ,\ \ X_R^{1a}\ \ ,\ X_R^{ab}\ ,\ H \tag{8.31}$$

and are inherited symmetries (see also the last commutators). The Type II symmetries are the vectors

$$\bar{X}_C^1 - \frac{1}{2}tu\partial_u\ ,\ \bar{X}_C^x - \frac{1}{2}xu\partial_u\ ,\ \bar{X}_C^y - \frac{1}{2}yu\partial_u \tag{8.32}$$

that is the Type II hidden symmetries are generated from the sp.CKVs of M^3.

Reduction with the gradient HV

In this case it is better to switch to hyperspherical coordinates (r,θ,ϕ,ζ).

In these coordinates the metric (8.27) is written

$$ds^2 = dr^2 - r^2\left[d\theta^2 + \cosh^2\theta\left(d\phi^2 + \cosh^2\phi d\zeta^2\right)\right] \tag{8.33}$$

and the wave equation (8.28) becomes

$$u_{rr} - \frac{1}{r^2}\left(u_{\theta\theta} + \frac{u_{\phi\phi}}{\cosh^2\theta} + \frac{u_{\zeta\zeta}}{\cosh^2\theta\cosh\phi^2}\right) + \frac{3}{r}u_r - 2\frac{\tanh\theta}{r^2}u_\theta - \frac{\tanh\phi}{r^2\cosh^2\theta}u_\phi = 0. \tag{8.34}$$

According to the analysis of section 8.3.2 the reduced equation is (8.9) which is the Laplacian in the three dimensional space of the variables (θ,ϕ,ζ) :

$$w_{\theta\theta} + \frac{w_{\phi\phi}}{\cosh^2\theta} + \frac{w_{\zeta\zeta}}{\cosh^2\theta\cosh\phi^2} + 2\frac{\tanh\theta}{r^2}w_\theta + \frac{\tanh\phi}{r^2\cosh^2\theta}w_\phi = 0. \tag{8.35}$$

This space is a space of constant curvature. The conformal algebra[4] of a 3D space of constant non-vanishing curvature consists of 6 non-gradient KVs and 4 proper CKVs[5] [23]. The conformal factors of the CKVs do not satisfy the condition ${}_h\Delta\psi = 0$ (see Theorem 6.4.2); hence, they do not generate Lie point symmetries for the reduced equation (8.35) whereas for the same reason the KVs are Lie symmetries of (8.35). Therefore, all point Lie point symmetries are inherited and we do not have Type II hidden symmetries.

We note that the proper CKVs in a space of constant non-vanishing curvature are gradient and their conformal factor satisfies the relation [23]

$$\psi_{;ab} = C\psi g_{ab} \to g^{ab}\psi_{;ab} = nC\psi \to_h \Delta\psi = nC\psi$$

which implies that they are Lie symmetries of the conformally invariant operator but not of the Laplace equation (8.35).

Reduction with a sp.CKV

Following the steps of section 8.3.3, we consider the transformation to axi-symmetric coordinates (t,R,θ,ϕ) in which (8.28) takes the form

$$u_{tt} - u_{RR} - \frac{1}{R^2}\left(u_{\theta\theta} + \frac{u_{\phi\phi}}{\cosh^2\theta}\right) - \frac{2}{R}u_r - \frac{\tanh\theta}{R^2}u_\theta = 0. \tag{8.36}$$

Applying the transformation (8.15) $t = \sqrt{\frac{R(\tau R-1)}{\tau}}$ we find (note that this is the case $m=3$) that (8.36) is written as (8.17) and the reduced equation is the Laplacian ${}_{(m=3)}\Delta w$ for the 3D metric

$$ds^2 = \frac{1}{\tau^4}d\tau^2 - \frac{1}{\tau^2}\left(d\theta^2 + \cosh^2\theta d\phi^2\right). \tag{8.37}$$

[4] All spaces of constant curvature are conformally flat, hence, they admit the same conformal algebra with the flat space but not the same subalgebras, i.e. the same conformal factors.

[5] The rotations and the sp.CKVs of the flat space are KVs for the space of constant curvature, the rest are proper gradient CKVs

The metric (8.37) under the coordinate transformation $\tau = \frac{1}{T}$ is written

$$ds^2 = dT^2 - T^2\left(d\theta^2 + \cosh^2\theta d\phi^2\right) \tag{8.38}$$

which is the flat 3D Lorentzian metric, which does not admit proper CKVs. This implies that the Lie symmetries of the reduced equation are generated from the KVs/HV/sp.CKVs of the flat M^3 metric and all are inherited. Therefore we do not have Type II hidden symmetries for the reduction with a sp.CKV.

As we shall show in the next section this is not the case for the reduction of Laplace equation in M^3.

8.4.2 Laplacian in M^3

We consider the reduction of Laplace equation in the 3d Minkowski M^3 spacetime [100], i.e. the wave equation in E^2

$$u_{tt} - u_{xx} - u_{yy} = 0. \tag{8.39}$$

As we have seen in section 8.4.1 the extra Lie point symmetries of (8.39) are the ten vectors (8.31) and (8.32).

Reduction with a gradient KV

We choose the vector ∂_y and the reduction gives the reduced equation

$$w_{tt} - w_{xx} = 0 \tag{8.40}$$

which is the one dimensional wave equation. The 2d space (t, x) has an infinite number of CKVs therefore (8.40) has infinite Lie point symmetries [1]. From these symmetries the KVs and the HV are inherited symmetries and the CKVs are Type II symmetries.

Reduction with the gradient HV

In order to do the reduction with the gradient HV we introduce spherical coordinates (r, θ, ϕ) and find that (8.39) becomes

$$u_{rr} - \frac{1}{r^2}\left(u_{\theta\theta} + \frac{u_{\phi\phi}}{\cosh^2\theta}\right) + \frac{2}{r}u_r - \frac{\tanh\theta}{r^2}u_\theta = 0. \tag{8.41}$$

According to the results of section 8.3.2 the reduced equation is equation (8.9) which is the Laplace equation in the 2d space of the variables (ϕ, θ)

$$w_{\theta\theta} + \frac{w_{\phi\phi}}{\cosh^2\theta} + \tanh\theta w_\theta = 0. \tag{8.42}$$

By making the transformation $\theta = \ln\left(\tan\frac{\rho}{2}\right)$ equation (8.42) becomes

$$\sin(x)^2\left(w_{\rho\rho} + w_{\phi\phi}\right) = 0 \tag{8.43}$$

which is the wave equation (8.40) with $t = \rho\ ,\ x = -i\phi$.

We obtain the results, concerning the Lie symmetries of (8.43) from section 8.4.2 with the difference that the Lie symmetry due to the HV of the two dimensional metric is not inherited but it is a Type II hidden symmetry.

The reduction of the wave equation in the 4D and in 3D Minkowski space has been done previously by Abraham-Shrauner et. all [100] and our results coincide with theirs. For example in the 3d case equation (17) of [100] is our equation (8.42) in other variables. However there are two differences (a) in the case of the 2D space they do not obtain that the Lie symmetries are infinite and (b) they use algebraic computing programs to find the Lie symmetry generators whereas our approach is geometric and general and does not need algebraic computing programs to find the complete answer.

The reduction with a sp.CKV has been considered in section 8.3.3.

8.5 The Laplace equation in the Petrov type III and in the FRW-like spacetime

To complete our analysis, we have to reduce the Laplace equation using a non gradient homothetic vector and a proper (i.e. non special) CKV. In order to do this, we consider the reduction of Laplace equation in the Petrov type III spacetime and in the FRW-like spacetime.

8.5.1 The Laplace equation in the Petrov Type III spacetime

In this section we consider the reduction of Laplace equation in spaces which do not admit gradient KVs or a gradient HV. As it has been mentioned in section 8.2 we shall consider the algebraically special solutions of Einstein equations, that is the Petrov type D,N,II and III. In fact we restrict our discussion to Petrov Type III because both the method of work and the results are the same for all Petrov types.

The metric of the Petrov type III space-time is

$$ds^2 = 2d\rho dv + \frac{3}{2}xd\rho^2 + \frac{v^2}{x^3}\left(dx^2 + dy^2\right) \tag{8.44}$$

with conformal algebra

$$\begin{aligned} K^1 &= \partial_\rho\ ,\ K^2 = \partial_y\ ,\ K^3 = v\partial_v - \rho\partial_\rho + 2x\partial_x + 2y\partial_y \\ H &= v\partial_v + \rho\partial_\rho\ ,\ \psi = 1 \end{aligned}$$

where K^{1-4} are KVs and H is a non-gradient HV. (The space does not admit proper CKVs).

In this space-time Laplace equation (8.2) takes the form

$$-\frac{3}{2}xu_{vv} + 2u_{v\rho} + \frac{x^3}{v^2}\left(u_{xx} + u_{yy}\right) - 3\frac{x}{v}u_v + \frac{2}{v}u_\rho = 0. \tag{8.45}$$

From Theorem 6.4.2 we have that the extra Lie point symmetries are the vectors

$$X_{1-3}=K_{1-3}\ ,\quad X_4=H$$

with nonzero commutators:

$$[X_{2,},X_3]=2X_2$$

$$[X_3,X_1]=X_1\ ,\quad [X_1,X_4]=X_1.$$

We use X_4 to reduce the PDE because this is the Lie symmetry generated by the nongradient HV.

The zero order invariants of X_4 are $\sigma=\frac{\rho}{v},x,y,w$. We choose σ,x,y as the independent variables and $w=w\left(\sigma,x,y\right)$ as the dependent variable and we find the reduced equation

$$-\sigma\left(\frac{3}{2}x\sigma+2\right)w_{\sigma\sigma}+x^3\left(w_{xx}+w_{yy}\right)=0. \tag{8.46}$$

Equation (8.46) can be written

$${}_{III}\Delta^{*}w-\left(\frac{3x\sigma}{2}+1\right)w_{\sigma}-\frac{3x^3\sigma}{2\left(3x\sigma+4\right)}w_x=0 \tag{8.47}$$

where ${}_{III}\Delta^{*}$ is the Laplacian for the metric

$$ds^2=-\frac{1}{\sigma\left(\frac{3}{2}x\sigma+2\right)}d\sigma^2+\frac{1}{x^3}\left(dx^2+dy^2\right). \tag{8.48}$$

The Lie symmetries of (8.47) will be generated from the conformal algebra of (8.48) with some extra conditions (see equations (6.21)-(6.25)). Finally, we find that equation (8.47) admits as Lie point symmetries the vectors ∂_y , $x\partial_x+y\partial_x-\sigma\partial_\sigma$ which are inherited symmetries. Therefore we do not have Type II hidden symmetries.

8.5.2 The Laplace equation in the n dimensional FRW-like spacetime.

We consider the n dimensional FRW-like space ($n>2$) with metric

$$ds^2=e^{2t}\left[dt^2-\left(\delta_{AB}dy^Ady^B\right)\right] \tag{8.49}$$

where δ_{AB} is the $n-1$ dimensional Euclidian metric. The reduction of Laplace equation in this space (for $n=4$) has been studied previously in [122]. The metric (8.49) is conformally flat hence admits the same CKVs with the flat space but with different subalgebras. More precisely the space admits

a. $(n-1)+\frac{(n-2)(n-3)}{2}$ KVs the $K_G^A,\ X_R^{AB}$

b. 1 gradient HV the $K_G^1=\partial_t$

the rest vectors being proper CKVs [45]. In this space Laplace equation (8.2) becomes

$$e^{-2t}\left[u_{tt}-\left(\delta^{AB}u_{AB}\right)+(n-2)\,u_t\right]=0 \tag{8.50}$$

and the extra Lie symmetries are

$$K_G^A,\ X_R^{AB}\ ,\ K_G^1\ ,\ X_R^{1A}-2pY^a u\partial_u$$

where $2p=\frac{2-n}{2}$. The algebra of the Lie point symmetries is the same with that of section 7.55. We consider the reduction with a proper CKV.

Reduction with a proper CKV

We may take any of the vectors X_R^{1A} (because as one can see in the Appendix there is a symmetry between the coordinates y^A). We choose the vector

$$X_R^{1x}=x\partial_t+t\partial_x+2pxu\partial_u.$$

whose zero order invariants are

$$R=t^2-x^2\ ,\ y^C\ ,\ w=e^{-2pt}u.$$

We take the dependent variable to be the $w=w\left(R,y^C\right)$ and find the reduced equation

$$4Rw_{RR}-\delta^{ab}w_{ab}+4w_R-4p^2w=0 \tag{8.51}$$

where $a=1,\ldots,n-2$. We consider cases.

Case $n>3$.

For $n>3$ equation (8.51) is

$$_C\Delta w-4p^2f\left(R\right)w=0 \tag{8.52}$$

where $_C\Delta$ is the Laplace operator for the $(n-1)$ dimensional metric

$$ds_C^2=\frac{1}{f\left(R\right)}\left(\frac{1}{4R}dR^2-\delta_{ab}dy^a dy^b\right) \tag{8.53}$$

and $f\left(R\right)=R^{-\frac{1}{n-3}}$. The metric (8.53) is conformally flat hence we know its conformal algebra. Application of theorem 7.3.3 gives that the Lie point symmetries of (8.52) are the vectors

$$\begin{aligned} X_u &= u\partial_u\ ,\ X_b=b\partial_u \\ X_K^a &= \partial_{y^a}\ ,\ X_R^{ab}\ =y^b\partial_a-y^a\partial_b. \end{aligned}$$

These are inherited symmetries (this result agrees with the commutators). We conclude that for this reduction we do not have Type II hidden symmetries.

Case $n=3$.

For $n=3$ the reduced equation is a two dimensional equation (that is $\delta_{AB}=\delta_{yy}$)

$$4Rw_{RR}-w_{yy}+4w_R-\frac{1}{4}w=0 \tag{8.54}$$

and admits as Lie point symmetry the KV ∂_y which is an inherited symmetry. Hence, we do not have Type II hidden symmetries.

We conclude that the reduction of Laplace equation in an n dimensional FRW like space with the proper CKV does not produce Type II hidden symmetries and in fact the inherited symmetries of the reduced equation are the KVs of the flat metric.

Reduction with the gradient HV

The gradient HV $K_G^1 = \partial_t$ is a Lie symmetry of the Laplacian (8.50) hence we consider the reduction by this vector. The zero order invariants are y^A, w and lead to the reduced equation

$$\delta^{AB} u_{AB} = 0 \tag{8.55}$$

which is Laplace equation in the flat space E^{n-1}. We consider again cases.

Case $n > 3$

In this case the Lie symmetries of (8.55) are given by the vectors

$$K_G^A \ , \ X_R^{AB} \ , \ {}_{n-1}H \ , X_C^A - y^A u \partial_u . \tag{8.56}$$

From these the K_G^A, X_R^{AB} are inherited symmetries and the rest - which are produced by the HV and the sp.CKVs of the space E^{n-1} - are Type II hidden symmetries.

If $n = 3$, the reduced equation (8.55) is the Laplacian in E^2, hence, admits infinite Lie symmetries. Type II hidden symmetries are generated from the HV and the CKVs of E^2.

In the following sections, we study the reduction of the homogeneous heat equation (8.1) in certain Riemannian spaces.

8.6 Reduction of the homogeneous heat equation in certain Riemannian spaces

In a general Riemannian space with metric g_{ij} the heat conduction equation with flux is

$$\Delta u - u_t = q \tag{8.57}$$

where Δ is the Laplace operator $\Delta = \frac{1}{\sqrt{g}} \frac{\partial}{\partial x^i} \left(\sqrt{g} g^{ij} \frac{\partial}{\partial x^j} \right)$ and $q = q(t, x, u)$. Equation (8.57) can also be written

$$g^{ij} u_{ij} - \Gamma^i u_i - u_t = q \tag{8.58}$$

where $\Gamma^i = \Gamma^i_{jk} g^{jk}$ and Γ^i_{jk} are the Christofell Symbols of the metric g_{ij}.

For $q=0$, equation (8.58) admits the Lie point symmetries

$$X_t = \partial_t \ \ , \ X_u = u\partial_u \ , \ X_b = b(t,x)\,\partial_u \tag{8.59}$$

where $b(t,x)$ is a solution of the heat equation. These symmetries are too general to provide sound reductions and consequently reduced PDEs which can give Type II hidden symmetries. However, in Chapter 6 it has been shown that there is a close relation between the Lie point symmetries of the heat equation and the collineations of the metric. Specifically it has been shown that the Lie point symmetries of the heat equation are generated from the HV and the KVs of g_{ij}. This implies that if we want to have new Lie point symmetries which will allow for sound reductions of the heat equation eqn (8.58) we have to restrict our considerations to spaces which admit a homothetic algebra. Our intention is to keep the discussion as general as possible therefore we consider spaces in which the metric g_{ij} can be written in generic form. The spaces we shall consider are:

a. Spaces which admit a gradient KV (8.3).

b. Spaces which admit a gradient HV (8.7).

c. Space which admits a nongradient HV acting simply transitive, i.e. Petrov Type III (8.44).

In what follows all spaces are of dimension $n \succeq 2$. The case $n=1$ although relatively trivial for our approach in general it is not so and has been studied for example in [123, 124].

8.6.1 The heat equation in a space which admits a gradient KV

In the $1+n$ decomposable space with line element (8.3) the heat equation (8.1) takes the form

$$u_{zz} + h^{AB}u_{AB} - \Gamma^{A}u_{B} - u_t = 0. \tag{8.60}$$

Application of Theorem 6.5.2 gives that (8.60) admits the following *extra* Lie point symmetries generated by the gradient KV ∂_x :

$$X_1 = \partial_z \ , \ X_2 = t\partial_z - \frac{1}{2}zu\partial_u$$

with nonvanishing commutators

$$[X_t, X_2] = X_1 \ , \ [X_2, X_1] = \frac{1}{2}X_u. \tag{8.61}$$

We reduce (8.60) using the zero order invariants of the extra Lie point symmetries X_1, X_2.

Reduction by X_1

The zero order invariants of X_1 are

$$\tau = t \ , \ y^A \ , \ w = u.$$

Taking these invariants as new coordinates eqn (8.60) reduces to

$${}_h\Delta w - w_t = 0 \tag{8.62}$$

where ${}_h\Delta$ is the Laplace operator in the $n-$dimensional space with metric h_{AB} :

$$ {}_h\Delta w = h^{AB} w_{AB} - \Gamma^A w_B. \tag{8.63}$$

Equation (8.62) is the homogeneous heat eqn (8.1) in the n dimensional space with metric h_{AB}. According to the Theorem 6.5.2 (see Chapter 6), the Lie symmetries of this equation are the homothetic algebra of h_{AB}. As it has been already mentioned the homothetic algebras of the n and the $1+n$ metrics are related as follows [44]:

a. The KVs of the $n-$ metric are identical with those of the $1+n$ metric.

b. The $1+n$ metric admits a HV if the n metric admits one and if ${}_nH^A$ is the HV of the n - metric then the HV of the $1+n$ metric is given by the expression

$$ {}_{1+n}H^{\mu} = x\delta_z^{\mu} +_n H^A \delta_A^{\mu} \qquad \mu = z, 1, ..., n. \tag{8.64}$$

The above imply that equation (8.62) inherits all symmetries which are generated from the KVs/HV of the $n-$metric h_{AB}. Hence we do not have Type II symmetries in this reduction.

Reduction by X_2

The zero order invariants of X_2 are

$$ \tau = t\ ,\ y^A\ ,\ w = ue^{\frac{z^2}{4t}}. $$

Taking these invariants as new coordinates eqn (8.60) reduces to

$$ h^{AB} w_{AB} - \Gamma^A w_B - w_\tau - \frac{1}{2\tau} w = 0 \tag{8.65}$$

or

$$ {}_h\Delta w - w_\tau = \frac{1}{2\tau} w. $$

This is the nonhomogeneous heat equation with flux $q\left(\tau, y^A, w\right) = \frac{1}{2\tau} w$. Application of Theorem 6.5.1 gives the following result[6].

Proposition 8.6.1 *The Lie point symmetries of the heat equation (8.65) in an $n-$dimensional Riemannian space with metric h_{AB} are constructed form the homothetic algebra of the metric as follows:*

a. Y^i is a HV/KV.

The Lie symmetry is

$$ X = \left(2c_2\psi\tau + c_1\right)\partial_\tau + c_2 Y^i \partial_i + \left[\left(-\frac{c_1}{2\tau} + a_0\right) w + b\left(\tau, x\right)\right]\partial_w \tag{8.66}$$

[6] For details see Appendix 8.A.

b. $Y^i = S_J^{;i}$ *is a gradient HV/KV (the index* J *counts gradient KVs).*
The Lie symmetry is

$$X = \left(\psi T_0 \tau^2\right)\partial_\tau + T_0 \tau S_J^{;i}\partial_i - \left(\frac{1}{2}T_0 S_J + T_0 \psi \tau\right) w\partial_w \tag{8.67}$$

where $b(\tau, x)$ *is a solution of the heat equation (8.65).*

We infer that for this reduction we have the Type II hidden symmetry $\partial_\tau - \frac{1}{2t}w\partial_w$. The rest of the Lie point symmetries are inherited.

8.6.2 The heat equation in a space which admits a gradient homothetic vector

For the spacetime with line element

$$ds^2 = dr^2 + r^2 h_{AB} dy^A dy^B$$

the homogeneous heat equation becomes

$$u_{rr} + \frac{1}{r^2}h^{AB}u_{AB} + \frac{(n-1)}{r}u_r - \frac{1}{r^2}\Gamma^A u_A - u_t = 0 \tag{8.68}$$

where $\Gamma^A = \Gamma^A_{BC}h^{BC}$ and Γ^A_{BC} are the connection coefficients of the Riemannian metric h_{AB} $(A, B, C = 1, 2, ..., n)$. Application of Theorem 6.5.2 gives that the heat equation (8.68) admits the following *extra* Lie point symmetries generated by the gradient homothetic vector

$$\bar{X}_1 = 2t\partial_t + r\partial_r \quad , \quad \bar{X}_2 = t^2\partial_t + tr\partial_r - \left(\frac{1}{4}r^2 + \frac{n}{2}t\right)u\partial_u \tag{8.69}$$

with nonzero commutators

$$\left[X_t, \bar{X}_1\right] = 2X_t \quad , \quad \left[\bar{X}_1, \bar{X}_2\right] = 2X_t$$
$$\left[X_t, \bar{X}_2\right] = \bar{X}_1 - \frac{n}{2}X_u.$$

We consider again the reduction of (8.68) using the zero order invariants of these extra Lie point symmetries.

Reduction by $\bar{X}_1$

The zero order invariants of $\bar{X}_1$ are

$$\phi = \frac{r}{\sqrt{t}} \ , \ w = u, \ y^A.$$

We choose $w = w\left(\phi, y^A\right)$ as the dependent variable.

Replacing in (8.68) we find the reduced PDE

$$w_{\phi\phi} + \frac{1}{\phi^2}h^{AB}w_{AB} + \frac{(n-1)}{\phi}w_\phi + \frac{\phi}{2}w_\phi - \frac{1}{\phi^2}\Gamma^A w_A = 0. \tag{8.70}$$

Consider a nonvanishing function $N^2(\phi)$ and divide (8.70) with $N^2(\phi)$ to get:

$$\frac{1}{N^2}w_{\phi\phi} + \frac{1}{\phi^2 N^2}h^{AB}w_{AB} + \frac{(n-1)}{\phi N^2}w_\phi + \frac{\phi}{2N^2}w_\phi - \frac{1}{\phi^2 N^2}\Gamma^A w_A = 0 \tag{8.71}$$

It follows that (for $n>2$) equation (8.71) can be written as

$$_{\bar{g}}\Delta w=0 \tag{8.72}$$

where $_{\bar{g}}\Delta$ is the Laplace operator if $N^{2}(\phi)=\exp\left(\frac{\phi^{2}}{2(n-2)}\right)$ and $\bar{g}_{ij}$ is the conformally related metric

$$d\bar{s}^{2}=\exp\left(\frac{\phi^{2}}{2(n-2)}\right)\left(d\phi^{2}+\phi^{2}h_{AB}dy^{A}dy^{B}\right). \tag{8.73}$$

According to Theorem 6.4.2 the Lie point symmetries of (8.72) are the CKVs of the metric (8.73) whose conformal factor satisfies the condition $_{\bar{g}}\Delta\psi=0$. Therefore, Type II hidden symmetries will be generated from the proper CKVs. The existence and the number of these vectors depends mainly on the n metric h_{AB}.

Reduction by $\bar{X}_2$

For $\bar{X}_2$ the zero order invariants are

$$\phi=\frac{r}{t}\ ,\ w=ut^{\frac{n}{2}}e^{\frac{r^{2}}{4t}},\ y^{A}.$$

We choose $w=w\left(\phi,y^{A}\right)$ as the dependent variable and we have the reduced equation

$$_{g}\Delta w=0 \tag{8.74}$$

where

$$_{g}\Delta w=w_{\phi\phi}+\frac{(n-1)}{\phi}w_{\phi}+\frac{1}{\phi^{2}}h^{AB}w_{AB}-\frac{1}{\phi^{2}}\Gamma^{A}w_{A}. \tag{8.75}$$

Equation (8.74) is the Laplace equation in the space $\left(\phi,y^{A}\right)$ with metric

$$ds^{2}=d\phi^{2}+\phi^{2}h_{AB}dy^{A}dy^{B}. \tag{8.76}$$

The Lie point symmetries of Laplace equation (8.74) are given in Theorem 6.4.2. As in the last case the existence and the number of these vectors depends mainly on the n metric h_{AB}.

We note that both vectors $\bar{X}_1,\bar{X}_2$ are generated form the gradient HV and in both cases the heat equation is reduced to Laplace equation. This gives the following

Proposition 8.6.2 *The reduction of the heat equation (8.1) in a space with metric (8.7) $(n>2)$ by means of the Lie symmetries generated by the gradient HV leads to Laplace equation $\Delta u=0$, where Δ is the Laplace operator for the metric (8.73) if the reduction is done by $\bar{X}_1$ and for the metric (8.76) if the reduction is done by $\bar{X}_2$.*

8.7 Applications of the reduction of the homogeneous heat equation

In this section, we consider applications of the general results of section 8.6 in various spacetimes.

8.7.1 The heat equation in a $1+n$ decomposable space

Consider the $1+n$ decomposable space

$$ds^2 = dx^2 + N^{-2}\left(y^C\right)\delta_{AB}y^A y^B \tag{8.77}$$

where $N\left(y^C\right) = \left(1+\frac{K}{4}y^C y_C\right)$, that is, the n space is a space of constant non vanishing ($K \neq 0$) curvature. The space (8.77) does not admit proper HV, however, admits $\frac{n(n-1)}{2}+n$ nongradient KVs and 1 gradient KV as follows [44]

$$\begin{aligned} &\text{1 gradient KV:} && \partial_x \\ &n \text{ nongradient KVs} &:& \quad K_V = \frac{1}{N}\left[(2N-1)\,\delta_I^i + \frac{K}{2}N x_I x^i\right]\partial_i \\ &\frac{n(n-1)}{2}\text{nongradient KVs} &:& \quad X_{IJ} = \delta^j_{[I}\delta^i_{J]}\partial_i \end{aligned}$$

In a space with metric (8.77), the homogeneous heat equation takes the form

$$u_{xx} + N^2\left(y^C\right)\delta^{AB}u_{AB} - \frac{N}{2}Ky^A u_A - u_t = 0 \tag{8.78}$$

Applying Theorem 6.5.2 we find that equation (8.78) admits the extra Lie point symmetries

$$\partial_x\ ,\ t\partial_x - \frac{1}{2}xu\partial_x\ ,K_V\ ,X_{IJ}. \tag{8.79}$$

The Lie point symmetries which are generated by the gradient KV are[7] $\partial_x\ ,\ t\partial_x - \frac{1}{2}xu\partial_x$.

Reduction of (8.78) by means of the gradient KV ∂_x results in the special form of equation (8.62)

$$\frac{1}{N^2\left(y^C\right)}\delta^{AB}u_{AB} - \frac{N}{2}Ky^A u_A - u_t = 0. \tag{8.80}$$

This is the homogeneous heat equation in an n- dimensional space of constant curvature. The Lie point symmetries of this equation have been determined in section 6.5.1 and are inherited symmetries. Hence, in this case, we do not have Type II hidden symmetries.

Reduction of (8.78) with the Lie symmetry $t\partial_x - \frac{1}{2}xu\partial_x$ gives that the reduced equation (8.65) is

$$N^2\left(y^C\right)\delta^{AB}w_{AB} - \frac{N}{2}Ky^A w_A - w_\tau = \frac{1}{2\tau}w\ ,\ w = ue^{\frac{x^2}{4t}} \tag{8.81}$$

which is the heat equation with flux. By Proposition 8.6.1, the Lie point symmetries of (8.81) are:

$$X = c_1\partial_\tau + \left(K_V\ + X_{IJ}\right) + \left[\left(-\frac{c_1}{2\tau}+a_0\right)w + b\left(\tau, y^C\right)\right]\partial_w \tag{8.82}$$

where c_1, a_0 are constants. From section 8.6.1 we have that Type II hidden symmetry is the one defined by the constant c_1.

[7] Here the algebra is the one given in section 8.6.1 and a separate algebra is the algebra of the KVs of the space of constant curvature. More specifically the KVs $K_V\ ,X_{IJ}$ commute with all other symmetries but not between themselves

8.7.2 FRW space-time with a gradient HV

Consider the spatially flat FRW metric

$$ds^2 = d\sigma^2 - \sigma^2\left(dx^2 + dy^2 + dz^2\right) \tag{8.83}$$

This metric admits the gradient HV [45]

$$H = \sigma\partial_\sigma \quad (\psi_H = 1)$$

and six nongradient KVs

$$X_{1-3} = \partial_{y^A} \quad , \quad X_{4-6} = y^B\partial_A - y^A\partial_B.$$

where $y^A = (x, y, z)$.

In this space the heat equation takes the form

$$u_{\sigma\sigma} - \frac{1}{\sigma^2}\left(u_{xx} + u_{yy} + u_{zz}\right) + \frac{3}{\sigma}u_\sigma - u_t = 0. \tag{8.84}$$

The Lie point symmetries of (8.84) are

$$\begin{aligned} &\partial_t \ , \ u\partial_u \ , \ b\left(\tau, y^A\right)\partial_u \ , \ X_{1-3} \ , \ X_{4-6} \ , \\ H_1 \ = \ &2t\partial_t + \sigma\partial_\sigma \ , \ H_2 = t^2\partial_t + t\sigma\partial_\sigma - \left(\frac{1}{4}\sigma^2 + 2t\right)u\partial_u \ . \end{aligned}$$

The Lie point symmetries H_1, H_2 are produced by the gradient HV therefore we use them to reduce (8.84). We note that this case is a special case of the one we considered in section 8.6.2 for $h_{AB} = \delta_{AB}$.

Reduction by H_1 gives that (8.84) becomes:

$$w_{\phi\phi} - \frac{1}{\phi^2}\left(w_{xx} + w_{yy} + w_{zz}\right) + \left(\frac{3}{\phi} + \frac{\phi}{2}\right)w_\phi = 0 \tag{8.85}$$

where $\phi = \frac{r}{\sqrt{\sigma}}$, $w = u$. This is a special form of (8.70).

Dividing with $N^2(\phi) = \exp\left(\frac{\phi^2}{4}\right)$ we find that (8.85) is written as

$$_{\bar{g}}\Delta w = 0 \tag{8.86}$$

where the metric $\bar{g}_{ij}$ is the conformally related metric of (8.83):

$$d\bar{s}^2 = e^{\frac{\phi^2}{4}}\left(d\phi^2 - \phi^2\left(dx^2 + dy^2 + dz^2\right)\right) \tag{8.87}$$

We have that the Lie symmetries of (8.86) are generated from elements of the conformal algebra of the space whose conformal factors satisfy condition $_{\bar{g}}\Delta\psi = 0$. The metric (8.86) is conformally flat therefore its conformal group is the same with that of the flat space [44, 45], however with different subgroups. We find that these vectors (i.e. the Lie symmetries) are the vectors

$$X_{1-3} \ , \ X_{4-6} \ , \partial_t, \ w\partial_w \ , \ b_0\left(\phi, y^A\right)\partial_w. \tag{8.88}$$

We conclude that there are no Type II symmetries for this reduction.

Using reduction by H_2 we find that (8.84) reduces to :

$$w_{\phi\phi} - \frac{1}{\phi^2}\left(w_{xx} + w_{yy} + w_{zz}\right) + \frac{3}{\phi}w_{\phi} = 0 \tag{8.89}$$

where $\phi = \frac{\tau}{\tau}$, $w = ut^2 e^{\frac{\tau^2}{4t}}$. This is a special form of (8.74) which is the Laplace equation. In this case the results of [90] apply and we infer that the Lie point symmetries of (8.89) are:

$$\begin{aligned} & X_{1-3}\ ,\ X_{4-6}\ ,\ \ w\partial_w\ ,\ b_1\left(\phi, y^A\right)\partial_w \\ X_7 &= \phi\partial_\phi\ ,\ X_{8-10} = \phi y^A\partial_\phi + \ln\phi\partial_A - y^A w\partial_w. \end{aligned} \tag{8.90}$$

The vector X_7 is the proper HV of the metric and the vectors X_{8-10} the proper CKVs which are not special CKVs, therefore these vectors are Type II hidden symmetries. A further analysis of (8.89) can be found in [122].

8.8 The Heat equation in Petrov type III spacetime

In this section we consider the special class of Petrov type III spacetime which admits a nongradient HV which acts simply transitively.

The metric of the Petrov type III space-time is

$$ds^2 = 2d\rho dv + \frac{3}{2}xd\rho^2 + \frac{v^2}{x^3}\left(dx^2 + dy^2\right) \tag{8.91}$$

with Homothetic algebra

$$\begin{aligned} K^1 &= \partial_\rho\ ,\ K^2 = \partial_y\ ,\ K^3 = v\partial_v - \rho\partial_\rho + 2x\partial_x + 2y\partial_y \\ H &= v\partial_v + \rho\partial_\rho\ \ (\psi_H = 1) \end{aligned}$$

where K^{1-4} are KVs and H is a nongradient HV.

In this space-time equation (8.1) takes the form:

$$-\frac{3}{2}xu_{vv} + 2u_{v\rho} + \frac{x^3}{v^2}\left(u_{xx} + u_{yy}\right) - 3\frac{x}{v}u_v + \frac{2}{v}u_\rho - u_t = 0. \tag{8.92}$$

From Theorem 6.5.2 we have that the extra Lie point symmetries are the vectors

$$X_{1-3} = K_{1-3}\ ,\ \ X_4 = 2t\partial_t + H$$

with nonzero commutators:

$$[X_t, X_4] = 2X_t\ ,\ \ [X_{2,}, X_3] = 2X_2$$

$$[X_3, X_1] = X_1\ ,\ \ [X_1, X_4] = X_1\ .$$

We use X_4 for reduction because this is the Lie point symmetry generated by the HV.

The zero order invariants of X_4 are

$$\alpha = \frac{v}{\sqrt{t}} \ , \ \beta = \frac{\rho}{\sqrt{t}} \ , \ \gamma = x \ , \ \delta = y \ , \ w = u.$$

We choose $w = w(\alpha, \beta, \gamma, \delta)$ as the dependent variable and we find that the reduced PDE is

$$_{III}\Delta w + \frac{\alpha}{2} w_\alpha + \frac{\beta}{2} w_\beta = 0 \tag{8.93}$$

where $_{III}\Delta$ is the Laplace operator for metric (8.91).

It is clear that the second order PDE (8.93) admits only the Lie symmetries X_2, X_3. Therefore, we do not have Type II hidden symmetries and the symmetries X_1, X_4 are Type I hidden symmetries.

8.9 Conclusion

Up to now in the literature the study of Type II hidden symmetries has been done by counter examples or by considering very special PDEs and in low dimensional flat spaces. In this chapter we have improved this scenario and have studied the problem of Type II hidden symmetries of second order PDEs from a geometric of view in n dimensional Riemannian spaces. We have considered the reduction of the Laplace and of the homogeneous heat equation and the consequent possibility of existence of Type II hidden symmetries in some general classes of spaces which admit some kind of symmetry; hence, they admit nontrivial Lie symmetries .

For the Laplace equation, the conclusion of this study is that the Type II hidden symmetries are directly related to the transition of the CKVs from the space where the original equation is defined to the space where the reduced equation resides. In this sense, we related the Lie point symmetries of PDEs with the basic collineations of the metric i.e. the CKVs. Concerning the general results of the reduction of Laplace equation we can summarize them as follows:

- If we reduce the Laplace equation with a gradient KV the reduced equation is a Laplace equation in the non-decomposable space. In this case, the Type II hidden symmetries are generated from the special and the proper CKVs of the non-decomposable space.

- If we reduce the Laplace equation with a gradient HV the reduced equation is a Laplace equation for an appropriate metric. In this case, the Type II hidden symmetries are generated from the HV and the special/proper CKVs.

- If we reduce the Laplace equation with the symmetry generated by a sp.CKV, the reduced equation is the Klein Gordon equation for an appropriate metric that inherits the Lie point symmetry generated by the gradient HV. In this case, the Type II hidden symmetries are generated from the proper CKVs.

We also considered the reduction of Laplace equation in some spaces of interest in which the metric does not admit the symmetries of the previous cases. In this context, we showed that the reduction with the non-gradient HV in the Petrov type III does not give any Type II hidden symmetries. Also, it is of interest the reduction of Laplace equation (i.e. the wave equation) in Minkowski spaces M^4 and M^3 where we recover the results of [100] in a straightforward manner without the need of computer software. Finally we considered an $n-$dimensional flat FRW like space and showed that the reduction with the proper CKV does not produce Type II hidden symmetries.

Moreover, we applied the zero order invariants of the Lie symmetries in order to reduce the number of independent variables of the homogeneous heat equation in certain general classes of Riemannian spaces, which admit some type of basic symmetry. For each reduction, we determined the origin of Type II hidden symmetries. The spaces we considered are the spaces which admit a gradient KV, a gradient HV and finally spacetimes which admits a HV which acts simply and transitively. For the reduction of the homogeneous heat equation and the existence of Type II hidden symmetries, we found the following general geometric results:

- If we reduce the homogeneous heat equation via the symmetries which are generated by a gradient KV $\left(S^{,i}\right)$ the reduced equation is a heat equation in the nondecomposable space. In this case we have the Type II hidden symmetry $\partial_t - \frac{1}{2t}w\partial_w$ provided we reduce the heat equation with the symmetry $tS^{,i} - \frac{1}{2}Su\partial_u$.

- If we reduce the homogeneous heat equation via the symmetries which are generated by a gradient HV the reduced equation is Laplace equation for an appropriate metric. In this case the Type II hidden symmetries are generated from the proper CKVs.

- In Petrov type III spacetime, the reduction of the homogeneous heat equation via the symmetry generated from the nongradient HV gives a PDE which inherits the Lie point symmetries, hence no Type II hidden symmetries are admitted.

The above results can be used in many important space-times and help facilitate the solution of the heat equation in these space-times.

8.A Proof of Corollary

Proof of Corollary 8.6.1. Using Theorem 6.5.1 and replacing q we have

For case a)

$$\begin{aligned} -a_\tau w + H(b) - \frac{1}{2\tau}(aw+b) + \frac{a}{2\tau}w - \left(\psi c_2 w + \frac{1}{2\tau}wc_1\right)_\tau &= 0. && (8.94)\\ -a_\tau w + H(b) - \frac{1}{2\tau}b + \frac{1}{2\tau^2}wc_1 &= 0 && (8.95)\\ \left[-a_\tau + \frac{c_1}{2\tau^2}\right]w + \left[H(b) - \frac{1}{2\tau}b\right] &= 0 && (8.96)\end{aligned}$$

that is

$$a = -\frac{c_1}{2\tau} + a_0 \ , \quad H(b) - \frac{1}{2\tau}b = 0 \tag{8.97}$$

For case b)

$$0 = \left(-\frac{1}{2}T_{,\tau}\psi + \frac{1}{2}T_{,\tau\tau}S - F_{,\tau}\right)w - \left(2\psi q\int T d\tau\right)_\tau - Tq_{,i}S^{,i}.$$

then

$$0 = \left(-\frac{1}{2}T_{,\tau}\psi + \frac{1}{2}T_{,\tau\tau}S - F_{,\tau}\right)w + \frac{\psi}{\tau^2}\int T d\tau \ w - \frac{\psi}{\tau}Tw \tag{8.98}$$

from here we have

$$T_{,\tau\tau} = 0 \rightarrow \ T = T_0\tau + T_1 \tag{8.99}$$

and

$$F = -T_0\psi\tau.$$

■

8.B The homogeneous heat equation in the Petrov spacetimes

In the following subsections, we study the reduction of the homogeneous heat equation in the Petrov spacetimes of type N,D and II.

8.B.1 Petrov type N

The metric of the Petrov type N space-time is

$$ds^2 = dx^2 + x^2dy^2 + 2d\rho dv + \ln x^2 d\rho^2 \tag{8.100}$$

and has the homothetic algebra [120]

$$\begin{aligned} K^1 &= \partial_\rho \ , \ K^2 = \partial_v \ , \ K^3 = \partial_y \\ H &= x\partial_x + \rho\partial_\rho + (v-2\rho)\partial_v \quad (\psi_H = 1)\end{aligned}$$

where K^{1-3} are KVs and H is a nongradient HV.

The heat equation (8.1) in this space-time is

$$u_{xx}+\frac{1}{x^2}u_{yy}+2u_{\rho v}-2\ln x^2\ u_{vv}+\frac{1}{x}u_x-u_t=0. \tag{8.101}$$

Application of Theorem 6.5.2 gives that the extra Lie point symmetries of (8.101) are

$$X_{1-3}=K_{1-3}\ \ ,\ X_4=2t\partial_t+H$$

with nonzero commutators

$$[X_t,X_4]=2X_t$$

$$[X_1,X_4]=X_1-2X_2\ ,\ \ [X_2,X_4]=X_2.$$

We use X_4 to reduce the PDE because this is the Lie symmetry generated by the HV. The zero order invariants of X_4 are

$$\alpha=\frac{x}{\sqrt{t}}\ ,\ \beta=\frac{\rho}{\sqrt{t}}\ ,\ \gamma=\frac{v+\rho\ln(t)}{\sqrt{t}}\ ,\ \delta=y\ ,\ w=u. \tag{8.102}$$

Choosing $\alpha,\beta,\gamma,\delta$ as the independent variables and $w=w(\alpha,\beta,\gamma,\delta)$ as the dependent variable we find that the reduced PDE is

$$_N\Delta w+\left(\frac{1}{2}\alpha w_\alpha+\frac{1}{2}\beta w_\beta+\left(\frac{1}{2}\gamma-\beta\right)w_\gamma\right)=0. \tag{8.103}$$

where $_N\Delta$ is the Laplace operator for the metric (8.100).

Equation (8.103) is of the form (6.14) with

$$A_{ij}=g_{ij}\left(x^k\right),B^i\left(x^k\right)=\Gamma^i+\frac{1}{2}\alpha\delta^i_\alpha+\frac{1}{2}\beta\delta^i_\beta+\left(\frac{1}{2}\gamma-\beta\right)\delta^i_\gamma,\ f\left(x^k,u\right)=0$$

where g_{ij} is the metric (8.100). Replacing in equations (6.21)-(6.25) we obtain the Lie symmetry conditions for (8.103). Because $A_{ij,u}=0$ it follows from equation (6.23) that the Lie point symmetries are generated from the CKVs of the metric (8.100). However taking into consideration the rest of the symmetry conditions we find that the only Lie symmetry which remains is the one of the KV X_3. We conclude that in this reduction we do not have Type II hidden symmetries.

8.B.2 Petrov type D

The metric of the Petrov type D space-time is

$$ds^2=-dx^2+x^{-\frac{2}{3}}dy^2-x^{\frac{4}{3}}\left(d\rho^2+dz^2\right) \tag{8.104}$$

with Homothetic algebra

$$\begin{aligned}K^1 &= \partial_\rho\ ,\ K^2=\partial_z\ ,\ K^3=\partial_y\ ,\ K^4=z\partial_\rho-\rho\partial_z\\ H &= x\partial_x+\frac{4}{3}y\partial_y+\frac{z}{3}\partial_z+\frac{\rho}{3}\partial_\rho\quad(\psi_H=1)\end{aligned}$$

where K^{1-4} are KVs and H is a nongradient HV.

In this space-time the heat equation (8.1) takes the form:

$$-u_{xx}+x^{\frac{2}{3}}u_{yy}-x^{-\frac{4}{3}}\left(u_{\rho\rho}+u_{zz}\right)-\frac{1}{x}u_{x}-u_{t}=0. \tag{8.105}$$

From Theorem 6.5.2 we have that the extra Lie point symmetries are the vectors

$$X_{1-4}=K_{1-4}\ ,\quad X_{5}=2t\partial_{t}+H.$$

with nonzero commutators:

$$\begin{aligned}
\left[X_{t},X_{5}\right] &= 2X_{t}\\
\left[X_{1},X_{5}\right] &= \frac{1}{3}X_{1}\ ,\ \left[X_{4},X_{1}\right]=-X_{2}\\
\left[X_{2},X_{4}\right] &= X_{1}\ ,\ \left[X_{2},X_{5}\right]=\frac{1}{3}X_{2}\\
\left[X_{3},X_{5}\right] &= \frac{4}{3}X_{3}.
\end{aligned}$$

We use X_5 to reduce the PDE because this is the Lie symmetry generated by the HV. The zero order invariants of X_5 are

$$\alpha=\frac{x}{t^{\frac{1}{2}}}\ ,\ \beta=\frac{y}{t^{\frac{2}{3}}}\ ,\ \gamma=\frac{\rho}{t^{\frac{1}{6}}}\ ,\ \delta=\frac{z}{t^{\frac{1}{6}}}\ ,\ w=u.$$

We choose $\alpha,\beta,\gamma,\delta$ as the independent variables and $w=w\left(\alpha,\beta,\gamma,\delta\right)$ as the dependent variable and we find that the reduced PDE is

$$_{D}\Delta w+\left(\frac{1}{2}aw_{\alpha}+\frac{2}{3}\beta w_{\beta}+\frac{1}{6}\gamma w_{\gamma}+\frac{1}{6}\delta w_{\delta}\right)=0 \tag{8.106}$$

where $_{D}\Delta$ is the Laplace operator with metric (8.104).

Working again with the Lie symmetry conditions (6.21)-(6.25) we find that equation (8.106) admits as Lie point symmetry the vector X_4 only which is an inherited symmetry. Hence we do not have Type II hidden symmetries. Obviously the Lie point symmetries X_{1-4} are Type I hidden symmetries for equation (8.106) for the reduction by X_5.

8.B.3 Petrov type II

The metric of the Petrov type II space-time is

$$ds^{2}=\rho^{-\frac{1}{2}}\left(d\rho^{2}+dz^{2}\right)-2\rho dxdy+\rho\ln\rho\ dy^{2} \tag{8.107}$$

with homothetic algebra

$$\begin{aligned}
K^{1} &= \partial_{x}\ ,\ K^{2}=\partial_{y}\ ,\ K^{3}=\partial_{z}\\
H &= \frac{1}{3}\left(x+2y\right)\partial_{x}+\frac{1}{3}y\partial_{y}+\frac{4}{3}z\partial_{z}+\frac{4}{3}\rho\partial_{\rho}\ \left(\psi_{H}=1\right)
\end{aligned}$$

where K^{1-4} are KVs and H is a nongradient HV.

In this space-time equation (8.1) takes the form:

$$\rho^{\frac{1}{2}}\left(u_{\rho\rho}+u_{zz}\right)-\frac{1}{\rho}\varepsilon\ln\rho u_{xx}-\frac{2}{\rho}u_{xy}+\rho^{-\frac{1}{2}}u_{\rho}-u_{t}=0. \tag{8.108}$$

From Theorem 6.5.2 we have that the extra Lie point symmetries are the vectors

$$X_{1-3}=K_{1-3}\ ,\quad X_{4}=2t\partial_{t}+H$$

with nonzero commutators:

$$[X_t,X_4]=2X_t$$

$$\begin{aligned}[X_1,X_4] &= \frac{1}{3}X_1\ ,\ \ [X_3,X_4]=\frac{4}{3}X_3\\ [X_2,X_4] &= \frac{2}{3}X_1+\frac{1}{3}X_2.\end{aligned}$$

We use X_4 to reduce the PDE because this is the Lie symmetry generated by the HV. The zero order invariants of X_4 are

$$\alpha=\frac{\rho}{t^{\frac{2}{3}}}\ ,\ \beta=\frac{z}{t^{\frac{2}{3}}}\ ,\ \gamma=\frac{x-\frac{1}{3}y\ln(t)}{t^{\frac{1}{6}}}\ ,\ \delta=\frac{y}{t^{\frac{1}{6}}}\ \ ,\ w=u$$

We choose $\alpha,\beta,\gamma,\delta$ as the independent variables and $w=w(\alpha,\beta,\gamma,\delta)$ as the dependent variable and we find that the reduced PDE is

$$_{II}\Delta w+\frac{2}{3}aw_{\alpha}+\frac{2}{3}\beta w_{\beta}+\left(\frac{1}{3}\delta-\frac{1}{6}\gamma\right)w_{\gamma}+\frac{1}{6}\delta w_{\delta}=0 \tag{8.109}$$

where $_{II}\Delta$ is the Laplace operator for metric (8.107).

From the Lie symmetry conditions (6.21)-(6.25) it follows that (8.109) does not admit any Lie point symmetries. Hence, we do not have Type II hidden symmetries in this case.

Part IV

Noether symmetries and theories of gravity

Chapter 9

Noether symmetries in Scalar field Cosmology

9.1 Introduction

The detailed analysis of the cosmological data indicate that the Universe is spatially flat and has suffered two acceleration phases. An early acceleration phase (inflation), which occurred prior to the radiation-dominated era, and a recently initiated accelerated expansion [125, 126, 127, 128, 129, 130, 131]. The source for the late time cosmic acceleration has been attributed to an unidentified type of matter, the dark energy. Dark energy contrary to the ordinary baryonic matter has a negative pressure, i.e. a negative equation of state parameter, which counteracts the gravitational force and leads to the observed accelerated expansion.

The simplest dark energy probe is the cosmological constant leading to the ΛCDM cosmology [132, 133, 134]. However, it has been shown that ΛCDM cosmology suffers from two major drawbacks known as the fine tuning problem and the coincidence problem [135]. Besides ΛCDM cosmology, many other candidates have been proposed in the literature. Most are based either on the existence of new fields (i.e. a scalar field) or in some modification of the Einstein Hilbert action (see [136]).

In addition to dark energy and the ordinary baryonic matter, it is believed that the Universe contains a third type of matter, the dark matter. This type of matter is assumed to be pressureless (non-relativistic) and interacts very weakly with the standard baryonic matter. Therefore, its presence is mainly inferred from gravitational effects on visible matter.

In the following, we consider scalar field cosmology (minimally coupled scalar field and non minimally coupled scalar field) and we propose a geometric principle ('selection rule') for specifying the potential $V(\phi)$ and the coupling function $F(\phi)$ of the scalar field in order to solve analytically the system of the resulting field equations. We propose that the scalar field model should be selected by the geometric requirement that the dynamical system of the field equations admits Noether point symmetries. The main reason for the consideration of this hypothesis is that the Noether symmetries provide first integrals, which assist the integrability of the system. Furthermore, as we saw in chapter 4 the Noether symmetries are generated from the elements of the homothetic algebra of the kinetic metric of the Lagrangian of the theory. Therefore, with this assumption we let the theory to select the potential, i.e. the dark energy model.

The idea to use Noether symmetries in cosmology, either on scalar field models and on modified theories of gravity is not new and indeed a lot of attention has been paid in the literature [137, 138, 107, 139, 140, 141, 142, 105, 86, 109, 102, 143, 144, 145, 146, 147, 148, 149]. However our approach is geometric and more fundamental.

The structure of the present chapter is as follows. In sections 9.2 and 9.3 we discuss the conformal equivalence of Lagrangians for scalar fields in a Riemannian space of dimension 4 and n respectively. In particular we enunciate a theorem which proves that the field equations for a non-minimally coupled scalar field are the same at the conformal level with the field equations of the minimally coupled scalar field. The necessity to preserve Einstein's equations in the context of Friedmann–Robertson–Walker (FRW) space-time leads us to apply, in section 9.4, the current general analysis to the scalar field (quintessence or phantom) spatially flat FRWcosmologies.

In section 9.5 we apply the Noether symmetry approach in non minimally coupled scalar field in a spatially flat FRW spacetime and by using the Noether invariants we determine analytical solutions for the field equations. Furthermore in sections 9.6 and 9.7 we apply the same procedure for a minimally coupled scalar field in a spatially flat FRW spacetime and in Bianchi Class A homogeneous spacetimes.

9.2 Conformally equivalent Lagrangians and scalar field Cosmology

In this section we discuss the conformal equivalence of Lagrangians for scalar fields in a general V^4 Riemannian space. The field equations in the scalar field cosmology are derived from two different variational principles. In the first case the scalar field ϕ and the gravitational field are minimally coupled and the equations of motion follow form the action

$$S_M = \int d\tau dx^3 \sqrt{-g}\left[R + \frac{1}{2} g_{ij}\phi^{;i}\phi^{;j} - V\left(\phi\right)\right]. \tag{9.1}$$

In the second case the scalar field ψ (which is different from the minimally coupled scalar field ϕ) interacts with the gravitational field (non minimal coupling) and the field equations follow from the action

$$S_{NM} = \int d\tau dx^3 \sqrt{-g}\left[F\left(\psi\right)R + \frac{1}{2}\bar{g}_{ij}\psi^{;i}\psi^{;j} - \bar{V}\left(\psi\right)\right] \tag{9.2}$$

where $F(\psi)$ is the coupling function between the gravitational and the scalar field ψ. Below we state the following proposition.

Proposition 9.2.1 *The field equations for a non minimally coupled scalar field ψ with Lagrangian $\bar{L}\left(\tau, x^k, \dot{x}^k\right)$ and coupling function $F(\psi)$ in the gravitational field $\bar{g}_{ij}$ are the same with the field equations of the minimally coupled scalar field Ψ for a conformal Lagrangian $L\left(\tau, x^k, \dot{x}^k\right)$ in the conformal metric $g_{ij} = N^{-2}\bar{g}_{ij}$, where the conformal function $N = \frac{1}{\sqrt{-2F(\psi)}}$ with $F(\psi) < 0$. The inverse is also true, that is, to a minimally coupled scalar field it can be associated a unique non minimally coupled scalar field in a conformal metric and with a different potential function.*

Proof. The action for the non minimally coupled Lagrangian $\bar{L}\left(\tau, x^k, x'^k\right)$ is:

$$S_{NM} = \int d\tau dx^3 \sqrt{-\bar{g}} \left[F(\psi)\bar{R} + \frac{\varepsilon}{2}\bar{g}^{ij}\psi_{;i}\psi_{;j} - \bar{V}(\psi)\right] \tag{9.3}$$

where $\varepsilon = 1$ for a real field and $\varepsilon = -1$ for phantom field. Let g_{ij} be the conformally related metric (this is not a coordinate transformation!):

$$g_{ij} = N^{-2}\bar{g}_{ij}.$$

Then the action (9.3) becomes[1]:

$$S_{NM} = \int d\tau dx^3 N^4 \sqrt{-g} \left[F(\psi)\bar{R} + \frac{\varepsilon}{2}N^{-2}g^{ij}\psi_{;i}\psi_{;j} - \bar{V}(\psi)\right].$$

Replacing [150]

$$\bar{R} = N^{-2}R - 2(n-1)N^{-3}\Delta_2 N$$

where $\Delta_2 N = g_{ij}N^{;ij}$ we find:

$$\begin{aligned} S_{NM} &= \int d\tau dx^3 N^4 \sqrt{-g} \left[F(\psi)\left(N^{-2}R - 6N^{-3}\Delta_2 N\right) + \frac{\varepsilon}{2}N^{-2}\Delta_1\psi - \bar{V}(\psi)\right] \\ &= \int d\tau dx^3 N^4 \sqrt{-g} \left[F(\psi)N^{-2}R - 6F(\psi)N^{-3}\Delta_2 N + \frac{\varepsilon}{2}N^{-2}\Delta_1\psi - \bar{V}(\psi)\right] \\ &= \int d\tau dx^3 \sqrt{-g} \left[F(\psi)N^2 R - 6F(\psi)N\Delta_2 N + \frac{\varepsilon}{2}N^2\Delta_1\psi - N^4\bar{V}(\psi)\right]. \end{aligned}$$

Define the conformal function in terms of the coupling function $F(\psi)$ by the requirement ($F(\psi) < 0$):

$$N = \frac{1}{\sqrt{-2F(\psi)}}\ . \tag{9.5}$$

We compute

$$N_{;i} = \frac{F_\psi \psi_{;i}}{(-2F)^{\frac{3}{2}}}. \tag{9.6}$$

[1] We use the result tha if $A = (a_{ij})$ is a $4x4$ matrix the

$$\det A = \varepsilon^{ijkl}a_{ij}a_{kl}$$

hence

$$\bar{g} = \varepsilon^{ijkl}\bar{g}_{ij}\bar{g}_{kl} = N^4 g. \tag{9.4}$$

Then the first term in the integral becomes:

$$\int d\tau dx^3 \sqrt{-g} F(\psi) N^2 R = \int d\tau dx^3 \sqrt{-g} \left(-\frac{R}{2}\right).$$

The second term gives after integration by parts:

$$\begin{aligned}
\int d\tau dx^3 \sqrt{-g} \left(-6F(\psi) N \Delta_2 N\right) &= \int d\tau dx^3 \sqrt{-g} \left(-6\frac{F}{\sqrt{-2F}} N_{;ij} g^{ij}\right) \\
&= \int d\tau dx^3 \sqrt{-g} \left(-6\frac{F}{\sqrt{-2F}} \frac{1}{\sqrt{-g}} \left(\sqrt{-g} g^{ij} N_{,k}\right)_{,j}\right) \\
&= \int d\tau dx^3 \left(-6\frac{F}{\sqrt{-2F}} \left(\sqrt{-g} g^{ij} N_{,k}\right)_{,j}\right) \\
&= \int d\tau dx^3 \sqrt{-g} \left(3\frac{F_\psi}{\sqrt{-2F}} \psi_{;j} N_{;i} g^{ij}\right).
\end{aligned}$$

Replacing $N_{;i}$ from (9.6) we find:

$$\int d\tau dx^3 \sqrt{-g} \left(-6F(\psi) N \Delta_2 N\right) = \int d\tau dx^3 \sqrt{-g} \left(3\frac{F_\psi^2}{(-2F)^2} \psi_{;i} \psi_{;j} g^{ij}\right).$$

The third term gives:

$$\frac{\varepsilon}{2} N^2 \Delta_1 \psi = \frac{\varepsilon}{4F} \psi_{;i} \psi_{;j} g^{ij}$$

Collecting all the results we find:

$$\begin{aligned}
S_{MN} &= \int d\tau dx^3 \sqrt{-g} \left[-\frac{R}{2} + 3\frac{F_\psi^2}{(-2F)^2} \psi_{;i} \psi_{;j} g^{ij} - \frac{\varepsilon}{4F} \psi_{;i} \psi_{;j} g^{ij} - \frac{\bar{V}(\psi)}{4F^2}\right] \\
&= \int d\tau dx^3 \sqrt{-g} \left[-\frac{R}{2} + 3\frac{F_\psi^2}{4F^2} \psi_{;i} \psi_{;j} g^{ij} - \frac{\varepsilon}{4F} \psi_{;i} \psi_{;j} g^{ij} - \frac{\bar{V}(\psi)}{4F^2}\right]
\end{aligned}$$

or

$$S_{MN} = \int d\tau dx^3 \sqrt{-g} \left[-\frac{R}{2} + \frac{\varepsilon}{2} \left(\frac{3\varepsilon F_\psi^2 - F}{2F^2}\right) \psi_{;i} \psi_{;j} g^{ij} - \frac{\bar{V}(\psi)}{4F^2}\right]. \tag{9.7}$$

We introduce the scalar field Ψ with the requirement:

$$d\Psi = \sqrt{\left(\frac{3\varepsilon F_\psi^2 - F}{2F^2}\right)} d\psi \tag{9.8}$$

and the action becomes

$$S_{MN} = \int d\tau dx^3 \sqrt{-g} \left[-\frac{R}{2} + \frac{\varepsilon}{2} \Psi_{;i} \Psi_{;j} g^{ij} - \frac{\bar{V}(\Psi)}{4F(\Psi)^2}\right]. \tag{9.9}$$

■

We conclude that the scalar field Ψ is minimally coupled to the gravitational field. Therefore we have proved that to every non minimally coupled scalar field we may associate a unique minimally coupled scalar field in a conformally related space and an appropriate potential. Since all considerations are reversible, the result is reversible.

In the following section we extend the above proposition to a general V^n Riemannian space.

9.3 Generalization to dimension n

Consider the non minimally coupled scalar field ψ whose field equations are obtained from the action:

$$
\begin{aligned}
S_{NM} &= \int dx^n N^n \sqrt{-g}\left[F(\psi)\bar{R} + \frac{\varepsilon}{2}N^{-2}g^{ij}\psi_{,i}\psi_{,j} - \bar{V}(\psi)\right] \\
&= \int dx^n \sqrt{-g}\left[\begin{array}{c} F(\psi)N^{n-2}R - 2(n-1)F(\psi)N^{n-3}\Delta_2 N + \\ -F(\psi)N^n(n-1)(n-4)\Delta_1 N + \frac{\varepsilon}{2}N^{n-2}g^{ij}\psi_{,i}\psi_{,j} - N^n\bar{V}(\psi) \end{array}\right]
\end{aligned}
$$

where we have substituted [150]

$$\bar{R} = N^{-2}R - 2(n-1)N^{-3}\Delta_2 N - (n-1)(n-4)\Delta_1 N$$

and

$$
\begin{aligned}
\Delta_1 N &= g_{ij}N^{,i}N^{,j} \\
\Delta_2 N &= g_{ij}N^{;ij}.
\end{aligned}
$$

Define the function $N(x^i)$ in terms of the coupling function $F(\psi)$ by the requirement:

$$N^{n-2} = \frac{1}{-2F}. \; , \; F = -\frac{N^{2-n}}{2}.$$

For each term of the action S_{NM} we have the following:
The first term gives:

$$\int dx^n \sqrt{-g}\left(F(\psi)N^{n-2}R\right) = \int dx^n \sqrt{-g}\left(-\frac{R}{2}\right).$$

The second term gives:

$$
\begin{aligned}
\int dx^n \sqrt{-g}\left(-2(n-1)F(\psi)N^{n-3}N_{;ij}g^{ij}\right) &= \int dx^n \sqrt{-g}\left((n-1)N^{2-n}N^{n-3}N_{;ij}g^{ij}\right) \\
&= \int dx^n \sqrt{-g}\left((n-1)N^{-1}\frac{1}{\sqrt{-g}}\left(\sqrt{-g}g^{ij}N_{,k}\right)_{,j}\right) \\
&= \int dx^n \sqrt{-g}\left((n-1)N^{-1}N_{;ij}g^{ij}\right) \\
&= \int dx^n \sqrt{-g}\left(-(n-1)\left(N^{-1}\right)_{;j}N_{;i}g^{ij}\right).
\end{aligned}
$$

Furthermore we compute

$$
\begin{aligned}
N &= \frac{1}{(-2F)^{\frac{1}{n-2}}} \rightarrow N^{-1} = (-2F)^{\frac{1}{n-2}} \\
N_{;i}N^{-1}_{;j} &= -\frac{F_\psi^2}{(n-2)^2 F^2}\psi_{;i}\psi_{;j}.
\end{aligned}
$$

Replacing we find for the second term:

$$\int dx^n \sqrt{-g}\left(-2(n-1)F(\psi)N^{n-3}N_{;ij}g^{ij}\right) = \int dx^n \sqrt{-g}\left(\frac{(n-1)}{(n-2)^2}\frac{F_\psi^2}{F^2}\psi_{;i}\psi_{;j}g^{ij}\right).$$

(note that this is the same with the previous expression for $n=4$).

The third term gives:

$$\begin{aligned}\int dx^n \sqrt{-g}\left(F(\psi)N^n(n-1)(n-4)\Delta_1 N\right) &= \int dx^n \sqrt{-g}\left(-\frac{N^{2-n}}{2}N^n(n-1)(n-4)\Delta_1 N\right)\\ &= \int dx^n \sqrt{-g}\left(-\frac{1}{2}N^2(n-1)(n-4)\Delta_1 N\right)\\ &= \int dx^n \sqrt{-g}\left(-\frac{1}{2}\frac{(n-1)(n-4)}{(n-2)^2}\frac{F_\psi^2}{(-2F)^{\frac{4}{2-n}}F^2}\psi_{;i}\psi_{;j}g^{ij}\right).\end{aligned}$$

Finally the the fourth term gives:

$$\int dx^n \sqrt{-g}\left(\frac{\varepsilon}{2}N^{n-2}g^{ij}\psi_{,i}\psi_{,j}\right) = \int dx^n \sqrt{-g}\left(-\frac{\varepsilon}{4}\frac{1}{F}g^{ij}\psi_{;i}\psi_{;j}\right).$$

Collecting the results for the last three terms we find

$$\begin{aligned}&\int dx^n \sqrt{-g}\left(\frac{(n-1)}{(n-2)^2}\frac{F_\psi^2}{F^2}\psi_{;i}\psi_{;j}g^{ij} - \frac{1}{2}\frac{(n-1)(n-4)}{(n-2)^2}\frac{F_\psi^2}{(-2F)^{\frac{4}{2-n}}F^2}\psi_{;i}\psi_{;j}g^{ij} - \varepsilon\frac{1}{4F}g^{ij}\psi_{;i}\psi_{;j}\right)\\ =&\int dx^n \sqrt{-g}\left(\frac{(n-1)}{(n-2)^2}\frac{F_\psi^2}{F^2} - \frac{1}{2}\frac{(n-1)(n-4)}{(n-2)^2}\frac{F_\psi^2}{(-2F)^{\frac{4}{2-n}}F^2} - \varepsilon\frac{1}{4F}\right)\psi_{;i}\psi_{;j}g^{ij}\\ =&\int dx^n \sqrt{-g}\frac{\varepsilon}{2}\left(\frac{2\varepsilon(n-1)}{(n-2)^2}\frac{F_\psi^2}{F^2} - \varepsilon\frac{(n-1)(n-4)}{(n-2)^2}\frac{F_\psi^2}{(-2F)^{\frac{4}{2-n}}F^2} - \frac{1}{2F}\right)\psi_{,i}\psi_{,j}g^{ij}.\end{aligned}$$

We define the new scalar field Ψ with the requirement

$$d\Psi = \left(\frac{2\varepsilon(n-1)}{(n-2)^2}\frac{F_\psi^2}{F^2} - \varepsilon\frac{(n-1)(n-4)}{(n-2)^2}\frac{F_\psi^2}{(-2F)^{\frac{4}{2-n}}F^2} - \frac{1}{2F}\right)^{\frac{1}{2}} d\psi.$$

In terms of Ψ the action becomes

$$\begin{aligned}&\int dx^n \sqrt{-g}\frac{\varepsilon}{2}\left(\frac{2\varepsilon(n-1)}{(n-2)^2}\frac{F_\psi^2}{F^2} - \varepsilon\frac{(n-1)(n-4)}{(n-2)^2}\frac{F_\psi^2}{(-2F)^{\frac{4}{2-n}}F^2} - \frac{1}{2F}\right)\psi_{,i}\psi_{,j}g^{ij}.\\ =&\int dx^n \sqrt{-g}\left(\frac{\varepsilon}{2}\Psi_{;i}\Psi_{;j}g^{ij}\right).\end{aligned}$$

Collecting the above we find the action S_M of a minimally coupled scalar field

$$S_M = \int dx^n \sqrt{-g}\left(-\frac{R}{2} + \frac{\varepsilon}{2}\Psi_{;i}\Psi_{;j}g^{ij} - \frac{\bar{V}(\Psi)}{(-2F)^{\frac{n}{n-2}}}\right).$$

We note that the new scalar field Ψ is minimally coupled to the gravitational field g_{ij} and that the potential of Ψ is $\frac{\bar{V}(\Psi)}{(-2F)^{\frac{n}{n-2}}}$.

The above proof agrees with the one given in the paper of [151]. However it is obviously simpler, more direct and clear.

9.4 Conformal Lagrangians in scalar field cosmology

In this section we apply the conformal transformation in the Lagrangian of the field equations in a FRW spatially flat spacetime.

We consider the flat FRW ($K=0$) spacetime whose metric is

$$ds^2 = -dt^2 + a^2(t)\,\delta_{ij}dx^i dx^j \tag{9.10}$$

where δ_{ij} is the 3-space metric in Cartesian coordinates. The Lagrangian of a scalar field ϕ minimally coupled to gravity in these coordinates is

$$L_M\left(a,\dot{a},\phi,\dot{\phi}\right) = -3a\dot{a}^2 + \frac{\varepsilon}{2}a^2\dot{\phi}^2 - a^3V(\phi). \tag{9.11}$$

The Lagrangian for a non minimally coupled scalar field ψ is

$$L_{NM}\left(a,\dot{a},\psi,\dot{\psi}\right) = 6F(\psi)\,a\dot{a}^2 + 6F_{\psi}(\psi)\,a^2\dot{a}\dot{\psi} + \frac{\varepsilon}{2}a^3\dot{\psi}^2 - a^3V(\psi) \tag{9.12}$$

where $F(\psi)<0$ is the coupling function and $\varepsilon=\pm1$ where $\varepsilon=1$ for real scalar field and $\varepsilon=-1$ for phantom field. The Hamiltonian for the Lagrangian (9.12) is

$$E = 6F(\psi)\,a\dot{a}^2 + 6F_{\psi}(\psi)\,a^2\dot{a}\dot{\psi} + \frac{\varepsilon}{2}a^3\dot{\psi}^2 + a^3V(\psi). \tag{9.13}$$

We construct a conformal Lagrangian which corresponds to a minimally coupled scalar field.

To do that we consider first a change in the scale factor from $a(t)\to A(t)$ defined by the formula (see [152, 140])

$$A(t) = \sqrt{-2F}\,a(t) \tag{9.14}$$

Then the Lagrangian (9.12) takes the form:

$$L_{NM}\left(A,\dot{A},\psi,\dot{\psi}\right) = \frac{1}{\sqrt{-2F}}\left[-3A\dot{A}^2 + \frac{\varepsilon}{2}\left(\frac{3\varepsilon F_{\psi}^2 - F}{2F^2}\right)A^3\ \dot{\psi}^2\right] - \frac{A^3}{(-2F)^{\frac{3}{2}}}V(\psi) \tag{9.15}$$

that is, the cross term $\dot{a}\dot{\psi}$ disappears.

Next we consider the coordinate transformation:

$$d\Psi = \sqrt{\left(\frac{3\varepsilon F_{\psi}^2 - F}{2F^2}\right)}\,d\psi \tag{9.16}$$

and Lagrangian (9.15) becomes:

$$L_{NM}(A,A',\Psi,\Psi') = \frac{1}{\sqrt{-2F}}\left[-3A\dot{A}^2 + \frac{\varepsilon}{2}A^3\ \dot{\Psi}^2\right]\ ,\ \bar{V}(\psi) = \frac{V(\Psi)}{(-2F)^{\frac{3}{2}}} \tag{9.17}$$

The form of the Lagrangian (9.17) is (9.11) hence the previous result applies and under the conformal transformation

$$d\tau = \sqrt{-2F(\psi)}dt \tag{9.18}$$

the Lagrangian (9.17) becomes:

$$L_M\left(A,\dot{A},\Psi,\Psi'\right)=-3AA'^2+\frac{\varepsilon}{2}A^3\Psi'^2-\frac{A^3}{(-2F)^{\frac{3}{2}}}V(\Psi) \tag{9.19}$$

where a prime $'$ indicates derivative wrt the new coordinate τ.

We note that if in the new coordinates τ, x^i we consider the metric

$$d\bar{s}^2=-d\tau^2+A^2(\tau)\,\delta_{ij}dx^idx^j \tag{9.20}$$

then the term $3AA'^2$ equals the Ricci scalar $\bar{R}$ of the conformally flat metric $d\bar{s}^2$. Therefore the Lagrangian (9.19) can be seen as the Lagrangian of a scalar field Ψ of potential $\bar{V}(\Psi)$ minimally coupled to the gravitational filed $\bar{g}_{ij}$ in the space with metric $d\bar{s}^2$. Replacing the coordinate τ and the quantity $A(\tau)$ from (9.14), (9.18), we find:

$$d\bar{s}^2=\sqrt{-2F}(-dt^2+a^2(t)\,\delta_{ij}dx^idx^j)=\sqrt{-2F}ds^2 \tag{9.21}$$

that is, the metric $d\bar{s}^2$ is conformally related to the metric ds^2 with conformal function $\sqrt{-2F}$. This means that the non-minimally coupled scalar field in the gravitational field ds^2 is equivalent to a minimally coupled scalar field - with appropriate potential defined in terms of the conformal function - in the gravitational field $d\bar{s}$, the resutl is reversible. Equivalently the Lagrangians L_M, L_{NM} are conformally related and the field equations (the Euler-Lagrange equations) are invariant under the conformal transformation if the Hamiltonian constrains H_M, H_{NM} vanish (see Lemma 7.2.2).

In the following sections we apply the Noether symmetry approach as a geometric selection rule in order to determine the dark energy models; that is, we search for dark energy models by requiring the field equations to admit Noether point symmetries.

9.5 Noether point symmetries of a non minimally coupled Scalar field.

Consider a non-minimally coupled scalar field with action

$$S_{NM}=\int d\tau dx^3\sqrt{-g}\left[F(\psi)R+\frac{\varepsilon}{2}\bar{g}_{ij}\psi^{;i}\psi^{;j}-\bar{V}(\psi)\right]+\int L_m d\tau dx^3$$

in a flat FRW space-time, whose metric in Cartesian coordinates is

$$ds^2=-dt^2+a^2(t)\,\delta_{ij}dx^idx^j$$

and L_m is the Lagrangian of dust matter of density ρ_D (for commoving observers).

The Lagrangian of the field equations is (9.12) are the Hamiltonian (total energy density) (9.13) and the Euler-Lagrange equations

$$\frac{d}{dt}\frac{\partial L_{NM}}{\partial\left(\dot{a},\dot{\psi}\right)}-\frac{\partial L}{\partial(a,\psi)}=0.$$

If the Hamiltonian (9.13) is $E \neq 0$ then the space admits dust which, however does not interact with the scalar field and has energy density $\rho_D = \frac{|E|}{a^3}$.If $E = 0$ the space does not admit dust. In the following we determine all potentials $V(\psi)$ for which this dynamical system admits Noether point symmetries beyond the trivial one ∂_t. Subsequently we use the resulting Noether integrals to find analytical solutions for the field equations for each of these potentials.

In order to determine the Noether point symmetries of the Lagrangian (9.12) we shall follow the results of chapter 4. That is we brake the Lagrangian in the kinematic part which defines the kinematic metric and the remaining part which we consider to be the potential. Then we apply theorem 4.3.2 which states that the Noether point symmetries of the Lagrangian follow form the homothetic algebra of the kinematic metric. The kinematic metric admits a non-trivial homothetic (not necessarily proper homothetic) algebra if a given condition is satisfied which involves the symmetry vector and the potential. The solution of this relation provides all the potentials for which extra Noether symmetries are admitted. We use the Noether integrals of these extra Noether symmetries to find an analytic solution for each of the corresponding potentials..

From the Lagrangian (9.12) we define the kinematic metric:

$$ds_{KM}^2 = 12F(\psi)\, a\dot{a}^2 + 12F_{\psi}(\psi)\, a^2\dot{a}\dot{\psi} + \varepsilon a^3\dot{\psi}^2. \tag{9.22}$$

This is a two dimensional metric in the space (a, ψ). Because the homothetic algebra of a 2 dimensional metric is different for a flat and a non-flat (but conformally flat because all two dimensional spaces are conformally flat) we consider the case the metric (9.22) is maximally symmetric[2].

The Ricci scalar of the metric (9.22) is computed to be:

$$R_{(KM)} = \frac{\varepsilon}{4a^3}\frac{\left(2F_{\psi\psi}F - F_{\psi}^2\right)}{\left(\varepsilon F - 3F_{\psi}^2\right)^2} \tag{9.23}$$

Hence if the metric (9.22) is maximally symmetric then it follows that $R_{(KM)} = 0$, that is, the kinetic metric (9.22) must be flat.

9.5.1 Case A. $R_{KM} = 0$

Condition $R_{KM} = 0$ and (9.23) give:

$$2F_{\psi\psi}F - F_{\psi}^2 = 0 \tag{9.24}$$

provided

$$\varepsilon F - 3F_{\psi}^2 \neq 0. \tag{9.25}$$

The solution of (9.24) is

$$F(\psi) = -\frac{F_0\varepsilon}{12}(\psi + \psi_0)^2 \tag{9.26}$$

[2] The Ricci scalar is $R = K$ where K is a constant

where $\varepsilon = \pm 1$ and $F_0\varepsilon > 0$. We note that this $F(\psi)$ satisfies condition (9.25) therefore it is acceptable.

In order to determine the homothetic algebra of the kinematic metric (9.22) we write it in a more familiar form. We introduce the coordinates A, Ψ by the relations

$$A = \sqrt{-2F}a \tag{9.27}$$

$$d\Psi = \sqrt{\left(\frac{3\varepsilon F_\psi^2 - F}{2F^2}\right)}\, d\psi. \tag{9.28}$$

In the new coordinates A, Ψ the metric (9.22), the Lagrangian (9.12) and the non minimal coupling function $F(\Psi)$ take the following form

For $\varepsilon = 1$ and $F_0 > 0$

$$ds^2_{KM} = \frac{1}{\sqrt{-2F(\Psi)}}\left[-3A\dot{A}^2 + \frac{1}{2}A^3\dot{\Psi}^2\right] \tag{9.29}$$

$$L_M = \frac{1}{\sqrt{-2F(\Psi)}}\left[-3A\dot{A}^2 + \frac{1}{2}A^3\dot{\Psi}^2\right] - \frac{A^3}{(-2F(\Psi))^{\frac{3}{2}}}V(\Psi)\ . \tag{9.30}$$

$$F(\Psi) = -\frac{F_0}{12}\exp\left(\pm\frac{\sqrt{6}}{3}\sqrt{\frac{F_0}{F_0+1}}\Psi\right)\ ,\ F_0 > 0 \tag{9.31}$$

For $\varepsilon = -1$ and $-1 < F_0 < 0$

$$ds^2_{KM} = \frac{1}{\sqrt{-2F(\Psi^*)}}\left[-3A\dot{A}^2 - \frac{1}{2}A^3\dot{\Psi}^2\right] \tag{9.32}$$

$$L_{NM} = \frac{1}{\sqrt{-2F(\Psi^*)}}\left[-3A\dot{A}^2 - \frac{1}{2}A^3\dot{\Psi}^2\right] - \frac{A^3}{(-2F(\Psi))^{\frac{3}{2}}}V(\Psi) \tag{9.33}$$

$$F(\Psi) = -\frac{|F_0|}{12}\exp\left(\pm\frac{\sqrt{6}}{3}\sqrt{\frac{|F_0|}{1-|F_0|}}\Psi\right)\ ,\ -1 < F_0 < 0 \tag{9.34}$$

For $\varepsilon = -1$ and $F_0 < -1$ we introduce a new real field $\Psi^* = i\Psi$ and get the real field $F(\Psi^*)$

$$ds^2_{KM} = \frac{1}{\sqrt{-2F(\Psi^*)}}\left[-3A\dot{A}^2 + \frac{1}{2}A^3\dot{\Psi}^{*2}\right] \tag{9.35}$$

$$L_{NM} = \frac{1}{\sqrt{-2F(\Psi^*)}}\left[-3A\dot{A}^2 + \frac{1}{2}A^3\dot{\Psi}^{*2}\right] - \frac{A^3}{(-2F(\Psi^*))^{\frac{3}{2}}}V(\Psi^*) \tag{9.36}$$

$$F(\Psi^*) = -\frac{|F_0|}{12}\exp\left(\mp\frac{\sqrt{6}}{3}\sqrt{\frac{|F_0|}{|F_0|-1}}\Psi^*\right)\ ,\ F_0 < -1. \tag{9.37}$$

We see that in the region $-\infty < F_0 < -1$ starting from a phantom field Ψ we end up with a real scalar field Ψ^*.

In order to consider the three Lagrangians (9.30),(9.33),(9.36) at the same time we consider the Lagrangian

$$L = N^2(\Psi)\left[-3A\dot{A}^2 + \frac{\varepsilon_k}{2}A^3\dot{\Psi}^2\right] - A^3\bar{V}(\Psi) \tag{9.38}$$

where $N^2(\Psi) = \frac{1}{\sqrt{-2F(\Psi)}}$ and $\bar{V}(\Psi) = \frac{V(\Psi)}{(-2F(\Psi))^{\frac{3}{2}}}$ where $F(\Psi) < 0$. The constant $\varepsilon_k = \pm 1$ where the value $+1$ is for $\varepsilon = 1,\ F_0 > 0$ and $\varepsilon = -1$, $F_0 < -1$ and the value -1 is for $\varepsilon = -1$, $-1 < F_0 < 0$. Then the new kinematic metric is written as

$$ds^2_{KM} = N^2(\Psi)\left[-3A\dot{A}^2 + \frac{\varepsilon_k}{2}A^3\dot{\Psi}^2\right]. \tag{9.39}$$

We simplify this metric by introducing new coordinates r, θ defined by the transformation:

$$r = \sqrt{\frac{8}{3}}A^{\frac{3}{2}}\ ,\ \theta = \sqrt{\frac{3\varepsilon_k}{8}}\ \Psi \tag{9.40}$$

This step is necessary in order to deduce the homothetic algebra of the metric from well known previous results. In the new coordinates the metric (9.39) takes the simple form:

$$ds^2_{KM} = N^2(\theta)\left(-dr^2 + r^2 d\theta^2\right). \tag{9.41}$$

that is, it is directly related to the flat 2d Lorentzian space with metric

$$ds^2 = -dr^2 + r^2 d\theta^2$$

with conformal factor $N^2(\theta)$. In the new coordinates the curvature scalar is

$$R_{KM} = -\frac{2}{r^4 N^3(\theta)}\left(N_{,\theta\theta} - \frac{1}{N(\theta)}(N_{,\theta})^2\right) \tag{9.42}$$

hence the condition $R_{KM} = 0$ gives the function $N(\theta)$:

$$N(\theta) = N_0 e^{k\theta}\ ,\ k \in \mathbb{C}\ , N_0 \in \mathbb{R} \tag{9.43}$$

where k is a new constant.

Taking this into account we have that in the r, θ coordinates the Lagrangian (9.38) becomes:

$$L = N_0^2 e^{2k\theta}\left(-\frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\theta}^2\right) - r^2\bar{V}(\theta). \tag{9.44}$$

The constant k is related to the previous constant F_0 via the function $N(\theta)$. Using (9.40) we express $N(\theta)$ in terms of Ψ:

$$N(\Psi) = N_0 e^{k\sqrt{\frac{3\varepsilon_k}{8}}\Psi}. \tag{9.45}$$

Comparing with $F(\Psi)$ and eliminating $N(\Psi)$ we find:

$$F(\Psi) = -\frac{1}{2N_0^4}e^{-4k\ \sqrt{\frac{3\varepsilon}{8}}\ \Psi}. \tag{9.46}$$

This is a second expression of $F(\Psi)$ in terms of the constant k. Comparing with the previous expression (9.31),(9.34) and (9.37), which expresses $F(\Psi)$ in terms of F_0 (and holds for all ranges of values of F_0!), we find:

$$-\frac{F_0\varepsilon}{12}\exp\left(\pm\frac{\sqrt{6}}{3}\sqrt{\frac{\varepsilon F_0}{F_0+1}}\Psi\right) = -\frac{1}{2N_0^4}e^{-4k\ \sqrt{\frac{3\varepsilon_k}{8}}\ \Psi}. \tag{9.47}$$

This relation must hold identically which leads to the conditions

$$N_0 = \left(\frac{6}{\varepsilon F_0}\right)^{1/4} \tag{9.48}$$

and

$$|k| = \frac{1}{3}\sqrt{\frac{\varepsilon F_0}{F_0+1}}. \tag{9.49}$$

In Appendix 9.A we give the relation between the various ranges of the constant F_0 and the corresponding ranges of the constant $|k|$.

Case $|k| \neq 1$.

For $|k| \neq 1$ the homothetic algebra consists of the gradient KVs vectors

$$K^1 = \frac{e^{(1-k)\theta} r^k}{N_0^2}\left(-\partial_r + \frac{1}{r}\partial_\theta\right) \tag{9.50}$$

$$K^2 = \frac{e^{-(1+k)\theta} r^{-k}}{N_0^2}\left(\partial_r + \frac{1}{r}\partial_\theta\right) \tag{9.51}$$

the non gradient KV

$$K^3 = r\partial_r - \frac{1}{k}\partial_\theta \tag{9.52}$$

and the gradient HV

$$H^i = \frac{1}{N_0^2\left(k^2-1\right)}\left(-r\partial_r + k\partial_\theta\right)\ ,\ H(r,\theta) = \frac{1}{2}\frac{r^2 e^{2k\theta}}{k^2-1}. \tag{9.53}$$

Using the results of chapter 4 we find the following results

1. The gradient KV K^1 produces Noether symmetries for the following potentials

   a) For $V(\theta) = V_0 e^{2\theta}$ we have the Noether symmetries $K^1,\ tK^1$ with Noether integrals

   $$I_1 = \frac{d}{dt}\left(\frac{r^{1+k}e^{(1+k)\theta}}{(k+1)}\right)\ ,\ I_2 = t\frac{d}{dt}\left(\frac{r^{1+k}e^{(1+k)\theta}}{(k+1)}\right) - \left(\frac{r^{1+k}e^{(1+k)\theta}}{(k+1)}\right) \tag{9.54}$$

   b) For $V(\theta) = V_0 e^{2\theta} - \frac{mN_0^2}{2(k^2-1)}e^{2k\theta}$ we have the Noether symmetries $e^{\pm\sqrt{m}t}K^1$ m =constant, with Noether integrals

   $$I'_{\pm} = e^{\pm\sqrt{m}t}\left[\frac{d}{dt}\left(\frac{r^{1+k}e^{(1+k)\theta}}{(k+1)}\right) \mp \sqrt{m}\left(\frac{r^{1+k}e^{(1+k)\theta}}{(k+1)}\right)\right] \tag{9.55}$$

   From the Noether integrals we construct the time independent first integral $I_{K^1} = I_+ I_-$.

2. The gradient KV K^2 produces the following Noether symmetries for the following potentials

   a) For $V(\theta) = V_0 e^{-2\theta}$,we have the Noether symmetries $K^1,\ tK^1$ with Noether integrals

   $$J_1 = \frac{d}{dt}\left(\frac{r^{1-k}e^{-(1-k)\theta}}{k-1}\right)\ ,\ J_2 = t\frac{d}{dt}\left(\frac{r^{1-k}e^{-(1-k)\theta}}{k-1}\right) - \frac{r^{1-k}e^{-(1-k)\theta}}{k-1} \tag{9.56}$$

b) For $V(\theta) = V_0 e^{-2\theta} - \frac{mN_0^2}{2(k^2-1)} e^{2k\theta}$, we have the Noether symmetries $e^{\pm\sqrt{m}t} K^2$ m =constant, with Noether integrals

$$J'_{\pm} = e^{\pm\sqrt{m}t} \left[\frac{d}{dt} \left(\frac{r^{1-k} e^{-(1-k)\theta}}{k-1} \right) \mp \sqrt{m} \frac{r^{1-k} e^{-(1-k)\theta}}{k-1} \right] \tag{9.57}$$

From the Noether integrals we construct the time independent first integral $J_{K^2} = J'_+ J'_-$.

3. The non gradient KV K^3 produces a Noether symmetry for the potential $V(\theta) = V_0 e^{2k\theta}$ with Noether integral

$$I_3 = \frac{re^{2k\theta}}{k} \left(k\dot{r} + r\dot{\theta} \right). \tag{9.58}$$

4. The gradient HV produces the following Noether symmetries for the following potentials

a) For $V(\theta) = V_0 e^{-2\frac{\left(k^2-2\right)}{k}\theta}$, $k^2 - 2 \neq 0$ we have the Noether symmetries $2t\partial_t + H^i$, $t^2\partial_t + tH^i$ with Noether integrals

$$I_{H_1} = 2tE - \frac{d}{dt} \left(\frac{1}{2} \frac{r^2 e^{2k\theta}}{k^2-1} \right) \;,\; I_{H_2} = t^2 E - t\frac{d}{dt} \left(\frac{1}{2} \frac{r^2 e^{2k\theta}}{k^2-1} \right) + \frac{1}{2} \frac{r^2 e^{2k\theta}}{k^2-1}. \tag{9.59}$$

We note that in this case the system is the Ermakov-Pinney dynamical system (because it admits the Noether symmetry algebra the $sl(2,R)$, hence the Lie symmetry algebra is at least $sl(2,R)$) .

b) For $V(\theta) = V_0 e^{-2\frac{\left(k^2-2\right)}{k}\theta} - \frac{N_0^2 m}{k^2-1} e^{2k\theta}$, $k^2 - 2 \neq 0$ we have the Noether symmetries $\frac{2}{\sqrt{m}} e^{\pm\sqrt{m}t}\partial_t \pm e^{\pm\sqrt{m}t} H^i$, m =constant with Noether integrals

$$I_{+,-} = e^{\pm 2\sqrt{m}t} \left(\frac{1}{\sqrt{m}} E \mp \frac{d}{dt} \left(\frac{1}{2} \frac{r^2 e^{2k\theta}}{k^2-1} \right) + 2\sqrt{m} \left(\frac{1}{2} \frac{r^2 e^{2k\theta}}{k^2-1} \right) \right) \tag{9.60}$$

For this potential the Noether symmetries form the $sl(2,R)$ Lie algebra, i.e the dynamical system is the two dimensional Kepler-Ermakov system Therefore it admits the Ermakov - Pinney invariant which we may construct with the use of the Noether symmetries or with the use of the corresponding Killing Tensor (see Proposition 5.6.2)

5. The case $V(\theta) = 0$ corresponds the free particle (see chapter 2).

Case $|k| = 1$

We have to consider two cases i.e. $k = 1$ and $k = -1$.

Case $k = 1$ The KVs of the kinematic metric are the $K^{1,2}_{k=1}$ of (9.50,9.51) and the vector

$$K^3_{k=1} = -r \left(\ln\left(re^{-\theta} \right) - 1 \right) \partial_r + \ln\left(re^{-\theta} \right) \partial_\theta. \tag{9.61}$$

The vectors $K^{1,2}_{k=1}$ are gradient and K^3 is non-gradient. The HV is gradient and it is given by

$$H^i = \frac{1}{4} r \left(2\ln\left(re^{-\theta} \right) + 3 \right) \partial_r - \frac{1}{2} \left(\ln\left(re^{-\theta} \right) + \frac{1}{2} \right) \partial_\theta \tag{9.62}$$

Case $k=-1$ The KVs of the kinematic metric are $K_{k=-1}^{1,2}$ of (9.50,9.51) and the vector

$$\bar{K}^{3}=r\left(\ln\left(re^{\theta}\right)-1\right)\partial_{r}+\ln\left(re^{\theta}\right)\partial_{\theta}. \tag{9.63}$$

The vectors $\bar{K}^{1,2}$ are gradient and $\bar{K}^{3}$ is non-gradient. The gradient HV is given by

$$\bar{H}^{i}=\frac{1}{4}r\left(2\ln\left(re^{\theta}\right)+3\right)\partial_{r}+\frac{1}{2}\left(\ln\left(re^{\theta}\right)+\frac{1}{2}\right)\partial_{\theta} \tag{9.64}$$

In the following we consider only the case $k=1$. The results for the case $k=-1$ are found if we make the change $\theta_{(k=-1)}=-\bar{\theta}$.

Using theorem 4.3.2 and making simple calculations we find the following results

1. Noether symmetries generated by the KV K^{1}.

   a) If $V(\theta)=V_{0}e^{2\theta}$ then we have the extra Noether symmetries K^{1} , tK^{1} with Noether integrals the (9.54) with $k=1$.

   b) If $V(\theta)=\left(V_{0}e^{2\theta}-\frac{m}{2}\theta e^{2\theta}\right)$, then we have the Noether symmetries $e^{\pm\sqrt{m}t}K^{1}$ with Noether integrals (9.55) with $k=1$.

2. Noether symmetries generated by the KV K^{2}.

   a) If $V(\theta)=V_{0}e^{-2\theta}$ then we have the Noether symmetries K^{2} , tK^{2} with Noether integrals

$$I_{2}^{\prime}=\frac{d}{dt}\left(\theta-\ln r\right)\ ,\ I_{2}^{\prime}=t\left[\frac{d}{dt}\left(\theta-\ln r\right)\right]-\left(\theta-\ln r\right) \tag{9.65}$$

   b) If $V(\theta)=V_{0}e^{-2\theta}-\frac{1}{4}pe^{2\theta}$ then we have the Noether symmetries K^{2} , tK^{2} with Noether integrals

$$I_{1}^{\prime}=\frac{d}{dt}\left(\theta-\ln r\right)-pt\ ,\ I_{2}^{\prime}=t\left[\frac{d}{dt}\left(\theta-\ln r\right)\right]-\left(\theta-\ln r\right)-\frac{1}{2}pt^{2} \tag{9.66}$$

3. If $V(\theta)=0$ then the system becomes the free particle and admits seven extra Noether symmetries.

As we have remarked the results for $k=-1$ are obtained directly from those for $k=1$ if we make the substitution $\theta_{(k=-1)}=-\bar{\theta}$. Therefore there is no need to state them explicitly.

In the next section using the Noether symmetries for the potentials we have found, we determine the analytic solution for each case.

9.5.2 Case A: Analytic solutions for $k=1$

We introduce new coordinates u,v by the relations

$$u=\left(\theta-\ln r\right)\ ,\ v=\frac{1}{2}e^{2\theta}r^{2} \tag{9.67}$$

with inverse relations:

$$r = \left(2ve^{-2u}\right)^{\frac{1}{4}} \quad , \quad \theta = \frac{1}{4}\ln\left(2ve^{2u}\right). \tag{9.68}$$

In the new coordinates u, v the Lagrangian (9.44) of the field equations becomes

$$L\left(u,v,\dot{u},\dot{v}\right) = \frac{N_0^2}{2}\left(\dot{u}\dot{v}\right) - U\left(u,v\right) \tag{9.69}$$

and the field equations are

$$E = \frac{N_0^2}{2}\left(\dot{u}\dot{v}\right) + U\left(u,v\right)$$

$$\begin{aligned} \frac{N_0^2}{2}\ddot{u} + U_{,v} &= 0 \\ \frac{N_0^2}{2}\ddot{v} + U_{,u} &= 0 \end{aligned}$$

where the potential $U\left(u,v\right)$ is one of the potentials we have found in the last section. In the new coordinates we have (p, m are constants; recall that in general $U(r,\theta) = r^2 V\left(\theta\right)$):

- $$U(r,\theta) = V_0 r^2 e^{-2\theta} \rightarrow U\left(u,v\right) = V_0 e^{-2u} \tag{9.70}$$

- $$U(r,\theta) = V_0 r^2 e^{2\theta} \rightarrow U\left(u,v\right) = 2V_0 v \tag{9.71}$$

- $$U(r,\theta) = r^2\left(V_0 e^{-2\theta} - \frac{1}{4}pe^{2\theta}\right) \rightarrow U\left(u,v\right) = V_0 e^{-2u} - \frac{p}{2}v \tag{9.72}$$

- $$U(r,\theta) = r^2\left(V_0 e^{2\theta} - \frac{m}{2}\theta e^{2\theta}\right) \rightarrow U\left(u,v\right) = 2\left(V_0 - m\frac{\ln 2}{8}\right)v - \frac{m}{4}v\ln v - \frac{m}{2}uv \tag{9.73}$$

- $$U(r,\theta) = 0 \rightarrow U\left(u,v\right) = 0 \tag{9.74}$$

The Hamiltonian equals

$$E = \frac{N_0^2}{2}\left(\dot{u}\dot{v}\right) + U\left(u,v\right). \tag{9.75}$$

We write Lagrange equations for each potential and solve them taking into consideration the first integrals for each Noether symmetry we have found and the constraint imposed by the Hamiltonian. For each of the potentials above we find the corresponding analytic solution given below.

- $U(u,v) = V_0 e^{-2u}$

$$u(t) = u_1 t + u_2,\ v(t) = \frac{e^{-2u_2}}{u_1^2}\frac{V_0}{N_0^2}e^{-2u_1 t} + v_1 t + v_2 \tag{9.76}$$

with Hamiltonian constraint

$$E = \frac{u_1 v_1}{2N_0^2}. \tag{9.77}$$

- $U(u,v) = 2V_0 v$

$$u(t) = -\frac{2V_0}{N_0^2}t^2 + u_1 t + u_2, v(t) = v_1 t + v_2 \tag{9.78}$$

with Hamiltonian constrain

$$E = 2V_0 v_2 + N_0^2 \frac{u_1 v_1}{2}. \tag{9.79}$$

- $U(u,v) = V_0 e^{-2u} - \frac{p}{2}v$

where

$$v_1 = \frac{2V_0\sqrt{\pi}}{p^{\frac{3}{2}}N_0}\exp\left(\frac{u_1^2 N_0^2}{p} - 2u_2\right)\ ,\ v_2 = \frac{2V_0}{p}e^{-2u_2} \tag{9.80}$$

with Hamiltonian constraint

$$E = N_0^2\frac{v_3 u_1}{2} - \frac{v_4 p}{2}. \tag{9.81}$$

- $U(u,v) = 2\left(V_0 - m\frac{\ln 2}{8}\right)v - \frac{m}{4}v\ln v - \frac{m}{2}uv$

$$u(t) = u_1 e^{\frac{1}{N_0}\sqrt{m}t} + u_2 e^{-\frac{1}{N_0}\sqrt{m}t} - \frac{\sqrt{m}}{2N_0}t + \frac{4V_0}{m} - \ln\left(\sqrt{2}\right) - \frac{1}{2}\ , \tag{9.82}$$

$$v(t) = e^{\frac{1}{N_0}\sqrt{m}t}\ ,\ E = -u_2 m \tag{9.83}$$

$$u(t) = u_1 e^{\frac{1}{N_0}\sqrt{m}t} + u_2 e^{-\frac{1}{N_0}\sqrt{m}t} + \frac{\sqrt{m}}{2N_0}t + \frac{4V_0}{m} - \ln\left(\sqrt{2}\right) - \frac{1}{2}\ , \tag{9.84}$$

$$v(t) = e^{-\frac{1}{N_0}\sqrt{m}t}\ ,\ E = -u_1 m. \tag{9.85}$$

- $U(u,v) = 0$ (the free particle)

$$u(t) = u_1 t + u_2\ ,\ v(t) = v_1 t + v_2 \tag{9.86}$$

with Hamiltonian constraint

$$E = \frac{N_0^2}{2}u_1 v_1. \tag{9.87}$$

9.5.3 Case A: Analytic solutions for $|k| \neq 1$

When $|k| \neq 1$ we have to consider two cases $k^2 > 1$ and $k^2 < 1$. Both cases it is convenient to be discussed if we use as variables the functions $S_1(r,\theta)$, $S_2(r,\theta)$ which generate the Killing vectors (i.e. $K^{I,i} = S_I^{;i}$ $I = 1,2$). A standard calculation gives (see appendix B for details) :

$$K^1 = \frac{e^{(1-k)\theta} r^k}{N_0^2}\left(-\partial_r + \frac{1}{r}\partial_\theta\right) \ , \ S_1(r,\theta) = \frac{r^{1+k} e^{(1+k)\theta}}{(k+1)} \tag{9.88}$$

$$K^2 = \frac{e^{-(1+k)\theta} r^{-k}}{N_0^2}\left(\partial_r + \frac{1}{r}\partial_\theta\right) , \ S_2(r,\theta) = \frac{r^{1-k} e^{-(1-k)\theta}}{k-1}. \tag{9.89}$$

Because the metric is flat the new variables $S_1(r,\theta)$, $S_2(r,\theta)$ are Cartesian and will be denoted with x, y. So we write:

$$x = \frac{r^{1+k} e^{-(1+k)\theta}}{(k+1)} \ , \ y = \frac{r^{1-k} e^{(-1+k)\theta}}{k-1}. \tag{9.90}$$

The inverse transformation is:

for $k^2 > 1$

$$\theta = \frac{1}{2(1-k^2)} \ln\left(\frac{\left(k^2-1\right)^{1-k}}{(k-1)^2}\frac{x^{1-k}}{y^{1+k}}\right) \tag{9.91}$$

$$r = \sqrt{(k^2-1)\,xy}\left(\frac{\left(k^2-1\right)^{1-k}}{(k-1)^2}\frac{x^{1-k}}{y^{1+k}}\right)^{\frac{k}{2(k^2-1)}} \tag{9.92}$$

and

for $k^2 < 1$

$$\theta = \frac{1}{2(1-k^2)} \ln\left(\frac{\left(1-k^2\right)^{1-k}}{(1-k)^2}\frac{\bar{x}^{1-k}}{\bar{y}^{1+k}}\right) \tag{9.93}$$

$$r = \sqrt{(1-k^2)\,\bar{x}\bar{y}}\left(\frac{\left(1-k^2\right)^{1-k}}{(1-k)^2}\frac{\bar{x}^{1-k}}{\bar{y}^{1+k}}\right)^{\frac{k}{2(k^2-1)}} \tag{9.94}$$

Note that in the second case we have written $\bar{x}, \bar{y}$ while we have kept the x, y notation for the case $k^2 > 1$.

The case $k^2 > 1$

Before we look for analytic solutions we transform the Lagrangian in the canonical coordinates x, y. Using the transformation relations (9.91), (9.92) we find that in the coordinates x, y the Lagrangian (9.42) takes the form

$$L(x, y, \dot{x}, \dot{y}) = \frac{N_0^2}{2}\dot{x}\dot{y} - U(x, y) \tag{9.95}$$

where $U(x, y) = r^2 V(\theta)$ where $V(\theta)$ is one of the potentials computed above. In the coordinates x, y these potentials are as follows:

- $$U_1(x, y) = V_0 r^2 e^{2k\theta} = V_0\left(k^2-1\right)xy \tag{9.96}$$

$$U_2(x, y) = V_0 r^2 e^{+2\theta} = V_0 (k+1)^{\frac{2}{1+k}} \ \ x^{\frac{2}{1+k}} \tag{9.97}$$

$$U_3(x,y) = V_0 r^2 e^{-2\theta} = V_0 (k-1)^{\frac{2}{1-k}} y^{\frac{2}{1-k}} \tag{9.98}$$

•

$$U_4(x,y) = V_0 r^2 e^{-2\frac{(k^2-2)}{k}\theta} = V_0 \frac{(k^2-1)^{\frac{2}{k}-1}}{(k-1)^{\frac{4}{k}}} \frac{1}{y^2} \left(\frac{x}{y}\right)^{\frac{2}{k}-1}.$$

To the potentials U_2, U_3, U_4 we have to add three more which we obtain by the addition of the potential of the harmonic oscillator. Therefore finally we have 7 potentials. The extra potentials are

•

$$U_5(x,y) = V_0 r^2 e^{+2\theta} + m r^2 e^{2k\theta} = \bar{V}_{0+}\, x^{\frac{2}{1+k}} + \bar{m} xy \tag{9.99}$$

$$U_6(x,y) = r^2 e^{-2\theta} + m r^2 e^{2k\theta} = \bar{V}_{0-} y^{\frac{2}{1-k}} + \bar{m} xy \tag{9.100}$$

where $\bar{V}_{0+} = V_0 (k+1)^{\frac{2}{1+k}}$, $\bar{V}_{0-} = (k-1)^{\frac{2}{1-k}}$, $\bar{m} = m(k^2-1)$

•

$$U_7(x,y) = V_0 r^2 e^{-2\frac{(k^2-2)}{k}\theta} + m r^2 e^{2k\theta} = \bar{V}_0 \frac{1}{y^2} \left(\frac{x}{y}\right)^{\frac{2}{k}-1} + \bar{m} xy \tag{9.101}$$

where $\bar{V}_0 = V_0 \frac{(k^2-1)^{\frac{2}{k}-1}}{(k-1)^{\frac{4}{k}}}$.

• And the free particle potential

$$U_8(x,y) = 0 \tag{9.102}$$

These expressions allow us to write for each potential the Lagrangian and the Hamiltonian constraint in the coordinates x, y. This means that we obtain the corresponding field equations in the coordinates x, y to determine their solution.

Analytic solutions The solution of the field equations for each potential is a formal and lengthy operation which adds nothing but unnecessary material to the matter. What is interesting is of course the final answer for each case and this is what we give below for each of the potentials above.

• $U_1(x,y) = V_0 (k^2-1) xy$

$$x(t) = x_1 \sin(\omega t) + x_2 \cos(\omega t) \tag{9.103}$$

$$y(t) = y_1 \sin(\omega t) + y_2 \cos(\omega t) \tag{9.104}$$

where $\omega^2 = \frac{2V_0(k^2-1)}{N_0^2}$ and the Hamiltonian is

$$E = V_0 (k^2-1)(x_1 y_1 + x_2 y_2)$$

- $U_2(x,y) = V_0(k+1)^{\frac{2}{1+k}}\ x^{\frac{2}{1+k}}$

  When $k \neq -3$

$$x(t) = x_1 t + x_2 \tag{9.105}$$

$$y(t) = -\frac{2\bar{V}(k+1)(x_1 t + x_2)^{\left(1+\frac{2}{1+k}\right)}}{x_1^2(3+k)N_0^2} + y_1 t + y_2 \tag{9.106}$$

  where $\bar{V} = V_0(k+1)^{\frac{2}{1+k}}$, and the Hamiltonian is

$$E = \frac{y_1 x_1 N_0^2}{2}.$$

  When $k = -3$

$$y(t) = -2\frac{\bar{V}}{N_0^2 x_1^2}\ln(x_1 t + x_2) + y_1 t + y_2. \tag{9.107}$$

  $x(t)$, E being the same.

- $U_3(x,y) = V_0(k-1)^{\frac{2}{1-k}}\ y^{\frac{2}{1-k}}$

  When $k \neq 3$

$$x(t) = \frac{2\bar{V}(k-1)(y_1 t + y_2)^{1+\frac{2}{k-1}}}{y_1^2(k-3)N_0^2} + x_1 t + x_2 \tag{9.108}$$

$$y(t) = y_1 t + y_2 \tag{9.109}$$

  where $\bar{V} = V_0(k-1)^{\frac{2}{1-k}}$, $k \neq 3$and the Hamiltonian is

$$E = \frac{y_1 x_1 N_0^2}{2}.$$

  When $k = 3$;

$$x(t) = -\frac{2\bar{V}}{N_0^2 y_1^2}\ln(y_1 t + y_2) + x_1 t + x_2. \tag{9.110}$$

  $y(t), E$ being the same

- $U_5(x,y) = \bar{V}_{0+}\ x^{\frac{2}{1+k}} + \bar{m}xy$

$$x(t) = x_1 \sin(\omega t + \omega_0) \tag{9.111}$$

$$y(t) = \cos(\omega t + \omega_0)\left(y_1 + 2\frac{\omega}{\bar{m}}\int \frac{y_2 - x_1\bar{V}_{0+}\sin(\omega t + \omega_0)^{\frac{2}{1+k}}}{x_1(\cos(\omega t + \omega_0) + 1)}dt\right) \tag{9.112}$$

  where $\omega^2 = \frac{2\bar{m}}{N_0^2}$ and $E = y_2$.

- $U_6(x,y) = r^2 e^{-2\theta} + m r^2 e^{2k\theta} = \bar{V}_{0-} y^{\frac{2}{1-k}} + \bar{m}xy$

$$x(t) = \cos(\omega t + \omega_0)\left(x_1 + 2\frac{\omega}{\bar{m}}\int \frac{x_2 - y_1\bar{V}_{0-}\sin(\omega t + \omega_0)^{\frac{2}{1-k}}}{y_1(\cos(\omega t + \omega_0) + 1)}dt\right) \tag{9.113}$$

$$y(t) = y_1 \sin(\omega t + \omega_0) \tag{9.114}$$

  where $\omega^2 = \frac{2\bar{m}}{N_0^2}$and $E = x_2$.

- $U_{4,7}(x,y) = \bar{V}_0 \frac{1}{y^2}\left(\frac{x}{y}\right)^{\frac{2}{k}-1} + \bar{m}xy$. ($U_4$ is for $\bar{m}=0$)

  This is the Ermakov Pinney system. To solve it, it is convenient to go to spherical coordinates. We consider the coordinate transformation

$$x = ze^{w} \ , \ y = ze^{-w} \tag{9.115}$$

and the Lagrangian takes the form

$$L(z,w,\dot{z},\dot{w}) = \frac{N_0^2}{2}\left(\dot{z}^2 - z^2\dot{w}^2\right) - \frac{\bar{V}_0}{z^2}e^{\frac{4}{k}w} - \bar{m}z^2 \tag{9.116}$$

whereas the Hamiltonian becomes

$$E = \frac{N_0^2}{2}\left(\dot{z}^2 - z^2\dot{w}^2\right) + \frac{\bar{V}_0}{z^2}e^{\frac{4}{k}w} + \bar{m}z^2. \tag{9.117}$$

This system admits the Ermakov-Lewis invariant, which is

$$J_{EL} = z^4\dot{w}^2 - 2\frac{\bar{V}_0}{N_0^2}e^{\frac{4}{k}w}. \tag{9.118}$$

Using the Ermakov invariant the Hamiltonian becomes

$$E = \frac{N_0^2}{2}\dot{z}^2 - N_0^2\frac{J_{EL}}{2z^2} + \bar{m}z^2. \tag{9.119}$$

This is the Hamiltonian of the Ermakov Pinney equation:

$$\ddot{z} + 2\bar{m}z + N_0^2\frac{J_{EL}}{z^3} = 0 \tag{9.120}$$

whose solution is

$$z(t) = \left(l_0 z_1(t) + l_1 z_2(t) + l_3\right)^{\frac{1}{2}} \tag{9.121}$$

$$e^{\frac{4}{k}w(t)} = \frac{N_0^2 J_{EL}}{2\bar{V}_0}\left[\tanh^2\left(\frac{2\sqrt{J_{EL}}}{k}\left(\int\frac{dt}{z^2(t)} + l_4\right)\right) - 1\right] \tag{9.122}$$

where $z_{1,2}(t)$ are solutions of the differential equation $\ddot{z} + 2\bar{m}z = 0$ and l_{0-4} are constants.

- $U_8(x,y) = 0$

  This is the free particle whose solution is

$$x(t) = x_1 t + x_2 \ , \ y(t) = y_1 t + y_2 \tag{9.123}$$

$$E = \frac{N_0^2}{2}x_1 y_1 \tag{9.124}$$

The case $k^2 < 1$

In this case the canonical coordinates are the $\bar{x}, \bar{y}$. Using the transformation equations (9.93), (9.94) we write the Lagrangian (9.42) as follows:

$$L\left(\bar{x}, \bar{y}, \dot{\bar{x}}, \dot{\bar{y}}\right) = -\frac{N_0^2}{2}\dot{\bar{x}}\dot{\bar{y}} - \bar{U}\left(\bar{x}, \bar{y}\right) \tag{9.125}$$

where $U\left(\bar{x}, \bar{y}\right) = r^2 V\left(\theta\right)$ the potentials $V\left(\theta\right)$ being as above. In the coordinates $\bar{x}, \bar{y}$ the potentials $U\left(\bar{x}, \bar{y}\right)$ are:

- $$\bar{U}_1\left(\bar{x}, \bar{y}\right) = V_0 r^2 e^{2k\theta} = V_0\left(1 - k^2\right)\bar{x}\bar{y} \tag{9.126}$$

$$\bar{U}_2\left(\bar{x}, \bar{y}\right) = V_0 r^2 e^{+2\theta} = V_0\left(k+1\right)^{\frac{2}{1+k}}\ \ \bar{x}^{\frac{2}{1+k}} \tag{9.127}$$

$$\bar{U}_3\left(\bar{x}, \bar{y}\right) = V_0 r^2 e^{-2\theta} = V_0\left(1-k\right)^{\frac{2}{1-k}}\ \bar{y}^{\frac{2}{1-k}} \tag{9.128}$$

- $$\bar{U}_4\left(\bar{x}, \bar{y}\right) = V_0 r^2 e^{-2\frac{\left(k^2-2\right)}{k}\theta} = V_0\frac{\left(1-k^2\right)^{\frac{2}{k}-1}}{\left(1-k\right)^{\frac{4}{k}}}\frac{1}{\bar{y}^2}\left(\frac{\bar{x}}{\bar{y}}\right)^{\frac{2}{k}-1} \tag{9.129}$$

  As before we have three more potentials

- $$\begin{aligned}\bar{U}_5\left(\bar{x}, \bar{y}\right) &= V_0 r^2 e^{+2\theta} = \bar{V}_{0+}\ \bar{x}^{\frac{2}{1+k}} + \bar{m}\bar{x}\bar{y} &(9.130)\\ \bar{U}_6\left(\bar{x}, \bar{y}\right) &= V_0 r^2 e^{-2\theta} = \bar{V}_{0-}\bar{y}^{\frac{2}{1-k}} + \bar{m}\bar{x}\bar{y} &(9.131)\end{aligned}$$

  where $\bar{V}_{0+} = V_0\left(k+1\right)^{\frac{2}{1+k}}$, $\bar{V}_{0-} = \left(1-k\right)^{\frac{2}{1-k}}$, $\bar{m} = m\left(1-k^2\right)$.

- $$\bar{U}_7\left(\bar{x}, \bar{y}\right) = V_0 r^2 e^{-2\frac{\left(k^2-2\right)}{k}\theta} + m r^2 e^{2k\theta} = \bar{V}_0\frac{1}{\bar{y}^2}\left(\frac{\bar{x}}{\bar{y}}\right)^{\frac{2}{k}-1} + \bar{m}\bar{x}\bar{y} \tag{9.132}$$

  where $\bar{V}_0 = V_0\frac{\left(1-k^2\right)^{\frac{2}{k}-1}}{\left(1-k\right)^{\frac{4}{k}}}$.

- and the free particle potential.

$$\bar{U}_8\left(\bar{x}, \bar{y}\right) = 0 \tag{9.133}$$

Analytic solutions Working as in the case $k^2 > 1$ we find the following analytic solutions and the associated Hamiltonian constraint for each of the potentials above.

- $\bar{U}_1(\bar{x},\bar{y}) = V_0\left(1-k^2\right)\bar{x}\bar{y}$.

$$\bar{x}(t) = x_1\cosh(\omega t) + x_2\sinh(\omega t) \tag{9.134}$$
$$\bar{y}(t) = y_1\cosh(\omega t) + y_2\sinh(\omega t) \tag{9.135}$$

and the Hamiltonian $E = V_0 m(x_1y_1 - x_2y_2)$ where $\omega^2 = \frac{2V_0\left(1-k^2\right)}{N_0^2}$.

- $\bar{U}_2(\bar{x},\bar{y}) = V_0(k+1)^{\frac{2}{1+k}}\ \bar{x}^{\frac{2}{1+k}}$

$$\bar{x}(t) = x_1t + x_2 \tag{9.136}$$
$$\bar{y}(t) = \frac{2\bar{V}(k+1)(x_1t+x_2)^{\left(1+\frac{2}{1+k}\right)}}{x_1^2(3+k)N_0^2} + y_1t + y_2 \tag{9.137}$$

where $\bar{V} = V_0(k+1)^{\frac{2}{1+k}}$ and the Hamiltonian $E = -\frac{N_0^2}{2}x_1y_1$.

- $\bar{U}_3(\bar{x},\bar{y}) = V_0(1-k)^{\frac{2}{1-k}}\ \bar{y}^{\frac{2}{1-k}}$

$$\bar{x}(t) = \frac{2\bar{V}(1-k)(y_1t+y_2)^{1+\frac{2}{k-1}}}{y_1^2(k-3)N_0^2} + x_1t + x_2 \tag{9.138}$$
$$\bar{y}(t) = y_1t + y_2 \tag{9.139}$$

where $\bar{V} = V_0(k+1)^{\frac{2}{1+k}}$ and the Hamiltonian constrain $E = -\frac{N_0^2}{2}x_1y_1$.

- $\bar{U}_5(\bar{x},\bar{y}) = \bar{V}_{0+}\ \bar{x}^{\frac{2}{1+k}} + \bar{m}\bar{x}\bar{y}$

$$\bar{x}(t) = \bar{x}_1\sinh(\omega t+\omega_0)$$
$$\bar{y}(t) = \cosh(\omega t+\omega_0)\left(\bar{y}_1 - \frac{2\omega}{\bar{m}}\int\frac{E-\bar{x}_1\bar{V}_{0+}\sinh(\omega t+\omega_0)^{\frac{2}{1+k}}}{\bar{x}_1(\cosh(\omega t+\omega_0)+1)}\right)$$

where $\omega^2 = \frac{2\bar{m}}{N_0^2}$

- $\bar{U}_6(\bar{x},\bar{y}) = \bar{V}_{0-}\bar{y}^{\frac{2}{1-k}} + \bar{m}\bar{x}\bar{y}$

$$\bar{x}(t) = \cosh(\omega t+\omega_0)\left(\bar{x}_1 - \frac{2\omega}{\bar{m}}\int\frac{E-\bar{y}_1\bar{V}_{0-}\sinh(\omega t+\omega_0)^{\frac{2}{1-k}}}{\bar{y}_1(\cosh(\omega t+\omega_0)+1)}\right)$$
$$\bar{y}(t) = \bar{y}_1\sinh(\omega t+\omega_0)$$

where $\omega^2 = \frac{2\bar{m}}{N_0^2}$

- $\bar{U}_{4,7}(\bar{x},\bar{y}) = \bar{V}_0\frac{1}{\bar{y}^2}\left(\frac{\bar{x}}{\bar{y}}\right)^{\frac{2}{k}-1} + \bar{m}\bar{x}\bar{y}$. $\left(\bar{U}_4 \text{ is for } \bar{m}=0\right)$.

  This is again the Ermakov Piney potential. To solve it we go to spherical coordinates

$$\bar{x} = ze^{w}\ \ ,\ \bar{y} = ze^{-w} \tag{9.140}$$

in which the Lagrangian takes the form

$$L(z,w,\dot{z},\dot{w}) = \frac{N_0^2}{2}\left(-\dot{z}^2 + z^2\dot{w}^2\right) - \frac{\bar{V}_0}{z^2}e^{\frac{4}{k}w} - \bar{m}z^2 \tag{9.141}$$

and the Hamiltonian

$$E = \frac{N_0^2}{2}\left(-\dot{z}^2 + z^2\dot{w}^2\right) + \frac{\bar{V}_0}{z^2}e^{\frac{4}{k}w} + \bar{m}z^2. \tag{9.142}$$

The Ermakov-Lewis invariant is

$$J_{EL} = z^4\dot{w}^2 + 2\frac{\bar{V}_0}{N_0^2}e^{\frac{4}{k}w} \tag{9.143}$$

which when replaced in the Hamiltonian gives

$$E = -\frac{N_0^2}{2}\dot{z}^2 + N_0^2\frac{J_{EL}}{2z^2} + \bar{m}z^2. \tag{9.144}$$

This is the Hamiltonian of the Ermakov Pinney equation:

$$\ddot{z} - 2\bar{m}z - N_0^2\frac{J_{EL}}{z^3} = 0 \tag{9.145}$$

whose solution is

$$z(t) = \left(l_0 z_1(t) + l_1 z_2(t) + l_3\right)^{\frac{1}{2}} \tag{9.146}$$

$$e^{\frac{4}{k}w(t)} = \frac{N_0^2 J_{EL}}{2\bar{V}_0}\left[1 - \tanh^2\left(\frac{2\sqrt{J_{EL}}}{k}\left(\int\frac{dt}{z^2(t)} + l_4\right)\right)\right] \tag{9.147}$$

where $z_{1,2}(t)$ are solutions of the ode $\ddot{z} - 2\bar{m}z = 0$ and l_{0-4} are constants.

- $\bar{U}_8(\bar{x},\bar{y}) = 0$
- This is the free particle whose solution is

$$\bar{x}(t) = x_1 t + x_2\ ,\ \bar{y}(t) = y_1 t + y_2 \tag{9.148}$$

$$E = -\frac{N_0^2}{2}x_1 y_1. \tag{9.149}$$

9.5.4 Case B: The 2d metric is conformally flat

In this case it is preferable to work with (9.39) which under the coordinate transformation (9.40) becomes:

$$L = N^2(\theta)\left[-\frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\theta}^2\right] - r^2 V(\theta). \tag{9.150}$$

The kinetic metric in this case is not flat (i.e. $R_{(2)} \neq 0$) but of course it is conformally flat being a two dimensional metric. Its conformal algebra is infinity dimensional however it has a closed subalgebra consisting of the following vectors (this is the special conformal algebra of M^2):

$$\begin{aligned} X^1 &= \cosh\theta\partial_r - \frac{1}{r}\sinh\theta\partial_\theta\ ,\ X^2 = \sinh\theta\partial_r - \frac{1}{r}\cosh\theta\partial_\theta \\ X^3 &= \partial_\theta\ ,\ X^4 = r\partial_r\ ,\ X^5 = \frac{1}{2}r^2\cosh\theta\partial_r + \frac{1}{2}r\sinh\theta\partial_\theta \\ X^6 &= \frac{1}{2}r^2\sinh\theta\partial_r + \frac{1}{2}r\cosh\theta\partial_\theta \end{aligned} \tag{9.151}$$

Writing $L_{X^I} g_{ij} = 2C_I(r,\theta) g_{ij}$ we find the conformal factors of the CKVs X^I $I = 1,...6$ above in terms of the the conformal function. The result is:

$$\begin{aligned} C_1(r,\theta) &= -\frac{1}{r}\sinh(\theta)\frac{N_{,\theta}}{N} \ , \ C_2(r,\theta) = -\frac{1}{r}\cosh(\theta)\frac{N_{,\theta}}{N} \\ C_3(r,\theta) &= \frac{N_{,\theta}}{N} \ , \ C_4(r,\theta) = 1 \ , \ C_5(r,\theta) = \frac{r}{2}\left(\frac{2N\cosh\theta + \sinh\theta N_\theta}{N}\right) \\ C_6(r,\theta) &= \frac{r}{2}\left(\frac{2N\sinh\theta + \cosh\theta N_\theta}{N}\right) \end{aligned} \tag{9.152}$$

The $N(\theta) \neq e^{c\theta}$, otherwise the kinetic metric of the Lagrangian (9.150) is flat (the Ricci scalar vanishes) and we return to the previous case A. This means that the vectors X^I $I = 1,...6$ except the $I = 4$ are proper CKVs therefore they do not give (in general) a Noether symmetry. The vector X_4 is a non-gradient HV which does not also produce a Noether symmetry. Therefore according to theorem 4.3.2 only Killing vectors are possible to serve as Noether symmetries. KVs do not exist but for special forms of the conformal function $N(\theta)$. Each such form of $N(\theta)$ results in a potential $V(\theta)$ hence to a scalar field potential which admits Noether point symmetries. In the following we determine the possible $N(\theta)$ which lead to a KV and give the corresponding Noether symmetry and the corresponding Noether integral which will be used for the solution of the field equations.

1. If $N(\theta) = \frac{N_0}{\cosh(2\theta)-1}$ then X^5 is a non gradient KV and a Noether symmetry for the Lagrangian (9.150) for the potential

$$V(\theta) = \frac{V_0}{\cosh(2\theta) - 1} \text{ or } V(\theta) = 0 \tag{9.153}$$

The corresponding Noether integral is

$$I_{X^5} = \frac{N_0^2 r^2}{(\cosh(2\theta) - 1)^2}\left(r\dot{\theta}\sinh\theta - \dot{r}\cosh\theta\right). \tag{9.154}$$

2. If $N(\theta) = \frac{N_0}{\cosh(2\theta)+1}$ then X^6 is a non gradient KV, X^6 and a Noether symmetry for the Lagrangian (9.150) if

$$V(\theta) = \frac{V_0}{\cosh(2\theta) + 1} \text{ or } V(\theta) = 0 \tag{9.155}$$

The corresponding Noether integral is

$$I_{X^6} = \frac{N_0^2 r^2}{(\cosh(2\theta) + 1)^2}\left(r\dot{\theta}\cosh\theta - \dot{r}\sinh\theta\right) \tag{9.156}$$

3. If $N(\theta) = \frac{N_0}{\cosh^2(\theta+\theta_0)}$ then the linear combination $X^{56} = c_1 X^5 + c_2 X^6$ where $c_1 = \sinh(\theta_0)$ and $c_2 = \cosh(\theta_0)$. X^{56} is a Noether symmetry for the Lagrangian (9.150) if

$$V(\theta) = \frac{V_0}{\cosh^2(\theta + \theta_0)} \text{ or } V(\theta) = 0 \tag{9.157}$$

with Noether integral

$$I_{X^{56}} = \frac{N_0^2 r^2}{\cosh^4(\theta + \theta_0)}\left(r\dot{\theta}\cosh(\theta + \theta_0) - \dot{r}\sinh(\theta + \theta_0)\right)$$

Obviously case 3 is the most general and contains cases 1 and 2 (and the trivial case) as special cases. Therefore in the following we look for analytic solutions for the vector X^{56}only.

We recall that $\frac{1}{\sqrt{-2F(\theta)}} = N^2(\theta)$ from which follows:

$$F(\theta) = -\frac{1}{2N_0^4}\cosh^8(\theta+\theta_0) \ , N_0 \in \mathbb{R}. \tag{9.158}$$

We may consider $\theta_0 = 0$ (e.g. by introducing the new variable $\Theta = \theta + \theta_0$).

For the potential (9.157) the Lagrangian (9.150) becomes

$$L = \frac{N_0^2}{\cosh^4\theta}\left[-\frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\theta}^2\right] - r^2\frac{V_0}{\cosh^2\theta} \tag{9.159}$$

and the Hamiltonian

$$E = \frac{N_0^2}{\cosh^4\theta}\left[-\frac{1}{2}\dot{r}^2 + \frac{1}{2}r^2\dot{\theta}^2\right] + r^2\frac{V_0}{\cosh^2\theta}. \tag{9.160}$$

The equations of motion are:

$$\ddot{r} + r\dot{\theta}^2 - 4\tanh\theta\,\dot{r}\dot{\theta} - 2\frac{V_0}{N_0^2}r\cosh^2\theta = 0 \tag{9.161}$$

$$\ddot{\theta} - 2\tanh\theta\left(\frac{1}{r^2}\dot{r}^2 + \dot{\theta}^2\right) + \frac{2}{r}\dot{r}\dot{\theta} - 2\frac{V_0}{N_0^2}\cosh\theta\sinh\theta = 0 \tag{9.162}$$

and the Noether integral $I_{X^{56}}$ for $\theta_0 = 0$ becomes:

$$I_{X^{56}} = \frac{N_0^2 r^2}{\cosh^4(\theta+\theta_0)}\left(r\dot{\theta}\cosh(\theta) - \dot{r}\sinh(\theta)\right). \tag{9.163}$$

In order to proceed with the solution of the system of equations (9.161), (9.162) we change to the coordinates x, y which we define by the relations

$$r = \frac{x}{\sqrt{1-x^2y^2}} \ , \ \theta = \arctan h\,(xy)\,. \tag{9.164}$$

In the coordinates x, y the Lagrangian and the Hamiltonian become:

$$L = \frac{N_0^2}{2}\left(-\dot{x}^2 + x^4\dot{y}\right) - V_0 x^2 \tag{9.165}$$

$$E = \frac{N_0^2}{2}\left(-\dot{x}^2 + x^4\dot{y}^2\right) + V_0 x^2 \tag{9.166}$$

and the Noether integral (we write I for $I_{X^{56}}$)

$$I = x^4\dot{y}. \tag{9.167}$$

Let us assume that $I \neq 0$.In the new variables the Euler - Lagrange equations read:

$$\ddot{x} + 2x^3\dot{y}^2 - 2V_0 x = 0 \tag{9.168}$$

$$\ddot{y} + \frac{4}{x}\dot{x}\dot{y} = 0. \tag{9.169}$$

From the Noether integral we have $\dot{y}=\frac{I}{x^4}$ which upon substitution in the field equations gives the equation

$$\ddot{x}-2V_0x+\frac{2I^2}{x^5} = 0 \tag{9.170}$$

$$\frac{1}{2}\left(-\dot{x}^2+\frac{I^2}{x^4}\right)+V_0x^2 = E. \tag{9.171}$$

from which we compute

$$\dot{x}=\sqrt{\frac{I^2}{x^4}+2V_0x^2-2E} \tag{9.172}$$

and finally the analytic solution

$$\int\frac{dx}{\sqrt{\frac{I^2}{x^4}+2V_0x^2-2E}}=t-t_0. \tag{9.173}$$

From the Noether integral we find

$$y(t)-y_0=\int\frac{I}{x^4}dt. \tag{9.174}$$

If $I=0$ then the analytic solution is

$$x=x_0\sinh\left(\sqrt{2V_0}t+x_1\right),\ y=y_0 \tag{9.175}$$

with Hamiltonian constrain $E=-x_0^2V_0$.

If $V_0=0$ (i.e. free particle) and $I=0$ the analytic solution is

$$x=x_0t+x_1\ ,\ y=y_0 \tag{9.176}$$

with Hamiltonian constrain $E=-\frac{1}{2}x_0^2$.

9.6 Noether point symmetries of a minimally coupled Scalar field.

In this section we study the Noether point symmetries of a minimally coupled scalar field in a spatially flat FRW spacetime. The action of the field equations is

$$S_M=\int d\tau dx^3\sqrt{-g}\left[R+\frac{1}{2}g_{ij}\phi^{;i}\phi^{;j}-V(\phi)\right]+\int L_m d\tau dx^3. \tag{9.177}$$

where L_m is the Lagrangian of the dust matter fluid. For a spatially flat FRW spacetime the Ricciscalar is

$$R=6\left(\frac{\ddot{a}}{a}+\frac{\dot{a}^2}{a^2}\right)$$

hence, the Lagrangian of the field equation is (9.11) and the field equations are

$$E=-3a\dot{a}^2+\frac{\varepsilon}{2}a^2\dot{\phi}^2+a^3V(\phi) \tag{9.178}$$

$$\ddot{a}+\frac{1}{2a}\dot{a}^2+\frac{\varepsilon}{4}\dot{\phi}^2-\frac{1}{2}aV=0 \tag{9.179}$$

$$\ddot{\phi}+\frac{3}{a}\dot{a}\dot{\phi}+\varepsilon V_{,\phi}=0 \tag{9.180}$$

From the kinetic term of (9.11) we define the two dimensional metric

$$ds^2=-6a\ da^2+\varepsilon a^3 d\phi^2. \tag{9.181}$$

We find that the curvature of the $\{a,\phi\}$ space is $R=0$ implying flatness (since all 2 dimensional spaces are Einstein spaces hence $\hat{R}=0$ implies that the space is flat). Also, the signature of the metric eq.(9.181) is 0, hence the space is the 2-d Minkowski space. In order to simplify the field equations we apply the following coordinate transformation

$$r=\sqrt{\frac{8}{3}}a^{3/2}\qquad \theta=\sqrt{\frac{3k\epsilon}{8}}\phi\ , \tag{9.182}$$

in the new coordinates the two dimensional metric (9.181) is given by

$$d\hat{s}^2=-dr^2+r^2d\theta^2 \tag{9.183}$$

that is, (r,θ) are hyperbolic spherical coordinates in the two dimensional Minkowski space $\{a,\phi\}$. Next we introduce the new coordinates (x,y) with the transformation:

$$x=r\cosh\left(\theta\right)$$
$$y=r\sinh\left(\theta\right) \tag{9.184}$$

which implies that the metric (9.183) becomes $d\hat{s}^2=-dx^2+dy^2$. We also point here that

$$r^2=x^2-y^2\qquad \theta=\operatorname{arctanh}(y/x)\ . \tag{9.185}$$

The scale factor $(a(t)>0)$ is now given by:

$$a=\left[\frac{3(x^2-y^2)}{8}\right]^{1/3} \tag{9.186}$$

which means that the new variables have to satisfy the following inequality: $x\geq|y|$.

In the new coordinate system (x,y) the Lagrangian (9.11) and the Hamiltonian (9.178) are written:

$$L=\frac{1}{2}\left(\dot{y}^2-\dot{x}^2\right)-V_{eff}(x,y) \tag{9.187}$$

$$E=\frac{1}{2}\left(\dot{y}^2-\dot{x}^2\right)+V_{eff}(x,y) \tag{9.188}$$

where

$$V_{eff}(x,y)=\left(x^2-y^2\right)\tilde{V}\left(\frac{y}{x}\right). \tag{9.189}$$

Note that we have used

$$\tilde{V}\left(\theta\right)=\frac{3k}{8}V(\theta)\ . \tag{9.190}$$

We now proceed in an attempt to provide the Noether point symmetries of the current dynamical problem using the results of chapter 4.

Since the Lagrangian (9.187) is autonomous admits the Noether point symmetry ∂_t with Noether integral the Hamiltonian (9.188). Lagrangian (9.187) admits extra Noether point symmetries in the following cases.

9.6.1 Hyperbolic - UDM Potential

Hyperbolic - UDM Potential: Generically, we use the following potential:

$$\tilde{V}(\theta) = \frac{\omega_1 \cosh^2(\theta) - \omega_2 \sinh^2(\theta)}{2} \tag{9.191}$$

or

$$V_{eff}(x,y) = r^2 \tilde{V}(\theta) = \frac{\omega_1 x^2 - \omega_2 y^2}{2} \tag{9.192}$$

The corresponding Noether symmetries, X_n, are known (see for example [5]). These are:

$$X_{n_1} = \partial_t \ , \ X_{n_2} = \sinh(\sqrt{\omega_1} t)\, \partial_x \ , \ X_{n_3} = \cosh(\sqrt{\omega_1} t)\, \partial_x$$
$$X_{n_4} = \sinh(\sqrt{\omega_2} t)\, \partial_y \ , \ X_{n_5} = \cosh(\sqrt{\omega_2} t)\, \partial_y$$

The Noether integrals are the Hamiltonian and the quantities:

$$I_{n_2} = \sinh(\sqrt{\omega_1} t)\, \dot{x} - \sqrt{\omega_1} \cosh(\sqrt{\omega_1} t)\, x$$
$$I_{n_3} = \cosh(\sqrt{\omega_1} t)\, \dot{x} - \sqrt{\omega_1} \sinh(\sqrt{\omega_1} t)\, x$$
$$I_{n_4} = \sinh(\sqrt{\omega_2} t)\, \dot{y} - \sqrt{\omega_2} \cosh(\sqrt{\omega_2} t)\, y$$
$$I_{n_5} = \cosh(\sqrt{\omega_2} t)\, \dot{y} - \sqrt{\omega_2} \sinh(\sqrt{\omega_2} t)\, y$$

Obviously the UDM potential is a particular case of the current general hyperbolic potential. Indeed for $\omega_1 = 2\omega_2$ and with the aid of eqs.(9.182), (9.190) we fully recover the UDM potential [153, 154, 155, 142]

$$V(\phi) = V_0 \left[1 + \cosh^2\left(\frac{3k\epsilon}{8}\phi\right)\right] \tag{9.193}$$

where $V_0 = \frac{4\omega_2}{3k}$ modulus a constant.

9.6.2 Exponential Potential

Exponential Potential: The exponential potential

$$V_{eff}(r,\theta) = r^2 \tilde{V}(\theta) = r^2 e^{-d\theta} \ .$$

admits the extra Noether symmetry

$$X_n = 2t\partial_t + \left(x + \frac{4}{d}y\right)\partial_x + \left(y + \frac{4}{d}x\right)\partial_y \ . \tag{9.194}$$

In general the Noether integral for the vector $X_n = 2t\partial_t + \eta^i \partial_i$ is

$$I = 2tE + \left(x + \frac{4}{d}y\right)\dot{x} - \left(y + \frac{4}{d}x\right)\dot{y} \tag{9.195}$$

where E is the Hamiltonian. Using $\tilde{V}=e^{-d\theta}$ together with eq.(9.182) and eq.(9.190) we write the potential to its nominal form [156] which is

$$V(\phi)=V_0\exp\left(-d\sqrt{\frac{3k\epsilon}{8}}\phi\right) \tag{9.196}$$

where $V_0=\frac{8}{3k}$ modulus a constant[3].

We note that the analytic solutions of sections 9.5.3 and 9.5.4 are also solutions for the minimally coupled Lagrangian (9.11) if and only if the space does not admit dust.

9.7 The Lie and Noether symmetries of Bianchi class A homogeneous cosmologies with a scalar field.

The class of Bianchi spatially homogeneous cosmologies contains many important cosmological models, including the standard FRW model. In these models the spacetime manifold is foliated along the time axis, with three dimensional homogeneous hypersurfaces. Bianchi has classified all three dimensional real Lie algebras and has shown that there are nine of them. This results in nine types (two of them being families of spacetimes) of Bianchi spatially homogeneous spacetimes. The principal advantage of Bianchi cosmological models is that, in these models the physical variables depend on the time only, reducing the Einstein and other governing equations to ordinary differential equations.

The Bianchi models are studied in the well known ADM decomposition ([157, 158]) according to which the metric is written

$$ds^2=-N^2(t)dt^2+g_{\mu\nu}\omega^{\mu}\otimes\omega^{\nu} \tag{9.197}$$

where $N(t)$ is the lapse function and $\{\omega^a\}$ is the canonical basis of 1-forms which satisfy the Lie algebra

$$d\omega^i=C^i_{jk}\omega^j\wedge\omega^k. \tag{9.198}$$

C^i_{jk} are the structure constants of the algebra. The spatial metric $g_{\mu\nu}$ splits so that

$$g_{\mu\nu}=\exp(2\lambda)\exp(-2\beta)_{\mu\nu} \tag{9.199}$$

where $\exp(2\lambda)$ is the scale factor of the universe and $\beta_{\mu\nu}$ is a 3×3 symmetric, traceless matrix, which can be written in a diagonal form with two independent quantities, the so called anisotropy parameters β_+,β_-, as follows:

$$\beta_{\mu\nu}=diag\left(\beta_+,-\frac{1}{2}\beta_++\frac{\sqrt{3}}{2}\beta_-,-\frac{1}{2}\beta_+-\frac{\sqrt{3}}{2}\beta_-\right). \tag{9.200}$$

The Bianchi models are grouped in classes A and B by means of a vector a^{μ} and a symmetric 3×3 metric $n_{\mu\nu}$ which are constrainted by the condition $n_{\mu\nu}a^{\nu}=0$. Class A is defined by $a^{\mu}=0$ and Class B by $a^{\mu}\neq0$. Each

[3] In the special case of $d=2$, the system admits an additional Lie symmetry $\partial_x+\partial_y$, with Noether integral $I=\dot{x}-\dot{y}$.

Class is divided into several types according to the rank and (the modulus of the) signature of $n^{\mu\nu}$. Because of the difficulty in formulating the class B Bianchi models in the ADM formalism, it is usually the case that one confines attention to the class A models. Furthermore it is well known that for the class A models there is a Lagrangian [159] whereas for the class B models, to the author's knowledge, no such Lagrangian seems to exist. Details on the structure and the Physics of the Bianchi models can be found e.g. in [157, 159].

Research in Physics on inflationary models has shown the importance of scalar fields in cosmology [160]. This has raised interest in the dynamics of Bianchi spacetimes filled with a scalar field, with an arbitrary self interaction potential, minimally coupled to the gravitational field [161]. The Lagrangian leading to the full Bianchi scalar field dynamics is

$$L = e^{3\lambda}\left[R^* + 6\lambda - \frac{3}{2}(\dot{\beta}_1^2 + \dot{\beta}_2^2) - \dot{\phi}^2 + V(\phi)\right] \tag{9.201}$$

where R^* is the Ricci scalar of the 3 dimensional spatial hypersurfaces given by the expression:

$$\begin{aligned} R^* = &-\frac{1}{2}e^{-2\lambda}\left[N_1^2 e^{4\beta_1} + e^{-2\beta_1}\left(N_2 e^{\sqrt{3}\beta_2} - N_3 e^{-\sqrt{3}\beta_2}\right)^2 - 2N_1 e^{\beta_1}\left(N_2 e^{\sqrt{3}\beta_2} - N_3 e^{-\sqrt{3}\beta_2}\right)\right] \\ &+ \frac{1}{2}N_1 N_2 N_3 (1 + N_1 N_2 N_3). \end{aligned}$$

The constants N_1, N_2, and N_3 are the components of the classification vector n^μ and $\beta_1 = -\frac{1}{2}\beta_+ + \frac{\sqrt{3}}{2}\beta_-$, $\beta_2 = -\frac{1}{2}\beta_+ - \frac{\sqrt{3}}{2}\beta_-$. It is important to note that the curvature scalar R^* does not depend on the derivatives of the anisotropy parameters β_+, β_- , equivalently on β_1, β_2.

The Euler Lagrange equations due to the Lagrangian (9.201) are [86]:

$$\begin{aligned} \ddot{\lambda} + \frac{3}{2}\dot{\lambda}^2 + \frac{3}{8}(\dot{\beta}_1^2 + \dot{\beta}_2^2) + \frac{1}{4}\dot{\phi}^2 - \frac{1}{12}e^{-3\lambda}\frac{\partial}{\partial\lambda}\left(e^{3\lambda}R^*\right) - \frac{1}{2}V(\phi) &= 0 \\ \ddot{\beta}_1 + 3\dot{\lambda}\dot{\beta}_1 + \frac{1}{3}\frac{\partial R^*}{\partial \beta_1} &= 0 \\ \ddot{\beta}_2 + 3\dot{\lambda}\dot{\beta}_2 + \frac{1}{3}\frac{\partial R^*}{\partial \beta_2} &= 0 \\ \ddot{\phi} + 3\dot{\phi}\dot{\lambda} + \frac{\partial V}{\partial \phi} &= 0 \end{aligned}$$

where a dot over a symbol indicates derivative with respect to t.

In the following we apply Theorem 4.2.2 and Theorem 4.3.2 and compute the Lie and the Noether point symmetries of class A Bianchi models. The Lie and the Noether point symmetries of Bianchi class A models with a scalar field have also been computed in [86, 143, 139]. However, as it will be shown, these studies are not complete, in the sense that they have not found all Noether symmetries. Furthermore our approach is entirely different than the classical Lie approach employed in these works. Finally it is general and can be applied without difficulty to Class B spacetimes.

We consider the four dimensional Riemannian space with coordinates $x^i = (\lambda, \beta_1, \beta_2, \phi)$ and metric

$$ds^2 = e^{3\lambda}\left(12d\lambda^2 - 3d\beta_1^2 - 3d\beta_2^2 - 2d\phi^2\right). \tag{9.202}$$

The metric (9.202) is the conformally flat FRW spacetime whose special projective algebra consists of the non gradient KVs [46, 45]

$$Y^1 = \partial_{\beta_1},\ Y^2 = \partial_{\beta_2},\ Y^3 = \partial_\phi,\ Y^4 = \beta_2\partial_{\beta_1} - \beta_1\partial_{\beta_2}$$
$$Y^5 = \phi\partial_{\beta_1} - \frac{3}{2}\beta_1\partial_\phi,\ Y^6 = \phi\partial_{\beta_2} - \frac{3}{2}\beta_2\partial_\phi$$

and the gradient HV

$$H^i = \frac{2}{3}\partial_\lambda\ ,\ \psi = 1.$$

The Lagrangian (9.201) is written:

$$L = T - U(x^i)$$

where $T = \frac{1}{2}g_{ij}\dot{x}^i\dot{x}^i$ is the geodesic Lagrangian, $U(x^i)$ is the potential function

$$U(x^i) = -e^{3\lambda}\left(V(\phi) + R^*\right) \tag{9.203}$$

and we have used the fact that the curvature scalar does not depend on the derivatives of the coordinates β_1, β_2. Now we apply Theorem 4.2.2 and Theorem 4.3.2 to determine the Lie and the point Noether symmetries of the dynamical system with Lagrangian (9.201).

In order to compute the potential $U(x^i)$ we need the expression of R^* for each Bianchi type. We find for the Class A models

Bianchi I: $R^* = 0$

Bianchi II: $R^* = -e^{(2\beta_1 - \lambda)}$

Bianchi VI$_0$: Class A. , $R^* = -\frac{1}{2}e^{-2\lambda}\left(e^{4\beta_1} + e^{-2\left(\beta_1 - \sqrt{3}\beta_2\right)} + 2e^{\beta_1 + \sqrt{3}\beta_2}\right)$

Bianchi VII$_0$: Class A., $R^* = -\frac{1}{2}e^{-2\lambda}\left(e^{4\beta_1} + e^{-2\left(\beta_1 - \sqrt{3}\beta_2\right)} - 2e^{\beta_1 + \sqrt{3}\beta_2}\right)$

Bianchi VIII: $R^* = -\frac{1}{2}e^{-2\lambda}\left(e^{4\beta_1} + e^{-2\beta_1}\left(e^{\sqrt{3}\beta_2} + e^{-\sqrt{3}\beta_2}\right)^2 - 2e^{\beta_1}\left(e^{\sqrt{3}\beta_2} - e^{-\sqrt{3}\beta_2}x\right)\right)$

Bianchi IX: $R^* = -\frac{1}{2}e^{-2\lambda}\left(e^{4\beta_1} + e^{-2\beta_1}\left(e^{\sqrt{3}\beta_2} + e^{-\sqrt{3}\beta_2}\right)^2 - 2e^{\beta_1}\left(e^{\sqrt{3}\beta_2} - e^{-\sqrt{3}\beta_2}x\right)\right) + 1.$

We determine the Lie and the Noether point symmetries in the following cases:

Case 1. Vacuum. In this case ϕ =constant and the metric (9.202) reduces to the three dimensional FRW metric.

Case 2. Zero potential $V(\phi) = 0$, $\dot{\phi} \neq 0$

Case 3. Constant Potential $V(\phi)$ =constant, $\dot{\phi} \neq 0$

Case 4. Arbitrary Potential $V(\phi)$, $\dot{\phi} \neq 0$.

9.7.1 Bianchi I

Case 1.

In this case $\dot{\phi} = 0$, $V(\phi) = 0$ and the Lagrangian becomes $L = e^{3\lambda}\left[6\dot{\lambda}^2 - \frac{3}{2}\left(\dot{\beta}_1^2 + \dot{\beta}_2^2\right)\right]$ hence the potential $U(x^\mu) = 0$ where $x^\mu = (\lambda, \beta_1, \beta_2)$. The auxiliary metric is $ds^2 = e^{3\lambda}\left(12d\lambda^2 - 3d\beta_1^2 - 3d\beta_2^2\right)$. The special PCs

of this metric are the non gradient KVs $Y^1,\ Y^2,\ Y^4$ and the gradient HV H^i .
From Theorem 4.2.2 we find that the Lie point symmetries are the vectors

$$\partial_t,\ t\partial_t,\ Y^1,\ Y^2,\ Y^4, H^i,\ t^2\partial_t + tH^i$$

which coincide with those found in [86]. From Theorem 4.3.2 we find that the Noether point symmetries are

$$\partial_t,\ Y^1,\ Y^2,\ Y^4,\ 2t\partial_t + H^i, t^2\partial_t + tH^i$$

i.e. we find two more Noether symmetries than [86].

Case 2. $V(\phi) = 0\,,\ \dot{\phi} \neq 0$

In this case the Lagrangian is $L = e^{3\lambda}\left[6\dot{\lambda}^2 - \frac{3}{2}\left(\dot{\beta}_1^2 + \dot{\beta}_2^2\right) - \dot{\phi}^2\right]$ and the potential function $U(x^i) = 0$. The auxiliary metric is (9.202). From Theorem 4.2.2 we find that the Lie point symmetries are

$$\partial_t,\ \ t\partial_t,\ Y^1,\ Y^2,\ Y^3,\ Y^4,\ Y^5,\ Y^6,\ H^i,\ t^2\partial_t + tH^i$$

and coincide with those found in [86]. Application of Theorem 4.3.2 gives that the Noether point symmetries are

$$\partial_t,\ Y^1,\ Y^2,\ Y^3,\ Y^4,\ Y^5,\ Y^6,\ 2t\partial_t + H^i, t^2\partial_t + tH^i$$

i.e. two more from the ones found in [86].

Case 3. $V(\phi) = C_0,\ \dot{\phi} \neq 0$

The Lagrangian is $L = e^{3\lambda}\left[6\dot{\lambda}^2 - \frac{3}{2}\left(\dot{\beta}_1^2 + \dot{\beta}_2^2\right) - \dot{\phi}^2 + C_0\right]$ hence the potential $U(x^i) = -C_0 e^{3\lambda}$. Using Theorem 4.2.2 we find that the Lie point symmetries are

$$\partial_t,\ Y^1,\ Y^2,\ Y^3,\ Y^4,\ Y^5,\ Y^6,\ H^i,\ \frac{1}{C}e^{\pm Ct}\partial_t \pm e^{\pm Ct}H^i$$

where $C = \frac{\sqrt{6C_0}}{2}$, and coincide with those found in [86]. Application of Theorem 4.3.2 gives the Noether point symmetries

$$\partial_t,\ Y^1,\ Y^2,\ Y^3,\ Y^4,\ Y^5,\ Y^6,\ \frac{1}{C}e^{\pm Ct}\partial_t \pm e^{\pm Ct}H^i.$$

Again we find two more Noether symmetries than [86].

Case 4. $V(\phi) =$arbitrary $\dot{\phi} \neq 0$

In this case the Lagrangian is $L = e^{3\lambda}\left[6\dot{\lambda}^2 - \frac{3}{2}\left(\dot{\beta}_1^2 + \dot{\beta}_2^2\right) - \dot{\phi}^2 + V(\phi)\right]$ and the potential $U(x^i) = -e^{3\lambda}V(\phi)$. Application of Theorem 4.2.2 gives the Lie point symmetries ∂_t , Y^1 , Y^2 , Y^4 , H^i and application of Theorem 4.3.2 gives the Noether symmetries ∂_t , Y^1 , Y^2 , Y^4.

Working in a similar manner we compute the Lie and the Noether point symmetries of all Bianchi class A homogenous spacetimes. The results of the calculations are collected in the following Tables.

Table 9.1: Lie and Noether Symmetries of Bianchi I scalar field

Bianchi I	Noether Symmetries	Lie Symmetries
Case 1	$\partial_t,\ Y^1,\ Y^2,\ Y^4$	$\partial_t,\ t\partial_t,\ Y^1,\ Y^2,\ Y^4,\ H^i$
	$2t\partial_t + H^i,\ t^2\partial_t + tH^i$	$t^2\partial_t + tH^i$
Case 2	$\partial_t,\ Y^1,\ Y^2,\ Y^3,\ Y^4,\ Y^5,\ Y^6$	$\partial_t,\ t\partial_t,\ Y^1,\ Y^2,\ Y^3,\ Y^4, Y^5,\ Y^6$
	$2t\partial_t + H^i,\ t^2\partial_t + tH^i$	$H^i,\ t^2\partial_t + tH^i$
Case 3	$\partial_t,\ Y^1,\ Y^2,\ Y^3,\ Y^4,\ Y\ ,\ Y^6$	$\partial_t,\ Y^1,\ Y^2,\ Y^3,\ Y^4,\ Y^5,\ Y^6,\ H^i$
	$\frac{1}{C}e^{\pm Ct}\partial_t \pm e^{\pm Ct}H^i$	$\frac{1}{C}e^{\pm Ct}\partial_t \pm e^{\pm Ct}H^i$
Case 4	$\partial_t,\ Y^1,\ Y^2,\ Y^4$	$\partial_t,\ Y^1,\ Y^2,\ Y^4\ ,\ H^i$

Table 9.2: Lie and Noether Symmetries of Bianchi II scalar field

Bianchi II	Noether Symmetries	Lie Symmetries
Case 1	$\partial_t,\ Y^2,\ 6t\partial_t + 3H^i - 5Y^1$	$\partial_t,\ Y^2,\ \frac{1}{3}t\partial_t + H^i,\ t\partial_t - Y^1$
Case 2	$\partial_t,\ Y^2,\ Y^3,\ Y^6,\ 6t\partial_t + 3H^i - 5Y^1$	$\partial_t,\ Y^2,\ Y^3,\ Y^6,\ \frac{1}{3}t\partial_t + H^i,\ t\partial_t - Y^1$
Case 3	$\partial_t,\ Y^2,\ Y^3,\ Y^6$	$\partial_t,\ Y^2,\ Y^3,\ Y^6,\ 3H^i + Y^1$
Case 4	$\partial_t,\ Y^2$	$\partial_t,\ Y^2,\ 3H^i + Y^1$

Table 9.3: Lie and Noether Symmetries of Bianchi VI/VII scalar field

Bianchi VI$_0$ / VII$_0$	Noether Symmetries	Lie Symmetries
Case 1	$\partial_t,\ \ 6t\partial_t + 3H^i - 2Y^1 - 2\sqrt{3}Y^2$	$\partial_t,\ H^i + \frac{1}{3}Y^1 + \frac{\sqrt{3}}{3}Y^2,\ 2t\partial_t - Y^1 - \sqrt{3}Y^2$
Case 2	$\partial_t,\ Y^3,\ 6t\partial_t + 3H^i - 2Y^1 - 2\sqrt{3}Y^2$	$\partial_t,\ Y^3,\ H^i + \frac{1}{3}Y^1 + \frac{\sqrt{3}}{3}Y^2,\ 2t\partial_t - Y^1 - \sqrt{3}Y^2$
Case 3	$\partial_t,\ Y^3$	$\partial_t,\ Y^3,\ H^i + \frac{1}{3}Y^1 + \frac{\sqrt{3}}{3}Y^2$
Case 4	∂_t	$\partial_t,\ H^i + \frac{1}{3}Y^1 + \frac{\sqrt{3}}{3}Y^2$

Table 9.4: Lie and Noether Symmetries of Bianchi VIII/IX scalar field

Bianchi VIII	Noether Symmetries	Lie Symmetries
Case 1	∂_t	$\partial_t,\ \frac{2}{3}t\partial_t\ + H^i$
Case 2	$\partial_t,\ Y^3$	$\partial_t,\ Y^3,\ \frac{2}{3}t\partial_t\ + H^i$
Case 3	$\partial_t,\ Y^3$	$\partial_t,\ Y^3$
Case 4	∂_t	∂_t

Bianchi IX	Noether Symmetries	Lie Symmetries
Case 1	∂_t	∂_t
Case 2	$\partial_t,\ Y^3$	$\partial_t,\ Y^3$
Case 3	$\partial_t,\ Y^3$	$\partial_t,\ Y^3$
Case 4	∂_t	∂_t

From the above tables we infer that the Lie point symmetries we found coincide with those of [86]. Some differences which appear are due to linear combinations of symmetries from the other set. The same does not apply to the Noether symmetries, for which we found a larger number than in [86].

We note that in Case 1 the Noether symmetry $2t + H$ is a combination of two Lie point symmetries, which is peculiar since the Noether point symmetries are considered to be a direct subset of Lie symmetries. This is explained as follows. The addition of a Killing vector to a homothetic vector retains a homothetic vector. Therefore a Lie symmetry due to a Killing vector and one due to a homothetic vector is possible to give a Lie point symmetry due to a homothetic vector. Concerning [143] from the examination of the Tables 9.1-9.4 and the results they present it can be seen that they loose the Noether symmetries which have a component along ∂_t direction.

The Bianchi I model with scalar field and exponential potential

We consider a scalar field described by an exponential potential $V(\phi) = e^{-d\phi}$ in a Bianchi class A spacetime. For this potential all the models admit the extra Lie symmetry $t\partial_t + \frac{2}{d}Y^3$. Concerning the Noether symmetries we have an extra Noether symmetry only for the types I. II, VI_0, VII $_0$ as follows:

Type I

$$t\partial_t + \frac{1}{2}H^i + \frac{2}{d}Y^3$$

Type II

$$t\partial_t + \frac{1}{2}H^i - \frac{5}{6}Y^1 + \frac{2}{d}Y^3$$

Type VI_0 / VII_0

$$6t\partial_t + 3H^i - 2Y^1 - 2\sqrt{3}Y^2 + \frac{6}{d}Y^3.$$

In the following we concentrate on the Bianchi I model and make use of the extra Noether integral to define a transformation [162] which allows the determination of the analytic form of the metric.

The Lagrangian describing a scalar field with exponential potential in an empty Bianchi I spacetime is

$$L = e^{3\lambda}\left[6\dot{\lambda}^2 - \frac{3}{2}\left(\dot{\beta}_1^2 + \dot{\beta}_2^2\right) - \dot{\phi}^2 + V_0 e^{-d\phi}\right] \tag{9.204}$$

and the corresponding Hamiltonian vanishes. The metric defined by this Lagrangian is (9.202).

Using the transformation $\lambda = \frac{1}{3}\ln\left(\frac{a^3}{2}\right)$ the Lagrangian becomes

$$L = 3a\dot{a}^2 - \frac{1}{2}a^3\dot{\phi} - \frac{3}{4}a^3\left(\dot{\beta}_1^2 + \dot{\beta}_2^2\right) + \frac{1}{2}V_0 a^3 e^{-d\phi}.$$

We change variables by means of the transformation

$$u = \frac{\sqrt{6}}{4}\phi + \frac{1}{2}\ln\left(a^3\right) \ , \ v = -\frac{\sqrt{6}}{4}\phi + \frac{1}{2}\ln\left(a^3\right)$$

and the Lagrangian takes the form

$$L\left(u,v,\dot{u},\dot{v}\right)=e^{(u+v)}\left(\frac{8}{3}\dot{u}\dot{v}-\frac{3}{2}\left(\dot{\beta}_1^2+\dot{\beta}_2^2\right)+V_0e^{-2K(u-v)}\right).$$

Next we change the time coordinate as follows

$$\frac{d\tau}{dt}=\sqrt{\frac{3V_0}{8}}e^{-K(u-v)}.$$

We make one more change $\beta_1=\sqrt{\frac{9}{16}}B_1$, $\beta_2=\sqrt{\frac{9}{16}}B_2$ and in the coordinates τ,u,v,B_1,B_2 the Lagrangian is:

$$L\left(\tau,u,v,B_1,B_2\right)=e^{(u+v)}e^{-K(u-v)}\left(u'v'-\left(B_1'^2+B_2'^2\right)+1\right) \tag{9.205}$$

where $u'=\frac{du}{d\tau}$, $v'=\frac{dv}{d\tau}$, $B_{1,2}'=\frac{dB_{1,2}}{d\tau}$. The equations of motion are

$$u''+(1-K)\,u'^2+(1+K)\,B_1^{'2}+(1+K)\,B_2^{'2}-(1+K)=0 \tag{9.206}$$

$$v''+(1+K)\,v'^2+(1-K)\,B_1^{'2}+(1-K)\,B_2^{'2}-(1-K)=0 \tag{9.207}$$

$$B_1^{''}+(1-K)\,B_1'u'+(1+K)\,B_1^{'}v'=0 \tag{9.208}$$

$$B_2^{''}+(1-K)\,B_2^{'}u'+(1+K)\,B_2'v'=0 \tag{9.209}$$

with constrain (the zero Hamiltonian) $u'v'-\left(B_1'^2+B_2'^2\right)-1=0$. These expressions are symmetric in B_1,B_2 therefore we set $B_1=B_{2.}=\frac{1}{2}B$ and the system of equations of motion becomes

$$u''+(1-K)\,u'^2+(1+K)\,B^{'2}-(1+K)=0 \tag{9.210}$$

$$v''+(1+K)\,v'^2+(1-K)\,B^{'2}-(1-K)=0 \tag{9.211}$$

$$B^{''}+(1-K)\,B'u'+(1+K)\,B^{'}v'=0 \tag{9.212}$$

with constraint

$$u'v'-B'^2-1=0. \tag{9.213}$$

We consider two cases $K=1$ and $K\neq 1$.

For $K=1$ the metric is $ds^2=2e^{2v}\left(dudv-dB^2\right)$. The potential is the gradient KV $V\left(u,v\right)=-e^{2v}$ and the solution of the system is

$$u\left(\tau\right)=\tau^2+2c_1^2\ln\left(\tau+c_2\right)+2tc_2 \tag{9.214}$$

$$v\left(\tau\right)=\frac{1}{2}\ln\left(2\tau+c_2\right) \tag{9.215}$$

$$B\left(\tau\right)=c_1\ln\left(\tau+c_2\right). \tag{9.216}$$

For $K\neq 1$ we make use of the extra Noether integral $e^{(u+v)}e^{-K(u-v)}\dot{B}=C$ and solve the system. We find that $K=\frac{1-C^2}{1+C^2}$ and that the system has two solutions. The first is:

$$u(\tau)=\frac{1}{2(1-K)}\ln\left(\frac{K-1}{1+K}\sin\left(2\sqrt{K^2-1}\tau\right)\right) \tag{9.217}$$

$$v(\tau)=\frac{1}{2(1+K)}\ln\left(\sin\left(2\sqrt{K^2-1}\tau\right)\right) \tag{9.218}$$

$$B(\tau)=\frac{i}{2\sqrt{K^2-1}}\operatorname{arctanh}\left(\cos\left(2\sqrt{K^2-1}\tau\right)\right) \tag{9.219}$$

and the second:

$$u(\tau)=\frac{1}{2(1-K)}\ln\left(\frac{K-1}{1+K}\cos\left(2\sqrt{K^2-1}\tau\right)\right) \tag{9.220}$$

$$v(\tau)=\frac{1}{2(1+K)}\ln\left(\cos\left(2\sqrt{K^2-1}\tau\right)\right) \tag{9.221}$$

$$B(\tau)=\frac{i}{2\sqrt{K^2-1}}\operatorname{arctanh}\left(\frac{1}{\sin\left(2\sqrt{K^2-1}\tau\right)}\right). \tag{9.222}$$

These solutions complement the results of [163]

9.8 Conclusion

In this chapter we have studied conformally related metrics and Lagrangians in the context of scalar–tensor cosmology. We have found that to every non-minimally coupled scalar field we can associate a unique minimally coupled scalar field in a conformally related space with an appropriate potential. The existence of such a connection can be used in order to study the dynamical properties of the various cosmological models, since the field equations of a non-minimally coupled scalar field are the same, at the conformal level, of the field equations of the minimally coupled scalar field. The above propositions can be extended to general Riemannian spaces in n-dimensions. Furthermore, we have identified the Noether point symmetries and the analytic solutions of the equations of motion in the context of a minimally coupled and a non minimally coupled scalar field in a FRW spacetime and we have classified the Noether symmetries of the field equations in Bianchi class A models with a minimally coupled scalar field. We found that there is a rather large class of hyperbolic and exponential potentials which admit extra (beyond the ∂_t) Noether pont symmetries which lead to integrals of motions.

In general, the Noether point symmetries play an important role in physics because they can be used to simplify a given system of differential equations as well as to determine the integrability of the system. The latter will provide the necessary platform in order to solve the equations of motion analytically and thus to obtain the evolution of the physical quantities. In cosmology, such a method is extremely relevant in order to compare cosmographic parameters, such as scale factor, Hubble expansion rate, deceleration parameter, density parameters with observational constrains.

However, since the Noether point symmetries are generated from the kinetic metric of the Lagrangian, they are not only a criterion for the integrability of the system and a method to determine analytical solutions of the field equations, but they are also a geometric criterion since by demanding the existence of Noether symmetries we let the geometry to select the dynamics, i.e. the dark energy model.

In the following chapters we study the Noether symmetries of the $f(R)$ and the $f(T)$ theories of gravity.

9.A Relating the ranges of the constants F_0 and $|k|$

We consider the following ranges for the constants F_0 and $|k|$

a. $F_0 > 0$. In this case we have $|k| = \frac{1}{3}\sqrt{\frac{F_0}{F_0+1}}$ from which follows

$$|k|^2 < 1 \ , \ F_0 > 0$$

b. $-1 < F_0 < 0$. In this case we have $|k| = \frac{1}{3}\sqrt{\frac{|F_0|}{F_0+1}}$ from which follows.

$$|k|^2 = 1 \ , \ F_0 = -\frac{9}{10} \ ; \ |k|^2 < 1, \ F_0 > -\frac{9}{10}; \ |k|^2 > 1 \ , \ F_0 \in \left(-1, -\frac{9}{10}\right)$$

c. $F_0 < 1$. Then $|k| = \frac{1}{3}\sqrt{\frac{|F_0|}{|F_0|-1}}$ from which follows

$$|k|^2 = 1 \ , \ F_0 = -\frac{9}{8} \ ; \ |k|^2 < 1 \ , \ F_0 < -\frac{9}{8}; |k|^2 > 1 \ , \ F_0 \in \left(-\frac{9}{8}, -1\right).$$

The ranges of $|k|$ are needed because they select different groups of Killing vectors of the metric (9.41).

9.B Computation of the gradient functions $S_1(r,\theta)$, $S_2(r,\theta)$

The functions $S_1(r,\theta)$, $S_2(r,\theta)$ are the canonical coordinates x, y for the KVs K^1, K^2. The canonical coordinates are defined with the requirement $K^1 = \frac{\partial}{\partial x}, K^2 = \frac{\partial}{\partial y}$ and are computed as follows. We have the system of differential equations:

$$\begin{aligned} \frac{\partial}{\partial y} &= \frac{e^{(1-k)\theta} r^k}{N_0^2}\left(-\partial_r + \frac{1}{r}\partial_\theta\right) \\ \frac{\partial}{\partial x} &= \frac{e^{-(1+k)\theta} r^{-k}}{N_0^2}\left(\partial_r + \frac{1}{r}\partial_\theta\right). \end{aligned}$$

To solve it we consider the associated Lagrange system and write:

$$\frac{dy}{1} = -N_0^2 \frac{dr}{r^k e^{(1-k)\theta}} = N_0^2 \frac{d\theta}{r^{k-1} e^{(1-k)\theta}}$$

The first equation gives:

$$y = -N_0^2 e^{(-1+k)\theta} \int \frac{dr}{r^k} = -N_0^2 e^{(-1+k)\theta} \frac{r^{1-k}}{1-k} + \Phi(\theta)$$

The second equation gives:

$$y = N_0^2 r^{-k+1} \int \frac{d\theta}{e^{(1-k)\theta}} = N_0^2 \frac{r^{1-k}}{-1+k} e^{(1-k)\theta} + \Phi_1(r)$$

hence we have (this is the $S_2(r,\theta)$):

$$y = N_0^2 \frac{r^{1-k}}{k-1} e^{-(1-k)\theta}.$$

For the other coordinate we have:

$$\frac{dx}{1} = N_0^2 \frac{dr}{r^{-k} e^{-(1+k)\theta}} = N_0^2 \frac{d\theta}{r^{-k-1} e^{-(1+k)\theta}}$$

The first equation gives:

$$x = N_0^2 e^{(1+k)\theta} \int \frac{dr}{r^{-k}} = N_0^2 e^{(1+k)\theta} \frac{1}{1+k} r^{1+k} + \Phi(\theta)$$

and the second equation gives:

$$x = N_0^2 r^{1+k} \int \frac{d\theta}{e^{-(1+k)\theta}} = \frac{1}{1+k} N_0^2 r^{1+k} e^{(1+k)\theta} + \Phi_1(r)$$

hence (this is the $S_1(r,\theta)$):

$$x = \frac{1}{1+k} N_0^2 r^{1+k} e^{(1+k)\theta}.$$

Therefore, we have the canonical coordinates

$$x = \frac{r^{1+k} e^{-(1+k)\theta}}{k+1} \;,\; y = \frac{r^{1-k} e^{(-1+k)\theta}}{k-1}$$

Chapter 10

Using Noether point symmetries to specify $f(R)$ gravity

10.1 Introduction

In chapter 9 we used the Noether point symmetries of the scalar tensor theories in order to constrain the dark energy models. Except the sclarar field cosmology there are other possibilities to explain the present accelerating stage. For instance, one may consider that the dynamical effects attributed to dark energy can be resembled by the effects of a nonstandard gravity theory. In other words, the present accelerating stage of the universe can be driven only by cold dark matter, under a modification of the nature of gravity. Such a reduction of the so-called dark sector is naturally obtained in the $f(R)$ gravity theories [87]. In the original nonstandard gravity models, one modifies the Einstein-Hilbert action with a general function $f(R)$ of the Ricci scalar R. The $f(R)$ approach is a relatively simple but still a fundamental tool used to explain the accelerated expansion of the universe. A pioneering fundamental approach was proposed long ago with $f(R) = R + mR^2$ [164]. Later on, the $f(R)$ models were further explored from different points of view in [89, 88, 165] and indeed a large number of functional forms of $f(R)$ gravity is currently available in the literature [140, 166, 167, 168, 169].

In the following, we will use the Lie and the Noether point symmetries in order to specify the $f(R)$ gravity in a FRW spacetime and use the first integrals of these models to determine analytic solutions of their field equations.

The structure of this chapter is as follows. The basic theoretical elements of the problem are presented in section 10.2, where we also introduce the basic FRW cosmological equations in the framework of $f(R)$ models. The Noether point symmetries and their relevance to the $f(R)$ models are discussed in section 10.4. In section 10.5 we provide analytical solutions for those $f(R)$ models which are Liouville integrable via Noether point symmetries. In section 10.6 we study the Noether point symmetries in spatially non-flat $f(R)$ cosmological models. Finally, we draw our main conclusions in section 10.7.

10.2 Cosmology with a modified gravity

Consider the modified Einstein-Hilbert action:

$$S = \int d^4x\sqrt{-g}\left[\frac{1}{2k^2}f\left(R\right) + \mathcal{L}_m\right] \tag{10.1}$$

where $\mathcal{L}_m$ is the Lagrangian of dust-like ($p_m = 0$) matter and $k^2 = 8\pi G$. Varying the action with respect to the metric[1] we arrive at

$$(1+f')G^{\mu}_{\nu} - g^{\mu\alpha}f_{R,\alpha;\nu} + \left[\frac{2\Box f' - (f - Rf')}{2}\right]\delta^{\mu}_{\nu} = k^2\,T^{\mu}_{\nu} \tag{10.2}$$

where the prime denotes derivative with respect to R, G^{μ}_{ν} is the Einstein tensor and T^{μ}_{ν} is the ordinary energy-momentum tensor of matter. Based on the matter era we treat the expanding universe as a dust fluid which includes only cold dark matter with comoving observers $U^{\mu} = \delta^{\mu}_{0}$. Thus the energy momentum tensor becomes $T_{\mu\nu} = \rho_m U_{\mu}U_{\nu}$, where ρ_m is the energy density of the cosmic fluid.

Now, in the context of a flat FRW model the metric is

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2). \tag{10.3}$$

The components of the Einstein tensor are computed to be:

$$G^0_0 = -3H^2,\; G^a_b = -\delta^a_b\left(2\dot{H} + 3H^2\right)\;. \tag{10.4}$$

Inserting (10.4) into the modified Einstein's field equations (10.2), for comoving observers, we derive the modified Friedman's equation

$$3f^{'}H^2 = k^2\rho_m + \frac{f^{'}R - f}{2} - 3Hf^{''}\dot{R} \tag{10.5}$$

$$2f^{'}\dot{H} + 3f^{'}H^2 = -2Hf^{''}\dot{R} - \left(f^{'''}\dot{R}^2 + f^{''}\ddot{R}\right) - \frac{f - Rf^{'}}{2}. \tag{10.6}$$

The contraction of the Ricci tensor provides the Ricci scalar

$$R = g^{\mu\nu}R_{\mu\nu} = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2}{a^2}\right) = 6(2H^2 + \dot{H})\;. \tag{10.7}$$

The Bianchi identity $\bigtriangledown^{\mu}T_{\mu\nu} = 0$ leads to the matter conservation law:

$$\dot{\rho}_m + 3H\rho_m = 0 \tag{10.8}$$

whose solution is

$$\rho_m = \rho_{m0}a^{-3}. \tag{10.9}$$

[1] We use the metric i.e. the Hilbert variational approach.

Note that the over-dot denotes derivative with respect to the cosmic time t and $H \equiv \dot{a}/a$ is the Hubble parameter.

If we consider $f(R) = R$ then the field equations (10.2) boil down to the Einstein's equations. On the other hand, the concordance Λ cosmology is fully recovered for $f(R) = R - 2\Lambda$.

From the current analysis it becomes clear that unlike the standard Friedman equations in Einstein's GR, the modified equations of motion (10.5) and (10.6) are complicated and thus it is difficult to solve them analytically.

We would like to stress here that within the context of the metric formalism the above $f(R)$ cosmological models must obey simultaneously some strong conditions [136]. These are: (i) $f^{'} > 0$ for $R \geq R_0 > 0$, where R_0 is the Ricci scalar at the present time. If the final attractor is a de Sitter point we need to have $f^{'} > 0$ for $R \geq R_1 > 0$, where R_1 is the Ricci scalar at the de Sitter point, (ii) $f^{''} > 0$ for $R \geq R_0 > 0$, (iii) $f(R) \approx R - 2\Lambda$ for $R \gg R_0$ and finally (iv) $0 < \frac{Rf^{''}}{f'}(r) < 1$ at $r = -\frac{Rf^{'}}{f} = -2$

10.3 Modified gravity versus symmetries

In the last decade a large number of experiments have been proposed in order to constrain dark energy and study its evolution. Naturally, in order to establish the evolution of the dark energy (DE) (”geometrical” in the current work) equation of state parameter a realistic form of $H(a)$ is required while the included free parameters must be constrained through a combination of independent DE probes (for example SNIa, BAOs, CMB etc). However, a weak point here is the fact that the majority of the $f(R)$ models appeared in the literature are plagued with no clear physical basis and/or many free parameters. Due to the large number of free parameters many such models could fit the data. The proposed additional criterion of Noether point symmetry requirement is a physically meaning-full geometric ansatz.

According to the theory of general relativity, the space-time Killing and homothetic symmetries via the Einstein's field equations, are also symmetries of the energy momentum tensor. Due to the fact that the $f(R)$ models provide a natural generalization of GR one would expect that the theories of modified gravity must inherit the symmetries of the space-time as the usual gravity (GR) does.

Furthermore, besides the geometric symmetries we have to consider the dynamical symmetries, which are the symmetries of the field equations (Lie symmetries). If the field equations are derived from a Lagrangian then there is the special class of Lie symmetries, the Noether symmetries, which lead to conserved currents or, equivalently, to first integrals of the equations of motion. The Noether integrals are used to reduce the order of the field equations or even to solve them. Therefore a sound requirement, which is possible to be made in Lagrangian theories, is that they admit extra Noether symmetries. This assumption is model independent, because it is imposed after the field equations have been derived, therefore it does not lead to conflict with the geometric symmetries while, at the same time, serves the original purpose of a selection rule. Of course, it is possible that a different method could be assumed and select another subset of viable models. However, symmetry has always played a dominant role in Physics and this gives an aesthetic and a physical priority to our proposal.

In the Lagrangian context, the main field equations (10.5) and (10.6), described in section 10.2, can be produced by the following Lagrangian:

$$L\left(a,\dot{a},R,\dot{R}\right)=6af^{'}\ \dot{a}^{2}+6a^{2}f^{''}\ \dot{a}\dot{R}+a^{3}\left(f^{'}R-f\right) \tag{10.10}$$

in the space of the variables $\{a,R\}$. Using eq.(10.10) we obtain the Hamiltonian of the current dynamical system

$$E=6af^{'}\ \dot{a}^{2}+6a^{2}f^{''}\ \dot{a}\dot{R}-a^{3}\left(f^{'}R-f\right) \tag{10.11}$$

or

$$E=6a^{3}\left[f^{'}H^{2}-\frac{1}{6f^{'}}\left(\left(f^{'}R-f\right)-6\dot{R}Hf^{''}\right)\right]\ . \tag{10.12}$$

Combining the first equation of motion (10.5) with eq.(10.12) we find

$$\rho_{m}=\frac{E}{2k^{2}}\ a^{-3}\ . \tag{10.13}$$

The latter equation together with $\rho_{m}=\rho_{m0}a^{-3}$ implies that

$$\rho_{m0}=\frac{E}{2k^{2}}\Rightarrow\Omega_{m}\rho_{cr,0}=\frac{E}{2k^{2}}\Rightarrow E=6\Omega_{m}H_{0}^{2} \tag{10.14}$$

where $\Omega_{m}=\rho_{m0}/\rho_{cr,0}$, $\rho_{cr,0}=3H_{0}^{2}/k^{2}$ is the critical density at the present time and H_0 is the Hubble constant.

We note that the current Lagrangian eq.(10.10) is time independent implying that the dynamical system is autonomous hence the Hamiltonian E is conserved.

10.4 Noether point symmetries of $f\left(R\right)$ gravity

The Noether condition for the Lagrangian (10.10) is equivalent with the following system of eight equations

$$\xi_{,a}=0 \tag{10.15}$$

$$\xi_{,R}=0 \tag{10.16}$$

$$a^{2}f''\eta_{,R}^{(1)}=0 \tag{10.17}$$

$$f'\eta^{(1)}+af''\eta^{(2)}+2af'\eta_{,a}^{(1)}+a^{2}f''\eta_{,a}^{(2)}-\frac{1}{2}af'\xi_{,t}=0 \tag{10.18}$$

$$2af''\eta^{(1)}+a^{2}f'''\eta^{(2)}+a^{2}f''\eta_{,a}^{(1)}+2af'\eta_{,R}^{(1)}+a^{2}f''\eta_{,R}^{(2)}-\frac{1}{2}a^{2}f''\xi_{,t}=0 \tag{10.19}$$

$$-3a^{2}Rf'\eta^{(1)}+3a^{2}f\eta^{(1)}-a^{3}Rf''\eta^{(2)}+a^{3}\left(f-f^{'}R\right)\xi_{,t}+g_{,t}=0 \tag{10.20}$$

$$12af'\eta_{,t}^{(1)}+6a^{2}f''\eta_{,t}^{(2)}+a^{3}\left(f^{'}R-f\right)\xi_{,a}-g_{,a}=0 \tag{10.21}$$

$$6a^2 f''\eta_{,t}^{(1)} + a^3\left(f^{'}R - f\right)\xi_{,R} - g_{,R} = 0 \tag{10.22}$$

The solution of the system (10.15)-(10.22) determines the Noether symmetries.

Since the Lagrangian (10.10) is in the form $L = T\left(a, \dot{a}, R, \dot{R}\right) - V(a, R)$, the results of chapter 4 can be used [2]. The kinematic term defines a two dimensional metric in the space of $\{a, R\}$ with line element

$$d\hat{s}^2 = 12af'da^2 + 12a^2 f''da\ dR \tag{10.23}$$

while the "potential" is

$$V(a, R) = -a^3(f^{'}R - f)\ . \tag{10.24}$$

The Ricci scalar of the two dimensional metric (10.23) is computed to be $\hat{R} = 0$, therefore the space is a flat space[3] with a maximum homothetic algebra. The homothetic algebra of the metric (10.23) consists of the vectors

$$\mathbf{K}^1 = a\partial_a - 3\frac{f'}{f''}\partial_R\ ,\ \mathbf{K}^2 = \frac{1}{a}\partial_a - \frac{1}{a^2}\frac{f'}{f''}\partial_R$$
$$\mathbf{K}^3 = \frac{1}{a}\frac{1}{f''}\partial_R\ ,\ \mathbf{H} = \frac{a}{2}\ \partial_a + \frac{1}{2}\frac{f'}{f''}\partial_R$$

where $\mathbf{K}$ are Killing vectors ($\mathbf{K}^{2,3}$ are gradient) and $\mathbf{H}$ is a gradient Homothetic vector.

Therefore applying theorem 4.3.2 we have the following cases:

Case 1: If $f(R)$ is arbitrary the dynamical system admits as Noether symmetry the $X^1 = \partial_t$ with Noether integral the Hamiltonian E.

Case 2: If $f(R) = R^{\frac{3}{2}}$ the dynamical system admits the extra Noether symmetries

$$X^2 = \mathbf{K}^2,\ X^3 = t\mathbf{K}^2 \tag{10.25}$$

$$X^4 = 2t\partial_t + \mathbf{H} + \frac{5}{6}\mathbf{K}^1. \tag{10.26}$$

with corresponding Noether Integrals

$$I_2 = \frac{d}{dt}\left(a\sqrt{R}\right) \tag{10.27}$$

$$I_3 = t\frac{d}{dt}\left(a\sqrt{R}\right) - a\sqrt{R} \tag{10.28}$$

$$I_4 = 2tE - 6a^2\dot{a}\sqrt{R} - 6\frac{a^3}{\sqrt{R}}\dot{R}. \tag{10.29}$$

the non vanishing commutators of the Noether algebra being

$$\left[X^1, X^3\right] = X^2 \qquad \left[X^1, X^4\right] = 2X^1$$

[2] Where T is the "kinetic" term and V is the "potential

[3] All two dimensional Riemannian spaces are Einstein spaces implying that if $\hat{R} = const$ the space is maximally symmetric [23] and if $\hat{R} = 0$, the space admit gradient homothetic vector, i.e. is flat.

$$\left[X^2, X^4\right] = \frac{8}{3}X^2 \qquad \left[X^3, X^4\right] = \frac{2}{3}X^3$$

Case 3: If $f(R) = R^{\frac{7}{8}}$ the dynamical system admits the extra Noether symmetries

$$X^5 = 2t\partial_t + \mathbf{H}\ ,\ X^6 = t^2\partial_t + t\mathbf{H} \tag{10.30}$$

with corresponding Noether Integrals

$$I_5 = 2tE - \frac{21}{8}\frac{d}{dt}\left(a^3 R^{-\frac{1}{8}}\right) \tag{10.31}$$

$$I_6 = t^2 E - \frac{21}{8}t\frac{d}{dt}\left(a^3 R^{-\frac{1}{8}}\right) + \frac{21}{8}a^3 R^{-\frac{1}{8}}. \tag{10.32}$$

with non vanishing commutators

$$\left[X^1, X^5\right] = 2X^1 \qquad \left[X^1, X^6\right] = X^5 \qquad \left[X^5, X^6\right] = 2X^6$$

From the time dependent integrals (10.31),(10.32) and the Hamiltonian we construct the Ermakov-Lewis invariant (see chapter 5)

$$\Sigma = 4I_6 E - I_5^2 \tag{10.33}$$

Case 4: If $f(R) = (R - 2\Lambda)^{\frac{3}{2}}$ the dynamical system admits the extra Noether symmetries

$$\bar{X}^2 = e^{\sqrt{m}t}\mathbf{K}^2\ ,\ \bar{X}^3 = e^{-\sqrt{m}t}\mathbf{K}^2 \tag{10.34}$$

with corresponding Noether Integrals

$$\bar{I}_2 = e^{\sqrt{m}t}\left(\frac{d}{dt}\left(a\sqrt{R-2L}\right) - 9\sqrt{m}a\sqrt{R-2\Lambda}\right) \tag{10.35}$$

$$\bar{I}_3 = e^{-\sqrt{m}t}\left(\frac{d}{dt}\left(a\sqrt{R-2L}\right) + 9\sqrt{m}a\sqrt{R-2\Lambda}\right) \tag{10.36}$$

where $m = \frac{2}{3}\Lambda$. The non vanishing commutators of the Noether algebra are

$$\left[X^1, \bar{X}^2\right] = \sqrt{m}\bar{X}^2 \qquad \left[\bar{X}^3, X^1\right] = \sqrt{m}\bar{X}^3$$

From the time dependent integrals (10.35),(10.36) we construct the time independent integral $\bar{I}_{23} = \bar{I}_2\bar{I}_3$.

Case 5: If $f(R) = (R - 2\Lambda)^{\frac{7}{8}}$ the dynamical system admits the extra Noether symmetries

$$\bar{X}^5 = \frac{1}{\sqrt{m}}e^{2\sqrt{m}t}\partial_t + e^{2\sqrt{m}t}\ \mathbf{H} \tag{10.37}$$

$$\bar{X}^6 = -\frac{1}{\sqrt{m}}e^{-2\sqrt{m}t}\partial_t + e^{-2\sqrt{m}t}\ \mathbf{H} \tag{10.38}$$

with corresponding Noether Integrals

$$\bar{I}_5 = e^{2\sqrt{m}t}\left[\frac{1}{\sqrt{m}}E - \frac{21}{8}\frac{d}{dt}\left(a^3(R-2\Lambda)^{-\frac{1}{8}}\right) + \frac{21}{4}\sqrt{m}a^3(R-2\Lambda)^{-\frac{1}{8}}\right] \tag{10.39}$$

$$\bar{I}_6 = e^{-2\sqrt{m}t}\left[\frac{1}{\sqrt{m}}E + \frac{21}{8}\frac{d}{dt}\left(a^3\left(R-2\Lambda\right)^{-\frac{1}{8}}\right) + \frac{21}{4}\sqrt{m}a^3\left(R-2\Lambda\right)^{-\frac{1}{8}}\right] \tag{10.40}$$

and the non vanishing commutators of the Noether algebra are

$$\left[X^1, \bar{X}^5\right] = 2\sqrt{m}\bar{X}^5 \qquad \left[\bar{X}^6, X^1\right] = 2\sqrt{m}\bar{X}^6$$

$$\left[\bar{X}^5, \bar{X}^6\right] = \frac{4}{\sqrt{m}}X^1$$

From the time dependent integrals (10.31),(10.32) and the Hamiltonian we construct the Ermakov-Lewis invariant.

$$\phi = E^2 - \bar{I}_5\bar{I}_6 \tag{10.41}$$

Case 6: If $f(R) = R^n$ (with $n \neq 0, 1, \frac{3}{2}, \frac{7}{8}$) the dynamical system admits the extra Noether symmetry

$$X^7 = 2t\partial_t + \mathbf{H} + \left(\frac{4n}{3} - \frac{7}{6}\right)\mathbf{K}^1 \tag{10.42}$$

with corresponding Noether Integral

$$I_7 = 2tE - 8na^2R^{n-1}\dot{a}\left(2-n\right) - 4na^3R^{n-2}\dot{R}\left(2n-1\right)\left(n-1\right). \tag{10.43}$$

and the commutator of the Noether algebra is $\left[X^1, X^7\right] = 2X^1$.

We note that the Noether subalgebra of case 2, $\left\{X^1, X^2, X^3\right\}$ and the algebra of case 4 $\left\{X^1, \bar{X}^2, \bar{X}^3\right\}$ is the same Lie algebra but in different representation. The same observation applies to the subalgebra of case 3 $\left\{X^1, X^5, X^6\right\}$ and the algebra of case 5 $\left\{X^1, \bar{X}^5, \bar{X}^6\right\}$. This connection between the Lie groups is useful because it reveals common features in the dynamic systems, as is the common transformation to the normal coordinates of the systems.

For the cosmological viability of the models see [169, 136]

10.5 Analytic Solutions

Using the Noether symmetries and the associated Noether integrals we solve analytically the differential eqs.(10.5), (10.6) and (10.7) for the cases where the dynamical system is Liouville integrable, that is for cases 2-5. Case 6 (i.e. $f(R) = R^n$) is not Liouville integrable via Noether point symmetries, since the Noether integral (10.43) is time dependent[4].

10.5.1 Power law model R^μ with $\mu = \frac{3}{2}$

In this case the Lagrangian eq.(10.10) of the $f(R) = R^{\frac{3}{2}}$ model is written as

$$L = 9a\sqrt{R}\dot{a}^2 + \frac{9a^2}{2\sqrt{R}}\dot{a}\dot{R} + \frac{a^3}{2}R^{\frac{3}{2}} \tag{10.44}$$

[4] In the appendix 10.A we present special solutions for the $f(R) = R^n$ model, using the zero order invariants.

Changing the variables from (a,R) to (z,w) via the relations:

$$a=\left(\frac{9}{2}\right)^{-\frac{1}{3}}\sqrt{z}\quad R=\frac{w^2}{z} \tag{10.45}$$

the Lagrangian (10.44) and the Hamiltonian (9.178) become

$$L=\dot{z}\dot{w}+V_0w^3 \tag{10.46}$$

$$E=\dot{z}\dot{w}-V_0w^3 \tag{10.47}$$

where $V_0=\frac{1}{9}$. The equations of motion in the new coordinate system are

$$\ddot{w} = 0 \tag{10.48}$$

$$\ddot{z}-3V_0w^2 = 0 \tag{10.49}$$

The Noether integrals (10.27),(10.28) in the coordinate system $\{z,y\}$ are

$$I_1^{'}=\dot{w}\ \ ,\ \ I_2^{'}=t\dot{w}-w \tag{10.50}$$

The general solution of the system is:

$$y\left(t\right)=I_1^{\prime}t-I_2^{'} \tag{10.51}$$

$$z\left(t\right)=\frac{1}{36\left(I_1^{'}\right)^2}\left(I_1^{'}t-I_2^{'}\right)^4+z_1t+z_0 \tag{10.52}$$

The Hamiltonian constraint gives $E=z_1I_1^{'}$ where $z_{0,1}$ are constants and the singularity condition results in the constraint

$$\frac{1}{36\left(I_1^{'}\right)^2}\left(I_2^{'}\right)^4+z_0=0. \tag{10.53}$$

10.5.2 Power law model R^{μ} with $\mu=\frac{7}{8}$

In this case the Lagrangian eq.(10.10) is written as

$$L=\frac{21a}{4R^{\frac{1}{8}}}\dot{a}^2-\frac{21}{16}\frac{a^2}{R^{\frac{9}{8}}}\dot{a}\dot{R}-\frac{1}{8}a^3R^{\frac{7}{8}}. \tag{10.54}$$

Changing now the variables from (a,R) to (ρ,σ) via the relations:

$$a=\left(\frac{21}{4}\right)^{-\frac{1}{3}}\sqrt{\rho e^{\sigma}}\quad R=\frac{e^{12\sigma}}{\rho^4}. \tag{10.55}$$

The Lagrangian (10.82) and the Hamiltonian (9.178) become

$$L=\frac{1}{2}\dot{\rho}^2-\frac{1}{2}\rho^2\dot{\sigma}^2+V_0\frac{e^{12\sigma}}{\rho^2} \tag{10.56}$$

$$E=\frac{1}{2}\dot{\rho}^2-\frac{1}{2}\rho^2\dot{\sigma}^2-V_0\frac{e^{12\sigma}}{\rho^2}. \tag{10.57}$$

where $V_0 = -\frac{1}{42}$. The Euler-Lagrange equations provide the following equations of motion:

$$\ddot{\rho} + \rho\dot{\sigma}^2 + 2V_0\frac{e^{12\sigma}}{\rho^3} = 0 \tag{10.58}$$

$$\ddot{\sigma} + \frac{2}{\rho}\dot{\sigma}\dot{\rho} + 12V_0\frac{e^{12\sigma}}{\rho^2} = 0. \tag{10.59}$$

The Noether integrals (10.31), (10.32) and the Ermakov-Lewis invariant 10.33 in the coordinate system $\{u, v\}$ are

$$I_5^{'} = 2tE - \rho\dot{\rho} \tag{10.60}$$

$$I_6^{'} = t^2E - t\rho\dot{\rho} + \frac{1}{2}\rho^2. \tag{10.61}$$

$$\Sigma = \rho^4\dot{\sigma}^2 + 4V_0e^{12\sigma}. \tag{10.62}$$

Using the Ermakov-Lewis Invariant, the Hamiltonian (10.56) and equation (10.58) are written:

$$\frac{1}{2}\dot{\rho}^2 - \frac{1}{2}\frac{\Sigma}{\rho^2} = E \tag{10.63}$$

$$\ddot{\rho} + \frac{\Sigma}{\rho^3} = 0. \tag{10.64}$$

And the analytical solution of the system is

$$\rho(t) = \left(\rho_2t^2 + \rho_1t + \frac{\left((\rho_1)^2 - 4\Sigma\right)}{4\rho_2}\right)^{\frac{1}{2}} \tag{10.65}$$

$$\exp(\sigma(t)) = \left\{\frac{21}{2}\Sigma\left[\left(\tanh\left[\sigma_0\rho_2\sqrt{\Sigma} - 6\arctan h\left(\frac{2\rho_2t + \rho_1}{2\sqrt{\Sigma}}\right)\right]\right)^2 - 1\right]\right\}^{\frac{1}{12}} \tag{10.66}$$

where $B(t) = \left(\frac{1}{2}\frac{2\rho_2t+\rho_1}{\sqrt{\Sigma}}\right)$ and $\rho_{1,2}$, σ_0 are constants with Hamiltonian constrain $E = \frac{1}{2}\rho_2$. The singularity constraint gives $(\rho_1)^2 = 4\Sigma$

In the case $\Sigma = 0$ the analytical solution is

$$\rho(t) = \left(\rho_2t^2 + \rho_1t + \frac{1}{2}\frac{(\rho_1)^2}{\rho_2}\right)^{\frac{1}{2}} \tag{10.67}$$

$$\exp\sigma(t) = \left[\frac{1}{24\sqrt{V_0}}\frac{(2\rho_2t + \rho_1)}{(4\sigma_0\rho_2^2t + 2\sigma_0\rho_2\rho_1 - 1)}\right]^{\frac{1}{6}} \tag{10.68}$$

The singularity constraint gives $\rho_1 = 0$, then the solution is

$$a(t) = \frac{a_0t^{\frac{7}{6}}}{(a_2t - 1)^{\frac{1}{6}}} \tag{10.69}$$

In contrast with the claim of [170] this model is analytically solvable and there exists models which admit Noether integrals with time dependent gauge functions.

10.5.3 Λ_{bc}CDM model with $(b,c)=(1,\frac{3}{2})$

Inserting $f(R)=(R-2\Lambda)^{3/2}$ into eq.(10.10) we obtain

$$L = 9a\sqrt{R-2\Lambda}\dot{a}^2 + \frac{9a^2}{2\sqrt{R-2\Lambda}}\dot{a}\dot{R} + \frac{a^3}{2}\left(R+4\Lambda\right)\sqrt{R-2\Lambda} \tag{10.70}$$

Changing the variables from (a,R) to (x,y) via the relations:

$$a = \left(\frac{9}{2}\right)^{-\frac{1}{3}}\sqrt{x} \quad R = 2\Lambda + \frac{y^2}{x} \tag{10.71}$$

the Lagrangian (10.70) and the Hamiltonian (9.178) become

$$L = \dot{x}\dot{y} + V_0\left(y^3 + \bar{m}xy\right) \tag{10.72}$$

$$E = \dot{x}\dot{y} - V_0\left(y^3 + \bar{m}xy\right) \tag{10.73}$$

where $V_0=\frac{1}{9}$ and $\bar{m}=6\Lambda$.

The equations of motion, using the Euler-Lagrange equations, in the new coordinate system are

$$\ddot{x} - 3V_0y^2 - \bar{m}V_0x = 0 \tag{10.74}$$

$$\ddot{y} - \bar{m}V_0y = 0. \tag{10.75}$$

The Noether integrals (10.35),(10.36) in the coordinate system $\{x,y\}$ are

$$\bar{I}_1' = e^{\omega t}\dot{y} - \omega e^{\omega t}y \tag{10.76}$$

$$\bar{I}_2' = e^{-\omega t}\dot{y} + \omega e^{-\omega t}y. \tag{10.77}$$

where $\omega=\sqrt{2\Lambda/3}$. From these we construct the time independent first integral

$$\Phi = I_1I_2 = \dot{y}^2 - \omega^2y^2. \tag{10.78}$$

The constants of integration are further constrained by the condition that at the singularity $(t=0)$, the scale factor has to be exactly zero, that is, $x(0)=0$.

The general solution of the system (10.74)-(10.75) is:

$$y\left(t\right) = \frac{I_2}{2\omega}e^{\omega t} - \frac{I_1}{2\omega}e^{-\omega t} \tag{10.79}$$

$$x\left(t\right) = x_{1G}e^{\omega t} + x_{2G}e^{-\omega t} + \frac{1}{4\bar{m}\omega^2}\left(I_2e^{\omega t} + I_1e^{-\omega t}\right)^2 + \frac{\Phi}{\bar{m}\omega^2}. \tag{10.80}$$

The Hamiltonian constrain gives $E=\omega\left(x_{1G}I_1 - x_{2G}I_2\right)$ where $x_{1G,2G}$ are constants and the singularity condition results in the constrain

$$x_{1G} + x_{2G} + \frac{1}{4\bar{m}\omega^2}\left(I_1+I_2\right)^2 + \frac{\Phi}{\bar{m}\omega^2} = 0. \tag{10.81}$$

At late enough times the solution becomes $a^2(t)\propto e^{2\omega t}$

10.5.4 Λ_{bc}CDM model with $(b,c)=(1,\frac{7}{8})$

In this case the Lagrangian eq.(10.10) of the $f(R)=(R-2\Lambda)^{7/8}$ model is written as

$$L=\frac{21a}{4\left(R-2\Lambda\right)^{\frac{1}{8}}}\dot{a}^{2}-\frac{21}{16}\frac{a^{2}}{\left(R-2\Lambda\right)^{\frac{9}{8}}}\dot{a}\dot{R}-\frac{1}{8}a^{3}\frac{\left(R-16\Lambda\right)}{\left(R-2\Lambda\right)^{\frac{1}{8}}}. \tag{10.82}$$

Changing the variables from (a,R) to (u,v) via the relations:

$$a=\left(\frac{21}{4}\right)^{-\frac{1}{3}}\sqrt{ue^{v}}\quad R=2\Lambda+\frac{e^{12v}}{u^{4}}. \tag{10.83}$$

the Lagrangian (10.82) and the Hamiltonian (9.178) become

$$L=\frac{1}{2}\dot{u}^{2}-\frac{1}{2}u^{2}\dot{v}^{2}+V_{0}\frac{\bar{m}}{4}u^{2}+V_{0}\frac{e^{12v}}{u^{2}} \tag{10.84}$$

$$E=\frac{1}{2}\dot{u}^{2}-\frac{1}{2}u^{2}\dot{v}^{2}-V_{0}\frac{\bar{m}}{4}u^{2}-V_{0}\frac{e^{12v}}{u^{2}}. \tag{10.85}$$

where $\bar{m}=-28\Lambda$, $V_0=-\frac{1}{42}$.

The Euler-Lagrange equations provide the following equations of motion:

$$\ddot{u}+u\dot{v}^{2}-\frac{V_{0}\bar{m}}{2}u+2V_{0}\frac{e^{12v}}{u^{3}}=0 \tag{10.86}$$

$$\ddot{v}+\frac{2}{u}\dot{u}\dot{v}+12V_{0}\frac{e^{12v}}{u^{4}}=0. \tag{10.87}$$

The Noether integrals (10.39),(10.40) and the Ermakov-Lewis invariant (10.41) in the coordinate system $\{u,v\}$ are

$$I_{+}=\frac{1}{\lambda}e^{2\lambda t}E-e^{2\lambda t}u\dot{u}+\lambda e^{2\lambda t}u^{2} \tag{10.88}$$

$$I_{-}=\frac{1}{\lambda}e^{-2\lambda t}E-e^{-2\lambda t}u\dot{u}+\lambda e^{-2\lambda t}u^{2}. \tag{10.89}$$

$$\phi=u^{4}\dot{v}^{2}+4V_{0}e^{12v}. \tag{10.90}$$

where $\lambda=\frac{1}{2}\sqrt{\frac{2}{3}\Lambda}$.

Using the Ermakov-Lewis Invariant (10.90), the Hamiltonian (10.85) and equation (10.86) are written:

$$\frac{1}{2}\dot{u}^{2}-V_{0}\frac{m}{8}u^{2}-\frac{1}{2}\frac{\phi}{u^{2}}=E \tag{10.91}$$

$$\ddot{u}-\frac{V_{0}m}{4}u+\frac{\phi}{u^{3}}=0. \tag{10.92}$$

The solution of (10.92) has been given by Pinney [79] and it is the following:

$$u\left(t\right)=\left(u_{1}e^{2\lambda t}+u_{2}e^{-2\lambda t}+2u_{3}\right)^{\frac{1}{2}} \tag{10.93}$$

where u_{1-3}. From the Hamiltonian constraint (10.91) and the Noether Integrals (10.88),(10.89) we find

$$E=-2\lambda u_{3}\ ,\ I_{+}=2\lambda u_{2}\ ,\ I_{-}=2\lambda u_{1}.$$

Replacing (10.93) in the Ermakov-Lewis Invariant (10.90) and assuming $\phi \neq 0$ we find:

$$\exp\left(v\left(t\right)\right) = 2^{\frac{1}{6}}\phi^{\frac{1}{12}}e^{-A(t)}\left(4V_0 + e^{-12A(t)}\right)^{-\frac{1}{6}} \tag{10.94}$$

where

$$A\left(t\right) = \arctan\left[\frac{2\lambda}{\sqrt{\phi}}\left(u_1 e^{2\lambda t} + u_3\right)\right] + 4\lambda^2 u_1\sqrt{\phi}. \tag{10.95}$$

Then the solution is

$$a^2\left(t\right) = 2^{-\frac{1}{3}}\phi^{\frac{1}{12}}e^{-A(t)}\left(4V_0 + e^{-12A(t)}\right)^{-\frac{1}{6}}\left(u_1 e^{2\lambda t} + u_2 e^{-2\lambda t} + 2u_3\right)^{\frac{1}{2}} \tag{10.96}$$

where from the singularity condition $x\left(0\right) = 0$ we have the constrain $u_1 + u_2 + 2u_3 = 0$, or

$$2E - \left(I_+ + I_-\right) = 0. \tag{10.97}$$

At late enough time we find $A\left(t\right) \simeq A_0$, which implies $a^2(t) \propto e^{\lambda t}$.

In the case where $\phi = 0$ equations (10.91),(10.92) describe the hyperbolic oscillator and the solution is

$$u\left(t\right) = \sinh\lambda t\ ,\ 2E = \lambda^2. \tag{10.98}$$

From the Ermakov-Lewis Invariant we have

$$\exp\left(v\left(t\right)\right) = \left(\frac{\lambda\sinh\lambda t}{\lambda v_1\sinh\lambda t - 12\sqrt{|V_0|}e^{-2\lambda t}}\right)^{\frac{1}{6}} \tag{10.99}$$

where v_1 is a constant. The analytic solution is

$$a^2\left(t\right) = \left(\frac{\lambda\sinh^7\lambda t}{\lambda v_1\sinh\lambda t - 12\sqrt{|V_0|}e^{-2\lambda t}}\right)^{\frac{1}{6}} \tag{10.100}$$

10.6 Noether point symmetries in spatially non-flat $f(R)$ models

In this section we study further the Noether point symmetries in non flat $f(R)$ cosmological models. In the context of a FRW spacetime the Lagrangian of the overall dynamical problem and the Ricci scalar are

$$L = 6f'a\dot{a}^2 + 6f''\dot{R}a^2\dot{a} + a^3\left(f'R - f\right) - 6Kaf' \tag{10.101}$$

$$R = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}^2 + K}{a^2}\right) \tag{10.102}$$

where K is the spatial curvature. Note that the two dimensional metric is given by eq.(10.23) while the "potential" in the Lagrangian takes the form

$$V_K(a,R) = -a^3(f'R - f) + Kaf'. \tag{10.103}$$

Based on the above equations and using the theoretical formulation presented in section 10.4, we find that the $f(R)$ models which admit non trivial Noether symmetries are the $f(R) = (R-2\Lambda)^{3/2}$, $f(R) = R^{3/2}$ and $f(R) = R^2$. The Noether symmetries can be found in section 10.4.

In particular, inserting $f(R) = (R-2\Lambda)^{3/2}$ into the Lagrangian (10.101) and changing the variables from (a, R) to (x, y) [see section 10.5.3] we find

$$L = \dot{x}\dot{y} + V_0\left(y^3 + \bar{m}xy\right) - \bar{K}y \tag{10.104}$$

$$E = \dot{x}\dot{y} - V_0\left(y^3 + \bar{m}xy\right) + \bar{K}y \tag{10.105}$$

where $\bar{K} = 3(6^{1/3}K)$. Therefore, the equations of motion are

$$\begin{aligned} \ddot{x} - 3V_0y^2 - \bar{m}V_0x + \bar{K} &= 0 \\ \ddot{y} - \bar{m}V_0y &= 0\,. \end{aligned}$$

The constant term $\bar{K}$ appearing in the first equation of motion is not expected to affect the Noether symmetries (or the integrals of motion). Indeed we find that the corresponding Noether symmetries coincide with those of the spatially flat $f(R) = (R-2\Lambda)^{3/2}$ model. However, in the case of $K \neq 0$ (or $\bar{K} \neq 0$) the analytic solution for the x-variable is written as

$$x_K(t) \equiv x(t) + \frac{\bar{K}}{\omega^2} \tag{10.106}$$

where $x(t)$ is the solution of the flat model $K = 0$ (see section 10.5.3). Note that the solution of the y-variable remains unaltered.

Similarly, for the $f(R) = R^{3/2}$ model the analytic solution is

$$z_K\left(t\right) = z\left(t\right) + \bar{K} \tag{10.107}$$

where $z\left(t\right)$ is the solution of the spatially flat model (see section 10.5.1).

10.7 Conclusion

In the literature the functional forms of $f(R)$ of the modified $f(R)$ gravity models are mainly defined on a phenomenological basis. In this article we use the Noether symmetry approach to constrain these models with the aim to utilize the existence of non-trivial Noether symmetries as a selection criterion that can distinguish the $f(R)$ models on a more fundamental level. Furthermore the resulting Noether integrals can be used to provide analytic solutions.

In the context of $f(R)$ models, the system of the modified field equations is equivalent to a two dimensional dynamical system moving in M^2 (mini superspace) under the constraint $\bar{E}$ =constant. Following the general methodology of chapter 4, we require that the two dimensional system admits extra Noether point symmetries. This requirement fixes the $f\left(R\right)$ function and the corresponding analytic solutions are computed. It is interesting that two well known dynamical systems appear in cosmology: the anharmonic oscillator and the Ermakov-Pinney system. We recall that the field equations of the $\Lambda-$cosmology is equivalent with that of the hyperbolic oscillator.

10.A Special solutions for the power law model R^n

The case $f(R) = R^n$ is not Liouville integrable via Noether point symmetries. However the zero order invariant will be used in order to find special solutions. Inserting $f(R) = R^n$ $\left(n \neq 0, 1, \frac{3}{2}, \frac{7}{8}\right)$ into eq.(10.10) we obtain

$$L\left(a, \dot{a}, R, \dot{R}\right) = 6naR^{n-1}\dot{a}^2 + 6n(n-1)a^2R^{n-2}\dot{a}\dot{R} + (n-1)a^3R^n \tag{10.108}$$

and this leads to the modified field equations

$$\ddot{a} + \frac{1}{a}\dot{a}^2 - \frac{1}{6}aR = 0 \tag{10.109}$$

$$\ddot{R} + \frac{n-2}{R}\dot{R}^2 - \frac{1}{n-1}\frac{R}{a^2}\dot{a}^2 + \frac{2}{a}\dot{a}\dot{R} - \frac{(n-3)}{6n(n-1)}R^2 = 0 \tag{10.110}$$

$$E = 6naR^{n-1}\dot{a}^2 + 6n(n-1)a^2R^{n-2}\dot{a}\dot{R} - (n-1)a^3R^n. \tag{10.111}$$

The Noether point symmetry (10.42) is also and a Lie symmetry, hence we have the zero order invariants

$$a_0 = at^{-N}\ ,\ R_0 = Rt^{-2}. \tag{10.112}$$

Applying the zero order invariants in the field equations (10.109)-(10.111) and in the Noether integral (10.43) we have the following results.

The dynamical system admits a special solution of the form

$$a = a_0t^N\ ,\ R = 6N(2N-1)t^{-2} \tag{10.113}$$

where the constants N, E and I_7 are

$$N = \frac{1}{2}\ ,\ E = 0\ ,\ I_7 = 0$$

or

$$N = -\frac{(2n-1)(n-1)}{n-2}\ ,\ E = 0\ ,\ I_7 = 0$$

or

$$N = \frac{2}{3}n\ ,\ E = \left(\frac{12n}{9}\right)^n (4n-3)^{n-1}\left(13n - 8n^2 - 3\right)a_0^3\ ,\ I_7 = 0.$$

Another special solution is the deSitter solution for $n = 2$

$$a = a_0e^{H_0t}\ ,\ R = 12H_0^2 \tag{10.114}$$

where $I_7 = 0$ and the spacetime is empty i.e. $E = 0$.

Chapter 11

Noether symmetries in $f(T)$ gravity

11.1 Introduction

In this chapter we continue our analysis on the application of Noether point symmetries in alternative theories of gravity and specifically the $f(T)$ modified theory of gravity. $f(T)$ gravity it is based on the old formulation of Teleparallel Equivalent of General Relativity (TEGR) [171, 172, 173, 174] which instead of the torsion-less Levi-Civita connection uses the curvatureless Weitzenbock connection [175] in which the corresponding dynamical fields are the four linearly independent vierbeins. Therefore, all the information concerning the gravitational field is included in the Weitzenbock tensor. Within this framework, considering invariance under general coordinate transformations, global Lorentz-parity transformations, and requiring up to second order terms of the torsion tensor, one can write down the corresponding Lagrangian density T by using some suitable contractions.

Furthermore $f(T)$ gravity which is based on the fact that we allow the Lagrangian to be a function of T [176, 177, 178], inspired by the extension of $f(R)$ Einstein-Hilbert action. However, $f(T)$ gravity does not coincide with $f(R)$ extension, but it rather consists of a different class of modified gravity models. It is interesting to mention that the torsion tensor includes only products of first derivatives of the vierbeins, giving rise to second-order field differential equations in contrast to the $f(R)$ gravity that provides fourth-order equations, which potentially may lead to some problems, for example in the well position and well formulation of the Cauchy problem [179]. Moreover, as we showed in chapter 10 the Lagrangian of the field equations in $f(R)$ gravity described a regular dynamical system; however, in $f(T)$ the dynamical system is singular and the T variable can be seen as a Lagrange multiplier.

In section 11.2 we discuss the role of unholonomic frames in the context of teleparallel gravity and its straightforward extension. In section 11.3, we briefly present $f(T)$ gravity, while in section 11.4 we construct the corresponding generalized Lagrangian formulation. In section 11.5, we analyze the main properties of the Noether Symmetry Approach for $f(T)$ gravity. Then, in section 11.6 and 11.7, we apply these results in the

FRW and the static spherically symmetric spaces.

11.2 Unholonomic frames and Connection Coefficient

In an $n-$dimensional manifold M consider a coordinate neighborhood U with a coordinate system $\{x^\mu\}$. At each point $P\epsilon U$ we have the resulting holonomic frame $\{\partial_\mu\}$. We define in U a new frame $\{e_a(x^\mu)\}$ which is related to the holonomic frame $\{\partial_a\}$ as follows:

$$e_a(x^\mu) = h_a^\mu \partial_\mu \quad a, \mu = 1, 2, ..., n \tag{11.1}$$

where the quantities $h_a^\mu(x)$ are in general functions of the coordinates (i.e. depend on the point P). Notice that Latin indexes count vectors, while Greek indexes are tensor indices. We assume that $\det h_a^\mu \neq 0$ which guaranties that the vectors $\{e_a(x^\mu)\}$ form a set of linearly independent vectors. We define the "inverse" quantities h_a^μ by means of the following "orthogonality" relations:

$$h_a^\mu h_\nu^a = \delta_\nu^\mu, h_b^\mu h_\mu^c = \delta_b^c. \tag{11.2}$$

The commutators of the vectors $\{e_a\}$ are not in general all zero. If they are, then there exists a new coordinate system in U, the $\{y^b\}$ say so that $e_b = \frac{\partial}{\partial y^b}$ i.e. the new frame is holonomic. If there are commutators $[e_a, e_b] \neq 0$ then the new frame $\{e_a\}$ is called unholonomic and the vectors e_a cannot be written in the form $e_b = \partial_b$. The quantities which characterize an unholonomic frame are the objects of unholonomicity or Ricci rotation coefficients $\Omega^a{}_{bc}$ defined by the relation

$$[e_a, e_b] = \Omega^c{}_{ab} e_c. \tag{11.3}$$

We compute:

$$[e_a, e_b] = [h_a^\mu \partial_\mu, h_b^\nu \partial_\nu] = \left[h_a^\mu h_{b,\mu}^\nu h_\nu^c - h_b^\nu h_{a,\nu}^\mu h_\mu^c\right] e_c$$

from which follows that the Ricci rotation coefficients of the frame $\{e_a\}$ are:

$$\Omega^a{}_{bc} = 2 h_{[b}^\mu h_{c],\mu}^\nu h_\nu^a. \tag{11.4}$$

The condition for $\{e_a\}$ to be a holonomic basis is $\Omega^a{}_{bc} = 0$ at all points $P\epsilon U$. This is a set of linear partial differential equations whose solution defines all holonomic frames and all coordinate systems in U. One obvious solution is $h_b^c = \delta_b^c$. The set of all coordinate systems in U equipped with the operation of composition of transformations has the structure of an infinite dimensional Lie group which is called the Manifold Mapping Group [**?**].

We consider now the special unholonomic frames which satisfy Jacobi's identity:

$$[[e_a, e_b], e_c] + [[e_b, e_c], e_a] + [[e_c, e_a], e_b] = 0. \tag{11.5}$$

These frames are the generators of a Lie Algebra, therefore they have an extra role to play. Replacing the commutator in terms of the unholonomicity objects we find the following identity:

$$\Omega^d{}_{ab,c} + \Omega^d{}_{ba,a} + \Omega^d{}_{ca,b} - \Omega^l{}_{ab}\Omega^d_{cl} - \Omega^l{}_{bc}\Omega^d_{al} - \Omega^l{}_{ca}\Omega^d_{bl} = 0. \tag{11.6}$$

Using the definition of the covariant derivative we write:

$$\nabla_{e_i} e_j = \Gamma^k_{ij} e_k \tag{11.7}$$

where Γ^k_{ij} are the connection coefficients in the frame $\{e_i\}$. Let us compute these Γ^k_{ij} assuming that

$$[e_i, e_j] = C^k_{.ij} e_k$$

from which follows

$$C^k_{.ij} = \Omega^k_{.jk}.$$

Consider three vector fields X, Y, Z and the covariant derivative of the metric wrt X. Then we have:

$$\nabla_X g(Y, Z) = X(g(Y, Z)) - g(\nabla_X Y, Z) - g(Y, \nabla_Y Z) \tag{11.8}$$

and by interchanging the role of X, Y, Z :

$$\nabla_Y g(Z, X) = Y(g(Z, X)) - g(\nabla_Y Z, X) - g(Z, \nabla_Z X) \tag{11.9}$$

$$\nabla_Z g(X, Y) = Z(g(X, Y)) - g(\nabla_Z X, Y) - g(X, \nabla_X Y). \tag{11.10}$$

Adding (11.8), (11.9) and subtracting (11.10) we obtain:

$$\begin{aligned} \nabla_X g(Y, Z) + \nabla_Y g(Z, X) - \nabla_Z g(X, Y) &= X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) + \\ &\quad - [g(\nabla_X Y, Z) + g(\nabla_Y Z, X) - g(\nabla_Z X, Y)] + \\ &\quad - [g(Y, \nabla_X Z) + g(Z, \nabla_Y X) - g(X, \nabla_Z Y)] \end{aligned}$$

then

$$\begin{aligned} \nabla_X g(Y, Z) + \nabla_Y g(Z, X) - \nabla_Z g(X, Y) &= X(g(Y, Z)) + Y(g(Z, X)) - Z(g(X, Y)) + \\ &\quad - [g(\nabla_X Y, Z) + g(Z, \nabla_Y X)] + \\ &\quad - [g(\nabla_Y Z, X) - g(X, \nabla_Z Y)] + \\ &\quad - [g(Y, \nabla_X Z) - g(\nabla_Z X, Y)] \, / \end{aligned}$$

that is

$$\begin{aligned}\nabla_X g(Y,Z)+\nabla_Y g(Z,X)-\nabla_Z g(X,Y) &= X(g(Y,Z))+Y(g(Z,X))-Z(g(X,Y))+\\ &-\left[g(Z,\nabla_X Y+\nabla_Y X)+g(X,\nabla_Y Z-\nabla_Z Y)+g(Y,\nabla_X Z-\nabla_Z X)\right].\end{aligned}$$

The term

$$g(Z,\nabla_X Y+\nabla_Y X)=2g\left(Z,\nabla_X Y\right)+g\left(Z,\nabla_Y X-\nabla_X Y\right).$$

Replacing in the last relation and solving for $2g\left(Z,\nabla_X Y\right)$ we find

$$\begin{aligned}2g\left(Z,\nabla_X Y\right) &= \left[X(g(Y,Z))+Y(g(Z,X))-Z(g(X,Y))\right]+\\ &-\left[\nabla_X g(Y,Z)+\nabla_Y g(Z,X)-\nabla_Z g(X,Y)\right]+\\ &-\left[g\left(Z,\nabla_Y X-\nabla_X Y\right)+g(X,\nabla_Y Z-\nabla_Z Y)+g(Y,\nabla_X Z-\nabla_Z X)\right]\end{aligned}$$

or

$$\begin{aligned}2g\left(Z,\nabla_X Y\right) &= \left[X(g(Y,Z))+Y(g(Z,X))-Z(g(X,Y))\right]+\\ &-\left[\nabla_X g(Y,Z)+\nabla_Y g(Z,X)-\nabla_Z g(X,Y)\right]+\\ &-\left[g\left(Z,\nabla_Y X-\nabla_X Y-[Y,X]\right)+g(X,\nabla_Y Z-\nabla_Z Y-[Y,Z])+g(Y,\nabla_X Z-\nabla_Z X-[X,Z])\right]+\\ &-\left[g\left(Z,[Y,X]\right)+g\left(X,[Y,Z]\right)+g\left(Y,[X,Z]\right)\right].\end{aligned}$$

Define the quantities

$$\begin{aligned}T_\nabla(X,Y) &= \nabla_X Y-\nabla_Y X-[X,Y]\\ A_\nabla(X,Y,Z) &= \nabla_X g(Y,Z)\end{aligned}$$

The tensors T_∇ and A_∇ are called the *torsion* and the *metricity* of the connection ∇ respectively. Last relation in terms of the fields T_∇ and A_∇ is written as follows:

$$\begin{aligned}2g\left(Z,\nabla_X Y\right) &= \left[X(g(Y,Z))+Y(g(Z,X))-Z(g(X,Y))\right]+\\ &-\left[A_\nabla\left(X,Y,Z\right)+A_\nabla\left(Y,Z,X\right)-A_\nabla\left(Z,X,Y\right)\right]+\\ &-\left[g\left(Z,T_\nabla\left(Y,X\right)\right)+g\left(X,T_\nabla\left(Y,Z\right)\right)+g\left(Y,T_\nabla\left(X,Z\right)\right)\right]\\ &-\left[g\left(Z,[Y,X]\right)+g\left(X,[Y,Z]\right)+g\left(Y,[X,Z]\right)\right].\end{aligned} \tag{11.11}$$

Let $X=e_l$, $Y=e_j$ and $Z=e_k$. Contracting with $\frac{1}{2}g^{il}$ we have

$$2g\left(Z,\nabla_X Y\right)\to\Gamma^i_{jk}$$

$$\left[X(g(Y,Z))+Y(g(Z,X))-Z(g(X,Y))\right]\to\left\{{i \atop jk}\right\}$$

$$g\left(X,T_\nabla\left(Y,Z\right)\right)\to Q^i_{.kj}$$

$$g\left(Z,T_{\nabla}\left(Y,X\right)\right)+g\left(Y,T_{\nabla}\left(X,Z\right)\right)\rightarrow g^{il}(g_{tj}Q^{t}_{kl}+g_{tk}Q^{t}_{jl})=-\bar{S}^{i}_{.kj}$$

$$g\left(X,[Y,Z]\right)\rightarrow\frac{1}{2}C^{i}_{.jk}$$

$$g\left(Z,[Y,X]\right)+g\left(Y,[X,Z]\right)=\frac{1}{2}g^{il}(g_{tj}C^{t}_{lk}+g_{tk}C^{t}_{jl})=-S^{i}_{.kj}$$

and

$$A_{\nabla}\left(X,Y,Z\right)+A_{\nabla}\left(Y,Z,X\right)-A_{\nabla}\left(Z,X,Y\right)\rightarrow\frac{1}{2}g^{il}\Delta_{jkl}$$

Replacing in (11.11) we find the connection coefficients in the frame $\{e_i\}$

$$\Gamma^{i}_{jk}=\left\{{}^{\,i}_{jk}\right\}+\bar{S}^{i}_{.kj}+S^{i}_{.kj}-\frac{1}{2}g^{il}\Delta_{jkl}+Q^{i}_{jk}-\frac{1}{2}C^{i}_{.jk} \tag{11.12}$$

where $\left\{{}^{\,i}_{jk}\right\}$ is the standard Levi Civita connection coefficients (Christofell symbols). This is the most general expression for the connection coefficients in terms of the fields $\left\{{}^{\,i}_{jk}\right\}$, T_{∇}, A_{∇} and C^{i}_{jk}.

Concerning the symmetric and antisymmetric part we have:

$$\Gamma^{i}_{.(jk)} = \left\{{}^{\,i}_{jk}\right\}+\bar{S}^{i}_{.jk}+S^{i}_{.jk}-\frac{1}{2}g^{il}\Delta_{jkl} \tag{11.13}$$

$$\Gamma^{i}_{.[jk]} = Q^{i}_{.jk}-\frac{1}{2}C^{i}_{.jk}. \tag{11.14}$$

From the above we draw the following conclusions:

1. The connection coefficients in a frame $\{e_i\}$ are determined from the metric, the torsion, the metricity and the unholonomicity objects (equivalently the commutators) of the frame vectors

2. The symmetric part $\Gamma^{i}_{.(jk)}$ of Γ^{i}_{jk} depends on all fields. This means that the geodesics and the autoparallels in a given frame depend on the geometric properties of the space (fields g_{ij}, $Q^{i}_{.kj}, g_{ij|k}$) and the unholonomicity of the frame (field $C^{i}_{.jk}$)

3. The antisymmetric part $\Gamma^{i}_{.[jk]}$ of Γ^{i}_{jk} depends only on the fields $Q^{i}_{.kj}$ and $C^{i}_{.jk}$.

4. The objects of unholonomicity $C^{i}_{.jk}$ behave in the same way as the components of the torsion. This means that even in a Riemannian space where $Q^{i}_{.kj}=0, g_{ij|k}=0$ in an unholonomic basis the antisymmetric part $\Gamma^{i}_{.[jk]}=-\frac{1}{2}C^{i}_{.jk}\neq 0$. This result has lead to the misunderstanding that when one works in an unholonomic frame then one has introduced torsion, which is not correct! This is the case with the TEGR. This misunderstanding has important consequences because the effects one will observe in an unholonomic frame will be frame dependent effects and not covariant effects. Therefore all conclusions made in a specific unholonomic frame are restricted to that frame only.

11.3 $f(T)$-gravity

Teleparallelism uses as dynamical objects special unholonomic frames in spacetime, called vierbeins, which are defined by the requirement

$$g(e_i,e_j)=e_i.e_j=\eta_{ij} \tag{11.15}$$

where $\eta_{ij} = \text{diag}(1, -1, -1, -1)$ is the Lorentz metric in canonical form. Obviously $g_{\mu\nu}(x) = \eta_{ij} h^i_\mu(x) h^j_\nu(x)$ where $e^i(x) = h^i_\mu(x) dx^i$ is the dual basis. Differing from GR, which uses the torsionless Levi-Civita connection, Teleparallelism utilizes the curvatureless Weitzenböck connection [175], where Weitzenböck non-null torsion is

$$T^\beta_{\mu\nu} = \hat{\Gamma}^\beta_{\nu\mu} - \hat{\Gamma}^\beta_{\mu\nu} = h^\beta_i (\partial_\mu h^i_\nu - \partial_\nu h^i_\mu) \, . \tag{11.16}$$

Notice that we assume that the Ricci rotation coefficients obey $\Omega^i_{jk} = T^i_{jk}$ and encompass all the information concerning the gravitational field. The TEGR Lagrangian for the gravitational field equations (Einstein equations) is taken to be:

$$T = S^{\mu\nu}_\beta T^\beta_{\mu\nu} \tag{11.17}$$

where

$$S^{\mu\nu}_\beta = \frac{1}{2}(K^{\mu\nu}_\beta + \delta^\mu_\beta T^{\theta\nu}_\theta - \delta^\nu_\beta T^{\theta\mu}_\theta) \tag{11.18}$$

and $K^{\mu\nu}_\beta$ is the tensor

$$K^{\mu\nu}_\beta = -\frac{1}{2}(T^{\mu\nu}_\beta - T^{\nu\mu}_\beta - T^{\mu\nu}_\beta), \tag{11.19}$$

which equals the difference of the Levi Civita connection in the holonomic and the unholonomic frame.

In this work the gravitational field will be driven by a Lagrangian density which is a function of T. Therefore, the corresponding action of $f(T)$ gravity reads as

$$\mathcal{A} = \frac{1}{16\pi G} \int d^4x e f(T) \tag{11.20}$$

where $e = det(e^i_\mu \cdot e^i_\nu) = \sqrt{-g}$ and G is Newton's constant. TEGR and thus General Relativity is restored when $f(T) = T$. First of all, in order to construct a realistic cosmology we have to incorporate the matter and radiation sectors too. Therefore, the total action is written as

$$S_{\text{tot}} = \mathcal{A} + \frac{1}{16\pi G} \int d^4x e \, (L_m + L_r) \, , \tag{11.21}$$

If matter couples to the metric in the standard form then the variation of the action (11.21) with respect to the vierbein leads to the equations [180]

$$\begin{aligned} &e^{-1}\partial_\mu(e S^{\mu\nu}_i) f'(T) - h^\lambda_i T^\beta_{\mu\lambda} S^{\nu\mu}_\beta f'(T) \\ &+ S^{\mu\nu}_i \partial_\mu(T) f''(T) + \frac{1}{4} h^\nu_i f(T) = 4\pi G h^\beta_i T^\nu_\beta \end{aligned} \tag{11.22}$$

where a prime denotes differentiation with respect to T, ${S_i}^{\mu\nu} = {h_i}^\beta S^{\mu\nu}_\beta$ and $T_{\mu\nu}$ is the matter energy-momentum tensor.

11.4 Generalized Lagrangian formulation of $f(T)$ gravity

In this section, we provide a generalized Lagrangian formulation in order to construct a theory of $f(T)$ gravity. Specifically, the gravitational field is driven by the Lagrangian density $f(T)$ in (11.20), which can be generalized through the use of a Lagrange multiplier. In particular, we can write it as

$$L\left(x^k, x'^k, T\right) = 2f_T\bar{\gamma}_{ij}\left(x^k\right)x'^i x'^j + M\left(x^k\right)\left(f - Tf_T\right) \tag{11.23}$$

where $x' = \frac{dx}{d\tau}$, $M(x^k)$ is the Lagrange multiplier and $\bar{\gamma}_{ij}$ is a second rank tensor which is related to the frame [one can use $eT(x^k, x'^k)$] of the background spacetime. In the same lines, the Hamiltonian of the system is written as

$$H\left(x^k, x'^k, T\right) = 2f_T\bar{\gamma}_{ij}\left(x^k\right)x'^i x'^j - M\left(x^k\right)\left(f - Tf_T\right) = 0\ . \tag{11.24}$$

In this case, the system is autonomous hence ∂_τ is a Noether symmetry with corresponding Noether integral the Hamiltonian H.

In this framework, considering $\{x^k, T\}$ as the canonical variables of the configuration space, we can derive, after some algebra, the general field equations of $f(T)$ gravity. Indeed, starting from the Lagrangian (11.23), the Euler-Lagrange equations

$$\frac{\partial L}{\partial T} = 0, \quad \frac{d}{d\tau}\left(\frac{\partial L}{\partial x'^k}\right) - \frac{\partial L}{\partial x^k} = 0 \tag{11.25}$$

give rise to

$$f_{TT}\left(2\bar{\gamma}_{ij}x'^i x'^j - MT\right) = 0, \tag{11.26}$$

$$x^{i\prime\prime} + \bar{\Gamma}^i_{jk}x^{j\prime}x^{k\prime} + \frac{f_{TT}}{f_T}x^{i\prime}T' - M^{,i}\frac{(f - Tf_T)}{4f_T} = B^i_m\ . \tag{11.27}$$

The functions $\bar{\Gamma}^i_{jk}$ are considered to be the Christoffel symbols for the metric $\bar{\gamma}_{ij}$. Therefore, the system is determined by the two independent differential equations (11.26),(11.27) and the Hamiltonian constraint $H = C_m$ where H is given by equation (11.24) and C_m, B^i_m are the components of the energy momentum tensor $T_{\mu\nu}$..

The point-like Lagragian (11.23) determines completely the related dynamical system in the minisuperspace $\{x^k, T\}$, implying that one can easily recover some well known cases of cosmological interest. In brief, these are:

- The static spherically symmetric spacetime:

  $$ds^2 = -a^2(\tau)\,dt^2 + \frac{1}{N^2\left(a(\tau), b(\tau)\right)}d\tau^2 + b^2(\tau)\left(d\theta^2 + \sin^2\theta d\phi^2\right) \tag{11.28}$$

  arising from the diagonal vierbein [1]

  $$e^A_i = \left(a(\tau), \frac{1}{N\left(a(\tau), b(\tau)\right)}, b(\tau), b(\tau)\sin\theta\right) \tag{11.29}$$

  where $a(\tau)$ and $b(\tau)$ are functions which need to be determined. Therefore, the line element of $\bar{\gamma}_{ij}$ and $M\left(x^k\right)$ are given by

  $$ds^2_{\bar{\gamma}} = N\left(2b\ da\ db + a\ db^2\right)\ ,\quad M(a,b) = \frac{ab^2}{N}. \tag{11.30}$$

[1] Note that, in general, one can choose a non-diagonal vierbein, giving rise to the same metric through (11.15).

- The flat FRW spacetime with Cartesian coordinates:

$$ds^2 = -dt^2 + a^2(t)\left(dx^2 + dy^2 + dz^2\right) \tag{11.31}$$

arising from the vierbein

$$e_i^A = (1, a(t), a(t), a(t)) \tag{11.32}$$

where t is the cosmic time and $a(t)$ is the scale factor of the universe. In this case we have

$$ds_{\bar{\gamma}}^2 = 3a\ da^2\ ,\ M(a) = a^3(t). \tag{11.33}$$

- The Bianchi type I spacetime:

$$ds^2 = -\frac{1}{N^2(a(t), \beta(t))}dt^2 + a^2(t)\left[e^{-2\beta(t)}dx^2 + e^{\beta(t)}\left(dy^2 + dz^2\right)\right] \tag{11.34}$$

arising from the vierbein

$$e_i^A = \left(\frac{1}{N(a(t), \beta(t))}, a(t)e^{-\beta(t)}, a(t)^{\frac{\beta(t)}{2}}, a(t)^{\frac{\beta(t)}{2}}\right)\ . \tag{11.35}$$

In this case, we obtain

$$ds_{\bar{\gamma}}^2 = N\left(-4a da^2 + a^3 d\beta^2\right)\quad ,\quad M(a, \beta) = \frac{a^3(t)}{N}. \tag{11.36}$$

In the present work we will focus on the static spherically-symmetric metric deriving new spherically symmetric solutions for $f(T)$ gravity. In particular, we look for Noether symmetries in order to reveal the existence of analytical solutions.

11.5 The Noether Symmetry Approach for $f(T)$ gravity

The aim is now to apply the Noether Symmetry Approach to a general class of $f(T)$ gravity models where the corresponding Lagrangian of the field equations is given by equation (11.23). First of all, we perform the analysis for arbitrary spacetimes, and then we focus on the spatially flat FRW spacetime and on the static spherically-symmetric spacetime.

11.5.1 Searching for Noether point symmetries in general spacetimes

The Noether symmetry condition for Lagrangian (11.23) is given by

$$X^{[1]}L + L\xi' = g'\ ,\ g = g\left(\tau, x^i\right). \tag{11.37}$$

Notice that the Lagrangian (11.51) is a singular Lagrangian (the Hessian vanishes), hence the jet space is $\bar{B}_M = \left\{\tau, x^i, T, \dot{x}^i\right\}$ and thus the first prolongation of X in the jet space $\bar{B}_M$ is [103, 181, 182]

$$\begin{aligned} X^{[1]} = &\ \xi\left(\tau, x^k, T\right)\partial_\tau + \eta^k\left(\tau, x^k, T\right)\partial_i \\ &+ \mu\left(\tau, x^k, T\right)\partial_T + \left(\eta'^i - \xi' x'^i\right)\partial_{x'^i}. \end{aligned} \tag{11.38}$$

For each term of the Noether condition (11.37) for the Lagrangian (11.23) we obtain

$$\begin{aligned} X^{[1]}L &= 2f_T\bar{g}_{ij,k}\eta^k x'^i x'^j + M_{,k}\eta^k\left(f - Tf_T\right) \\ &+2f_{TT}\mu\bar{g}_{ij}x'^i x'^j - Mf_{TT}\mu \\ &+4f_T\bar{g}_{ij}x'^i\left(\eta^j_{,\tau} + \eta^j_{,k}x'^k + \eta^j_{,T}T'\right. \\ &\left.-\xi_{,\tau}x'^j - \xi_{,k}x'^j x'^k - \xi_{,T}x'^j T'\right), \end{aligned}$$

$$L\xi' = \left[2f_T\bar{g}_{ij}x'^i x'^j + M\left(x^i\right)\left(f - Tf_T\right)\right]\left(\xi_{,\tau} + \xi_{,k}x'^k + \xi_{,T}T'\right),$$

$$g' = g_{,\tau} + g_{,k}x'^k + g_{,T}T' .$$

Inserting these expressions into (11.37) we find the Noether symmetry conditions

$$\xi_{,k} = 0\ ,\ \xi_{,T} = 0\ ,\ g_{,T} = 0\ ,\ \eta_{,T} = 0, \tag{11.39}$$

$$4f_T\bar{\gamma}_{ij}\eta^k_{,\tau} = g_{,k}, \tag{11.40}$$

$$M_{,k}\eta^k\left(f - Tf_T\right) - MTf_{TT}\mu + \xi_{,\tau}M\left(f - Tf_T\right) - g_{,\tau} = 0, \tag{11.41}$$

$$2f_T\bar{\gamma}_{ij,k}\eta^k + 2f_{TT}\mu\bar{\gamma}_{ij} + 4f_T\bar{\gamma}_{ij}\eta^j_{,k} - 2f_T\bar{\gamma}_{ij}\xi_{,\tau} = 0 . \tag{11.42}$$

Conditions $\eta_{,T} = g_{,T} = 0$ imply, through equation (11.40), that $\eta^k_{,\tau} = g_{,k} = 0$. Also, equation (11.42) takes the form

$$L_\eta\bar{\gamma}_{ij} = \left(\xi_{,\tau} - \frac{f_{TT}}{f_T}\mu\right)\bar{\gamma}_{ij}, \tag{11.43}$$

where $L_\eta\bar{\gamma}_{ij}$ is the Lie derivative with respect to the vector field $\eta^i(x^k)$. Furthermore, from (11.43) we deduce that η^i is a CKV of the metric $\bar{\gamma}_{ij}$, with conformal factor

$$2\bar{\psi}\left(x^k\right) = \xi_{,\tau} - \frac{f_{TT}}{f_T}\mu = \xi_{,\tau} - S(\tau, x^k) . \tag{11.44}$$

Finally, utilizing simultaneously equations (11.41), (11.43), (11.44) and the condition $g_{,\tau} = 0$, we rewrite (11.41) as

$$M_{,k}\eta^k + \left[2\bar{\psi} + \left(1 - \frac{Tf_T}{f - Tf_T}\right)S\right]M = 0 . \tag{11.45}$$

Considering that $S = S(x^k)$ and using the condition $g_{,\tau} = 0$, we acquire $\xi_{,\tau} = 2\bar{\psi}_0, \bar{\psi}_0 \in \mathbb{R}$ with $S = 2(\bar{\psi}_0 - \bar{\psi})$. At this point, we have to deal with the following two situations:

Case 1. In the case of $S = 0$, the symmetry conditions are

$$\begin{aligned} &L_\eta\bar{\gamma}_{ij} = 2\bar{\psi}_0\bar{\gamma}_{ij} \\ &M_{,k}\eta^k + 2\bar{\psi}_0 M = 0 \end{aligned} \tag{11.46}$$

implying that the vector $\eta^i(x^k)$ is a homothetic Vector of the metric $\bar{\gamma}_{ij}$. The latter means that for arbitrary $f(T) \neq T^n$ functional forms, the dynamical system could possibly admit extra (time independent) Noether symmetries.

Case 2. If $S \neq 0$ then equation (11.45) leads to the differential equation

$$\frac{Tf_T}{f - Tf_T} = C \tag{11.47}$$

which has the solution

$$f(T) = T^n, \quad C \equiv \frac{n}{1-n} . \tag{11.48}$$

In this context, $\eta^i(x^k)$ is a CKV of $\bar{\gamma}_{ij}$, and the symmetry conditions become

$$\begin{aligned} L_\eta \bar{\gamma}_{ij} &= 2\bar{\psi}\bar{\gamma}_{ij}, \\ M_{,k}\eta^k &+ \left[2\bar{\psi} + (1-C)\, S\right] = 0, \end{aligned} \tag{11.49}$$

with $S = 2(\bar{\psi}_0 - \bar{\psi})$.

Collecting the above results we have the following result

Lemma 11.5.1 *The general autonomous Lagrangian*

$$L\left(x^k, x'^k, T\right) = 2f_T\bar{\gamma}_{ij}\left(x^k\right)x'^i x'^j + M\left(x^k\right)(f - Tf_T)$$

admits extra Noether point symmetries as follows

a) If $f(T)$ is an arbitrary function of T, then the symmetry vector is written as

$$X = (2\psi_0\tau + c_1)\,\partial_\tau + \eta^i\left(x^k\right)\partial_i$$

where $\eta^i\left(x^k\right)$ is a HV/KV of the metric $\bar{\gamma}_{ij}$ and the following condition holds

$$M_{,k}\eta^k + 2\bar{\psi}_0 M = 0.$$

b) If $f(T)$ is a power law, i.e. $f(T) = T^n$, then we have the extra symmetry vector

$$X = \left(2\bar{\psi}_0\tau\right)\partial_\tau + \eta^i\left(x^k\right)\partial_i + \frac{2\bar{\psi}_0 - 2\bar{\psi}\left(x^k\right)}{C}T\partial_T$$

where $C = \frac{n}{1-n}$, $\eta^i\left(x^k\right)$ is a CKV of the metric $\bar{\gamma}_{ij}$ with conformal factor $\bar{\psi}\left(x^k\right)$ and the following condition holds

$$M_{,k}\eta^k + \left[2\bar{\psi} + (1-C)\,S\right]M = 0$$

where $S = 2\left(\bar{\psi}_0 - \bar{\psi}\right)$.

In both cases the corresponding gauge function is a constant.

In the following we apply the above Lemma in the case of FRW cosmology and static spherical symmetric spacetimes.

11.6 Spatially flat FRW

The FRW in the holonomic (commoving) frame $\{\partial t, \partial x, \partial y, \partial z\}$ has the form

$$ds^2 = -dt^2 + a^2(t)(dx^2 + dy^2 + dz^2)$$

where $a(t)$ is the cosmological scale factor. In this spacetime we define the vierbein (unholonomic frame) $\{e_i\}$ with the requirement:

$$h^i_\mu(t) = \mathrm{diag}(-1, a(t), a(t), a(t)). \tag{11.50}$$

In order to derive the cosmological equations in a FRW metric, we need to deduce a point-like Lagrangian from the action (11.20). As a consequence, the infinite number of degrees of freedom of the original field theory will be reduced to a finite number. In this framework considering (a, T) as canonical variables the corresponding $f(T)$ Lagrangian becomes:

$$\mathcal{L} = a^3 \left[f(T) - Tf'(T)\right] - 6\dot{a}^2 a f'(T)\,. \tag{11.51}$$

Therefore the field equations are

$$T = -6\left(\frac{\dot{a}^2}{a^2}\right) = -6H^2 \tag{11.52}$$

$$12H^2 f'(T) + f(T) = 16\pi G\rho \tag{11.53}$$

$$48H^2 f''(T)\dot{H} - f'^2 + 4\dot{H} - f(T) = 16\pi G p \tag{11.54}$$

where H is the Hubble parameter, $\rho = \rho_m + \rho_r$ and $p = p_m + p_r$ are the total energy density and (isotropic) pressure respectively, which are measured in the unholonomic frame. It is interesting to mention that using the conservation equation $\dot{\rho} + 3H(\rho + p) = 0$ one can rewrite equations (11.53) and (11.54) in the usual form

$$H^2 = \frac{8\pi G}{3}(\rho + \rho_T) \tag{11.55}$$

$$2\dot{H} + 3H^2 = -\frac{8\pi G}{3}(p + p_T) \tag{11.56}$$

where

$$\rho_T = \frac{1}{16\pi G}\left[2Tf'(T) - f(T) - T\right] \tag{11.57}$$

$$p_T = \frac{1}{16\pi G}[2\dot{H}(4Tf''(T) + 2f'(T) - 1)] - \rho_T. \tag{11.58}$$

are the unholonomicity contributions to the energy density and pressure. Finally, a basic question here is the following: under which circumstances $f(T)$ gravity can resemble that of the scalar field dark energy? In order to address this crucial question we need to calculate the effective equation-of-state parameter $w(a)$ for the $f(T)$ cosmology. Indeed, utilizing equations (11.57) and (11.58), we can easily obtain the effective unholonomicity equation of state as

$$\omega_T \equiv \frac{p_T}{\rho_T} = -1 + \frac{4\dot{H}(4Tf''(T) + 2f'(T) - 1)}{4Tf'(T) - 2f(T) - T}\,. \tag{11.59}$$

11.6.1 Noether symmetries

From Lemma 11.5.1 for the Lagrangian (11.51) we find that for

- For arbitrary $f(T)$ the Lagrangian (11.51) admits only the Noether symmetry ∂_t

- For $f(T) = f_0 T^n$ where f_0 is the integration constant we have the following extra Noether symmetries:
  - For $n \neq \frac{1}{2}, \frac{3}{2}$ the Noether point symmetry vector is

$$\begin{aligned} X_1 &= \left(\frac{3C}{2n-1}t\right)\partial_t + \left(Ca + c_3 a^{1-\frac{3}{2n}}\right)\partial_a + \\ &+ \left[\frac{1}{n}\left((c-m)\,n + 3c_3 a^{-\frac{3}{2n}}\right) + \frac{3C}{2n-1} + c\right] T\partial_T \end{aligned}$$

with corresponding Noether integral

$$I_1 = \left(\frac{3C}{2n-1}t\right)\mathcal{H} - 12 f_0 n\left(Ca^2 + c_3 a^{2-\frac{3}{2n}}\right) T^{n-1}\dot{a}$$

where $C = \frac{m(1-n)+nc}{3}$.

- For $n = \frac{3}{2}$, the Noether point symmetry is

$$\begin{aligned} X_2 &= \frac{1}{5}(3c-2m)\,t\partial_t + \left[\left(\frac{c}{2} - \frac{m}{6}\right)a + c_4\right]\partial_a + \\ &+ \left[(m+11c) - \frac{c_4}{a} + \frac{2}{5}(8c-2m)\right] T\partial_T \end{aligned} \tag{11.60}$$

with corresponding Noether integral

$$I_2 = \frac{1}{5}(3c-2m)\,t\mathcal{H} - 18 f_0\left[\left(\frac{c}{2} - \frac{m}{6}\right)a^2 + c_4 a\right] T^{\frac{1}{2}}\dot{a}\ .$$

- For $n = \frac{1}{2}$ the Noether point symmetry becomes

$$X_3 = c_1 t\partial_t + \left(-2c_1 + c_3 a^{\frac{1}{4}}\right)\partial_a + \left(4c_1 + c_2 + \frac{3c_3}{2}a^{-\frac{3}{4}}\right) T\partial_T$$

with Noether integral

$$I_3 = c_1 t\mathcal{H} - 6 f_0\left(-2c_1 a + c_3 a^{\frac{3}{4}}\right) T^{-\frac{1}{2}}\dot{a}.$$

We would like to stress that our results are in agreement with those of [147] but they are richer because we have considered the term $\xi\partial_t$ in the generator which is not done in [147]. To this end it becomes evident that $f(T) = f_0 T^n$ is the only form that admits extra Noether point symmetries implying the existence of exact analytical solutions.

11.6.2 Exact cosmological solutions

In this section we proceed in an attempt to analytically solve the basic cosmological equations of the $f(T) = f_0 T^n$ gravity model. In particular from the Lagrangian (11.51) we obtain the main field equation

$$\ddot{a} + \frac{1}{2a}\dot{a}^2 + \frac{f''}{f'}\dot{a}\dot{T} - \frac{1}{4}a\frac{f'T - f}{f'} = 0 \, . \tag{11.61}$$

Also differentiating equation (11.52) we find

$$\dot{T} = 12\left[\left(\frac{\dot{a}}{a}\right)^3 - \frac{\dot{a}\ddot{a}}{a^2}\right] \, . \tag{11.62}$$

Finally, inserting $f(T) = f_0 T^n$, $H = \dot{a}/a$, equation (11.52) and equation(11.62) into equation(11.61) we derive after some algebra that

$$(2n-1)\left[\ddot{a} - \frac{\dot{a}^2}{2a}\frac{(2n-3)}{n}\right] = 0 \tag{11.63}$$

a solution of which is

$$a(t) = a_0 t^{2n/3} \quad H(t) = \frac{2n}{3t} \tag{11.64}$$

or

$$H = H_0 a^{-3/2n} = H_0(1+z)^{3/2n} \tag{11.65}$$

where $n \in \mathcal{R}_+^\star - \{\frac{1}{2}\}$, $a(z) = (1+z)^{-1}$ and H_0 is the Hubble parameter. We note that the above analytic solution confirms that of [147].

From equation(11.64) it is evident that this cosmological models have no inflection point (that is the deceleration parameter does not change sign). Therefore, the main drawback of the $f(T) = f_0 T^n$ gravity model is that the deceleration parameter preserves sign, and therefore the universe always accelerates or always decelerates depending on the value of n. Indeed, if we consider $n = 1$ (TEGR) then the above solution boils down to the Einstein de Sitter model as it should. On the other hand, the accelerated expansion of the universe ($q < 0$) is recovered for $n > \frac{3}{2}$. The latter means that even if we admit $n > \frac{3}{2}$ as a mere phenomenological possibility, we would be also admitting that the universe has been accelerating forever, which is of course difficult to accept.

11.6.3 Cosmological analogue to other models

In this section (assuming flatness) we present the cosmological equivalence at the background level between the current $f(T)$ gravity with $f(R)$ modified gravity and dark energy, through a specific reconstruction of the $f(R)$ and vacuum energy density namely, $f(R) = R^n$ and $\Lambda(H) = 3\gamma H^2$. In particular, in the case of $f(R) = R^n$ it has been found (see chapter 10) that the corresponding scale factor obeys equation (11.64), where $n \in \mathcal{R}_+^\star - \{2, \frac{3}{2}, \frac{7}{8}\}$.[2]

[2] The Lagrangian here is $\mathcal{L}_R = 6naR^{n-1}\dot{a}^2 + 6n(n-1)a^2R^{n-2}\dot{a}\dot{R} + (n-1)a^3R^n$, where R is the Ricci scalar. For $n = 1$ the solution of the Euler-Lagrange equations is the Einstein de-Sitter model [$a(t) \propto t^{2/3}$] as it should. Note, that for $n = 2$ one can find a de-Sitter solution [$a(t) \propto e^{H_0 t}$, see chapter 10].

On the other hand, considering a spatially flat FRW metric and in the context of GR the combination of the Friedmann equations with the total (matter+vacuum) energy conservation in the matter dominated era, provides (for more details see [183, 184])

$$\dot{H} + \frac{3}{2}H^2 = \frac{\Lambda}{2} \ . \tag{11.66}$$

Solving equation (11.66) for $\Lambda(H) = 3\gamma H^2$ (see [185, 186, 187]) we end up with

$$H = H_0 a^{-3(1-\gamma)/2} = H_0(1+z)^{3(1-\gamma)/2} \ . \tag{11.67}$$

Now, comparing equations (11.65), (11.67) and connecting the above coefficients as $n^{-1} = 1-\gamma$, we find that the $f(T) = f_0 T^n$ and the flat $\Lambda(H) = 3\gamma H^2$ models can be viewed as equivalent cosmologies as far as the Hubble expansion is concerned, despite the fact that the time varying vacuum model is inside GR. However, when the $\Lambda(H) = 3\gamma H^2$ cosmological model is confronted with the current observations it provides a poor fit [183, 184]. Because the current time varying vacuum model shares exactly the same Hubble parameter with the $f(T) = f_0 T^n$ gravity model, it follows that the latter is also under observational pressure when compared against the background cosmological data. The same observational situation holds also for the $f(R) = R^n$ modified gravity.

11.7 Static spherically symmetric spacetimes

We apply now the results of the general Noether analysis of the previous subsection, to the specific case of static spherically-symmetric geometry given by the metric (11.28), that is the vierbein (11.29). Armed with the general expressions provided above, we can deduce the Noether algebra of the metric (11.28).

In this metric the Lagrangian (11.23) and the Hamiltonian (11.24) become

$$L = 2f_T N\left(2ba'b' + ab'^2\right) + M(a,b)\left(f - f_T T\right) \tag{11.68}$$

$$H = 2f_T N\left(2ba'b' + ab'^2\right) - M(a,b)\left(f - f_T T\right) \equiv 0 \tag{11.69}$$

where $M(a,b)$ is given by (11.30). As one can immediately deduce, TEGR and thus General Relativity is restored when $f(T) = T$, while if $N = 1$, $\tau = r$ and $ab = 1$ we fully recover the standard Schwarzschild solution.

Applying Lemma 11.5.1 in the case of static spherically-symmetric geometry, we determine all the functional forms of $f(T)$ for which the above dynamical system admits Noether point symmetries beyond the trivial one ∂_τ. We summarize the results in Tables 11.1, 11.2 and 11.3. Furthermore, we can use the obtained Noether integrals in order to classify the analytic solutions for each case.

In the case of $f(T) = T^n$ we have the additional extra Noether symmetries of Table 11.1

11.7.1 Exact Solutions

Using the Noether symmetries and the corresponding integral of motions obtained in the previous section, we can extract all the static spherically-symmetric solutions of $f(T)$ gravity. Without loss of generality, we choose

Table 11.1: Noether Symmetries and Noether Integrals for arbitary f(T)

$\mathbf{N}(a,b)$	**Noether Symmetry**	**Noether Integral**
$\frac{1}{a^3}N_1\left(a^2b\right)$	$X_1=-\frac{a}{2b^3}\partial_a+\frac{1}{b^2}\partial_b$	$I_1=\frac{N_1\left(a^2b\right)}{2a^3b^2}\left(2ba'+ab'\right)f_T$
$N_2\left(b\sqrt{a}\right)$	$X_2=-2a\partial_a+b\partial_b$	$I_2=N_2\left(b\sqrt{a}\right)\left(b^2a'-abb'\right)f_T$
$aN_3(b)$	$X_3=\frac{1}{ab}\partial_a$	$I_3=N_3(b)\,b'f_T$

Table 11.2: Extra Noether Symmetries and Noether Integrals for $f(T)=T^n$

$\mathbf{N}(a,b)$	**Noether Symmetry**[3]	**Noether Integral**
arbitrary	$X_4=2\bar{\psi}_0\tau+\frac{2\bar{\psi}_0(C-1)}{2C+1}a\partial_a+\frac{2\bar{\psi}_0-2\bar{\psi}_4}{C}T\partial_T$	$I_4=2\psi_0 n\frac{C-1}{1+2C}abN(a,b)\,T^{n-1}b'$
arbitrary	$X_5=-2a\partial_a+b\partial_b-\frac{2\bar{\psi}_5}{C}T\partial_T$	$I_5=nN(a,b)\,T^{n-1}\left(b^2a'-abb'\right)$
arbitrary	$X_6=-\frac{a}{2}b^{-\frac{3(1+2C)}{4C}}\partial_a+b^{-\frac{3+2C}{4C}}\partial_b-\frac{2\bar{\psi}_6}{C}T\partial_T$	$I_6=\frac{n}{2}N(a,b)\,T^{n-1}\left(2b^{\frac{2C-3}{4C}}a'+ab^{-\frac{3+2C}{4C}}b'\right)$
arbitrary	$X_7=a^{-\frac{1}{2C}}b^{-\frac{1+2C}{4C}}\partial_a-\frac{2\bar{\psi}_7}{C}T\partial_T$	$I_7=N(a,b)\,na^{-\frac{1}{2C}}b^{-\frac{1+2C}{4C}}T^{n-1}b'$

Table 11.3: Extra Noether Symmetries and Noether Integrals for $f(T)=T^{\frac{1}{2}}$

$\mathbf{N}(a,b)$	**Noether Symmetry**	**Noether Integral**
arbitrary	$\bar{X}_4=2\bar{\psi}_0\tau+\frac{3\bar{\psi}_0}{2}a\ln\left(a^2b\right)\partial_a+\frac{2\bar{\psi}_0-2\bar{\psi}_4'}{C}T\partial_T$	$\bar{I}_4=\frac{3}{2}\psi_0N(a,B)\,T^{-\frac{1}{2}}ab\ln\left(a^2b\right)b'$
arbitrary	$\bar{X}_5=b\partial_b-\frac{2\bar{\psi}_5}{C}T\partial_T$	$\bar{I}_5=\frac{1}{2}N(a,b)\,T^{-\frac{1}{2}}\left(b^2a'+abb'\right)$
arbitrary	$\bar{X}_6=-a\ln(ab)\,\partial_a+b\ln b\partial_b-\frac{2\bar{\psi}_6}{C}T\partial_T$	$\bar{I}_2=\frac{1}{2}N(a,B)\,T^{-\frac{1}{2}}b\left(b\ln b\ a'-a\ln a\ b'\right)$
arbitrary	$\bar{X}_7=a\partial_a-\frac{2\bar{\psi}_7}{C}T\partial_T$	$\bar{I}_3=\frac{1}{2}N(a,b)\,T^{-\frac{1}{2}}ab\ b'$

the conformal factor $N(a,b)$ as $N(a,b)=ab^2$ [or equivalently[4] $M(a,b)=1$]. In order to simplify the current dynamical problem, we consider the coordinate transformation

$$b=(3y)^{\frac{1}{3}} \qquad a=\sqrt{\frac{2x}{(3y)^{\frac{1}{3}}}}\ . \tag{11.70}$$

Substituting the above variables into the field equations (11.26), (11.27), (11.69) we immediately obtain

$$x''+\frac{f_{TT}}{f_T}x'T'=0 \tag{11.71}$$
$$y''+\frac{f_{TT}}{f_T}y'T'=0 \tag{11.72}$$
$$H=4f_Tx'y'-(f-Tf_T) \tag{11.73}$$

while the torsion scalar is given by

$$T=4x'y'\ . \tag{11.74}$$

Finally, the generalized Lagrangian (11.23) acquires the simple form

$$L=4f_Tx'y'+(f-Tf_T)\ . \tag{11.75}$$

Since the analysis of the previous subsection revealed two classes of Noether symmetries, namely for arbitrary $f(T)$, and $f(T)=T^n$, in the following subsections we investigate them separately.

Arbitrary $f(T)$

In the case where $f(T)$ is arbitrary, a special solution of the system (11.71)-(11.74) is

$$x(\tau) = c_1\tau+c_2 \tag{11.76}$$
$$y(\tau) = c_3\tau+c_4 \tag{11.77}$$

and the Hamiltonian constraint ($H=0$) reads

$$4c_1c_3\frac{df}{dT}|_{T=4c_1c_3}-f+T\frac{df}{dT}|_{T=4c_1c_3}=0 \tag{11.78}$$

where $T=4c_1c_3$, and c_{1-4} are integration constants. Utilizing (11.70), (11.76) and (11.77), we get

$$b(\tau)=3^{\frac{1}{3}}(c_3\tau+c_4)^{\frac{1}{3}}$$
$$a(\tau)=\frac{\sqrt{6}}{3^{\frac{2}{3}}}(c_1\tau+c_4)^{\frac{1}{2}}(c_3\tau+c_4)^{\frac{1}{6}}\ . \tag{11.79}$$

For convenience, we can change variables from $b(\tau)$ to r according to the transformation $b(\tau)=r$, where r denotes the radial variable. Inserting this into the above equations, we conclude that the spacetime (11.28) in

[4] Since the space is empty, the field equations are conformally invariant, therefore the resutls are similar for an arbitrary function $N(a,b)$ (see chapter 9)

the coordinates (t, r, θ, ϕ) can be written as

$$ds^2 = -A(r)\,dt^2 + \frac{1}{c_3^2}\frac{1}{A(r)}dr^2 + r^2\left(d\theta^2 + \sin^2\theta d\phi^2\right), \tag{11.80}$$

with

$$A(r) = \frac{2c_1}{3c_3}r^2 - \frac{2c_\mu}{c_3 r} = \lambda_A(1 - \frac{r_\star}{r})R(r), \tag{11.81}$$

and

$$R(r) = \left(\frac{r}{r_\star}\right)^2 + \frac{r}{r_\star} + 1. \tag{11.82}$$

In these expressions we have $c_\mu = c_1c_4 - c_2c_3$, $\lambda_A = \left(\frac{8c_1c_\mu^2}{3c_3^3}\right)^{1/3}$ and $r_\star = (\frac{3c_\mu}{c_1})^{1/3} = (\frac{3c_3\lambda_A}{2c_1})^{1/2}$ is a characteristic radius with the restriction $c_\mu c_1 > 0$.

We observe that if we select the constant $c_3 \equiv 1$ then we retain the Schwarzschild-like metric. On the other hand, the function $R(r)$ can be viewed as a distortion factor which quantifies the smooth deviation from the pure Schwarzschild solution. Thus, the $f(T)$ gravity on small spherical scales ($r \to r_\star^+$) tends to create a Schwarzschild solution. In particular, the $f(T)$ spherical solution admits singularity only at $r = r_\star$, which is also the case with the usual Schwarzschild solution.

Within this framework, for commoving observers, $u^iu_i = A^2(r)$, it is easy to show that the Einstein's tensor becomes

$$G^i_j = \text{diag}\left(2c_1c_3 - \frac{1}{r^2}, 2c_1c_3 - \frac{1}{r^2}, 2c_1c_3, 2c_1c_3\right).$$

Therefore, from the 1+3 decomposition of G_{ij}we define the fluid physical parameters

$$\rho_T = \frac{1}{(u^iu_i)}G_{ij}u^iu^j = 2c_1c_3 - \frac{1}{r^2} \tag{11.83}$$

$$p_T = \frac{1}{3}h_{ij}G_{ij} = 2c_1c_3 - \frac{1}{3r^2} \tag{11.84}$$

$$q^i = h^{ij}G_{jk}u^k = 0 \tag{11.85}$$

$$\pi^\theta_\theta = \pi^\phi_\phi = -\frac{\pi^r_r}{2} = \frac{1}{3r^2} \tag{11.86}$$

where

$$\pi_{ij} = (h^r_ih^s_j - \frac{1}{3}h_{ij}h^{rs})G_{rs} \tag{11.87}$$

is the anisotropic stress tensor and h_{ij} is the u^i projection tensor defined by

$$h_{ij} = g^{ij} - \frac{1}{(u^iu_i)}u^iu^j. \tag{11.88}$$

Furthermore the fluid is also anisotropic ($\pi_{ij} \neq 0$) but not heat conducting ($q^i = 0$).

In order to apply the above considerations for specific $f(T)$ forms, we consider the following viable $f(T)$ models, motivated by cosmology[5]

[5] The $f(T)$ models of Refs.[188, 189] are consistent with the cosmological data.

- Exponential $f(T)$ gravity [189]:

$$f(T) = T + f_0 e^{-f_1 T},$$

where f_0 and f_1 are the two model parameters which are connected via (11.78)

$$f_0 = \frac{4c_1c_3}{8f_1c_1c_3+1}\exp\left(4f_1c_1c_3\right).$$

- A sum of two different power law $f(T)$ gravity:

$$f(T) = T^m + f_0T^n$$

where from (11.78) we have

$$f_0 = \frac{1-2m}{2n-1}\left(4c_1c_3\right)^{m-n} \ .$$

Note that in the case of $m = 1$ we recover the $f(T)$ model by Bengochea & Ferraro [188].

$f(T) = T^n$

In the $f(T) = T^n$ case, the field equations (11.26), (11.27), (11.69) and the torsion scalar (11.74) give rise to the following dynamical system:

$$T = 4x'y', \tag{11.89}$$

$$4nT^{n-1}x'y' - (1-n)\,T^n = 0, \tag{11.90}$$

$$x'' + (n-1)\,x'T^{-1}T' = 0, \tag{11.91}$$

$$y'' + (n-1)\,y'T^{-1}T' = 0\ . \tag{11.92}$$

It is easy to show that combining equation (11.89) with the Hamiltonian (11.90), we can impose constraints on the value of n, namely $n = 1/2$. Under this condition, solving the system of equations (11.91) and (11.92) we arrive at the solutions

$$x(\tau) = \frac{\sigma(\tau)^3}{3} + c_\sigma \tag{11.93}$$

$$y(\tau) = \frac{\sigma(\tau)^3}{3} \tag{11.94}$$

where c_σ is the integration constant. Now using (11.70) we derive a,b as

$$b(\tau) = \sigma(\tau) \tag{11.95}$$

$$a(\tau) = \sqrt{\frac{2\left[\sigma^3(\tau)+3c_\sigma\right]}{3\sigma(\tau)}}. \tag{11.96}$$

Using the coordinate transformation $\sigma(\tau) = r$, which implies $\tau = F(r)$ [with $F(\sigma(\tau)) = \tau$], and using simultaneously (11.95), the spherical metric (11.28) can be written as

$$ds^2 = -A(r)dt^2 + B(r)dr^2 + r^2\left(d\theta + \sin^2\theta d\phi^2\right) \tag{11.97}$$

where

$$A(r)=\frac{2}{3}r^{2}+\frac{2c}{r} \tag{11.98}$$

and

$$B(r)=\frac{F_{,r}^{2}}{A(r)r^{4}}. \tag{11.99}$$

Furthermore, considering the commoving observers $\left(u^{i}u_{i}\right)=-\frac{2\left(r^{3}+3c_{\sigma}\right)}{3r}$, we can write the Einstein tensor components as

$$\begin{aligned}
G_{t}^{t} &= -\frac{r}{3F_{r}^{3}}\left[4rF_{,rr}\left(r^{3}+3c_{\sigma}\right)-2F_{,r}\left(7r^{3}+12c_{\sigma}\right)+\frac{3}{r^{3}}F_{,r}^{3}\right] \\
G_{r}^{r} &= \frac{2r^{4}}{F_{,r}^{2}}-\frac{1}{r^{2}} \\
G_{\theta}^{\theta} &= G_{\phi}^{\phi}=-\frac{1}{3}\frac{r}{F_{,r}^{3}}\left[rF_{,rr}\left(4r^{3}+3c_{\sigma}\right)-F_{,r}\left(14r^{3}+6c_{\sigma}\right)\right]
\end{aligned}$$

where $F_{,r}=dF/dr$ and $F_{,rr}=d^{2}F/dr^{2}$.

Similarly, based on the equalities (11.83)-(11.87), we compute the following fluid parameters

$$\rho_{T}=\frac{4r^{2}F_{,rr}}{F_{,r}^{3}}\left(\frac{1}{3}r^{3}+c_{\sigma}\right)-\frac{2r}{F_{,r}^{2}}\left(\frac{7}{3}r^{3}-4c_{\sigma}\right)+\frac{1}{r^{2}} \tag{11.100}$$

$$p_{T}=-\frac{2}{3}\frac{r^{2}F_{,rr}}{F_{,r}^{3}}\left(\frac{4}{3}r^{3}+c_{\sigma}\right)+\frac{2}{3}\frac{r}{F_{,r}^{2}}\left(\frac{17}{3}r^{3}+2c_{\sigma}\right)-\frac{1}{3r^{2}} \tag{11.101}$$

$$\pi_{,r}^{r}=\frac{2}{3}\frac{r^{3}F_{,rr}}{F_{,r}^{2}}\left(\frac{4}{3}r^{3}+c_{\sigma}\right)-\frac{2}{3}\frac{1}{F_{,r}^{2}}\left(\frac{8}{3}r^{3}+2c_{\sigma}\right)-\frac{2}{3r^{2}}$$

$$\pi_{\theta}^{\theta}=\pi_{\phi}^{\phi}=-\frac{1}{2}\pi_{r}^{r}$$

$$q^{i}=0$$

11.8 Conclusion

In this chapter we studied the Noether symmetries of $f(T)$ gravity. We proved that for some diagonal frames the Lagrangian of the field equations admits Noether symmetries for arbitrary $f(T)$ function. However, in the case of power law $f(T)$, i.e. $f(T)=T^{n}$ it is possible the Lagrangian to admits extra Noether symmetries. We applied this results in order to classify the Noether symmetries of the field equations in a spatially flat FRW spacetime and in a static spherical symmetric spacetime. For each background spacetime we found analytical solutions of the field equations.

Chapter 12

Discussion

12.1 Discussion

In this thesis we study the Lie point symmetries and the Noether point symmetries of second order differential equations usinng a geometric approach and we apply the results to systems which are relevant to relativistic physics.

In particular, we have studied the point symmetries of the equations of motion of dynamical systems in a Riemannian space with Lagrangian

$$L\left(x^{i},\dot{x}^{k}\right)=\frac{1}{2}g_{ij}\dot{x}^{i}\dot{x}^{j}-V\left(x^{k}\right) \tag{12.1}$$

where $g_{ij}=g_{ij}\left(x^{k}\right)$ is the metric of the space and we proved that the Lie point symmetries of the Euler-Lagrange equations, i.e. $E^{i}\left(L\right)=0$, of Lagrangian (12.1) are generated from the elements of the special Projective algebra of the Riemannian manifold with metric g_{ij} whereas the Noether point symmetries are generated from the homothetic algebra of the space with metric g_{ij}. Therefore we have transfer the problem of determination of the Lie/Noether symmetries of differential equations to the determination of the collineations of the underlying manifold; hence, we are able to use the plethora of existing results of differential geometry.

We have applied this geometric approach in many directions. In particular, we have classified the Lie and the Noether symmetries of the geodesic Lagrangian for some important spacetimes, such as the FRW spacetime, the Gödel space, the Taub space and the 1+3 decomposable spacetimes. Moreover we proved that for Einstein spaces the point symmetries of the geodesic equations are generated from the elements of the Killing algebra of the metric.

Furthermore we have determined all the two and the three dimensional Newtonian systems which admit Lie and Noether point symmetries. We note that, due to the geometric derivation and the tabular presentation, the results can be extended easily to higher dimensional flat spaces. We applied these results in the study of the symmetries of the Hènon - Heiles potential and of the Kepler-Ermakov potential in a two dimensional space.

Moreover, we determined the potentials which admit Noether symmetries in a two dimensional sphere S^2.

We proved that a dynamical system admits as Lie symmetries the $sl(2,R)$ Lie algebra if and only if the underlying manifold admits a gradient Homothetic vector. The Newtonian system which is invariant under the Lie group $sl(2,R)$ is the well known Kepler-Ermakov system. Therefore, the requirement for a dynamical system with Lagrangian of the form of (12.1) to admit as Lie and Noether symmetries the generators of the $sl(2,R)$ Lie algebra leads us to the generalization of the Newotian Kepler-Ermakov system in a Riemannian manifold; that is, we found that the general autonomous Kepler-Ermakov system follows from the Lagrangian

$$L\left(u,\dot{u},y^{A},\dot{y}^{A}\right)=\frac{1}{2}\left(\dot{u}^{2}+\dot{u}^{2}h_{AB}\dot{y}^{A}\dot{y}^{B}\right)+\frac{\mu^{2}}{2}u^{2}+\frac{1}{u^{2}}V\left(y^{C}\right) \tag{12.2}$$

In additionally, we studied the Liouville integrability of the three dimensional Newtonian Kepler-Ermakov via Noether point symmetries and we investigated the application of Lagrangian (12.2) in dynamical systems emerging from alternative theories of gravity. In particular we showed that the field equations in a Bianchi I spacetime for an exponential scalar field and for $f(R)$ gravity when $f(R)=(R-2\Lambda)^{\frac{7}{8}}$ follow from Lagrangians of the form of (12.2) and for these models we proved that the field equations are Liouville integrable.

Concerning the second order partial differential equations we considered equations of the generic form

$$A^{ij}\left(x^{k}\right)u_{ij}-B^{i}(x^{k},u)u_{i}-f(x^{k},u)=0 \tag{12.3}$$

and we proved a theorem which relates the Lie pont symmetries of equation (12.3) with the elements of the Conformal Killing vectors of the second order tensor $A_{ij}\left(x^{k}\right)$ (considered to be a metric). We have applied this result in order to study the Lie point symmetries of the Heat equation and the Poisson equation. It has been shown that the Lie symmetries of the Heat equation follow from the Killing and the homothetic algebras of $A_{ij}\left(x^{k}\right)$, whereas the Lie symmetries of the Poisson equation follow from the Killing, the homothetic and the conformal algebras of $A_{ij}\left(x^{k}\right)$. In each case we have determoned the form of the Lie symmetry vectors.

Furthermore, we have determined the Lie symmetries of the Schrödinger equation

$$\Delta u-u_{,t}=V\left(x^{k}\right)u \tag{12.4}$$

and the Klein Gordon equation

$$\Delta u=V\left(x^{k}\right)u \tag{12.5}$$

in a general Riemannian space. It has been shown that these symmetries are related to the Noether symmetries of the classical Lagrangian for which the metric g_{ij} is the kinematic metric. More precisely, for the Schrödinger equation (12.4) it has been shown that if a KV or a HV of the metric g_{ij} produces a Lie symmetry for the Schrödinger equation, then it produces a Noether symmetry for the Classical Lagrangian in the space with metric g_{ij} and potential $V(x^k)$. For the Klein Gordon equation (12.5) the situation is different; the Lie symmetries of the Klein Gordon are generated by the elements of the conformal group of the metric g_{ij}. The KVs and the HV of this group produce a Noether symmetry of the classical Lagrangian with a constant gauge function. However the proper CKVs produce a Noether symmetry for the conformal Lagrangian if there exists a conformal factor $N\left(x^{k}\right)$ such that the CKV becomes a KV/HV of g_{ij}.

We have applied these results to three cases of practical interest: the motion in a central potential, the classification of all potentials in two and three dimensional Euclidian spaces for which the Schrödinger equation and the Klein Gordon equation admit a Lie symmetry and finally we have considered the Lie symmetries of the Klein Gordon equation in the static, spherically symmetric empty spacetime. In the last case, we have demonstrated the role of the Lie symmetries and that of the conformal Lagrangians in the determination of the closed form solution of Einstein equations. Furthermore, we investigated the Lie point symmetries of the null Hamilton Jacobi equation and we proved that if a CKV generates a point symmetry for the Klein Gordon equation, then it also generates a point symmetry for the null Hamilton Jacobi equation.

We also studied the problem of Type II hidden symmetries of second order partial differential equations in n dimensional Riemannian spaces from a geometric of view. We have considered the reduction of the Laplace and of the homogeneous heat equation and the consequent possibility of existence of Type II hidden symmetries in some general classes of spaces which admit some kind of symmetry; hence, they admit nontrivial Lie symmetries.

The Type II hidden symmetries of Laplace equation are directly related to the transition of the CKVs from the space where the original equation is defined to the space where the reduced equation resides. In this sense, we related the Lie symmetries of PDEs with the basic collineations of the metric i.e. the CKVs.

Concerning the Type II hidden symmetries of the homogeneous heat equation we considered the problem in the spaces which admit a gradient KV or a gradient HV and finally spacetime which admits a HV which acts simply and transitively. For the reduction of the homogeneous heat equation and the existence of Type II hidden symmetries, we found the following general geometric results: (a) If we reduce the homogeneous heat equation via the symmetries which are generated by a gradient KV $\left(S^{,i}\right)$ the reduced equation is a heat equation in the nondecomposable space. In this case we have the Type II hidden symmetry $\partial_{t}-\frac{1}{2t}w\partial_{w}$ provided if we reduce the heat equation with the symmetry $tS^{,i}-\frac{1}{2}Su\partial_{u}$. (b) If we reduce the homogeneous heat equation via the symmetries which are generated by a gradient HV the reduced equation is Laplace equation for an appropriate metric. In this case the Type II hidden symmetries are generated from the proper CKVs and (c) in Petrov type III spacetime, the reduction of the homogeneous heat equation via the symmetry generated from the nongradient HV gives PDE that inherit the Lie symmetries, hence no Type II hidden symmetries are admitted.

Finally, we applied the point symmetries and especially the Noether point symmetries in modified theories of gravity in order to probe the nature of dark energy. We used the Noether symmetries as a geometric criterion or "selection rule" in order to select the scalar field potential in scalar-tensor theories, and the functions $f\left(R\right)$ and $f\left(T\right)$ in the corresponding alternative theories of gravity.

In the context of scalar-tensor cosmology we have found that to every non-minimally coupled scalar field, we can associate a unique minimally coupled scalar field in a conformally related space with an appropriate potential. This result can be used in order to study the dynamical properties of the various cosmological models, since the field equations of a non-minimally coupled scalar field are the same, at the conformal level, with the field equations of the minimally coupled scalar field. Furthermore, we have identified the Noether point symmetries and the analytic solutions of the equations of motion in the context of a minimally coupled and a non minimally coupled scalar field in a FRW spacetime and we have classified the Noether point symmetries of the field equations in Bianchi class A models with a minimally coupled scalar field. We find that there is a rather large class of hyperbolic and exponential potentials which admit extra (beyond the ∂_t) Noether symmetries which lead to integral of motions. For these potentials we used the corresponding Noether integrals in order to solve analytically the field equations and find the functional form of the scalar factor.

Concerning the $f(R)$ models we applied the Noether point symmetries with the aim to utilize the existence of non-trivial Noether symmetries as a selection criterion that can distinguish the $f(R)$ models on a more fundamental level. We proved that in a spatially flat FRW background the $f(R)$ theories which admit Noether point symmetries are the R^n, $R^{\frac{7}{8}}$, $(R-2\Lambda)^{\frac{7}{8}}$, $R^{\frac{3}{2}}$ and $(R-2\Lambda)^{\frac{3}{2}}$. The last two functional forms of the $f(R)$ function admit Noether point symmetries also in the case of a non spatially flat FRW background. It is interesting to note that the $\frac{7}{8}$ models are equivalent with the Newtonian Kepler-Ermakov system whereas the $\frac{3}{2}$ models are equivalent with the anisotropic hyperbolic oscillator. For these functional forms we use the Noether integrals in order to find exact solutions of the modified field equations.

Furthermore, in $f(T)$ gravity we have proved a Lemma that for diagonal frames the only functional form of $f(T)$ which admits extra Noether symmetries is the $f(T)=T^n$. We applied this functional form in a spatially flat FRW background and we determined the analytic solution of the field equations for each case. Finally, we studied the field equations for the $f(T)=T^n$ model in a static spherically symmetric spacetime and we determined a family of analytic solutions.

The geometric approach is a new method for studying the symmetries of differential equations and has shown that gives directly results by using only the results (usually existing) of diferential geometry without the need to use of a computer library in order to detrmine the Lie or the Noether point symmetries. This approach implies a better understanding of the nature of symmetries and of the conservation laws and can be used in order to find analogues of classical Newtonian systems in relativistic physics. It is of interest that this method would be extendented in other classes of differential equations and in other transformations which are not necessary point transformations. Concerning the applications in Cosmology, it is of interest the classification of modified theories of gravity with geometric selection rules. In this thesis we studied some of the basic modified theories of gravity; however there are other more such theories which could be studied further either by means of point symmetries or by some new geometric criteria.

Bibliography

[1] H. Stephani. *Differential Equations: Their Solutions Using Symmetry*. Cambridge University Press, New York, 1989.

[2] L. Serge. *Nonlinear Dynamics*. Springer-Verlag, New York, 2010.

[3] G.W. Bluman and S. Kumei. *Symmetries of Differential equations*. Springer-Verlag, New York, 1989.

[4] H.N. Ibragimov. *Transformation Groups Applied to Mathematical Physics*. Reidel Publishing Co., Dordrecht, 1985.

[5] P.G.L. Leach. The complete symmetry group of a foced harmonic oscillator. *J. Austral. Math. Soc.*, 22 (Series B):12, 1980.

[6] G.E. Prince and C.J. Eliezer. Symmetries of the time-dependent n-dimensional oscillator. *Journal of Physics A: Mathematical and General*, 13(3):815, 1980.

[7] A. Karasu and P.G.L. Leach. Nonlocal symmetries and integrable ordinary differential equations: $x'' + 3x' + x^3 = 0$ and its generalizations. *J. Math. Phys.*, 50:073509, 2009.

[8] F.M. Mahomed and A. Qadir. Linearization of second order quadratically semi-linear systems of ordinary differential equations. *Nonlinear Dynamics*, 48:417, 2007.

[9] E Fredericks, F.M. Mahomed, E Momoniat, and A. Qadir. Constructing a space from the geodesic equations. *Com. Phys. Commun. 179 438*, 179:438, 2008.

[10] A.D. Polyanin and V.F. Zaitsev. *Handbook of Exact Solutions for Ordinary Differential Equations*. CRC Press, New York, 1999.

[11] P.G.L. Leach. First integrals for the modified emden equation $q + a(t)q + q^n = 0$. *J. Math. Phys.*, 26:510, 1985.

[12] C.M. Khalique and P. Ntsime. Exact solutions of lane-emden equation. *New Astronomy*, 13:476, 2008.

[13] G.E. Prince and C.J. Eliezer. On the lie symmetries of the classical kepler problem. *J. Phys. A.: Math. Gen.*, 14:587, 1981.

[14] P.G.L. Leach, K.S. Govinder, and B. Abraham-Shrauner. Symmetries of first integrals and their associated differential equations. *J. Math. Anal. Appl.*, 235:58, 1999.

[15] M.C. Nucci and P.G.L. Leach. Jacobi's last multiplier and lagrangians for multidimensional systems. *J. Math. Phys.*, 49:073517, 2008.

[16] M.C. Nucci and P.G.L. Leach. An old method of jacobi to find lagrangians. *J. Nonlinear Math. Phys.*, 16:431, 2009.

[17] P.G.L. Leach. Lie symmetries and noether symmetries. *Appl. Anal. Discrete Math.*, 6:298, 2012.

[18] R. Miron. Noether theorem in higher-order lagrangian mechanics. *Int. J. Theor. Phys.*, 34:1123, 1995.

[19] C.E. Wulfman and B.G. Wybourne. The lie group of newton's and lagrange's equations for the harmonic oscillator. *J. Phys. A: Math. Gen.*, 9:507, 1976.

[20] G.W. Bluman, G.J. Reid, and S. Kumei. New classes of symmetries for partial differential equations. *J. Math. Phys.*, 29:806, 1988.

[21] K. Yano. *The Theory of Lie Derivatives and Its Applications*. North Holland Publishing Co., Amsterdam, 1956.

[22] G.S. Hall. Special conformal symmetries in general relativity. *Gen. Relativ. Gravit.*, 22:103, 1993.

[23] A. Barnes. Projective collineations in einstein spaces. *Class. Quantum Grav.*, 10:1139, 1993.

[24] G.S. Hall and J. da Costa. Affine collineations in space-time. *J. Math. Phys*, 29:2465, 1988.

[25] G.S. Hall and I.M. Roy. Some remarks on special conformal and special projective symmetries in general relativity. *Gen. Relativ. Gravit.*, 29:827, 1997.

[26] G.H. Katzin, J. Levine, and W.R. Davis. Curvature collineations: A fundamental symmetry property of the space-times of general relativity defined by the vanishing lie derivative of the riemann curvature tensor. *J. Math. Phys.*, 10:617, 1969.

[27] G.H. Katzin and J. Levine. Dynamical symmetries and constants of motion for classical particle systems. *J. Math. Phys.*, 15:1460, 1974.

[28] G.H. Katzin and J. Levine. A gauge invariant formulation of time dependent dynamical symmetry mappings and associated constant of motion for lagrangian particle mechanics i. *J. Math. Phys.*, 17:1345, 1976.

[29] G.H. Katzin and J. Levine. Geodesic first integrals with explicit path-parameter dependence in riemannian space-times. *J. Math. Phys.*, 22:1878, 1981.

[30] A.V. Aminova. Automorphisms of geometric structures as symmetries of diferential equations (in russian). *Izv. - Vyssh. - Uchebn. - Zaved. - Mat.*, 2:3, 1994.

[31] A.V. Aminova. Projective transformations and symmetries of differential equations. *Sbornik Mathematics*, 186:1711, 1995.

[32] A.V. Aminova. Projective transformations and symmetries of differential equations. *Tensors N.S.*, 65:86, 2000.

[33] A.V. Aminova and N.A. Aminov. Projective geometry of systems of second-order differential equations. *Sbornik Mathematics*, 197:951, 2006.

[34] A.V. Aminova and N.A. Aminov. The projective geometric theory of systems of second-order differential equations: straightening and symmetry theorems. *Sbornik Mathematics*, 201:631, 2010.

[35] G.E. Prince and M. Crampin. Projective transformations and symmetries of differential equations. *Gen. Relativ. Gravit.*, 16:921, 1984.

[36] T. Feroze, F.M. Mahomed, and A. Qadir. The connection between isometries and symetries of geodesic equations of the underlying spaces. *Nonlinear Dynamics*, 45:65, 2006.

[37] A.H. Bokhari, A.H. Kara, A.R. Kashif, and F.D. Zaman. The connection between isometries and symetries of geodesic equations of the underlying spaces. *Int. J. Theor. Phys.*, 45:1063, 2006.

[38] A.H. Bokhari and A.H. Kara. Noether versus killing symmetry of conformally flat friedmann metric. *Gen. Relativ. Gravit.*, 39:2053, 2007.

[39] T. Feroze and I. Hussain. Noether symmetries and conserved quantities for spaces with a section of zero curvature. *Journal of Geometry and Physics*, 61:658, 2011.

[40] I. Hussain. A note on noether symmetries and conformal killing vectors. *Gen. Relativ. Gravit.*, 42:1791, 2010.

[41] T. Feroze. New conserved quantities for the spaces of different curvatures. *Modern Phys. Lett. A.*, 25:1107, 2010.

[42] H. Stephani, D. Kramer, M. MacCallum, and E. Herlt. *Exact Solutions of Einstein's Field Equations*. Cambridge University Press, New York, 2003.

[43] B.O.J. Tupper. Conformal symmetries of conformal-reducible spacetimes with non-zero weyl tensor. *Class. Quantum Grav.*, 13:1679, 1996.

[44] M. Tsamparlis, D. Nikolopoulos, and P.S. Apostolopoulos. Computation of the conformal algebra of 1+3 decomposable spacetimes. *Class. Quantum Grav.*, 15:2909, 1998.

[45] E Maartens and S.D. Maharj. Conformal killing vectors in robertson-walker spacetimes. *Class. Quantum Grav.*, 3:1005, 1986.

[46] R. Maartens. Affine collineations in robertsonâĂŞwalker spaceâĂŘtime. *J. Math. Phys.*, 28:2051, 1987.

[47] J. Butcher, J. Carminati, and K.T. Vu. A comparative study of some computer algebra packages which determine the lie point symmetries of differential equations. *Com. Phys. Commun.*, 115:92, 2003.

[48] T. Sen. Lie symmetries and integrability. *Phys. Lett. A.*, 122:327, 1987.

[49] P.A. Damianou and C. Sophocleous. Symmetries of hamiltonian systems with two degrees of freedom. *J. Math. Phys.*, 40:210, 1999.

[50] P.A. Damianou and C. Sophocleous. Classification of noether symmetries for lagrangians with three degrees of freedom. *Nonlinear Dynamics*, 36:3, 2004.

[51] A. Karasu and H.b Yildirim. On the lie symmetries of kepler–ermakov systems. *J. Nonlinear Math. Phys.*, 9:475, 2002.

[52] P.G.L. Leach and A. Karasu. The lie algebra sl(2, r) and so-called kepler-ermakov systems. *J. Nonlinear Math. Phys.*, 11:475, 2004.

[53] S. Moyo and P.G.L. Leach. A note on the construction of the ermakov–lewis invariant. *J. Math. Phys. A.: Math. and Gen.*, 35:5333, 2002.

[54] P.G.L. Leach. A further note on the hÃľnon–heiles problem. *J. Math. Phys.*, 22:679, 1981.

[55] B. Abraham-Shrauner. Lie group symmetries and invariants of the hÃľnon–heiles equations. *J. Math. Phys.*, 31:1627, 1990.

[56] P.G.L Leach. Generalized ermakov systems. *Physics Letters A*, 158(3âĂŞ4):102 – 106, 1991.

[57] W. Sarlet and F. Cantrijin. Higher-order noether symmetries and constants of the motion. *J. Phys. A.: Math. and Gen.*, 14:479, 1981.

[58] T.M. Kalotas and B.G. Wybourne. Dynamical noether symmetries. *J. Phys. A: Math. and Gen.*, 15:2077, 1982.

[59] M. Crampin. Hidden symmetries and killing tensors. *Rep. Math. Phys.*, 20:31, 1984.

[60] P.L. Eisenhart. *Riemannian Geometry.* Princeton, NJ: Princeton University Press, 1949.

[61] J.F. Carinena, M.F. Ranada, and Santander M. Central potentials on spaces of constant curvature: The kepler problem on the two-dimensional sphere s^2 and the hyperbolic plane h^2. *J. Math. Phys.*, 46:052702, 2005.

[62] V.V Kozlov and A.O. Harin. Kepler's problem in constant curvature spaces. *Celest. Mech. Dyn. Astr.*, 54:393, 1992.

[63] T.G. Vozmishecheva. Integrable problems of celestial mechanics in spaces of constant curvature. *J. Math. Sci.*, 125:419, 2005.

[64] D. Voyatzi and S. Ichtiaroglou. Motion on the sphere: Integrability and families of orbits. *Celest. Mech. Dyn. Astr.*, 86:209, 2005.

[65] C. Athorne. Kepler-ermakov problems. *J. Phys. A.: Math. and Gen.*, 24:L1385, 1991.

[66] V. Ermakov. Second order differential equations. conditions of complete integrability. *Univerita Izvestia Kiev Series III (translated by A.O. Harin)*, 9:1, 1880.

[67] H. R. Lewis. Classical and quantum systems with time-dependent harmonic-oscillator-type hamiltonians. *Phys. Rev. Lett.*, 18:510–512, Mar 1967.

[68] J. R. Ray and J. L. Reid. More exact invariants for the time-dependent harmonic oscillator. *Physics Letters A*, 71(4):317 – 318, 1979.

[69] P.G.L. Leach and S.K. Andriopoulos. The ermakov equation: A commentary. *Appl. Anal. Discrete Math.*, 2:146, 2008.

[70] C. Rogers, C. Hoenselaers, and J.R. Ray. On (2+1)-dimensional ermakov systems. *Journal of Physics A: Mathematical and General*, 26(11):2625, 1993.

[71] W.K. Schief, C. Rogers, and A.P. Bassom. Ermakov systems of arbitrary order and dimension: structure and linearization. *J. Phys. A.: Math. Gen.*, 29:903, 1996.

[72] J. Goedert and F. Haas. On the lie symmetries of a class of generalized ermakov systems. *Physics Letters A*, 239:348, 1998.

[73] K.S. Govinder and P.G.L. Leach. Ermakov systems: a group theoretic approach. *Phys. Lett. A.*, 186:391, 1994.

[74] F. Haas and J. Goedert. Dynamical symmetries and the ermakov invariant. *Phys. Lett. A.*, 279:181, 2001.

[75] J.R. Ray and J.L. Reid. Ermakov systems, velocity dependent potentials, and nonlinear superposition. *J. Math. Phys.*, 22:91, 1981.

[76] C. Athorne. On generalized ermakov systems. *Phys. Lett. A.*, 159:375, 1991.

[77] K.S. Govinder, C. Athorne, and P.G.L. Leach. The algebraic structure of generalized ermakov systems in three dimensions. *J. Phys. A.: Math. Gen.*, 26:4035, 1993.

[78] K.S. Govinder and P.G.L. Leach. Generalized ermakov systems in terms of sl(2,r) invariants. *Quaestiones Mathematicae*, 16:405, 1993.

[79] E. Pinney. The nonlinear differential equation $y'' + p(x)y + c/y^3 = 0$. *Proc. Am. Math. Soc.*, 1:681, 1950.

[80] S. Wojciechowski. On the integrability of the calogero-moser system in an external quartic potential and other many-body systems. *Phys. Lett. A.*, 102:85, 1984.

[81] F. Haas and J. Goedert. On the hamiltonian structure of ermakov systems. *J. Phys. A.: Math. Gen.*, 29:4083, 1996.

[82] M. Ranada. Superintegrability of the calogero moser system: Constants of motion, master symmetries, and time-dependent symmetries. *J. Math. Phys.*, 40:236, 1999.

[83] A.A. Colley and B.O.J. Tupper. Special conformal killing vector spaceâĂŘtimes and symmetry inheritance. *J. Math. Phys.*, 30:2616, 1989.

[84] A.A. Colley and B.O.J. Tupper. Spacetimes admitting special affine conformal vectors. *J. Math. Phys.*, 31:649, 1990.

[85] M.P. Ryan and L.C. Shepley. *Homogeneous Relativistic Cosmologies*. Princeton: Princeton University Press, 1975.

[86] S. Cotsakis, P.G.L. Leach, and H. Pantazi. Symmetries of homogeneous cosmologies. *Grav. Cosm.*, 4:314, 1998.

[87] T.P. Sotiriou and V. Faraoni. f(r) theories of gravity. *Rev. Mod. Phys.*, 82:451, 2010.

[88] L. Amendola, D. Polarski, and S. Tsujikawa. Power-laws f(r) theories are cosmologically unacceptable. *Int. J. Mod. Phys. D*, 16:1557, 2007.

[89] L. Amendola and Tsujikawa S. Phantom crossing, equation-of-state singularities and local gravity constraints in f(r) models. *Phys. Lett. B.*, 660:125, 2008.

[90] Y. Bozhkov and I.L. Freite. Special conformal groups of a riemannian manifold and lie point symmetries of the nonlinear poisson equation. *2010*, 249:872, 2010.

[91] P.J. Olver. *Application of Lie groups to differential equations*. Springer Graduate texts in Mathematics, New York: Springer, 1986.

[92] G.W. Bluman. Simplifying the form of lie groups admitted by a given differential equation. *J. Math. Anal. Applic.*, 145:52, 1990.

[93] J. M. Lee and T.H. Parker. The yamabe problem. *Bull. Amer. Math. Soc.*, 17:37, 1987.

[94] I.L. Freire. On the paper "symmetry analysis of wave equation on sphere" by h. azad and m.t. mustafa. *J. Math. Anal. Appl.*, 367:716, 2010.

[95] G.S. Hall and J.D. Steele. Conformal vector fields in general relativity. *J. Math. Phys.*, 32:1847, 1990.

[96] Jr. Miller. The technique of variable separation for partial differential equations. *Lectures Notes in Physics*, 189:184, 1983.

[97] N. Kamran and R.G. McLenaghan. Separation of variables and symmetry operators for the conformally invariant klein-gordon equation on curved space-time. *Lett. Math. Phys.*, 9:65, 1985.

[98] V. Daftardar and N. Dadhich. Gradient conformal killing vectors and exact solutions. *Gen. Relativ. Gravit.*, 26:859, 1994.

[99] B. Abraham-Shrauner, K.S. Govinder, and P.G.L. Leach. Integration of second order ordinary differential equations not possessing lie point symmetries. *Phys. Lett. A.*, 203:169, 1995.

[100] B. Abraham-Shrauner, K.S. Govinder, and D. Arrigo. Type-ii hidden symmetries of the linear 2d and 3d wave equations. *J. Math. Phys. A.: Math. and Gen.*, 39:5739, 2006.

[101] A.A. Belavin, A.M. Polyakov, and A.B. Zamolodchikov. Infinite conformal symmetry in two-dimensional quantum field theory. *Nuc. Phys. B*, 241:333, 1984.

[102] B. Vakili. Quantization of the schwarzschild black hole: A noether symmetry approach. *International Journal of Theoretical Physics*, 51(1):133, 2012.

[103] T. Christodoulakis, N. Dimakis, Petros A. Terzis, G. Doulis, Th. Grammenos, et al. Conditional symmetries and the canonical quantization of constrained minisuperspace actions: the schwarzschild case. *J.Geom.Phys.*, 71:127–138, 2013.

[104] TarcÃŋsio M. Rocha Filho and Annibal Figueiredo. [sade] a maple package for the symmetry analysis of differential equations. *Comp. Phys. Comm.*, 182:469, 2011.

[105] S. Capozziello, M. De Laurentis, and S. D. Odintsov. Hamiltonian dynamics and noether symmetries in extended gravity cosmology. *EPJC*, 72:2068, 2012.

[106] S.P. Kim. New asymptotic expansion method for the wheeler-dewitt equation. *Phys. Rev. D*, 52:3382, 1995.

[107] S. Capozziello, R. De Ritis, and P. Scudellaro. Noether symmetries in quantum cosmology. *IJMPD*, 3:609, 1994.

[108] T. Christodoulakis, G. Kofinas, and G. O. Papadopoulos. Conditional symmetries and phase space reduction towards gct invariant wave functions, for the class a bianchi type vi and vii vacuum cosmologies. *Phys. Lett. B.*, 514:149, 2001.

[109] B. Vakili. Noether symmetric $f(r)$ quantum cosmology and its classical correlations. *Phys. Lett. B.*, 669:209, 2008.

[110] B. Vakili and F. Khazale. Noether symmetric classical and quantum scalar field cosmology. *Class. Quantum Grav.*, 29:035015, 2012.

[111] B.S. De Witt. Quantum theory of gravity. i. the canonical theory. *Phys. Rev.*, 160:1113, 1967.

[112] B. Abraham-Shrauner and A. Guo. Hidden symmetries of differential equations. *AMS Providence Cont. Math*, 160:1, 1994.

[113] B. Abraham-Shrauner and K.S. Govinder. Provenance of type ii hidden symmetries from nonlinear partial differential equations. *Journal of Nonlinear Math. Phys.*, 13:612, 2006.

[114] B. Abraham-Shrauner. Type ii hidden symmetries of the second heavenly equation. *Phys. Lett. A.*, 369:299, 2007.

[115] R. J. Moitsheki, P. Broadbridge, and Eqwards M. P. Systematic construction of hidden nonlocal symmetries for the inhomogeneous nonlinear diffusion equation. *J. Phys. A: Math. Gen.*, 37:8279, 2004.

[116] P.G.L. Leach, K.S. Govinder, and K. Andriopoulos. Hidden and not so hidden symmetries. *Journal of Applied Mathematics*, 2012:890171, 2012.

[117] K.S. Govinder. Lie subalgebras, reduction of order and group invariant solutions. *J. Math. Anal. Appl.*, 258:720, 2001.

[118] K.S. Govinder and B. Abraham-Shrauner. On the origins of symmetries of partial differential equations: the example of the korteweg-de vries equation. *Journal of Nonlinear Math. Phys.*, 15:60, 2009.

[119] M. L. Gandarias. Type-ii hidden symmetries through weak symmetries for nonlinear partial differential equations. *J. Math. Anal. Appl.*, 348:752, 2008.

[120] J.D. Steele. Simply-transitive homothety groups. *Gen. Relativ. Gravit.*, 23:811, 1991.

[121] E.S. Cheb-Terrab and K. von Bulow. A computational approach for the analytical solving of partial differential equations. *Comp. Phys. Comm.*, 90:102, 1995.

[122] A.H. Bokhari, A.H. Kara, M. Karim, and F. D. Zaman. Invariance analysis and variational conservation laws for the wave equation on some manifolds. *Int. J.of Theor. Phys.*, 48:1919, 2009.

[123] P.A. Clarkson and E.L. Mansfield. Symmetry reductions and exact solutions of a class of nonlinear heat equations. *Physica D*, 70:250, 1993.

[124] N.M. Ivanova. Exact solutions of diffusion-convection equations. *Dynamics of PDE*, 5:139, 2008.

[125] J.A.S. Lima and J.S. Alcaniz. Constraining the cosmic equation of state from old galaxies at high redshift. *MNRAS*, 317:893, 2000.

[126] M. Tegmark and et al. The three-dimensional power spectrum of galaxies from the sloan digital sky survey. *The Astrophysical Journal*, 606(2):702, 2004.

[127] D.N. Spergel and et al. Three-year wilkinson microwave anisotropy probe (wmap) observations: Implications for cosmology. *The Astrophysical Journal Supplement Series*, 170(2):377, 2007.

[128] T.M. Davis and et al. Scrutinizing exotic cosmological models using essence supernova data combined with other cosmological probes. *The Astrophysical Journal*, 666(2):716, 2007.

[129] M. Kowalski and et al. Improved cosmological constraints from new, old, and combined supernova data sets. *The Astrophysical Journal*, 686(2):749, 2008.

[130] M. Hicken and et.al. Improved dark energy constraints from 100 new cfa supernova type ia light curves. *The Astrophysical Journal*, 700(2):1097, 2009.

[131] E. Komatsu and et al. Five-year wilkinson microwave anisotropy probe observations: Cosmological interpretation. *The Astrophysical Journal Supplement Series*, 180(2):330, 2009.

[132] S. Weinberg. The cosmological constant problem. *Rev. Mod. Phys.*, 61:1–23, Jan 1989.

[133] P. J. E. Peebles and B. Ratra. The cosmological constant and dark energy. *Rev. Mod. Phys.*, 75:559–606, Apr 2003.

[134] T. Padmanabhan. Cosmological constant - the weight of the vacuum. *Physics Reports*, 380(5):235, 2003.

[135] L. Perivolaropoulos. Six puzzles for lcdm cosmology. (Preprint arxXiv.0811.4684).

[136] L. Amendola and S. Tsujikawa. *Dark Energy Theory and Observations.* Cambridge University Press, Cambridge UK, 2010.

[137] R. de Ritis, G. Marmo, G. Platania, C. Rubano, P. Scudellaro, and C. Stornaiolo. New approach to find exact solutions for cosmological models with a scalar field,. *Phys. Rev. D*, 42:1091, 1990.

[138] S. Capozziello, R. de Ritis, and Scudellaro P. Noether's symmetries in (n+1)-dimensional nonminimally coupled cosmologies,. *Int. J. Mod. Phys. D*, 2:463, 1993.

[139] S. Capozzielo, G. Marmo, C. Rubano, and P. Scudellaro. Noether symmetries in bianchi universes. *Int. J. Mod. Phys. D*, 6:491, 1997.

[140] S. Capozziello, R. de Ritis, and A.A. Marino. Conformal equivalence and noether symmetries in cosmology. *Class. Quantum Grav.*, 14:3259, 1997.

[141] S. Capozziello, S. Nesseris, and L. Perivolaropoulos. Reconstruction of the scalar-tensor lagrangian from a lcdm background and noether symmetry. *JCAP*, 0712:009, 2007.

[142] S. Capozziello, E. Piedipalumbo, C. Rubano, and P. Scudellaro. Noether symmetry approach in phantom quintessence cosmology. *Phys. Rev. D*, 80:104030, Nov 2009.

[143] B. Vakili, N. Khosvravi, and H.R. Sepangi. Bianchi spacetimes in noncommutative phase space. *Class. Quantum Grav.*, 24:931, 2007.

[144] A.K. Sanyal, B. Modak, C. Rubano, and E. Piedipalumbo. Noether symmetry in the higher order gravity theory. *General Relativity and Gravitation*, 37(2):407–417, 2005.

[145] A.K. Sanyal. Study of symmetry in f(r) theory of gravity. *Modern Physics Letters A*, 25(31):2667–2676, 2010.

[146] M Jamil, F.M. Mahomed, and D. Momeni. Noether symmetry approach in f(r) tachyon model. *Physics Letters B*, 702(5):315 – 319, 2011.

[147] H. Wei, X.J. Guo, and L.F. Wang. Noether symmetry in f(t) theory. *Physics Letters B*, 707(2):298 – 304, 2012.

[148] H. Motavali, S. Capozziello, and M. Rowshan Almeh Jog. Scalar-tensor cosmology with r-1 curvature correction by noether symmetry. *Phys. Lett. B.*, 666:10, 2008.

[149] Y. Kucukakca. Scalar tensor teleparallel dark gravity via noether symmetry. *The European Physical Journal C*, 73(2):1–7, 2013.

[150] S.W. Hawking and G.F.R. Ellis. *The Large Scale Structure of Space-Time.* Cambridge University Press, Cambridge: New York, 1975.

[151] D. I. Kaiser. Conformal transformations with multiple scalar fields. *Phys. Rev. D*, 81:084044, Apr 2010.

[152] S. Capozziello, R. de Ritis, and A.A. Marino. Some aspects of the cosmological conformal equivalence between the jordan frame and the einstein frame. *Class. Quantum Grav.*, 14:3243, 1997.

[153] D. Bertacca, S. Matarrese, and M. Pietroni. Unified dark matter in scalar field cosmologies. *Modern Physics Letters A*, 22(38):2893–2907, 2007.

[154] V. Gorini, A. Kamenshchik, U. Moschella, V. Pasquier, and A. Starobinsky. Stability properties of some perfect fluid cosmological models. *Phys. Rev. D*, 72:103518, Nov 2005.

[155] Spyros Basilakos and Georgios Lukes-Gerakopoulos. Dynamics and constraints of the unified dark matter flat cosmologies. *Phys. Rev. D*, 78:083509, Oct 2008.

[156] J.L. Sievers and et al. Cosmological parameters from cosmic background imager observations and comparisons with boomerang, dasi, and maxima. *The Astrophysical Journal*, 591(2):599, 2003.

[157] M.P.Jr Rayan and L.C. Shepley. *Homogeneous Relativistic Cosmologies.* Princeton University Press, Princeton, 1975.

[158] C.W. Misner. Quantum cosmology i. *Phys. Rev. D*, 186:1319, 1969.

[159] M.A.H MacCallumn. In: Hawking, w.s.w., israel, w. general relativity: An einstein centenary survey. In ., 1979.

[160] M. Madsen and P. Coles. Chaotic inflation. *Nucl. Phys.*, B298:701, 1988.

[161] M. Demianski, R. de Ritis, C. Rubano, and P. Scudellaro. Scalar fields and anisotropy in cosmological models. *Phys. Rev. D.*, 46:1391, 1992.

[162] J.G. Russo. Exact solution of scalar field cosmology with exponential potentials and transient acceleration. *Phys. Lett. B.*, 600:185, 2004.

[163] T Christodoulakis and et al. Decoupling of the general scalar field mode and the solution space for bianchi type i and v cosmologies coupled to perfect fluid sources. *J. Math. Phys*, 47:042505, 2006.

[164] A.A. Starobinsky. A new type of isotropic cosmological models without singularity. *Physics Letters B*, 91(1):99 – 102, 1980.

[165] S. M. Carroll, V. Duvvuri, M. Trodden, and M. S. Turner. Is cosmic speed-up due to new gravitational physics? *Phys. Rev. D*, 70:043528, Aug 2004.

[166] W. Hu and I. Sawicki. Models of f(r) cosmic acceleration that evade solar system tests. *Phys. Rev. D*, 76:064004, Sep 2007.

[167] A.A. Starobinsky. Disappearing cosmological constant in f(r) gravity. *JETP Letters*, 86(3):157–163, 2007.

[168] S. Tsujikawa. Observational signatures of f(r) dark energy models that satisfy cosmological and local gravity constraints. *Phys. Rev. D*, 77:023507, Jan 2008.

[169] Luca Amendola and Shinji Tsujikawa. Phantom crossing, equation-of-state singularities, and local gravity constraints in models. *Physics Letters B*, 660(3):125 – 132, 2008.

[170] Sk. Nayem and A.K. Sanyal. Revisiting noether gauge symmetry for f(r) theory of gravity. *Astrophys. Space Sci.*, 342:549, 2012.

[171] A. Einstein. Riemannian geometry, while maintaining the notion of teleparallelism. *Sitzber. Preuss. Akad. Wiss.*, 17:217, 1928.

[172] A. Einstein. A new possibility for a unified field theory of gravitation and electromagnetism. *Sitzber. Preuss. Akad. Wiss.*, 17:228, 1928.

[173] Kenji Hayashi and Takeshi Shirafuji. New general relativity. *Phys. Rev. D*, 19:3524–3553, Jun 1979.

[174] JosÃl' W. Maluf. Hamiltonian formulation of the teleparallel description of general relativity. *Journal of Mathematical Physics*, 35(1):335–343, 1994.

[175] R. Weitzenbock. Differential invariants in einstein's theory of teleparallelism. *Sitzber. Preuss. Akad. Wiss.*, 17:466, 1928.

[176] Rafael Ferraro and Franco Fiorini. Modified teleparallel gravity: Inflation without an inflaton. *Phys. Rev. D*, 75:084031, Apr 2007.

[177] Eric V. Linder. EinsteinâĂŹs other gravity and the acceleration of the universe. *Phys. Rev. D*, 81:127301, Jun 2010.

[178] Shih-Hung Chen, James B. Dent, Sourish Dutta, and Emmanuel N. Saridakis. Cosmological perturbations in f(t) gravity. *Phys. Rev. D*, 83:023508, Jan 2011.

[179] S. Capozziello and S. Vignolo. The cauchy problem for metric-affine f (r) gravity in the presence of perfect-fluid matter. *Classical and Quantum Gravity*, 26(17):175013, 2009.

[180] Gabriel R. Bengochea and Rafael Ferraro. Dark torsion as the cosmic speed-up. *Phys. Rev. D*, 79:124019, Jun 2009.

[181] M. Havelkova. Symmetries of a dynamical system represented by singular lagrangians. *Communications in Mathematics*, 20:23, 2012.

[182] Zi-ping Li. Symmetry in phase space for a system with a singular higher-order lagrangian. *Phys. Rev. E*, 50:876–887, Aug 1994.

[183] Spyros Basilakos, Manolis Plionis, and Joan Solà. Hubble expansion and structure formation in time varying vacuum models. *Phys. Rev. D*, 80:083511, Oct 2009.

[184] Spyros Basilakos, Manolis Plionis, and Joan Solà. Spherical collapse model in time varying vacuum cosmologies. *Phys. Rev. D*, 82:083512, Oct 2010.

[185] K. Freese, F.C. Adams, J.A. Frieman, and E. Mottola. Cosmology with decaying vacuum energy. *Nuclear Physics B*, 287(0):797 – 814, 1987.

[186] J. C. Carvalho, J. A. S. Lima, and I. Waga. Cosmological consequences of a time-dependent <math><mi>ÎŻ</mi></math> term. *Phys. Rev. D*, 46:2404–2407, Sep 1992.

[187] R. C. Arcuri and I. Waga. Growth of density inhomogeneities in newtonian cosmological models with variable ÎŻ. *Phys. Rev. D*, 50:2928–2931, Aug 1994.

[188] Gabriel R. Bengochea and Rafael Ferraro. Dark torsion as the cosmic speed-up. *Phys. Rev. D*, 79:124019, Jun 2009.

[189] Eric V. Linder. Exponential gravity. *Phys. Rev. D*, 80:123528, Dec 2009.