Operator Spaces and Operator Systems: An introduction

Aristides Katavolos

December 1, 2023

Operator Spaces

Definition

An operator space E is a pair (E, j) where E is a linear space and $j : E \rightarrowtail \mathscr{B}(H)$ a linear embedding (i.e. 1-1).
For each n define

$$\left\|[x_{ij}]\right\|_{M_n(E)} \stackrel{def}{=} \left\|[j(x_{ij})]\right\|_{\mathscr{B}(H^n)} \ (x_{ij} \in E)\,.$$

The sequence of norms $\{\|\cdot\|_{M_n(E)} : n \in \mathbb{N}\}$ is called the operator space structure on E induced by j.

Definition

An operator space structure on a normed space $(E, \|\cdot\|)$ is the operator space structure induced by a linear isometric embedding $j : E \rightarrowtail \mathscr{B}(H)$ for some Hilbert space H.

(Thus $\|j(x)\|_{\mathscr{B}(H)} = \|x\|_E$ for all $x \in E$)

Completely bounded maps

Thus if $j_1 : E \rightarrowtail \mathscr{B}(H_1)$ and $j_2 : E \rightarrowtail \mathscr{B}(H_2)$ are two op. space structures, then $j_1(E)$ and $j_2(E)$ are isom. isomorphic (via $\phi := j_2 \circ j_1^{-1}$).
But are $M_2(j_1(E))$ and $M_2(j_2(E))$ isom. isomorphic?

Completely bounded maps

Notation Given a linear map $\phi : \mathscr{B}(H) \to \mathscr{B}(H')$, for any $n \in \mathbb{N}$ define $\phi_n : \mathscr{B}(H^n) \to \mathscr{B}(H'^n) : [a_{il}] \mapsto [\phi(a_{ij})]$.

Definition

A linear map $\phi : E \to F$ between operator spaces is said to be completely bounded if (each $\phi_n : M_n(E) \to M_n(F)$ is bounded and) $\|\phi\|_{cb} := \sup_n \|\phi_n\| < \infty$.
The map ϕ is said to be completely isometric if each $\phi_n : M_n(E) \to M_n(F)$ is an isometry.

Write $CB(E,F)$ for the space of all cb maps $E \to F$. These are the natural morphisms for the category of operator spaces. When F is complete, $(CB(E,F), \|\cdot\|_{cb})$ is a Banach space.

The minimal operator space structure on $(E, \|\cdot\|_E)$

Remark

Every normed space $(E, \|\cdot\|_E)$ admits an operator space structure.

Idea: Embed $E \hookrightarrow \ell_\infty(\Gamma)$ isometrically, then embed $\ell_\infty(\Gamma) \hookrightarrow \mathscr{B}(\ell_2(\Gamma)) = \mathscr{B}(H)$ as diagonal operators.

This op. space structure is called the minimal op. structure $\min E$ on $(E, \|\cdot\|_E)$.
It has the universal property:
For every op. space F, every bounded linear map $\phi : F \to \min E$ is completely bounded and in fact $\|\phi\|_{cb} = \|\phi\|$.

The maximal operator space structure on $(E, \|\cdot\|_E)$

Let $\mathscr{S}$ be the family of all isometric embeddings $\phi : E \to \mathscr{B}(H_\phi)$ (this is not empty since $\min E$ exists).

The max structure corresponds to the embedding given by the 'direct sum' (suitable defined) of all the embeddings $\phi : E \to \mathscr{B}(H_\phi)$.

Each ϕ induces a norm $\|\cdot\|_n^\phi$ on each $M_n(E)$ given by $\left\|[x_{ij}]\right\|_n^\phi = \left\|[\phi(x_{ij})]\right\|_{\mathscr{B}(H_\phi^n)}$. The max norm is defined to be the supremum of these norms:

$$\left\|[x_{ij}]\right\|_{\max} = \sup\{\left\|[\phi(x_{ij})]\right\|_{\mathscr{B}(H_\phi^n)} : (\phi, H_\phi) \in \mathscr{S}\}, \quad [x_{ij}] \in M_n(E)$$

(this supremum is finite, since $\left\|[\phi(x_{ij})]\right\|_{\mathscr{B}(H_\phi^n)} \leq \left(\sum_{i,j} \|\phi(x_{ij})\|^2_{\mathscr{B}(H_\phi)}\right)^{1/2} = \left(\sum_{i,j} \|x_{ij}\|_E^2\right)^{1/2}$).

The maximal and minimal operator space structures

The maximal operator space structure structure $\max E$ on $(E, \|\cdot\|_E)$ has the universal property:
For every op. space V, every bounded linear map $\psi : \max E \to V$ is completely bounded and in fact $\|\psi\|_{cb} = \|\psi\|$.

To compare:

For $n \in \mathbb{N}$ and $x = [x_{ij}] \in M_n(E)$,

$$\left\|[x_{ij}]\right\|_{\min} = \sup\{\left\|[\phi(x_{ij})]\right\|_{\mathscr{B}(H^n)} : \phi : E \to \mathbb{C} \text{ contraction}\}$$

$$\left\|[x_{ij}]\right\|_{\max} = \sup\{\left\|[\phi(x_{ij})]\right\|_{\mathscr{B}(H_\phi^n)} : \phi : E \to \mathscr{B}(H_\phi) \text{ contraction}\}$$

The column and row spaces C_n and R_n

$R_n \subseteq \mathscr{B}(\ell^2[n])$ consists of all $n \times n$ matrices having zeroes except for the first row $\simeq \mathscr{B}(\ell^2[n], \mathbb{C}) \simeq M_{1,n}$.

$C_n \subseteq \mathscr{B}(\ell^2[n])$ consists of all $n \times n$ matrices having zeroes except for the first column $\simeq \mathscr{B}(\mathbb{C}, \ell^2[n]) \simeq M_{n,1}$.

Both are isometric to $\ell^2[n]$, hence to one another.

The column and row spaces C_n and R_n

Now $M_{p,q}(C_n) \simeq M_{np,q}$ while $M_{p,q}(R_n) \simeq M_{p,nq}$.

Thus $M_{1,n}(C_n) \simeq M_{n,n}$ while $M_{1,n}(R_n) \simeq M_{1,n^2}$.

Take $[e_1, \ldots, e_n] \in M_{1,n}(C_n)$ to get $I_n \in M_n$ which has norm 1, while $[e_1, \ldots, e_n] \in M_{1,n}(R_n)$ gives a row vector of length n^2 with n 1's so its norm is $\sqrt{n}$.

Thus the identity mapping $\imath : C_n \to R_n$ has $\|\imath\| = 1$ but $\|\imath\|_{cb} \geq \sqrt{n}$.
It follows that the identity mapping $\imath : C \to R$ between the infinite-dimensional analogues, which is an isometry, is not completely bounded.

The dual of an operator space E

- A bounded linear map $\phi : E \to \mathbb{C}$ is automatically completely bounded with $\|\phi\|_{cb} = \|\phi\|$.

Proposition

The (Banach space) dual E^* of an operator space E has a natural operator space structure.

Idea: Let $D = \bigcup_n \mathrm{ball}(M_n(E))$ (disjoint union). For each $x \in D$ there is $n(x) \in \mathbb{N}$ with $x = [x_{ij}] \in \mathrm{ball}(M_{n(x)}(E))$. Define

$$v_x : E^* \to M_{n(x)}(\mathbb{C}) : \phi \in E^* \mapsto [\phi(x_{ij})] .$$

The map

$$J : E^* \to \bigoplus_{x \in D} M_{n(x)}(\mathbb{C}) : \phi \mapsto \bigoplus_{x \in D} v_x(\phi)$$

is an isometric embedding into a direct sum of matrix algebras, which consists of operators on the Hilbert space direct sum $\bigoplus_{x \in D} \ell^2[n(x)]$.

Positivity

An $A \in \mathscr{B}(H)$ is positive if ($A = A^*$ and) $\langle Ax, x \rangle \geq 0$ for all $x \in H$.
Equivalently if $\exists B \in \mathscr{B}(H)$ with $A = B^* B$ (!)

(For a *complex Hilbert space* H, $\langle Ax, x \rangle \geq 0$ for all $x \in H$ implies $A = A^*$ automatically.)

Συστήματα τελεστών (Operator Systems)

Σύστημα τελεστών (operator system) είναι ένας γραμμικός υπόχωρος $E \subseteq \mathscr{B}(H)$ (ή $E \subseteq \mathscr{B}$ όπου $\mathscr{B}$ μια C^* άλγεβρα με μονάδα) που είναι αυτοσυζυγής και περιέχει την μονάδα.

Ακριβέστερα, σύστημα τελεστών είναι ένα (E, j) (όπου E γραμμικός χώρος και $j : E \to \mathscr{B}(H)$ γραμμική εμφύτευση) ώστε ο $j(E) \subseteq \mathscr{B}(H)$ να είναι αυτοσυζυγής και να περιέχει την μονάδα.
Συνήθως ταυτίζουμε τον E με τον $j(E)$.
Αλλά η ταύτιση εξαρτάται απ την εμφύτευση!

Παρατήρηση Κάθε $x \in E$ γράφεται μοναδικά $x = x_1 + ix_2$ όπου $x_k \in E^h$ (Ερμιτιανά ή αυτοσυζυγή). Κάθε $y \in E^h$ γράφεται (όχι μοναδικά) ως διαφορά δυο θετικών στοιχείων του E^h, π.χ. $y = (\|y\|\mathbf{1} + y) - \|y\|\mathbf{1}$. Άρα $E = E^h + iE^h$ και $E^h = E^+ - E^+$.

Η μονάδα είναι μονάδα διάταξης (order unit) δηλαδή για κάθε $y \in E^h$ υπάρχει $r > 0$ με $-r\mathbf{1} \leq y \leq r\mathbf{1}$. Η μονάδα είναι επιπλέον Αρχιμήδεια, δηλαδή αν $-\varepsilon\mathbf{1} \leq y \leq \varepsilon\mathbf{1}$ για κάθε $\varepsilon > 0$ τότε $y = 0$.

Συστήματα τελεστών (Operator Systems)

Αν $E \subseteq \mathscr{B}$ είναι σύστημα τελεστών, τότε για κάθε $n \in \mathbb{N}$ ο χώρος $E_n := M_n(E) \subseteq M_n(\mathscr{B}(H))$ είναι σύστημα τελεστών, αν εφοδιασθεί με την δομή που κληρονομεί από την C^* άλγεβρα $M_n(\mathscr{B}(H)) \simeq \mathscr{B}(H^n)$ (δηλ. ενέλιξη, μονάδα και θετικό κώνο $M_n(E)^+ := M_n(\mathscr{B}(H))^+ \cap M_n(E)$).

Αν E, F είναι συστήματα τελεστών, μια γραμμική απεικόνιση $\phi : E \to F$ λέγεται θετική αν $\phi(E^+) \subseteq F^+$. Λέγεται n-θετική αν η απεικόνιση $\phi_n : M_n(E) \to M_n(F) : [x_{ij}] \mapsto [\phi(x_{ij})]$ είναι θετική και πλήρως θετική αν είναι n-θετική για κάθε n.
Μια γραμμική απεικόνιση $\phi : E \to \mathbb{C}$ λέγεται κατάσταση (state) αν είναι θετική και $\phi(\mathbf{1}) = 1$. Το σύνολο των καταστάσεων συμβολίζεται $\mathscr{S}(E)$.

Operator Systems

Every operator system is at the ground level an ordered *-vector space with an Archimedean order unit and, conversely, given any Archimedean order unit space, there are possibly many different operator systems that all have the given Archimedean order unit space as their ground level. (Paulsen, Todorov, Tomforde, [PTT11])

Thus, from each operator system there may be derived a sequence of order unit spaces $(E_n, E_n^+, \mathbf{1}_n)$. Here $\mathbf{1}_n = \mathrm{diag}(\mathbf{1}, \ldots, \mathbf{1})$ where $\mathbf{1}$ is the unit in E. Moreover, there is a natural family of connecting maps defined as follows. For $a \in M_{n,k}$ define

$$\mathrm{ad}(a) : E_n \to E_k : x \mapsto a^* x a.$$

Notice that $\mathrm{ad}(a)$ preserves the order structure.
(T. Sinclair, [Sin22])

Higher order cone determines norm

Exercise For $x \in \mathscr{B}(H)$ and $\lambda \geq 0$,

$$\|x\| \leq \lambda \iff \begin{bmatrix} \lambda \mathbf{1} & x \\ x^* & \lambda \mathbf{1} \end{bmatrix} \in \mathscr{B}(H^2)^+.$$

Proposition

Let $E \subset \mathscr{B}(H)$ be an operator system. For $x \in E_n$ we have that

$$\|x\|_n = \inf \left\{ \lambda > 0 : \begin{bmatrix} \lambda 1 & x \\ x^* & \lambda 1 \end{bmatrix} \in E_{2n}^+ \right\}.$$

Complete positivity and complete boundedness

Lemma

Κάθε μοναδιαία και 2-θετική απεικόνιση $\phi : E \to F$ είναι συστολή.

Κάθε μοναδιαία και θετική $\phi : E \to \mathbb{C}$ (κατάσταση) είναι συστολή.
Αλλά (όταν $\dim F > 1$) δεν αρκεί η θετικότητα:

Example

Έστω $E \subseteq C(\mathbb{T})$ η γραμμική θήκη των $\{\mathbf{1}, \zeta, \bar{\zeta}\}$ όπου $\zeta(z) = z$. Δηλαδή κάθε $f \in E$ είναι της μορφής $f(e^{it}) = a + be^{it} + ce^{-it}$ με $a, b, c \in \mathbb{C}$. Ορίζουμε

$$\phi : E \to M_2(\mathbb{C}) : f \mapsto \begin{bmatrix} a & 2b \\ 2c & a \end{bmatrix}.$$

Τότε η ϕ είναι θετική και μοναδιαία αλλά $\|\phi\| > 1$.

Corollary

Κάθε μοναδιαία και πλήρως θετική απεικόνιση $\phi : E \to F$ είναι πλήρης συστολή.

Complete positivity and complete boundedness

Proposition

Any unital, contractive linear map between operator systems is positive.

Compare Any complex measure μ on a space X with total variation $\|\mu\| = 1$ and $\mu(X) = 1$ is a positive (probability) measure.

Corollary

Every complete order embedding of operator systems $\phi : E \to F$ is a completely isometric embedding.
Every unital completely isometric embedding is a complete order embedding.

Choi-Kraus decomposition

Recall that if E is an operator system and $a \in M_{n,k}$ the map $\phi : E_n \to E_k : x \mapsto a^* x a$ is CP.

Proposition (Choi's Theorem)

Every completely positive map $\phi : M_n \to M_k$ is of the form $\phi(x) = \sum_{i=1}^{nk} a_i^* x a_i$ for some $a_1, \ldots, a_{nk} \in M_{n,k}$.

The minimal number of a_i's required is the Choi rank of ϕ.

Abstract Operator Systems

Definition

A matrix ordered vector space is a *-vector space E together with a family of proper cones $C_n \subseteq M_n(E)^h$ (i.e. $C_n \cap (-C_n) = \{0\}$) for each $n \in \mathbb{N}$, which are compatible in the sense that

$$a^* C_n a \subseteq C_k \text{ for every } \; a \in M_{n,k}(\mathbb{C}).$$

A matrix ordered vector space E is called an abstract operator system if E^h has an order unit e such that, for all $n \in \mathbb{N}$, $e_n := \mathrm{diag}(e, \ldots, e)$ is an *Archimedean order unit* for $M_n(E)^h$.

Η e είναι μονάδα διάταξης (order unit) δηλαδή για κάθε $y \in E^h$ υπάρχει $r > 0$ με $-re \leq y \leq re$. Η e είναι επιπλέον Αρχιμήδεια, αν ικανοποιεί την: αν $y \in E^h$ και $-\varepsilon e \leq y \leq \varepsilon e$ για κάθε $\varepsilon > 0$ τότε $y = 0$.

The Choi - Effros Theorem (1977)

Theorem ([CE77])

Every abstract operator system 'is' a concrete operator system: For every abstract operator system E there exists a Hilbert space H and a complete order embedding

$$J : E \to \mathscr{B}(H).$$

Idea: Let $D = \bigcup_n UCP(E, M_n)$ (unital CP maps: matrix states). For each $\phi \in D$ there is $n(\phi) \in \mathbb{N}$ with $\phi \in UCP(E, M_{n(\phi)})$. Define

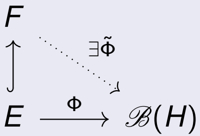
$$J : E \to \bigoplus_{\phi \in D} M_{n(\phi)} : x \mapsto \bigoplus_{\phi \in D} \phi(x).$$

This is a complete order embedding of E into a direct sum of matrix algebras, which consists of operators on the Hilbert space direct sum $\bigoplus_{\phi \in D} \ell^2[n(\phi)]$.

The Arveson Extension Theorem (1969)

Theorem (Arveson, [Arv69])

Αν $E \subseteq F$ είναι συστήματα τελεστών και $\Phi : E \to \mathscr{B}(H)$ μια πλήρως θετική γραμμική απεικόνιση, τότε η Φ έχει επέκταση σε μια γραμμική απεικόνιση $\tilde{\Phi} : F \to \mathscr{B}(H)$ που είναι πλήρως θετική.



The Dual of an operator system

Given an operator system E, consider the dual operator *space* $F = E^*$. We let $F^+ := \{\phi \in F : \phi(x) \geq 0 \, \forall x \in E^+\}$.
For $n \in \mathbb{N}$, each $\phi = [\phi_{ij}] \in M_n(F)$ defines a $\tilde{\phi} : E \to M_n : x \mapsto [\phi_{ij}(x)]$.
We put

$$M_n(E^*)^+ = F_n^+ := \{\phi = [\phi_{ij}] \in M_n(F) : \tilde{\phi} \in CP(E, M_n)\}.$$

This defines a matrix ordered vector space structure $(M_n(E^*), M_n(E^*)^+)$ on E^*.
However E^* is *not* in general an operator system, because it may fail to have an order unit.
In the special case $\dim E < \infty$ it is possible to choose an Archimedean order unit and thus to give an operator space structure to E^*.

The Minimal Operator System structure

Given (E, E_+, e) where e is an Archimedean order unit:

The state space $\mathscr{S}(E)$ is a w*-closed convex subset of the norm-dual of E.

The unital linear map $j_{\min} : E \to C(\mathscr{S}(E))$ defined by $j_{\min}(v)(s) = s(v)$, $s \in \mathscr{S}(E)$, is an order isomorphism onto its range.

Definition

The Minimal Operator System structure $\mathrm{OMIN}(E)$ on E is the one induced by $j_{\min}$.

Proposition (Universal property)

For every operator system F, every positive linear map $\phi : F \to \mathrm{OMIN}(E)$ is completely positive.

The Maximal Operator System structure

Given (E, E_+, e) where e is an Archimedean order unit:

Proposition

There exists an operator system structure on E having the following
universal property: For every operator system F, every positive linear map $\phi : \mathrm{OMAX}(E) \to F$ is completely positive.

(Suffices to take $F = \mathscr{B}(H)$.)

References

William B. Arveson.
Subalgebras of C^*-algebras.
Acta Math., 123:141–224, 1969.

Man Duen Choi and Edward G. Effros.
Injectivity and operator spaces.
J. Functional Analysis, 24(2):156–209, 1977.

Vern I. Paulsen, Ivan G. Todorov, and Mark Tomforde.
Operator system structures on ordered spaces.
Proc. Lond. Math. Soc. (3), 102(1):25–49, 2011.

Thomas Sinclair.
Model theory of operator systems and C*-algebras, 2022.
arXiv:2210.03862 [math.OA]