Operator systems, contextuality and nonlocality[1]

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Functional analysis and operator algebras seminar
Athens 2023-24

[1] Based on a joint work with M. Anoussis and I.G. Todorov.

Postulates of Quantum Mechanics

Physical system $\leftrightsquigarrow$ Hilbert space $\mathcal{H}$

State of the system $\leftrightsquigarrow$ unit vector $\psi \in \mathcal{H}$

Positive measurement (POVM) $\leftrightsquigarrow$ $\{E_i\}_{i=1}^n \subseteq \mathcal{B}(\mathcal{H})^+$, $\sum_{i=1}^n E_i = I_{\mathcal{H}}$

Projective measurement (PVM) $\leftrightsquigarrow$ $\{P_i\}_{i=1}^n \subseteq \mathcal{B}(\mathcal{H})$, P_i projections, $\sum_{i=1}^n P_i = I_{\mathcal{H}}$

Probability to observe i in state ψ $\leftrightsquigarrow$ $\langle T_i\psi, \psi\rangle$, where $\{T_i\}_{i=1}^n$ POVM or PVM

Composite systems

Tensor paradigm: The joint system, composed out of two others, $\mathcal{H}_A, \mathcal{H}_B$ is given by $\mathcal{H}_A \otimes \mathcal{H}_B$ and measurements are of the form $\{E_i \otimes F_j\}_{i,j}$, where $\{E_i\}_i \subseteq \mathcal{B}(\mathcal{H}_A)$ and $\{F_j\}_j \subseteq \mathcal{B}(\mathcal{H}_B)$.

Commutativity paradigm: The composite system consists of one Hilbert space $\mathcal{H}$ and the subsystems are specified by C^*-algebras $\mathcal{A}, \mathcal{B} \subseteq \mathcal{B}(\mathcal{H})$ that commute. The measurements are performed by $\{E_i\}_i \subseteq \mathcal{A}$ and $\{F_j\}_j \subseteq \mathcal{B}$ such that $E_i F_j = F_j E_i$.

- The tensor paradigm is a special case of the commutativity one, since we can set $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ and $\mathcal{A} = \mathcal{B}(\mathcal{H}_A) \otimes I_B$ and $\mathcal{B} = I_A \otimes \mathcal{B}(\mathcal{H}_B)$.

- For finite dimensions they coincide.

- Not necessarily for infinite dimensions.

Nonlocality

Fix A, B, X, Y finite sets.

Alice's lab:

Questions: X
Answers: A
Measurements: $\{E_{a,x}\}_{a\in A}$, $x \in X$

Bob's lab:

Questions: Y
Answers: B
Measurements: $\{F_{b,y}\}_{b\in B}$, $y \in Y$

$$\text{Correlations} \rightsquigarrow p = \{p(a,b|x,y)\}_{a,b,x,y}$$

Local correlations: Convex combinations of $p_A(a|x)$ and $p_B(b|y)$. Notation: $\mathcal{C}_{loc}$.

Quantum: Assuming the tensor paradigm $p(a,b|x,y) = \langle E_{a,x} \otimes F_{y,b}\psi, \psi\rangle$, with

$$\psi \in \mathcal{H}_A \otimes \mathcal{H}_B, \quad \{E_{a,x}\}_{a\in A} \subseteq \mathcal{B}(\mathcal{H}_A), \quad \{F_{b,y}\}_{b\in B} \subseteq \mathcal{B}(\mathcal{H}_B) \text{ POVM's.}$$

*We assume $\mathcal{H}_A, \mathcal{H}_B$ finite dimensional. Notation: $\mathcal{C}_q$.

Quantum commuting: Assuming the commutativity paradigm

$$p(a,b|x,y) = \langle E_{a,x} F_{y,b} \psi, \psi \rangle \text{ such that}$$
$$\psi \in \mathcal{H}, \quad \{E_{a,x}\}_{a \in A}, \{F_{b,y}\}_{b \in B} \subseteq \mathcal{B}(\mathcal{H}) \text{ POVM's}, \quad E_{a,x} F_{b,y} = F_{b,y} E_{a,x}.$$

Notation: $\mathcal{C}_{qc}$.

$$\mathcal{C}_{loc} \subseteq \mathcal{C}_q \subseteq \mathcal{C}_{qc}\ .$$

Nonlocality: Correlations p with $p \in \mathcal{C}_q \setminus \mathcal{C}_{loc}$ (Bell's Theorem, CHSH inequality)

Tsirelson's Problem (TP): Is $\overline{\mathcal{C}_q} = \mathcal{C}_{qc}$? (No, JNVWY 20')

We denote $\mathcal{C}_{qa} := \overline{\mathcal{C}_q}$.

Connes, Tsirelson, and Kirchberg's problems

Kirchberg's Problem (KP): Is $C^*(\mathbb{F}_2) \otimes_{min} C^*(\mathbb{F}_2) = C^*(\mathbb{F}_2) \otimes_{max} C^*(\mathbb{F}_2)$?

Tsirelson's Problem $\Leftrightarrow$ Kirchberg's Problem $\Leftrightarrow$ Connes Embedding Problem

KP $\Rightarrow$ TP : Passes through the following characterisation

Theorem [Fritz 10']: Set $\mathbb{F}_{X,A} = \underbrace{\mathbb{Z}_A * \cdots * \mathbb{Z}_A}_{X-times}$ (similarly $\mathbb{F}_{Y,B}$). A correlation p is in the set:

1. $\mathcal{C}_{qa}$ if and only if there exists a state s of $C^*(\mathbb{F}_{X,A}) \otimes_{min} C^*(\mathbb{F}_{Y,B})$ such that

$$p(a,b|x,y) = s(e_{x,a} \otimes e_{y,b})$$

2. $\mathcal{C}_{qc}$ if and only if there exists a state s of $C^*(\mathbb{F}_{X,A}) \otimes_{max} C^*(\mathbb{F}_{Y,B})$ such that

$$p(a,b|x,y) = s(e_{x,a} \otimes e_{y,b})$$

Via operator systems

Set $\mathcal{A}_{X,A} = \underbrace{\ell^\infty_A *_1 \cdots *_1 \ell^\infty_A}_{X-times}$ and $\mathcal{S}_{X,A} = \underbrace{\ell^\infty_A \oplus_1 \cdots \oplus_1 \ell^\infty_A}_{X-times}$ where $\mathcal{S}_{X,A} \subseteq \mathcal{A}_{X,A}$.

Using $C^*(\mathbb{F}_{X,A}) = \mathcal{A}_{X,A}$ and the theory of tensor products for operator systems:

Theorem [Paulsen-Todorov 13']: A correlation p is in the set:

1. $\mathcal{C}_{qa}$ if and only if there exists a state s of $\mathcal{S}_{X,A} \otimes_{min} \mathcal{S}_{Y,B}$ such that

$$p(a,b|x,y) = s(e_{x,a} \otimes e_{y,b})$$

2. $\mathcal{C}_{qc}$ if and only if there exists a state s of $\mathcal{S}_{X,A} \otimes_{c} \mathcal{S}_{Y,B}$ such that

$$p(a,b|x,y) = s(e_{x,a} \otimes e_{y,b})$$

Introduction

A hypergraph is a pair $\mathbb{G} = (V, E)$, where V is a finite set and E is a finite set of subsets of V.

Definition: A **contextuality scenario** is a hypergraph $\mathbb{G} = (V, E)$ such that $\cup_{e \in E} e = V$.

Vertices represent the "outcomes" and edges represent the "measurements".

Definition: Let $\mathbb{G} = (V, E)$ be a contextuality scenario. A **probabilistic model** on $\mathbb{G}$, is an assignment $p : V \to [0, 1]$ such that

$$\sum_{x \in e} p(x) = 1, \quad \text{for every} \quad e \in E.$$

Notation: $\mathcal{G}(\mathbb{G})$.

This hypergraph theoretic framework was introduced in [AFLS15] to study contextuality.

- A contextuality scenario that admits a probabilistic model will be called *non-trivial*.

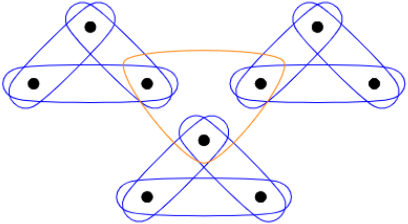


Figure 1: Example of a scenario that does not admit a probabilistic model.

Definition: Let $\mathbb{G} = (V, E)$ be a non-trivial contextuality scenario. A **Positive Operator Representation (POR)** of $\mathbb{G}$ on a Hilbert space $\mathcal{H}$ is a collection $(A_x)_{x \in V} \subseteq \mathcal{B}(\mathcal{H})^+$ such that

$$\sum_{x \in e} A_x = 1, \quad \text{for every} \quad e \in E.$$

A **Projective Representation (PR)**, is a POR such that A_x is a projection for every $x \in V$.

Consider a scenario $\mathbb{B}_{X,A}$ such that

$$V = X \times A \;\text{ and }\; E = \big\{\{x\} \times A : x \in X\big\},$$

then a POR $E = (E_{x,a})_{x \in X, a \in A}$ is in fact a family of POVM's. Such scenarios are called Bell scenarios.

Remark: A family of POVM's always dilates to a family of PVM's. It's not true for POR's as we will see.

Definition: Let $\mathbb{G} = (V, E)$ be a contextuality scenario. A probabilistic model $p \in \mathcal{G}(\mathbb{G})$ is called

1. **deterministic**, if $p(x) \in \{0, 1\}$, $\forall x \in V$.
2. **classical**, if it is a convex combination of deterministic ones. Notation: $\mathcal{C}(\mathbb{G})$
3. **quantum**, if there exists a Hilbert space $\mathcal{H}$, a PR $(P_x)_{x \in V}$ on $\mathcal{H}$ and a state $\psi \in \mathcal{H}$ such that
$$p(x) = \langle P_x \psi, \psi \rangle \quad \forall x \in V$$
Notation: $\mathcal{Q}(\mathbb{G})$

$$\mathcal{C}(\mathbb{G}) \subseteq \mathcal{Q}(\mathbb{G}) \subseteq \mathcal{G}(\mathbb{G})$$

Theorem [Kochen-Specker Theorem]: There exists a contextuality scenario $\mathbb{G}_{KS}$, such that $\mathcal{C}(\mathbb{G}_{KS}) = \emptyset$, while $\mathcal{Q}(\mathbb{G}_{KS}) \neq \emptyset$.

For finite sets X, A,

- ℓ_A^∞ is the universal operator system for POVM's:

$$\{E_a\}_{a\in A} \text{ POVM on } \mathcal{H} \longleftrightarrow \phi : \ell_A^\infty \to \mathcal{B}(\mathcal{H}) : \phi(e_a) = E_a \text{ is ucp.}$$

- $S_{X,A} := \underbrace{\ell_A^\infty \oplus_1 \cdots \oplus_1 \ell_A^\infty}_{X-times}$ is the universal operator system for families of POVM's:

$$\{E_{x,a}\}_{a\in A} \text{ POVM on } \mathcal{H}, \forall x \longleftrightarrow \phi : S_{X,A} \to \mathcal{B}(\mathcal{H}) : \phi(e_{x,a}) = E_{x,a} \text{ is ucp}$$

where $\{e_{x,a}\}_{a\in A}$ is the canonical basis of the x-th copy of ℓ_A^∞.

Aim: For hypergraph $\mathbb{G}$, we want to find the universal operator system for POR's.

The operator system for POR's

- Fix a scenario $\mathbb{G} = (V, E)$, and write $E = \{e_1, e_2, \ldots, e_d\}$. For each $e \in E$ we set

$$\mathcal{S} := \ell^\infty_{e_1} \oplus \cdots \oplus \ell^\infty_{e_d} .$$

For $x \in V$, denote by $\delta^e_x \in \ell^\infty_e$ the element with 1 in the x-th, and zero in the remaining ones.

- Define

$$\mathcal{J} := \mathrm{span}\{(1 \oplus -1 \oplus \cdots \oplus 0), (1 \oplus 0 \oplus -1 \oplus \cdots \oplus 0), \ldots, (1 \oplus 0 \oplus \cdots \oplus -1), \\ (0 \oplus \cdots \oplus \delta^{e_i}_x \oplus \cdots \oplus -\delta^{e_j}_x \oplus \cdots 0) : \forall i \neq j \in \{1, \ldots, n\} \text{ s.t. } x \in e_i \cap e_j\}.$$

- If $\mathcal{J}$ is a "kernel", i.e., $\mathcal{J} = \ker \phi$ for a ucp map ϕ from $\mathcal{S}$, then $\mathcal{S} \,/\, \mathcal{J}$ is an operator system. Otherwise we take $\mathcal{S} \,/\, \tilde{\mathcal{J}}$ where $\mathcal{J} \subseteq \tilde{\mathcal{J}}$ is an appropriate subspace so that the quotient is an operator system.

Remark: If the hyperedges in $\mathbb{G}$ are mutually disjoint, $\mathcal{S} \,/\, \mathcal{J}$ is simply the unital coproduct $\ell^\infty_{e_1} \oplus_1 \ell^\infty_{e_2} \oplus_1 \cdots \oplus_1 \ell^\infty_{e_d}$.

For $e \in E$, let $\iota_e : \ell_e^\infty \to \oplus_{f\in E} \ell_f^\infty$ be the natural embedding let $i_e : \ell_e^\infty \to \mathcal{S} \,/\, \tilde{\mathcal{J}}$ be the map given by

$$i_e(u) = |E|(q \circ \iota_e)(u), \quad u \in \ell_e^\infty.$$

The maps i_e are ucp but may not always be complete order embeddings so set

$$a_x := i_e(\delta_x^e), \quad x \in V$$

and thus $\mathcal{S} \,/\, \tilde{\mathcal{J}} = \mathrm{span}\{a_x : x \in V\}$.

Universal property: If $\Phi : \mathcal{S} \,/\, \tilde{\mathcal{J}} \to \mathcal{B}(\mathcal{H})$ is a unital completely positive map then $(\Phi(a_x))_{x\in V}$ is a POR of $\mathbb{G}$. Conversely, if $(A_x)_{x\in V} \subseteq \mathcal{B}(\mathcal{H})$ is a POR of $\mathbb{G}$ then there exists a unique unital completely positive map $\Phi : \mathcal{S} \,/\, \tilde{\mathcal{J}} \to \mathcal{B}(\mathcal{H})$ such that $\Phi(a_x) = A_x$, $x \in V$. Moreover, it is the unique operator system with this property.

We set $\mathcal{S}_{\mathbb{G}} := \mathcal{S} \,/\, \tilde{\mathcal{J}}$.

The operator system for dilatable POR's

The **free hypergraph C*-algebra** $C^*(\mathbb{G})$ [AFLS15] is the universal C*-algebra generated by orthogonal projections p_x, $x \in V$ such that $\sum_{x \in e} p_x = 1$ for every $e \in E$.

The *-representations $\pi : C^*(\mathbb{G}) \to \mathcal{B}(\mathcal{H})$ correspond precisely to PR's $(P_x)_{x \in V}$ of $\mathbb{G}$ on $\mathcal{H}$ via $\pi(p_x) = P_x$, $x \in V$.

Consider

$$\mathcal{T}_{\mathbb{G}} := \operatorname{span}\{p_x : x \in V\} \subseteq C^*(\mathbb{G}).$$

- We say that a POR $(A_x)_{x \in V} \subseteq \mathcal{B}(\mathcal{H})$ of $\mathbb{G}$ dilates to a PR, if there exist a Hilbert space $\mathcal{K}$, an isometry $V : \mathcal{H} \to \mathcal{K}$ and a PR $(P_x)_{x \in V}$ of $\mathbb{G}$ such that $A_x = V^* P_x V$, $x \in V$.

- $\mathcal{T}_{\mathbb{G}}$ is universal for dilatable POR's.

The operator system for classically dilatable POR's

Consider the C*-algebra $\mathcal{D} = \ell^\infty_{e_1} \otimes \cdots \otimes \ell^\infty_{e_d}$, and let $\tilde{\iota}_e : \ell^\infty_e \to \mathcal{D}$ be the natural unital embedding and let $\mathcal{I}$ be the two-sided ideal generated by the elements

$$\tilde{\iota}_{e_i}(\delta_x^{e_i}) - \tilde{\iota}_{e_j}(\delta_x^{e_j}), \quad x \in e_i \cap e_j, i, j \in [d].$$

The quotient $\mathcal{D}_{\mathbb{G}} := D/\mathcal{I}$ is a unital abelian C*-algebra; we let

$$d_x := 1 \otimes \cdots \otimes \delta_x^e \otimes \cdots \otimes 1 + \mathcal{I}, \quad x \in V$$

Set

$$\mathcal{R}_{\mathbb{G}} := \mathrm{span}\{d_x : x \in V\},$$

viewed as an operator subsystem of $\mathcal{D}_{\mathbb{G}}$.

Definition: We say that a POR $(A_x)_{x\in V}$ is **classically dilatable** if there exists a Hilbert space $\mathcal{K}$ and an isometry $V : \mathcal{H} \to \mathcal{K}$ and a PR $(P_x)_{x\in V}$ with commuting entries such that $A_x = V^* P_x V$, $x \in V$.

- The operator system $\mathcal{D}_{\mathbb{G}}$ is universal for classically dilatable POR's.

We have the following picture:

$$\mathcal{S}_{\mathbb{G}} \xrightarrow{\Phi} \mathcal{T}_{\mathbb{G}} \xrightarrow{\Psi} \mathcal{R}_{\mathbb{G}},$$

where Φ and Ψ are ucp maps that come from the universal properties.

As a corollary we have the following picture:

models	$\leftrightsquigarrow$	**states**
$\mathcal{G}(\mathbb{G})$	$\leftrightsquigarrow$	$\mathcal{S}_{\mathbb{G}}$
$\mathcal{Q}(\mathbb{G})$	$\leftrightsquigarrow$	$\mathcal{T}_{\mathbb{G}}$
$\mathcal{C}(\mathbb{G})$	$\leftrightsquigarrow$	$\mathcal{R}_{\mathbb{G}}$

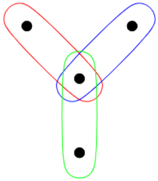
Dilations

Definition: We say that a scenario $\mathbb{G}$ is dilating (resp. classically dilating), if every POR of $\mathbb{G}$ dilates to a PR (resp. PR with commuting entries) of $\mathbb{G}$.

Theorem: Let $\mathbb{G} = (V, E)$ be a contextuality scenario. Then

- $\mathbb{G}$ is dilating if and only if $\mathcal{S}_{\mathbb{G}} = \mathcal{T}_{\mathbb{G}}$;
- $\mathbb{G}$ is classically dilating if and only if $\mathcal{S}_{\mathbb{G}} = \mathcal{R}_{\mathbb{G}}$

Proposition: Scenarios $\mathbb{G} = (V, E)$ such that $e' \cap e'' = \bigcap_{e \in E} e \neq \emptyset$ for all $e', e'' \in E$ with $e' \neq e''$ are dilating.



The relations between the sets of probabilistic models can be described via operator systems:

Proposition: Let $\mathbb{G} = (V, E)$ be a contextuality scenario. Then

1. $\mathcal{G}(\mathbb{G}) = \mathcal{Q}(\mathbb{G})$ if and only if $\mathrm{OMIN}(\mathcal{S}_{\mathbb{G}}) = \mathrm{OMIN}(\mathcal{T}_{\mathbb{G}})$;
2. $\mathcal{Q}(\mathbb{G}) = \mathcal{C}(\mathbb{G})$ if and only if $\mathrm{OMIN}(\mathcal{T}_{\mathbb{G}}) = \mathcal{R}_{\mathbb{G}}$;
3. $\mathcal{C}(\mathbb{G}) = \mathcal{G}(\mathbb{G})$ if and only if $\mathcal{R}_{\mathbb{G}} = \mathrm{OMIN}(\mathcal{S}_{\mathbb{G}})$

We recall [PTT11] that for an Archimedean order unit space $(\mathcal{V}, \mathcal{V}^+, e)$, $\mathrm{OMIN}(\mathcal{V})$ is the operator system obtained by the inclusion[2] of $\mathcal{V}$ into $C(S(\mathcal{V}))$;

$$v \in \mathcal{V} \mapsto \big(\Phi_v : s \mapsto s(v),\ s \in S(\mathcal{V})\big).$$

The positive elements are such that: $[v_{i,j}]_{i,j=1}^n \in C^n_{\min}(\mathcal{V})$ iff $[s(v_{i,j})]_{i,j=1}^n \in M_n^+$ for every state $s \in S(\mathcal{V})$.

[2] Kadison's Representation Theorem.

Quantum magic squares

Definition $A = [a_{i,j}] \in M_n(\mathcal{B}(\mathbb{C}^s))$ is called a **quantum magic square**, if $a_{i,j} \in \mathcal{B}(\mathbb{C}^s)^+$, $\forall i, j$ and all rows and columns sum to 1. It's called a **quantum permutation matrix** if moreover $a_{i,j}$ are projections.

Given $n \in \mathbb{N}$ define a hypergraph $\mathbb{G}_n$ by

$$V = [n] \times [n] \text{ and } E = \big\{\{i\} \times [n], [n] \times \{j\} : i, j = 1, \ldots, n\big\},$$

so that a quantum magic square $A = [a_{i,j}]_{i,j=1}^n$, is a POR $(a_{i,j})_{(i,j)\in V}$ (PR if A was a quantum permutation matrix).

In particular for $\mathbb{G}_n$, the Birkhoff von Neumann Theorem [3] implies that $\mathcal{C}(\mathbb{G}_n) = \mathcal{Q}(\mathbb{G}_n) = \mathcal{G}(\mathbb{G}_n)$.

[3] Every magic square is a convex combination of permutation matrices.

The non-commutative analogue of the Birkhoff von Neumann Theorem was proved not to be true in general.

[DICDN20]: There exist quantum magic squares that do not dilate to any quantum permutation matrix.

It is not clear if this automatically implies that they can't dilate into infinite dimensions.

By an adaptation of the considerations in [DICDN20] in infinite dimensions, we proved that their results extend to infinite dimensions. In other words,

Proposition: For every $n \geq 3$ there is a POR of $\mathbb{G}_n$, that doesn't admit a dilation into a PR. That is, $\mathbb{G}_n$ are not dilating for $n \geq 3$ and $\mathcal{S}_{\mathbb{G}_3} \neq \mathcal{T}_{\mathbb{G}_3}$.

Product scenarios

Let $\mathbb{G} = (V, E)$ and $\mathbb{H} = (W, F)$ and $\mathbb{G} \times \mathbb{H} = (V \times W, E \times F)$. A probabilistic model p on $\mathbb{G} \times \mathbb{H}$ is called:

- no-signalling, if

$$\sum_{x \in e} p(x, y) = \sum_{x \in e'} p(x, y) \text{ and } \sum_{y \in f} p(x, y) = \sum_{y \in f'} p(x, y).$$

Notation: $\mathcal{G}_{ns}(\mathbb{G}, \mathbb{H})$.

- deterministic, if $p(x, y) \in \{0, 1\}$ for all $(x, y) \in V \times W$.

- classical, if it's a convex combination of deterministic models

$$p(x, y) = p^1(x)p^2(y), \ \ x \in V, \ y \in W$$

where $p^1 \in \mathcal{G}(\mathbb{G})$, $p^2 \in \mathcal{G}(\mathbb{H})$.
Notation: $\mathcal{C}(\mathbb{G}, \mathbb{H})$.

- generalised tensor probabilistic model (resp. **tensor probabilistic models**), if

$$p(x,y) = \langle (A_x \otimes B_y)\psi, \psi \rangle, \quad (x,y) \in V \times W$$

for POR's (resp. **PR's**) $(A_x)_{x\in V} \subseteq \mathcal{B}(\mathcal{H}_{\mathbb{G}})$ and $(B_y)_{y\in W} \subseteq \mathcal{B}(\mathcal{H}_{\mathbb{H}})$, $\dim\mathcal{H}_{\mathbb{G}}$, $\dim\mathcal{H}_{\mathbb{H}} < \infty$ and $\psi \in \mathcal{H}_{\mathbb{G}} \otimes \mathcal{H}_{\mathbb{H}}$ unit vector.
Notation: $\tilde{\mathcal{Q}}_q(\mathbb{G},\mathbb{H})$ (resp. $\mathcal{Q}_q(\mathbb{G},\mathbb{H})$).

- generalised commuting probabilistic model (resp. **commuting probabilistic models**), if

$$p(x,y) = \langle (A_x B_y)\psi, \psi \rangle, \quad (x,y) \in V \times W$$

for POR's (resp. **PR's**) $(A_x)_{x\in V} \subseteq \mathcal{B}(\mathcal{H})$ and $(B_y)_{y\in W} \subseteq \mathcal{B}(\mathcal{H})$ that commute and $\psi \in \mathcal{H}$ unit vector.
Notation: $\tilde{\mathcal{Q}}_{qc}(\mathbb{G},\mathbb{H})$ (resp. $\mathcal{Q}_{qc}(\mathbb{G},\mathbb{H})$).

Bell scenarios and correlations

A no-signalling correlation $p = \{(p(a,b|x,y))_{a\in A, b\in B} : x \in X, y \in Y\}$ defines $\tilde{p} \in \mathcal{G}_{ns}(\mathbb{B}_{X,A}, \mathbb{B}_{Y,B})$ where $\tilde{p}((x,a),(y,b)) := p(a,b|x,y)$ and vice versa.

In particular,

Correlations	**Probabilistic models**
$\mathcal{C}_{ns}(X,Y,A,B) = \mathcal{G}_{ns}(\mathbb{B}_{X,A}, \mathbb{B}_{Y,B})$	
$\mathcal{C}_{loc}(X,Y,A,B) = \mathcal{C}(\mathbb{B}_{X,A}, \mathbb{B}_{Y,B})$	
$\mathcal{C}_{q}(X,Y,A,B) = \mathcal{Q}_{q}(\mathbb{B}_{X,A}, \mathbb{B}_{Y,B})$	
$\mathcal{C}_{qa}(X,Y,A,B) = \mathcal{Q}_{qa}(\mathbb{B}_{X,A}, \mathbb{B}_{Y,B})$	
$\mathcal{C}_{qc}(X,Y,A,B) = \mathcal{Q}_{qc}(\mathbb{B}_{X,A}, \mathbb{B}_{Y,B})$	

Where $\mathcal{Q}_{qa}(\mathbb{G}, \mathbb{H}) = \mathrm{cl}(\mathcal{Q}_{q}(\mathbb{G}, \mathbb{H}))$.

Characterisation in terms of states

Given a functional $s : \mathcal{S}_{\mathbb{G}} \otimes \mathcal{S}_{\mathbb{H}} \to \mathbb{C}$ define

$$p_s(x, y) := s(a_x \otimes b_y)$$

Prob. models	$\leftrightsquigarrow$	**states on**
$\mathcal{G}_{ns}(\mathbb{G}, \mathbb{H})$	$\leftrightsquigarrow$	$\mathcal{S}_{\mathbb{G}} \otimes_{max} \mathcal{S}_{\mathbb{H}}$
$\tilde{\mathcal{Q}}_{qc}(\mathbb{G}, \mathbb{H})$	$\leftrightsquigarrow$	$\mathcal{S}_{\mathbb{G}} \otimes_{c} \mathcal{S}_{\mathbb{H}}$
$\tilde{\mathcal{Q}}_{qa}(\mathbb{G}, \mathbb{H})$	$\leftrightsquigarrow$	$\mathcal{S}_{\mathbb{G}} \otimes_{min} \mathcal{S}_{\mathbb{H}}$
$\mathcal{Q}_{qc}(\mathbb{G}, \mathbb{H})$	$\leftrightsquigarrow$	$\mathcal{T}_{\mathbb{G}} \otimes_{\mathrm{e}} \mathcal{T}_{\mathbb{H}}$
$\mathcal{Q}_{qa}(\mathbb{G}, \mathbb{H})$	$\leftrightsquigarrow$	$\mathcal{T}_{\mathbb{G}} \otimes_{min} \mathcal{T}_{\mathbb{H}}$
$\mathcal{C}(\mathbb{G}, \mathbb{H})$	$\leftrightsquigarrow$	$\mathcal{R}_{\mathbb{G}} \otimes_{min} \mathcal{R}_{\mathbb{H}}$

Where $\tilde{\mathcal{Q}}_{qa}(\mathbb{G}, \mathbb{H}) = \mathrm{cl}(\tilde{\mathcal{Q}}_{q}(\mathbb{G}, \mathbb{H}))$.

The above generalise the works of [PT13], [PSS$^+$16], [LLM$^+$18] in the no-signalling correlation framework.

On Connes Embedding Problem

Theorem: The following are equivalent:

- CEP has an affirmative answer
- $\tilde{\mathcal{Q}}_{\rm qa}(\mathbb{G},\mathbb{G}) = \tilde{\mathcal{Q}}_{\rm qc}(\mathbb{G},\mathbb{G})$ for every scenario $\mathbb{G}$.
- $C_u^*(\mathcal{S}_{\mathbb{G}}) \otimes_{\min} C_u^*(\mathcal{S}_{\mathbb{G}}) = C_u^*(\mathcal{S}_{\mathbb{G}}) \otimes_{\max} C_u^*(\mathcal{S}_{\mathbb{G}})$ for every scenario $\mathbb{G}$.
- $\mathcal{S}_{\mathbb{G}} \otimes_{\min} \mathcal{S}_{\mathbb{G}} = \mathcal{S}_{\mathbb{G}} \otimes_{\rm c} \mathcal{S}_{\mathbb{G}}$ for every scenario $\mathbb{G}$.

and also

Theorem: The following are equivalent:

- CEP has an affirmative answer
- $\mathcal{Q}_{\rm qa}(\mathbb{G},\mathbb{G}) = \mathcal{Q}_{\rm qc}(\mathbb{G},\mathbb{G})$ for every dilating scenario $\mathbb{G}$.
- $C^*(\mathbb{G}) \otimes_{\min} C^*(\mathbb{G}) = C^*(\mathbb{G}) \otimes_{\max} C^*(\mathbb{G})$ for every dilating scenario $\mathbb{G}$.
- $\mathcal{T}_{\mathbb{G}} \otimes_{\min} \mathcal{T}_{\mathbb{G}} = \mathcal{T}_{\mathbb{G}} \otimes_{\rm c} \mathcal{T}_{\mathbb{G}}$ for every dilating scenario $\mathbb{G}$.

What we know on $\mathcal{S}_{\mathbb{G}}$ and $\mathcal{T}_{\mathbb{G}}$

- For any scenario $\mathbb{G}$, $C^*(\mathbb{G}) = C_e^*(\mathcal{T}_{\mathbb{G}})$.

- For any scenario $\mathbb{G}$, $C_u^*(\mathcal{S}_{\mathbb{G}})$ can be identified as the right C*-algebra of a ternary ring of operators (TRO) arising from the hypergraph $\mathbb{G}$.

- Let $\mathbb{G} = (V, E)$ with $|E| = n$, define

$$\mathcal{L}_{\mathbb{G}} = \Big\{ (\lambda_x^1)_{x \in e_1} \oplus \cdots \oplus (\lambda_x^n)_{x \in e_n} : \sum_{x \in e_i} \lambda_x^i = \sum_{x \in e_j} \lambda_x^j \text{ and } \lambda_x^i = \lambda_x^j \text{ for all } x \in e_i \cap e_j, i, j \in [n] \Big\},$$

then for uniform hypegraphs $\mathbb{G}$, we can identify $\mathcal{S}_{\mathbb{G}}^d = \mathcal{L}_{\mathbb{G}}$.

A few questions

- Other C^*-covers for $\mathcal{S}_{\mathbb{G}}$ and $\mathcal{T}_{\mathbb{G}}$?

For dilating scenarios $\mathbb{G}$ and $\mathbb{H}$ we can show that $\tilde{\mathcal{Q}}_{\mathrm{q}}(\mathbb{G},\mathbb{H}) = \mathcal{Q}_{\mathrm{q}}(\mathbb{G},\mathbb{H})$ and $\tilde{\mathcal{Q}}_{\mathrm{qa}}(\mathbb{G},\mathbb{H}) = \mathcal{Q}_{\mathrm{qa}}(\mathbb{G},\mathbb{H})$.

- Does it hold that $\tilde{\mathcal{Q}}_{\mathrm{qc}}(\mathbb{G},\mathbb{H}) = \mathcal{Q}_{\mathrm{qc}}(\mathbb{G},\mathbb{H})$ for dilating scenarios?

- Is it true that $\mathcal{T}_{\mathbb{G}} \otimes_{\mathrm{c}} \mathcal{T}_{\mathbb{H}} \subseteq C^*(\mathbb{G}) \otimes_{\max} C^*(\mathbb{H})$?

- When $\mathbb{G}$ is not uniform what is the $\mathcal{S}_{\mathbb{G}}^{d}$?

- Is it true that CEP is equivalent the equality $\mathcal{Q}_{\mathrm{qa}}(\mathbb{G},\mathbb{G}) = \mathcal{Q}_{\mathrm{qc}}(\mathbb{G},\mathbb{G})$ for all contextuality scenarios $\mathbb{G}$?

Thank You!

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