Morita equivalence of operator algebras and operator spaces

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A′: Strong Morita equivalence

Ternary rings of operators, Hestenes 1961

- Let H, K be Hilbert spaces and $\mathcal{M}$ be a subspace of $\mathcal{B}(H, K)$. We call $\mathcal{M}$ a ternary ring of operators (TRO's) if $X, Y, Z \in \mathcal{M} \Rightarrow XY^*Z \in \mathcal{M}$. In this case the spaces $[\mathcal{M}^*\mathcal{M}], [\mathcal{M}\mathcal{M}^*]$ are selfadjoint algebras.
- Examples:
  (i) If $\mathcal{A} \subseteq \mathcal{B}(H)$ a C^*-algebra and $P, Q \in \mathcal{A}$ projections then $P\mathcal{A}Q$ is a TRO.
  (ii) If H, K are Hilbert spaces then $\mathcal{B}(H, K)$ is TRO.
  (iii) If $\mathcal{X} \subseteq \mathcal{B}(H, K)$ a subspace then $\mathcal{M} = [\mathcal{X}C^*(\mathcal{X}^*\mathcal{X})]$ is a TRO. Here $C^*(\mathcal{X}^*\mathcal{X})$ is the C^* algebra generated by the set $\mathcal{X}^*\mathcal{X}$.

Rieffel, 1974

- The C^*-algebras $\mathcal{A}$ and $\mathcal{B}$ are called strongly Morita equivalent (SME) if there exist faithful $*-$homomorphisms

$$\alpha : \mathcal{A} \to \mathcal{B}(H), \beta : \mathcal{B} \to \mathcal{B}(K)$$

  and a TRO $\mathcal{M} \subseteq \mathcal{B}(H, K)$ such that

$$\alpha(\mathcal{A}) = \overline{[\mathcal{M}^*\mathcal{M}]}^{\|\cdot\|}, \;\; \beta(\mathcal{B}) = \overline{[\mathcal{M}\mathcal{M}^*]}^{\|\cdot\|}.$$

  We write $\mathcal{A} \sim_{SME} \mathcal{B}$. For example the algebras $\mathbb{C}, M_n, \mathcal{K}$, where $\mathcal{K}$ is the algebra of compact operators are SME.

- If $\mathcal{A}$ is a C^*-algebra, we say that the pair (H, π), where $\pi : \mathcal{A} \to \mathcal{B}(H)$ is a $*$-homomorphism is an object of the category ${}_{\mathcal{A}}Mod$. If $(H_i, \pi_i), i = 1, 2$ are objects the space of morphisms is;

$$Hom_{\mathcal{A}}(H_1, H_2) = \{T \in B(H_1, H_2) : T\pi_1(A) = \pi_2(A)T, \;\; \forall \in \mathcal{A}\}.$$

- We write $\mathcal{A} \sim_R B$ in case there exists a functor of equivalence

$$\mathcal{F}: \ {}_{\mathcal{A}}Mod \to \ {}_{\mathcal{B}}Mod$$

satisfying

$$\mathcal{F}(T^*) = \mathcal{F}(T)^*, \ \ \forall \in Hom_{\mathcal{A}}(H_1, H_2), \ \ \forall, H_1, H_2 \in \ {}_{\mathcal{A}}Mod.$$

- If $\mathcal{A}, \mathcal{B}$ are C^*-algebras then

$$\mathcal{A} \sim_{SME} \Rightarrow \mathcal{A} \sim_R B.$$

The converse does not hold always.

- Example: $A \sim_R B \quad \not\Rightarrow A \sim_{SME} B$. If $A = C([0,1])$ and $B = C(\mathcal{T})$, where $\mathcal{T}$ is the circle, then $A \sim_R B$. But they are not SME because the commutative C^*-algebras are SME iff they are isomorphic. This is not true in this case because $[0,1]$ and $\mathcal{T}$ are not homeomorphic.

Brown, Green, Rieffel, 1977

- The operator algebras $\mathcal{A}, \mathcal{B}$ are called strongly stably isomorphic if the algebras $\mathcal{A} \otimes_{min} \mathcal{K}, \mathcal{B} \otimes_{min} \mathcal{K}$ are $*$-isomorphic. Here $\mathcal{K}$ is the space of compact operators acting on l^2,
- If $\mathcal{A}, \mathcal{B}$ are SME C^*-algebras which possess countable approximate units then they are strongly stably isomorphic. There exist C^*-algebras which are SME but not strongly stably isomorphic.
- Let $\mathcal{A}$ be a II_1 factor, H be a non-separable Hilbert space and $\mathcal{B} = \mathcal{A}\bar{\otimes}\mathcal{B}(H)$. If $\mathcal{C}$ is the C^*-subalgebra of $\mathcal{B}$ generated by the finite projections of $\mathcal{B}$ then $\mathcal{A} \sim_{SME} \mathcal{C}$ but $\mathcal{A}$ and $\mathcal{C}$ are not strongly stably isomorphic.

Operator algebras

- An operator space $\mathcal{A}$ for which the space $M_n(\mathcal{A})$ is Banach algebra for all $n \in \mathbb{N}$ is called operator algebra. The C^*-algebras are operator algebras.
- If $\mathcal{A}$ is an operator algebra there exists a Hilbert space H and a completely isometric homomorphism $\pi : \mathcal{A} \to \mathcal{B}(H)$.

Blecher, Muhly, Paulsen,2000

- The operator algebras $\mathcal{A}$ and $\mathcal{B}$ are called strongly Morita equivalent if there exist completely isometric homomorphisms

$$\alpha : \mathcal{A} \to \mathcal{B}(H), \ \ \beta : \mathcal{B} \to \mathcal{B}(K)$$

and operator bimodules $\mathcal{U} \subseteq \mathcal{B}(H, K)$, $\mathcal{V} \subseteq \mathcal{B}(K, H)$ such that

$$\alpha(A) = \overline{[\mathcal{V}\mathcal{U}]}^{\|\cdot\|}, \beta(B) = \overline{[\mathcal{U}\mathcal{V}]}^{\|\cdot\|}$$

and there exist contractve rows

$$V_t^1 \in R_{n_t}(\mathcal{V}), \ \ U_s^2 \in R_{m_s}(\mathcal{U})$$

and columns

$$U_t^1 \in C_{n_t}(\mathcal{U}), \ \ V_s^2 \in C_{m_s}(\mathcal{V})$$

such that $(V_t^1 U_t^1)_t$ is approximate unit for $\alpha(A)$ and $(U_s^2 V_s^2)_s$ is approximate unit for $\beta(B)$. In this case we write $\mathcal{A} \sim_{BMP} \mathcal{B}$.

- If $\mathcal{A}$ is an operator algebra the objects of the category ${}_{\mathcal{A}}OMOD$ are operator spaces $\mathcal{U}$ for which there exist completely contractive maps

$$\mathcal{A} \times \mathcal{U} \to \mathcal{U}.$$

If $\mathcal{U}_1, \mathcal{U}_2 \in {}_{\mathcal{A}}OMOD$ the corresponding space of morphisms is

$$CB_{\mathcal{A}}(\mathcal{U}_1, \mathcal{U}_2) = \{T : \mathcal{U}_1 \to \mathcal{U}_2, T \text{ is completely bounded } \mathcal{A}-\text{module map}\}.$$

A functor

$$\mathcal{F} : {}_{\mathcal{A}}OMOD \to \mathcal{B}OMOD$$

is called completely contractive if the maps

$$\mathcal{F} : CB_{\mathcal{A}}(\mathcal{U}_1, \mathcal{U}_2) \to CB_{\mathcal{A}}(\mathcal{F}(\mathcal{U}_1), F(U_2))$$

are completely contractive for every $\mathcal{U}_1, \mathcal{U}_2 \in {}_{\mathcal{A}}OMOD$

Blecher, Muhly, Paulsen

- If $\mathcal{A} \sim_{BMP} \mathcal{B}$ then the categories ${}_{\mathcal{A}}OMOD$, ${}_{\mathcal{B}}OMOD$ are equivalent through a cc functor.
- **Blecher, 2001**, If the categories ${}_{\mathcal{A}}OMOD$, ${}_{\mathcal{B}}OMOD$ are equivalent through a cc functor then $\mathcal{A} \sim_{BMP} \mathcal{B}$.
- If $\mathcal{A}, \mathcal{B}$ are C^*-algebras then

$$\mathcal{A} \sim_{SME} \mathcal{B} \Longleftrightarrow \mathcal{A} \sim_{BMP} \mathcal{B}.$$

- $\mathcal{A} \sim_{BMP} \mathcal{B}$ iff there exists a $\mathcal{A} - \mathcal{B}$ operator bimodule $\mathcal{V}$ and a $\mathcal{B} - \mathcal{A}$ operator bimodule $\mathcal{U}$ such that

$$\mathcal{A} = \mathcal{V} \otimes^h_{\mathcal{B}} \mathcal{U}, \quad \mathcal{B} = \mathcal{U} \otimes^h_{\mathcal{A}} \mathcal{V}.$$

Here $\mathcal{V} \otimes^h_{\mathcal{B}} \mathcal{U}$ is the balanced Haagerup tensor product of $\mathcal{V}$ and $\mathcal{U}$ which linearizes completely bounded bilinear maps $f : \mathcal{V} \times \mathcal{U} \to \mathcal{B}(H)$ satisfying

$$f(TB, S) = f(T, BS), \ \ \forall\ T \in \mathcal{V},\ B \in \mathcal{B},\ S \in \mathcal{U}.$$

E, 2016

- Let $\mathcal{A}, \mathcal{B}$ be operator algebras acting on the Hilbert spaces H, K. We call them TRO equivalent if there exists a TRO $\mathcal{M} \subseteq \mathcal{B}(H, K)$ such that

$$\mathcal{A} = \overline{[\mathcal{M}^* \mathcal{B} \mathcal{M}]}^{\|\cdot\|}, \ \ \mathcal{B} = \overline{[\mathcal{M} \mathcal{A} \mathcal{M}^*]}^{\|\cdot\|}.$$

  In this case we write $\mathcal{A} \sim_{TRO} \mathcal{B}$.

- Let $\mathcal{A}, \mathcal{B}$ be operator algebras. We call them Δ equivalent if there exists completely isometric homomorphisms

$$\alpha : \mathcal{A} \to \alpha(A), \beta : \mathcal{B} \to \beta(\mathcal{B})$$

  such that

$$\alpha(\mathcal{A}) \sim_{TRO} \beta(\mathcal{B}).$$

  In this case we write $\mathcal{A} \sim_{\Delta} \mathcal{B}$.

- Let $\mathcal{A}, \mathcal{B}$ be operator algebras.
  (iii) In case $\mathcal{A}, \mathcal{B}$ possess countable approximate units then $\mathcal{A} \sim_\Delta \mathcal{B}$ iff $\mathcal{A}, \mathcal{B}$ are strongly stably isomorphic.
  (ii) If $\mathcal{A}, \mathcal{B}$ are C^*-algebras then

$$\mathcal{A} \sim_\Delta \mathcal{B} \Leftrightarrow A \sim_{BMP} \mathcal{B} \Leftrightarrow A \sim_{SME} \mathcal{B}.$$

  (i) $\mathcal{A} \sim_\Delta \mathcal{B}$ then $A \sim_{BMP} \mathcal{B}$. The converse does not hold: Example, of Blecher, Muhly, Paulsen of unital operator algebras which are $BMP-$equivalent but not stably isomorphic.
- A totally ordered set of projections $\mathcal{N}$ is called a nest in the Hibert space H if it is w^*-closed and contains the zero and the identity, operators. The algebra

$$Alg(\mathcal{N}) = \{T :\in \mathcal{B}(H), T(N(H)) \subseteq N(H), \forall N \in \mathcal{N}\}$$

  is called a nest algebra.
- Let $\mathcal{A}, \mathcal{B}$ be separably acting nest algebras. The following are equivalent:
  (i) $\mathcal{A} \cap \mathcal{K} \sim_\Delta \mathcal{B} \cap \mathcal{K}$
  (ii) $\mathcal{A} \cap \mathcal{K}$ and $\mathcal{B} \cap \mathcal{K}$ are strongly stably isomorphic.
  (iii) The algebras $\mathcal{A} \bar{\otimes} \mathcal{B}(l^2)$ and $\mathcal{B} \bar{\otimes} \mathcal{B}(l^2)$ are completely isometrically and w^*-homeomorphically isomorphic.

E, Kakariadis, 2017

- Let $\mathcal{X} \subseteq \mathcal{B}(H_1, H_2), \mathcal{Y} \subseteq \mathcal{B}(K_1, K_2)$ be operator spaces. We call them TRO equivalent if there exist TRO's

$$\mathcal{M}_i \subseteq \mathcal{B}(H_i, K_i), \ i = 1, 2$$

such that

$$\mathcal{Y} = \overline{[\mathcal{M}_2^* \mathcal{X} \mathcal{M}_1]}^{\|\cdot\|}, \ \ \mathcal{X} = \overline{[\mathcal{M}_2 \mathcal{Y} \mathcal{M}_1^*]}^{\|\cdot\|}.$$

We write $\mathcal{X} \sim_{TRO} \mathcal{Y}$. In case $\mathcal{M}_i$ are $\sigma-$ TRO's we write $\mathcal{X} \sim_{\sigma TRO} \mathcal{Y}$.

- Let $\mathcal{X}, \mathcal{Y}$ be operator spaces. We call them strongly Δ-equivalent if there exist completely isometric maps $\phi : \mathcal{X} \to \phi(\mathcal{X}), \psi : \mathcal{Y} \to \psi(\mathcal{Y})$ such that $\phi(\mathcal{X}) \sim_{TRO} \psi(\mathcal{Y})$. We write $\mathcal{X} \sim_{\Delta} \mathcal{Y}$. In case the TRO's are $\sigma-$ TRO's we write $\mathcal{X} \sim_{\sigma\Delta} \mathcal{Y}$.

- (i) $\mathcal{X} \sim_{\sigma\Delta} \mathcal{Y}$ iff $\mathcal{X}, \mathcal{Y}$ are strongly stably isomorphic.
(ii) If $\mathcal{X} \sim_{\Delta} \mathcal{Y}$ then the TRO envelopes of $\mathcal{X}, \mathcal{Y}$ are Δ-equivalent.
Therefore if $\mathcal{X}, \mathcal{Y}$ are unital operator spaces then their C^*-envelopes are strongly stably isomorphic.

B′: Weak Morita equivalence

Rieffel, 1974

- The W^*-algebras, (von Neumann algebras), $\mathcal{A}, \mathcal{B}$ are called weakly Morita equivalent if there exist faithful w^* continuous homomorphisms $\alpha : \mathcal{A} \to \alpha(\mathcal{A})$, $\beta : \mathcal{B} \to \beta(\mathcal{B})$ and a TRO $\mathcal{M}$ such that

$$\alpha(\mathcal{A}) = \overline{[\mathcal{M}^*\mathcal{M}]}^{w^*}, \ \ \beta(\mathcal{B}) = \overline{[\mathcal{M}\mathcal{M}^*]}^{w^*}.$$

  We write $\mathcal{A} \sim_{WME} \mathcal{B}$.
- The algebras $\mathbb{C}, M_n, \mathcal{B}(H)$ are weakly Morita equivalent for every Hilbert space H.
- If the von Neumann algebras $\mathcal{A}, \mathcal{B}$ are $*$-isomorphic then their commutants $\mathcal{A}', \mathcal{B}'$ are weakly Morita equivalent.
- If $\mathcal{A}$ is a W^*-algebra, we say that the pair (H, π), where $\pi : \mathcal{A} \to \mathcal{B}(H)$ is a w^* continuous $*$-homomorphism is an object of the category ${}_{\mathcal{A}}WMod$. If $(H_i, \pi_i), i = 1, 2$ are objects the space of morphisms is;

$$Hom_{\mathcal{A}}(H_1, H_2) = \{T \in B(H_1, H_2) : T\pi_1(A) = \pi_2(A)T, \ \ \forall \in \mathcal{A}\}.$$

- Let $\mathcal{A}, \mathcal{B}$ be W^* algebras. The following are equivalent:
  (i) $A \sim_{WME} \mathcal{B}$
  (ii) The categories ${}_{\mathcal{A}}WMod$, ${}_{\mathcal{B}}WMod$ are equivalent.
  (iii) There exist faithful w^*-continuous homomorphisms

$$\alpha : \mathcal{A} \to \alpha(\mathcal{A}), \ \beta : \mathcal{B} \to \beta(\mathcal{B})$$

  such that the commutants of $\alpha(\mathcal{A})$ and $\beta(\mathcal{B})$ to be $*-$isomorphic.
  (iv) There exists a Hilbert space H such that the algebras $\mathcal{A}\bar{\otimes}\mathcal{B}(H), \mathcal{B}\bar{\otimes}\mathcal{B}(H)$ are $*-$isomorphic.

Dual operator algebras

- An operator algebra which is dual operator space is called dual operator algebra. The C^*-algebras which are dual Banach spaces are dual operator algebras and they are called W^*-algebras.
- If $\mathcal{A}$ is a dual operator algebra there exists a Hilbert space H and a completely isometric w^*-continuous homomorphism $\pi : \mathcal{A} \to \mathcal{B}(H)$.

Blecher, Kashyap, 2007

- The dual operator algebras $\mathcal{A}$ and $\mathcal{B}$ are called weakly Morita equivalent if there exist completely isometric w^*-continuous homomorphisms $\alpha : \mathcal{A} \to \mathcal{B}(H)$, $\beta : \mathcal{B} \to \mathcal{B}(K)$ and operator bimodules $\mathcal{U} \subseteq \mathcal{B}(H, K)$, $\mathcal{V} \subseteq \mathcal{B}(K, H)$ such that $\alpha(A) = \overline{[\mathcal{V}\mathcal{U}]}^{w^*}, \beta(B) = \overline{[\mathcal{U}\mathcal{V}]}^{w^*}$ and there exist contractive rows $V_t^1 \in R_{n_t}(\mathcal{V})$, $U_s^2 \in R_{m_s}(\mathcal{U})$ and columns $U_t^1 \in C_{n_t}(\mathcal{U})$, $V_s^2 \in C_{m_s}(\mathcal{V})$ such that $(V_t^1 U_t^1)_t$ is approximate in w^*-topology unit for $\alpha(A)$ and $(U_s^2 V_s^2)_s$ is approximate unit for $\beta(B)$. In this case we write $\mathcal{A} \sim_{BK} \mathcal{B}$.
- $\mathcal{A} \sim_{BK} \mathcal{B}$ iff there exists a $\mathcal{A} - \mathcal{B}$ dual operator bimodule $\mathcal{V}$ and a $\mathcal{B} - \mathcal{A}$ dual operator bimodule $\mathcal{U}$ such that

$$\mathcal{A} = \mathcal{V} \otimes^{\sigma h}_{\mathcal{B}} \mathcal{U}, \quad \mathcal{B} = \mathcal{U} \otimes^{\sigma h}_{\mathcal{A}} \mathcal{V}.$$

  Here $\mathcal{V} \otimes^{\sigma h}_{\mathcal{B}} \mathcal{U}$ is the w^*-balanced Haagerup tensor product of $\mathcal{V}$ and $\mathcal{U}$ which linearizes w^*-continuous completely bounded bilinear maps $f : \mathcal{V} \times \mathcal{U} \to \mathcal{B}(H)$ satisfying

$$f(TB, S) = f(T, BS), \ \forall\, T \in \mathcal{V},\ B \in \mathcal{B},\ S \in \mathcal{U}.$$

- If $\mathcal{A}$ is a dual operator algebra the objects of the category ${}_{\mathcal{A}}WOMOD$ are dual operator spaces $\mathcal{U}$ for which there exist completely contractive w^*-continuous maps

$$\mathcal{A} \times \mathcal{U} \to \mathcal{U}.$$

If $\mathcal{U}_1, \mathcal{U}_2 \in {}_{\mathcal{A}}WOMOD$ the corresponding space of morphisms is

$$CB^w_{\mathcal{A}}(\mathcal{U}_1, \mathcal{U}_2) =$$

$$\{T : \mathcal{U}_1 \to \mathcal{U}_2, T \text{ is } w^* \text{ continuous completely bounded } \mathcal{A} - \text{module map}\}.$$

A functor

$$\mathcal{F} : {}_{\mathcal{A}}WOMOD \to {}_{\mathcal{B}}WOMOD$$

is called completely contractive if the maps

$$\mathcal{F} : CB^w_{\mathcal{A}}(\mathcal{U}_1, \mathcal{U}_2) \to CB^w_{\mathcal{A}}(\mathcal{F}(\mathcal{U}_1), F(U_2))$$

is completely contractive for every $\mathcal{U}_1, \mathcal{U}_2 \in {}_{\mathcal{A}}WOMOD$.

- $\mathcal{A} \sim_{BK} \mathcal{B}$ iff the categories ${}_{\mathcal{A}}WOMOD \to {}_{\mathcal{B}}WOMOD$ are equivalent through a cc functor.

E, 2008

- The dual operator spaces $\mathcal{X}, \mathcal{Y}$ are called weakly Δ equivalent if there exist w^*-continuous complete isometries $\phi : \mathcal{X} \to \phi(\mathcal{X}), \psi : \mathcal{Y} \to \psi(\mathcal{Y})$ and there exists TRO's $\mathcal{M}_i, i = 1, 2$ such that $\psi(\mathcal{Y}) = \overline{[\mathcal{M}_2^* \phi(\mathcal{X}) \mathcal{M}_1]}^{w^*}, \ \ \phi(\mathcal{X}) = \overline{[\mathcal{M}_2 \psi(\mathcal{Y}) \mathcal{M}_1^*]}^{w^*}$. In this case we write $\mathcal{X} \sim_{w\Delta} \mathcal{Y}$.

- If $\mathcal{D}$ is a dual operator algebra then the category ${}_{\mathcal{D}}\mathfrak{M}$ has objects pairs (H, π), where $\pi : \mathcal{D} \to \mathcal{B}(H)$ is a normal representation of $\mathcal{D}$. If $(H_i, \pi_i) \in {}_{\mathcal{D}}\mathfrak{M}, i = 1, 2$ the corresponding space of homomorphisms is

$$Hom_{\mathcal{D}}(H_1, H_2) = \{T \in \mathcal{B}(H_1, H_2), T\pi_1(A) = \pi_2(A)T, \ \forall\, A \in \mathcal{D}\}.$$

  Notice that if $\Delta(\mathcal{D}) = \mathcal{D} \cap \mathcal{D}^*$ then for $(H_i, \pi_i) \in {}_{\mathcal{D}}\mathfrak{M} \Rightarrow (H_i, \pi_i) \in {}_{\Delta(\mathcal{D})}\mathfrak{M}, i = 1, 2$ and

$$Hom_{\Delta(\mathcal{D})}(H_1, H_2) = \{T \in \mathcal{B}(H_1, H_2), T\pi_1(A) = \pi_2(A)T, \ \forall\, A \in \Delta(\mathcal{D})\} \supseteq$$

$$\supseteq Hom_{\mathcal{D}}(H_1, H_2).$$

- Let $\mathcal{A}, \mathcal{B}$ be dual operator algebras. Then a functor $\mathcal{F} : {}_{\Delta(\mathcal{A})}\mathfrak{M} \to {}_{\Delta(\mathcal{B})}\mathfrak{M}$ is called Δ-restricting if

$$\mathcal{F}(Hom_{\mathcal{A}}(H_1, H_2)) \subseteq Hom_{\mathcal{B}}(\mathcal{F}(H_1), \mathcal{F}(H_2))), \ \ \forall\, H_i \in {}_{\mathcal{A}}\mathfrak{M}.$$

  Theorem: $\mathcal{A} \sim_{W\Delta} \mathcal{B}$ iff there exists a Δ-restricting functor of equivalence

$$\mathcal{F} : {}_{\Delta(\mathcal{A})}\mathfrak{M} \to {}_{\Delta(\mathcal{B})}\mathfrak{M}.$$

E, Paulsen, Todorov, Kakariadis

- **E, Paulsen, 2008** Let $\mathcal{A}, \mathcal{B}$ be dual operator algebras. Then $\mathcal{A} \sim_{W\Delta} \mathcal{B}$. iff there exists a Hilbert space H such that the algebras $\mathcal{A}\bar{\otimes}\mathcal{B}(H), \mathcal{B}\bar{\otimes}\mathcal{B}(H)$ are completely isometric and w^* homeomorphic as algebras.
- **E, Paulsen, Todorov, 2010** Let $\mathcal{X}, \mathcal{Y}$ be dual operator spaces. Then $\mathcal{X} \sim_{W\Delta} \mathcal{Y}$ iff there exists a Hilbert space H such that the spaces $\mathcal{X}\bar{\otimes}\mathcal{B}(H), \mathcal{Y}\bar{\otimes}\mathcal{B}(H)$ are completely isometric and w^* homeomorphic.
- **E, Kakariadis, 2017** If $\mathcal{X}, \mathcal{Y}$ are $\Delta-$equivalent operator spaces then $\mathcal{X}^{**} \sim_{W\Delta} \mathcal{Y}^{**}$.

E, 2010

- The nest algebras $\mathcal{A} = Alg(\mathcal{N}_1), \mathcal{B} = Alg(\mathcal{N}_2)$ are weakly Morita equivalent in the sense of Blecher Kashyap iff the nests $\mathcal{N}_1, \mathcal{N}_2$ are isomorphic.
- The nest algebras $\mathcal{A} = Alg(\mathcal{N}_1), \mathcal{B} = Alg(\mathcal{N}_2)$ are weakly $\Delta-$ equivalent iff there exists a $*-$isomorphism $\theta : (\mathcal{N}_1)'' \to (\mathcal{N}_2)''$ such that $\theta(\mathcal{N}_1) = \mathcal{N}_2$.
- There exist isomorphic nests $\mathcal{N}_1, \mathcal{N}_2$ for which the von Neumann algebra $(\mathcal{N}_1)''$ is atomic but the algebra $(\mathcal{N}_2)''$ has continuous part. Therefore these algebras cant be $*-$isomorphic. Thus the nest algebras $\mathcal{A} = Alg(\mathcal{N}_1), \mathcal{B} = Alg(\mathcal{N}_2)$ are Morita equivalent in the sense of Blecher Kashyap but not weakly $\Delta-$ equivalent and so not stably isomorphic.

Davidson, book

- We consider the counting meaure μ on $\mathcal{Q}$ and the nest

$$\mathcal{N}_1 = \{Q_t, P_t : t \in \mathbb{R}\}$$

acting on the Hilbert space $L^2(\mathbb{R}, \mu)$, where

$$Q_t = \{f : f|_{(t,+\infty)} = 0\}, \;\; P_t = \{f : f|_{[t,+\infty)} = 0\}.$$

The atoms of this nest are the characteristic functions $\chi_{\{t\}}, t \in \mathcal{Q}$ which generate the von Neumann algebra $(\mathcal{N}_1)''$.

- If $t \in \mathbb{R}, N_t$ is the projection on $L^2((-\infty, t], \lambda), \lambda$ is the Lebesgue measure. We consider the nest $\mathcal{N}_2 = \{Q_t \oplus N_t, P_t \oplus N_t, t \in \mathbb{R}\}$ acting on the Hilbert space $L^2(\mathbb{R}, \mu) \oplus L^2(\mathbb{R}, \lambda)$. The map $\mathcal{N}_2 \to \mathcal{N}_1, Q_t \oplus N_t \to Q_t, \; P_t \oplus N_t \to P_t$ is a nest isomorphism. Thus, $\mathcal{A} = Alg(\mathcal{N}_1), \mathcal{B} = Alg(\mathcal{N}_2)$ are Morita equivalent in the sense of Blecher Kashyap. But the algebras $(\mathcal{N}_1)'', (\mathcal{N}_2)''$ cant be isomorphic, since $(\mathcal{N}_2)'' \sim l^\infty(\mathbb{N}) \oplus L^\infty(\mathbb{R})$ has continuous part. Thus $\mathcal{A} = Alg(\mathcal{N}_1), \mathcal{B} = Alg(\mathcal{N}_2)$ cant be Δ-equivalent.

E, 2018

- Let $\mathcal{A}, \mathcal{B}$ be dual operator algebras. We say that $\mathcal{B}$ TRO-embeds into $\mathcal{A}$ if there exists a TRO $\mathcal{M}$ such that
$$\mathcal{B} = \overline{[\mathcal{M}^*\mathcal{A}\mathcal{M}]}^{w^*}, \;\; \mathcal{M}\mathcal{B}\mathcal{M}^* \subseteq \mathcal{A}.$$
We write in this case $\mathcal{B} \subset_{TRO} \mathcal{A}$. If β is a completely isometric normal representation of $\mathcal{B}$ and α of $\mathcal{A}$ such that $\beta(\mathcal{B}) \subset_{TRO} \alpha(\mathcal{A})$ we say that the algebra $\mathcal{B}, \Delta-$embeds into $\mathcal{A}$ and we write $\mathcal{B} \subset_{\Delta} \mathcal{A}$.
- Let $\mathcal{L}$ be a lattice of projections acting on the Hilbert space H. Then
$$Alg(\mathcal{L}) = \{T \in \mathcal{B}(H) : \; T(L(H)) \subseteq L(H)\}$$
is a dual operator algebra.
- If $\mathcal{A} \subseteq \mathcal{B}(H)$ a unital algebra,
$$Lat(\mathcal{A}) = \{L \in \mathcal{B}(H) : L = projection, \; T(L(H)) \; \subseteq L(H) \, \forall \, T \; \in \; \mathcal{A}\}.$$
- If $\mathcal{L}$ is lattice satisfying
$$Lat(Alg(\mathcal{L})) = \mathcal{L},$$
we call $\mathcal{L}$ reflexive lattice.

- Let $\mathcal{A}, \mathcal{B}$ be dual operator algebras. The following are equivalent
  (i) $\mathcal{B} \subset_\Delta \mathcal{A}$.
  (ii) There exist reflexive lattices $\mathcal{L}_i, i = 1, 2$ and w^*continuous completely isometric onto homomorphisms

$$\alpha : \mathcal{A} \to Alg(\mathcal{L}_1), \quad \beta : \mathcal{B} \to Alg(\mathcal{L}_2)$$

  and an onto w^*continuous homomorphism

$$\theta : \Delta(\mathcal{A})' = \mathcal{L}_1'' \to \Delta(\mathcal{B})' = \mathcal{L}_2''$$

  such that

$$\theta(\mathcal{L}_1) = \mathcal{L}_2.$$

  (iii) There exists a Hilbert space H and a projection $Q \in \mathcal{A}$ such that the algebras

$$\mathcal{B}\bar{\otimes}\mathcal{B}(H), \;\; Q\mathcal{A}Q\bar{\otimes}\mathcal{B}(H)$$

  are completely isometrically and w^*- homeomorphically isomorphic.

- In case $\mathcal{A}, \mathcal{B}$ are von Neumann algebras we have that
  (i) $\mathcal{B} \subset_\Delta \mathcal{A}$ iff there exists a w^*-continuous onto homomorphism from $\mathcal{A}\bar{\otimes}\mathcal{B}(H)$ onto $\mathcal{B}\bar{\otimes}\mathcal{B}(H)$.
  (ii) $\mathcal{B} \subset_\Delta \mathcal{A}$ iff there exists a w^*-closed ideal $\mathcal{I} \subseteq \mathcal{A}$ such that $\mathcal{B} \sim_{W\Delta} \mathcal{A}/\mathcal{I}$
  (iii) If $\mathcal{B} \subset_\Delta \mathcal{A}$ and $\mathcal{A} \subset_\Delta \mathcal{B}$ then $\mathcal{B} \sim_{W\Delta} \mathcal{A}$

Bibliography on Δ-equivalence

- Morita equivalence of operator systems, Eleftherakis GK, Kakariadis ETA, Todorov I.G, submitted.
- MR4177870 -A Morita characterisation for algebras and spaces of operators on Hilbert spaces Eleftherakis, G. K.; Papapetros, E. Integral Equations Operator Theory 92 (2020), no. 6, Paper No. 51, 21 pp.
- MR3906459 - On stable maps of operator algebras Eleftherakis, G. K. J. Math. Anal. Appl. 472 (2019), no. 2, 1951–1975.
- MR3818327 - Morita embeddings for dual operator algebras and dual operator spaces Eleftherakis, G. K. Studia Math. 243 (2018), no. 3, 303–328.
- MR3722271 - Morita equivalence of C^*-correspondences passes to the related operator algebras Eleftherakis, George K.; Kakariadis, Evgenios T. A.; Katsoulis, Elias G. Israel J. Math. 222 (2017), no. 2, 949–972.
- MR3563051 - Strong Morita equivalence of operator spaces Eleftherakis, George K.; Kakariadis, Evgenios T. A. J. Math. Anal. Appl. 446 (2017), no. 2, 1632–1653.

- MR3609826 - Stable isomorphism and strong Morita equivalence of operator algebras Eleftherakis, G. K. Houston J. Math. 42 (2016), no. 4, 1245–1266.

- MR3105545 - Morita equivalence of nest algebras Eleftherakis, G. K. Math. Scand. 113 (2013), no. 1, 83–107.

- MR2917279 - TRO equivalent algebras Eleftherakis, G. K. Houston J. Math. 38 (2012), no. 1, 153–175.

- MR2669424 - Morita type equivalences and reflexive algebras Eleftherakis, G. K. J. Operator Theory 64 (2010), no. 1, 3–17.

- MR2557963 - Stable isomorphism of dual operator spaces Eleftherakis, G. K.; Paulsen, V. I.; Todorov, I. G. J. Funct. Anal. 258 (2010), no. 1, 260–278.

- MR2377471 - Stably isomorphic dual operator algebras Eleftherakis, G. K.; Paulsen, V. I. Math. Ann. 341 (2008), no. 1, 99–112.

- MR2387585 - A Morita type equivalence for dual operator algebras Eleftherakis, G. K. J. Pure Appl. Algebra 212 (2008), no. 5, 1060–1071.

THANK YOU