Fock covariance for product systems and the Hao–Ng isomorphism problem

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Coactions

- In this talk G will always denote a discrete group and $\otimes$ the spatial tensor product of C*-algebras.
- Set $\{u_g\}_{g\in G}$ to be the generators of the universal group C*-algebra $\mathrm{C}^*_{\max}(G)$ and let $\lambda\colon \mathrm{C}^*_{\max}(G) \to \mathrm{C}^*_{\lambda}(G)$ be the left regular representation.
- By the universal property of $\mathrm{C}^*_{\max}(G)$ there exists a faithful $*$-homomorphism

$$\Delta\colon \mathrm{C}^*_{\max}(G) \to \mathrm{C}^*_{\max}(G) \otimes \mathrm{C}^*_{\max}(G); u_g \mapsto u_g \otimes u_g.$$

- By Fell's absorption principle there exists a faithful $*$-homomorphism

$$\Delta_\lambda\colon \mathrm{C}^*_{\lambda}(G) \to \mathrm{C}^*_{\lambda}(G) \otimes \mathrm{C}^*_{\lambda}(G); \lambda_g \mapsto \lambda_g \otimes \lambda_g,$$

with the additional property that $\Delta_\lambda \circ \lambda = (\lambda \otimes \lambda) \circ \Delta$.

Coactions

- We say that a C*-algebra $\mathcal{C}$ *admits a coaction* δ *by* G if there is a faithful $*$-homomorphism $\delta\colon \mathcal{C} \to \mathcal{C} \otimes \mathrm{C}^*_{\max}(G)$ such that *the coaction identity*

$$(\delta \otimes \mathrm{id}) \circ \delta = (\mathrm{id} \otimes \Delta) \circ \delta$$

  is satisfied.
- Since G is discrete, the coaction identity is equivalent to the induced *spectral spaces*

$$\mathcal{C}_g := \{c \in \mathcal{C} \mid \delta(c) = c \otimes u_g\} \text{ for all } g \in G,$$

  together spanning a dense subset of $\mathcal{C}$.

Coactions

- If, in addition, the map $(\mathrm{id} \otimes \lambda) \circ \delta$ is faithful, then δ will be called *normal*.
- It follows that δ is normal if and only if $\mathcal{C}$ admits a *reduced* coaction by G, i.e., there is a faithful $*$-homomorphism $\delta_\lambda \colon \mathcal{C} \to \mathcal{C} \otimes \mathrm{C}^*_\lambda(G)$ such that

$$(\delta_\lambda \otimes \mathrm{id}) \circ \delta_\lambda = (\mathrm{id} \otimes \Delta_\lambda) \circ \delta_\lambda.$$

- If $\delta \colon \mathcal{C} \to \mathcal{C} \otimes \mathrm{C}^*_{\max}(G)$ is a coaction, then

$$E := (\mathrm{id} \otimes E_\lambda) \circ (\mathrm{id} \otimes \lambda) \circ \delta$$

defines a conditional expectation on $\mathcal{C}_e$, where E_λ is the faithful conditional expectation of $\mathrm{C}^*_\lambda(G)$.
- δ is normal if and only if E is faithful.

Gradings

- A collection $\{\mathcal{C}_g\}_{g\in G}$ of closed subspaces of a C*-algebra $\mathcal{C}$ is called a *C*-grading of* $\mathcal{C}$ if:
  1. $\sum_{g\in G}\mathcal{C}_g$ is dense in $\mathcal{C}$;
  2. $\mathcal{C}_g\mathcal{C}_h \subseteq \mathcal{C}_{gh}$; and
  3. $\mathcal{C}_g^* \subseteq \mathcal{C}_{g^{-1}}$.
- If there exists a conditional expectation $E\colon \mathcal{C} \to \mathcal{C}_e$, then the subspaces are linearly independent and in this case the C*-grading is called *topological*.
- By definition, a coaction on a C*-algebra induces a topological C*-grading.
- Gradings form the prototypical example of Fell bundles.

Fell bundles

- A *Fell bundle* over a group G is a collection $\mathcal{B} = \{\mathcal{B}_g\}_{g \in G}$ of Banach spaces, each of which is called a *fiber*, such that:
  1. there are bilinear and associative *multiplication maps* from $\mathcal{B}_g \times \mathcal{B}_{g'}$ to $\mathcal{B}_{gg'}$ such that $\|b_g b_{g'}\| \leq \|b_g\| \, \|b_{g'}\|$;
  2. there are conjugate linear *involution maps* from $\mathcal{B}_g$ to $\mathcal{B}_{g^{-1}}$ such that $(b_g^*)^* = b_g$ and $\|b_g^*\| = \|b_g\|$;
  3. $(b_g b_{g'})^* = (b_{g'})^* b_g^*$;
  4. $\|b_g^* b_g\| = \|b_g\|^2$;
  5. $b_g^* b_g \geq 0$ in $\mathcal{B}_e$.
- Note that conditions (i)–(iv) imply that $\mathcal{B}_e$ is in fact a C*-algebra and thus condition (v) makes sense.

Fell bundles

- A *representation* Ψ of a Fell bundle $\mathcal{B}$ over G is a family $\{\Psi_g\}_{g\in G}$ of linear maps each one defined on $\mathcal{B}_g$ such that:
  1. $\Psi_g(b_g)\Psi_h(b_h) = \Psi_{gh}(b_g b_h)$ for all $g, h \in G$; and
  2. $\Psi_g(b_g)^* = \Psi_{g^{-1}}(b_g^*)$ for all $g \in G$.
- It follows that Ψ_e is a $*$-homomorphism and thus contractive.
- A standard C*-trick shows that every Ψ_g is contractive.
- We say that a representation Ψ is *injective* if Ψ_e is injective; in this case every Ψ_g is isometric.
- A representation Ψ is called *equivariant* if there exists a $*$-homomorphism

$$\delta\colon \mathrm{C}^*(\Psi) \to \mathrm{C}^*(\Psi) \otimes \mathrm{C}^*_{\max}(G); \Psi_g(b_g) \mapsto \Psi_g(b_g) \otimes u_g.$$

- It follows that δ is a coaction of $\mathrm{C}^*(\Psi)$ by G.

Fell bundles

- There exists a universal C*-algebra $\mathrm{C}^*_{\max}(\mathcal{B})$ with respect to the representations of $\mathcal{B}$ and a canonical embedding

$$\hat{j}\colon \mathcal{B} \to \mathrm{C}^*_{\max}(\mathcal{B}).$$

- By universality, we have that $\{\hat{j}_g\}_{g\in G}$ is an equivariant repn of $\mathcal{B}$, and in particular the map $\hat{j}_g\colon \mathcal{B}_g \to [\mathrm{C}^*_{\max}(\mathcal{B})]_g$ is an isometric isomorphism.
- Hence, any Fell bundle arises as a C*-grading from some C*-algebra (conversely, any C*-grading defines a Fell bundle).
- The left regular representation is defined by considering the left creation operators

$$(\lambda_g(b_g)\xi)_{g'} = b_g \xi_{g^{-1}g'} \text{ for all } b_g \in \mathcal{B},$$

on the Hilbert $\mathcal{B}_e$-module direct sum $\ell^2(\mathcal{B}) := \sum^{\oplus}_{g\in G} \mathcal{B}_g$.
- We call $\mathrm{C}^*_\lambda(\mathcal{B}) := \mathrm{C}^*(\lambda)$ the *reduced C*-algebra of* $\mathcal{B}$.

Fell bundles

Theorem (Exel 1997)

If Ψ is an equivariant representation of $\mathcal{B}$ that is injective on $\mathcal{B}_e$, then there are equivariant canonical $*$-epimorphisms

$$\mathrm{C}^*_{\max}(\mathcal{B}) \longrightarrow \mathrm{C}^*(\Psi) \longrightarrow \mathrm{C}^*_\lambda(\mathcal{B}).$$

If, in addition, the coaction on $\mathrm{C}^*(\Psi)$ is normal, then $\mathrm{C}^*(\Psi) \simeq \mathrm{C}^*_\lambda(\mathcal{B})$.

We will be interested in graded quotients of C*-algebras of Fell bundles. If $\delta\colon \mathcal{C} \to \mathcal{C} \otimes \mathrm{C}^*_{\max}(G)$ is a coaction on a C*-algebra $\mathcal{C}$, then we say that an ideal $\mathcal{J} \lhd \mathcal{C}$ is *induced* if

$$\mathcal{J} = \left\langle \mathcal{J} \cap [\mathcal{C}]_e \right\rangle.$$

In that case, the canonical quotient map $q_{\mathcal{J}}$ is equivariant, i.e., the coaction δ descends to a coaction on $\mathcal{C}/\mathcal{J}$.

Product Systems and their representations

- P will always denote a unital subsemigroup of G.
- We say that a family $X = \{X_p\}_{p \in P}$ of closed operator spaces in a common $\mathcal{B}(H)$ is a *product system over* P if the following conditions are satisfied:
  1. $A := X_e$ is a C*-algebra;
  2. $X_p \cdot X_q \subseteq X_{pq}$ for all $p, q \in P$;
  3. $X_p^* \cdot X_{pq} \subseteq X_q$ for all $p, q \in P$.
- It follows that each X_p has a natural Hilbert A-module structure.

Example

Every Fell bundle over G defines a product system over G.

Example

Every C*-correspondence defines a product system over $\mathbb{Z}_+$. Indeed, recall that a (concrete) C*-correspondence (X, A) is a pair where $X \subseteq B(H)$ is an operator space and $A \subseteq \mathcal{B}(H)$ is a C*-algebra such that

$$A \cdot X \subseteq X \quad \text{and} \quad X \cdot A \subseteq X \quad \text{and} \quad X^* \cdot X \subseteq A.$$

By setting

$$X_0 := A \quad \text{and} \quad X_n := X^n \text{ for every } n \in \mathbb{N},$$

we obtain a product system over $\mathbb{Z}_+$.

Product Systems and their representations

- A *representation* $t = \{t_p\}_{p \in P}$ of X consists of a family of linear maps t_p of X_p such that:
  1. t_e is a $*$-representation of $A := X_e$;
  2. $t_p(\xi_p)t_q(\xi_q) = t_{pq}(\xi_p\xi_q)$ for all $\xi_p \in X_p$ and $\xi_q \in X_q$;
  3. $t_p(\xi_p)^* t_{pq}(\xi_{pq}) = t_q(\xi_p^*\xi_{pq})$ for all $\xi_p \in X_p$ and $\xi_{pq} \in X_{pq}$.
- We say that t is *injective* if t_e is injective.
- We use the notation

$$t^{(p)} \colon \mathcal{K}(X_p) := [X_p X_p^*] \to \mathrm{C}^*(t); \xi_p \eta_p^* \mapsto t_p(\xi_p) t_p(\eta_p)^*,$$

  for the induced $*$-representation of the compact operators $\mathcal{K}(X_p)$.
- t will be called *equivariant* if there exists a $*$-homomorphism

$$\delta \colon \mathrm{C}^*(t) \to \mathrm{C}^*(t) \otimes \mathrm{C}^*_{\max}(G); t_p(\xi_p) \mapsto t_p(\xi_p) \otimes u_p.$$

- We say that t *admits a normal coaction by* G if δ is normal.

Product Systems and their representations

- Consider the *Fock space* $\mathcal{F}X := \sum_{r\in P}^{\oplus} X_r$ as a Hilbert A-module and the *Fock representation* $\lambda\colon X \to \mathcal{L}(\mathcal{F}X)$ where

$$\lambda_p(\xi_p)\eta_r = \xi_p \cdot \eta_r \quad \text{and} \quad \lambda_p(\xi_p)^*\eta_r = \begin{cases} \xi_p^* \cdot \eta_r & \text{if } r = p \cdot s, \\ 0 & \text{otherwise.} \end{cases}$$

- We set $\mathcal{T}(X) := \mathrm{C}^*(\hat{t})$, where $\hat{t}$ is the universal representation of X. We also set $\mathcal{T}_\lambda(X) := \mathrm{C}^*(\lambda)$ for the *Fock C*-algebra* of X and $\mathcal{T}_\lambda(X)^+ := \overline{\mathrm{alg}}\{\lambda_p(X_p) \mid p \in P\}$ for the *tensor algebra* of X.
- Universality implies that $\hat{t}$ is equivariant.
- λ admits a reduced coaction and hence a normal coaction.

Indeed, consider the unitary

$$U\colon \mathcal{F}X\otimes \ell^2(G)\to \mathcal{F}X\otimes \ell^2(G); U(\xi_r\otimes \delta_g)=\xi_r\otimes \delta_{rg}.$$

Then, we have that

$$U\cdot(\lambda_p(\xi_p)\otimes I)=(\lambda_p(\xi_p)\otimes \lambda_p)\cdot U \text{ for all } p\in P,$$

and therefore the map

$$\mathcal{T}_\lambda(X)\xrightarrow{\simeq}\mathrm{C}^*(\lambda_p(\xi_p)\otimes I\mid p\in P)\xrightarrow{\mathrm{ad}_U}\mathrm{C}^*(\lambda_p(\xi_p)\otimes \lambda_p\mid p\in P)$$

defines a reduced coaction on $\mathcal{T}_\lambda(X)$.

- For a set $Z \subseteq P$ and $p \in P$ we write

$$pZ := \{px \mid x \in Z\} \quad \text{and} \quad p^{-1}Z := \{y \in P \mid py \in Z\}.$$

- We set $\mathcal{J} := \{q_n^{-1}p_n \ldots q_1^{-1}p_1 P \mid n \in \mathbb{N}; p_i, q_i \in P\} \cup \{\emptyset\}$. It follows that $\mathcal{J}$ is $\cap$-closed.
- We will write $\mathbf{x}, \mathbf{y}$ etc. for the elements in $\mathcal{J}$ and we will refer to them as *constructible ideals*.
- For $\mathbf{x} \in \mathcal{J}$ we define $\mathbf{K}_{\mathbf{x},t_*}$ to be the closed linear span of the spaces

$$\left(t_{p_1}(X_{p_1})^*\right)^{\varepsilon} t_{q_1}(X_{q_1}) \cdots t_{p_n}(X_{p_n})^* t_{q_n}(X_{q_n})^{\varepsilon'}$$

for any $p_1, q_1, \ldots, p_n, q_n \in P$ and $\varepsilon, \varepsilon' \in \{0,1\}$ that satisfy

$$p_1^{-\varepsilon} q_1 \cdots p_n^{-1} q_n^{\varepsilon'} = e \text{ and } q_n^{-\varepsilon'} p_n \ldots q_1^{-1} p_1^{\varepsilon} P = \mathbf{x}.$$

Fock covariant representations

- Let $\lambda_* : \mathcal{T}(X) \to \mathcal{T}_\lambda(X)$ be the induced $*$-epimorphism.
- Consider the induced ideal

$$\mathcal{J}_{\mathrm{c}}^{\mathrm{F}} := \langle \ker \lambda_* \cap [\mathcal{T}(X)]_e \rangle \lhd \mathcal{T}(X),$$

  and write $\mathcal{T}_{\mathrm{c}}^{\mathrm{F}}(X)$ for the quotient of $\mathcal{T}(X)$ by $\mathcal{J}_{\mathrm{c}}^{\mathrm{F}}$.
- $\mathcal{J}_{\mathrm{c}}^{\mathrm{F}}$ is an induced ideal of $\mathcal{T}(X)$ and hence $\mathcal{T}_{\mathrm{c}}^{\mathrm{F}}(X)$ inherits a coaction by G.
- The *Fock covariant* bundle of X is the Fell bundle

$$\mathcal{F}\mathcal{C}X := \left\{ [\mathcal{T}_{\mathrm{c}}^{\mathrm{F}}(X)]_g \right\}_{g \in G},$$

  defined by the coaction by G on $\mathcal{T}_{\mathrm{c}}^{\mathrm{F}}(X)$.

Fock covariant representations

- By universality and Fell bundle theory we obtain

$$\mathcal{T}_{\mathrm{c}}^{\mathrm{F}}(X) \simeq \mathrm{C}^*_{\max}(\mathcal{F}\mathcal{C}X).$$

- Since $\mathcal{T}_\lambda(X)$ admits a normal coaction we obtain

$$\mathcal{T}_\lambda(X) \simeq \mathrm{C}^*_\lambda(\mathcal{F}\mathcal{C}X).$$

- A representation of X that promotes to a representation of $\mathcal{F}\mathcal{C}X$ will be called a *Fock covariant representation of* X.

Fock covariant representations

Theorem (Kakariadis-P. 2024)

An equivariant injective representation t of X is Fock covariant if and only if t satisfies the following conditions:

1. $\mathbf{K}_{\emptyset, t_*} = (0)$.
2. *For any $\cap$-closed $\mathcal{F} = \{\mathbf{x}_1, \dots, \mathbf{x}_n\} \subseteq \mathcal{J}$ such that $\bigcup_{i=1}^n \mathbf{x}_i \neq \emptyset$, and any $b_{\mathbf{x}_i} \in \mathbf{K}_{\mathbf{x}_i, \hat{t}_*}$, with $i = 1, \dots, n$, the following property holds:*

$$\text{if } \sum_{i : r \in \mathbf{x}_i} t_*(b_{\mathbf{x}_i}) t_r(X_r) = (0) \text{ for all } r \in \bigcup_{i=1}^n \mathbf{x}_i, \text{ then}$$

$$\sum_{i=1}^n t_*(b_{\mathbf{x}_i}) = 0.$$

Fock covariant representations

- P is said to be a *right LCM semigroup* if $\mathcal{J} = \{pP \mid p \in P\} \bigcup \{\emptyset\}$.
- A p.s. X over a right LCM semigroup P is *compactly aligned* if

$$\lambda^{(p)}(\mathcal{K}(X_p))\lambda^{(q)}(\mathcal{K}(X_q)) \subseteq \lambda^{(w)}(\mathcal{K}(X_w)) \text{ when } pP \cap qP = wP.$$

- We say that a representation t of X is *Nica covariant* if

$$t^{(p)}(\mathcal{K}(X_p))t^{(q)}(\mathcal{K}(X_q)) \subseteq \begin{cases} t^{(w)}(\mathcal{K}(X_w))) & \text{if } pP \cap qP = wP, \\ (0) & \text{if } pP \cap qP = \emptyset. \end{cases}$$

Using our characterisation we give an alternative proof of:

Proposition (Dor-On–Kakariadis–Katsoulis–Laca–Li 2020)

A representation of X is Fock covariant if and only if it is Nica covariant.

Fock covariant representations

- $P \subseteq G$ is a unital subsemigroup and $X_p = \mathbb{C}$ for each $p \in P$.
- $\alpha = (p_1, q_1, \ldots, p_n, q_n)$ is called *neutral* if $p_1^{-1} q_1 \cdots p_n^{-1} q_n = e$.
- For a unital (i.e. $t_e(1) = 1$) representation t of X set $w_p := t_p(1)$.
- For a neutral word $\alpha = (p_1, q_1, \ldots, p_n, q_n)$ we write

$$K(\alpha) := q_n^{-1} p_n \ldots q_1^{-1} p_1 P \text{ and } \dot{w}_\alpha := w_{p_1}^* w_{q_1} \cdots w_{p_n}^* w_{q_n}.$$

Using our characterisation we give an alternative proof of:

Proposition (Laca-Sehnem 2021)

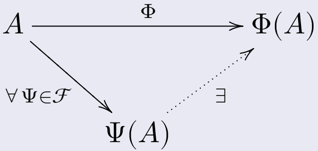
A unital (equivariant) representation t is Fock covariant if and only if

1. $w_e = 1$,
2. $\dot{w}_\alpha = 0$ *if* $K(\alpha) = \emptyset$ *for a neutral word* α, *and*
3. $\dot{w}_\alpha = \dot{w}_\beta$ *if* $K(\alpha) = K(\beta)$ *for neutral words* α *and* β,
4. $\prod_{\beta \in F} (\dot{w}_\alpha - \dot{w}_\beta) = 0$ *whenever* α *is a neutral word,* F *is a finite set of neutral words and* $K(\alpha) = \bigcup_{\beta \in F} K(\beta)$.

Fock covariant representations

Definition

Let $\mathcal{F}$ be a class of $*$-representations of a C*-algebra A. We say that a Φ in $\mathcal{F}$ is $\mathcal{F}$*-boundary* and denote it by $\partial\mathcal{F}$ if:



Example

Let A, B be C*-algebras and

$$\mathcal{F}^{\odot} := \{\Psi \colon A \otimes_{\max} B \to \mathcal{B}(H) \mid \Psi \text{ is injective on } A \odot B\}.$$

Then the $*$-homomorphism $A \otimes_{\max} B \to A \otimes B$ is $\mathcal{F}^{\odot}$-boundary.

Fock covariant representations

Example

Let δ be a coaction on a C*-algebra A and let $\mathcal{A} := \{\mathcal{A}_g\}_{g \in G}$ to be the induced Fell bundle. Let

$$\mathcal{F}_{\mathcal{A}^\delta} := \{ \text{ equivariant repn's of } \mathrm{C}^*_{\max}(\mathcal{A}) \text{ that are injective on } \mathcal{A}_e \}.$$

Then the canonical $*$-epimorphism $\mathrm{C}^*_{\max}(\mathcal{A}) \to \mathrm{C}^*_\lambda(\mathcal{A})$ is $\mathcal{F}_{\mathcal{A}^\delta}$-bdy.

Theorem (Hamana 1978, Dritschel-McCullough 2000)

Let $\mathfrak{A}$ be an operator algebra and let

$$\mathcal{F}_{\mathfrak{A}} := \{\Psi \colon \mathrm{C}^*_{\max}(\mathfrak{A}) \to \mathcal{B}(K) \mid \Psi|_{\mathfrak{A}} \text{ is c.is.}\}.$$

Then $\mathcal{F}_{\mathfrak{A}}$ admits an $\mathcal{F}_{\mathfrak{A}}$-boundary repn, called the C*-envelope of $\mathfrak{A}$, denoted by $\mathrm{C}^*_{\mathrm{env}}(\mathfrak{A})$.

Fock covariant representations

Let X be a product system over P. A key question in the theory of product systems was whether there exists a boundary representation for

$$\mathcal{F}^{\mathrm{F},G}_{\mathrm{c},X} := \{ \text{ equivariant Fock covariant injective representations of } X \},$$

and what is its relation to the C*-envelope of $\mathcal{T}_\lambda(X)^+$?

Fock covariant representations

Affirmative answers

- For $P = \mathbb{Z}_+$ Katsoulis-Kribs (2004) showed that $\mathrm{C}^*_{\text{env}}(\mathcal{T}_\lambda(X)^+) \simeq \mathcal{O}_X$ and Katsura (2007) showed that $\mathcal{O}_X \simeq \partial\mathcal{F}^{\mathrm{F},G}_{\mathrm{c},X}$.

- For (G, P) quasi-lattice and with assumptions on X Carlsen–Larsen–Sims–Vittadello (2011) showed that $\partial\mathcal{F}^{\mathrm{F},G}_{\mathrm{c},X} \simeq \mathcal{N}\mathcal{O}^r_X$.

- For $P = \mathbb{Z}^d_+$ and X arising from dynamics Davidson–Fuller–Kakariadis (2014) showed that $\partial\mathcal{F}^{\mathrm{F},G}_{\mathrm{c},X} \simeq \mathrm{C}^*_{\text{env}}(\mathcal{T}_\lambda(X)^+)$.

- For (G, P) an abelian lattice Dor-On–Katsoulis (2018) showed that $\partial\mathcal{F}^{\mathrm{F},G}_{\mathrm{c},X} \simeq \mathrm{C}^*_{\text{env}}(\mathcal{T}_\lambda(X)^+)$.

Fock covariant representations

Affirmative answers

- For P a right LCM Dor-On–Kakariadis–Katsoulis–Laca–Li (2020) showed that $\partial \mathcal{F}_{\mathrm{c},X}^{\mathrm{F},G} \simeq \mathrm{C}^*_{\mathrm{env}}(\mathcal{T}_\lambda(X)^+, G, \delta)$.

- For any P and X trivial Kakariadis–Katsoulis–Laca–Li (2021) showed that $\partial \mathcal{F}_{\mathrm{c},X}^{\mathrm{F},G} \simeq \mathrm{C}^*_{\mathrm{env}}(\mathcal{T}_\lambda(X)^+, G, \delta)$.

- For general product systems Sehnem (2021) showed that $\partial \mathcal{F}_{\mathrm{c},X}^{\mathrm{F},G} \simeq \mathrm{C}^*_{\mathrm{env}}(\mathcal{T}_\lambda(X)^+)$.

Fock covariant representations

Strong covariant relations (Sehnem 2018)

(1) Sehnem (2018) constructed a universal C*-algebra $A \times_X P$ of a product system X with the following property:

$X \hookrightarrow A \times_X P$ and if $t\colon A \times_X P \to \mathcal{B}(H)$ is a $*$-representation with $t|_A$ injective then t is injective on the fixed point algebra of $A \times_X P$.

(2) $A \times_X P$ admits a coaction by G and a Fell bundle, say $\mathcal{SC}_G X$. We then write

$$\mathrm{C}^*_{\max}(\mathcal{SC}_G X) = A \times_X P \text{ and } \mathrm{C}^*_\lambda(\mathcal{SC}_G X) = A \times_{X,\lambda} P.$$

Theorem (Sehnem 2021)

$$\partial\mathcal{F}^{\mathrm{F},G}_{\mathrm{c},X} \simeq \mathrm{C}^*_{\mathrm{env}}(\mathcal{T}_\lambda(X)^+) \simeq A \times_{X,\lambda} P. \qquad (1)$$

The reduced Hao–Ng isomorphism problem

- Let $\mathfrak{H}$ be a locally compact group.
- A *generalised gauge action* is an action of $\mathfrak{H}$ on $\mathcal{T}_\lambda(X)$ such that

$$\alpha_{\mathfrak{h}}(\lambda_p(X_p)) = \lambda_p(X_p) \text{ for all } p \in P \text{ and } \mathfrak{h} \in \mathfrak{H}.$$

- There is an induced product system $X \rtimes_{\alpha,\lambda} \mathfrak{H} := \{X_p \rtimes_{\alpha,\lambda} \mathfrak{H}\}_{p\in P}$.
- In the case where $\mathfrak{H}$ is discrete we have

$$X_p \rtimes_{\alpha,\lambda} \mathfrak{H} = \overline{\text{span}}\{\overline{\pi}(\lambda_p(\xi_p))U_{\mathfrak{h}} \mid \xi_p \in X_p, \mathfrak{h} \in \mathfrak{H}\},$$

where $\overline{\pi} \times U$ is a faithful representation of $\mathcal{T}_\lambda(X) \rtimes_{\alpha,\lambda} \mathfrak{H}$.

The reduced Hao-Ng isomorphism problem

Is α compatible with the reduced strong covariant functor, i.e., is there a canonical $*$-isomorphism

$$A \times_{X \rtimes_{\alpha,\lambda} \mathfrak{H}, \lambda} P \overset{?}{\simeq} (A \times_{X,\lambda} P) \rtimes_{\alpha,\lambda} \mathfrak{H}.$$

The reduced Hao–Ng isomorphism problem

Affirmative answers

- For $P = \mathbb{Z}_+$ and $\mathfrak{H}$ locally compact amenable (Hao-Ng 2008).

- For $P = \mathbb{Z}_+$ and $\mathfrak{H}$ discrete (Katsoulis 2017).

- For (G, P) is an abelian lattice and $\mathfrak{H}$ discrete (Dor-On–Katsoulis 2018).

- For (G, P) is an abelian lattice and $\mathfrak{H}$ locally compact abelian (Katsoulis 2020).

- For P a right LCM and $\mathfrak{H}$ discrete (Dor-On–Kakariadis–Katsoulis–Laca–Li 2020).

The reduced Hao–Ng isomorphism problem

Definition (Katsoulis-Ramsey 2019)

Let $\mathfrak{A}$ be an operator algebra and α an action of a locally compact group $\mathfrak{H}$ on $\mathfrak{A}$ by completely isometric isomorphisms. The *(nonselfadjoint) crossed product* $\mathfrak{A} \rtimes_{\alpha,\lambda} \mathfrak{H}$ is the norm-closed sublagebra generated by the copies of $\mathfrak{A}$ and $\mathfrak{H}$ inside $\mathrm{C}^*_{\text{env}}(\mathfrak{A}) \rtimes_{\dot{\alpha},\lambda} \mathfrak{H}$ where $\dot{\alpha}$ is the induced action on $\mathrm{C}^*_{\text{env}}(\mathfrak{A})$.

The next proposition, in the case $\mathfrak{A}$ admits a contractive approximate identity, was proved by Katsoulis (2017) when $\mathfrak{H}$ is discrete and by Katsoulis-Ramsey (2019) when $\mathfrak{H}$ is abelian. By using maximal representations we can remove the c.a.i. hypothesis, when $\mathfrak{H}$ is discrete.

Proposition

Let $\mathfrak{H}$ be a discrete group acting by α on an operator algebra $\mathfrak{A}$. Then

$$\mathrm{C}^*_{\text{env}}(\mathfrak{A} \rtimes_{\alpha,\lambda} \mathfrak{H}) \simeq \mathrm{C}^*_{\text{env}}(\mathfrak{A}) \rtimes_{\dot{\alpha},\lambda} \mathfrak{H}. \tag{2}$$

The reduced Hao–Ng isomorphism problem

Let $\mathfrak{H}$ be discrete. Using our characterisation of equivariant Fock covariant injective representations we obtain the following:

Proposition

The identity representation $\iota\colon X \rtimes_{\alpha,\lambda} \mathfrak{H} \to \mathcal{T}_\lambda(X) \rtimes_{\alpha,\lambda} \mathfrak{H}$ is a Fock covariant injective representation that admits a normal coaction.

We also need the following corollary of a result of Sehnem (2021):

Proposition

If t is a Fock covariant injective representation of X that admits a normal coaction, then the map

$$\mathcal{T}_\lambda(X)^+ \to \overline{\mathrm{alg}}\{t_p(X_p) \mid p \in P\}; \lambda_p(\xi_p) \mapsto t_p(\xi_p),$$

is a (well defined) completely isometric isomorphism.

The reduced Hao–Ng isomorphism problem

Combining the preceding propositions we obtain

$$\mathcal{T}_\lambda(X \rtimes_{\alpha,\lambda} \mathfrak{H})^+ \simeq \mathcal{T}_\lambda(X)^+ \rtimes_{\alpha,\lambda} \mathfrak{H},$$

then (2) implies that

$$\mathrm{C}^*_{\text{env}}(\mathcal{T}_\lambda(X \rtimes_{\alpha,\lambda} \mathfrak{H})^+) \simeq \mathrm{C}^*_{\text{env}}(\mathcal{T}_\lambda(X)^+) \rtimes_{\alpha,\lambda} \mathfrak{H},$$

and (1) finishes the proof.

Hence the reduced Hao–Ng isomorphism problem for generalised gauge actions by discrete groups has an affirmative answer:

Theorem (Kakariadis-P. 2024)

There exists a canonical $*$-isomorphism

$$(A \rtimes_{\alpha,\lambda} \mathfrak{H}) \times_{X \rtimes_{\alpha,\lambda} \mathfrak{H},\lambda} P \simeq (A \times_{X,\lambda} P) \rtimes_{\dot{\alpha},\lambda} \mathfrak{H}.$$

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Thank you!