C^*-algebras

M. Anoussis

November 2023

C^*-algebras

The space of all bounded linear operators $T : \mathcal{H} \to \mathcal{H}$ on a Hilbert space $\mathcal{H}$ is denoted $\mathcal{B}(\mathcal{H})$. It is complete under the norm

$$\|T\| = \sup\{\|Tx\| : x \in \mathrm{b}_1(\mathcal{H})\}$$

($\mathrm{b}_1(\mathcal{X})$ the closed unit ball of a normed space $\mathcal{X}$) and is an algebra under composition. Moreover, because it acts on a Hilbert space, it has additional structure: an *involution* $T \to T^*$ defined via

$$\langle T^*x, y\rangle = \langle x, Ty\rangle \quad \text{for all } x, y \in \mathcal{H}.$$

This satisfies

$$\|T^*T\| = \|T\|^2 \qquad \text{the } C^* \text{ property.}$$

C^*-algebras

These fundamental properties of $\mathcal{B}(\mathcal{H})$ (norm-completeness, involution, C^* property) motivate the definition of an abstract C*-algebra.

Definition

(a) A Banach algebra $\mathcal{A}$ is a complex algebra equipped with a complete norm which is sub-multiplicative:

$$\|ab\| \leq \|a\| \, \|b\| \qquad \text{for all } \; a, b \in \mathcal{A}.$$

(b) An involution is a map on $\mathcal{A}$ such that $(a + \lambda b)^* = a^* + \bar{\lambda} b^*$, $(ab)^* = b^* a^*$, $a^{**} = a$ for all $a, b \in \mathcal{A}$ and $\lambda \in \mathbb{C}$.

(c) A C^*-algebra $\mathcal{A}$ is a Banach algebra equipped with an involution $a \to a^*$ satisfying the C^*-condition

$$\|a^* a\| = \|a\|^2 \qquad \text{for all } \; a \in \mathcal{A}.$$

C^*-algebras

If $\mathcal{A}$ has a unit $\mathbf{1}$ then necessarily $\mathbf{1}^* = \mathbf{1}$ and $\|\mathbf{1}\| = 1$.

Definition

If $\mathcal{A}$ is a C*-algebra let

$$\mathcal{A}^{\sim} =: \mathcal{A} \oplus \mathbb{C}$$

$$(a, z)(b, w) =: (ab + wa + zb, zw)$$

$$(a, z)^* =: (a^*, \bar{z})$$

$$\|(a, z)\| =: \sup\{\|ab + zb\| : b \in \mathrm{b}_1 \mathcal{A}\}$$

Thus the norm of $\mathcal{A}^{\sim}$ is defined by identifying each $(a, z) \in \mathcal{A}^{\sim}$ with the operator $L_{(a,z)} : \mathcal{A} \to \mathcal{A} : b \to ab + zb$ acting on the Banach space $\mathcal{A}$.

C^*-algebras

$\mathbb{C}^2$ with norm

$$\|(x, y)\| = |x| + |y|$$

and pointwise multiplication is not a C^*-algebra.

$$a = (1, 1)$$

$$\|a^* a\| = \|(1, 1)(1, 1)\| = \|(1, 1)\| = 2$$

$$\|a\|^2 = \|(1, 1)\|^2 = 4$$

C^*-algebras

A morphism $\phi : \mathcal{A} \to \mathcal{B}$ between C*-algebras is a linear map that preserves products and the involution.

C^*-algebras

- $\mathbb{C}$, the set of complex numbers.
- $C(K)$, the set of all continuous functions $f : K \to \mathbb{C}$, where K is a compact Hausdorff space. With pointwise operations, $f^*(t) = \overline{f(t)}$ and the sup norm, $C(K)$ is an abelian, unital algebra.
- $C_0(X)$, where X is a locally compact Hausdorff space. This consists of all functions $f : X \to \mathbb{C}$ which are continuous and 'vanish at infinity': given $\varepsilon > 0$ there is a compact $K_{f,\varepsilon} \subseteq X$ such that $|f(x)| < \varepsilon$ for all $x \notin K_{f,\varepsilon}$. With the same operations and norm as above, this is an abelian C*-algebra.

C^*-algebras

- $M_n(\mathbb{C})$, the set of all $n \times n$ matrices with complex entries. With matrix operations, $A^* =$ conjugate transpose, and $\|A\| = \sup\{\|Ax\|_2 : x \in \ell^2(n), \|x\|_2 = 1\}$, this is a non-abelian, unital algebra.
- $\mathcal{B}(\mathcal{H})$ is a non-abelian, unital C*-algebra.
- $\mathcal{K}(\mathcal{H}) = \{A \in \mathcal{B}(\mathcal{H}) : \overline{A(\mathrm{b}_1(\mathcal{H}))}$ compact in $\mathcal{H}\}$: the compact operators. This is a closed selfadjoint subalgebra of $\mathcal{B}(\mathcal{H})$, hence a C*-algebra.

C^*-algebras

If X is an index set and $\mathcal{A}$ is a C*-algebra, the Banach space $\ell^\infty(X, \mathcal{A})$ of all bounded functions $a : X \to \mathcal{A}$ (with norm $\|a\|_\infty = \sup\{\|a(x)\|_\mathcal{A} : x \in X\}$) becomes a C*-algebra with pointwise product and involution.
Its subspace $c_0(X, \mathcal{A})$ consisting of all $a : X \to \mathcal{A}$ such that $\lim_{x \to \infty} \|a(x)\|_\mathcal{A} = 0$ is a C*-algebra. (for each $\varepsilon > 0$ there is a finite subset $X_\varepsilon \subseteq X$ s.t. $x \notin X_\varepsilon \Rightarrow \|a(x)\|_\mathcal{A} < \varepsilon$).

C^*-algebras

If $\mathcal{A}$ is a C*-algebra and $n \in \mathbb{N}$, the space $M_n(\mathcal{A})$ of all matrices $[a_{ij}]$ with entries $a_{ij} \in \mathcal{A}$ becomes a *-algebra with product $[a_{ij}][b_{ij}] = [c_{ij}]$ where $c_{ij} = \sum_k a_{ik} b_{kj}$ and involution $[a_{ij}]^* = [d_{ij}]$ where $d_{ij} = a_{ji}^*$.

Define a norm on $M_n(\mathcal{A})$ satisfying the C*-condition.

C^*-algebras

Suppose $\mathcal{A}$ is $\mathcal{B}(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. Identify $M_n(\mathcal{B}(\mathcal{H}))$ with $\mathcal{B}(\mathcal{H}^n)$: Given a matrix $[a_{ij}]$ of bounded operators a_{ij} on $\mathcal{H}$, we define an operator A on $\mathcal{H}^n$ by

$$A \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} = \begin{bmatrix} \sum_j a_{1j}\xi_j \\ \vdots \\ \sum_j a_{nj}\xi_j \end{bmatrix}$$

Conversely any $A \in \mathcal{B}(\mathcal{H}^n)$ defines an $n \times n$ matrix of operators a_{ij} on $\mathcal{H}$ by $\langle a_{ij}\xi, \eta \rangle_{\mathcal{H}} = \langle A\xi_j, \eta_i \rangle_{\mathcal{H}^n}$, where $\xi_j \in \mathcal{H}^n$ is the vector having ξ at the j-th entry and zeroes elsewhere (and η_i is defined analogously).

C^*-algebras

Hence one defines the norm $\|[a_{ij}]\|$ of $[a_{ij}] \in M_n(\mathcal{B}(\mathcal{H}))$ to be the norm $\|A\|$ of the corresponding operator on $\mathcal{H}^n$.

For $n = 2$:

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} \xi \\ \eta \end{bmatrix} = \begin{bmatrix} A\xi + B\eta \\ C\xi + D\eta \end{bmatrix}$$

This applies also if $\mathcal{A}$ is a C^*-subalgebra of $\mathcal{B}(\mathcal{H})$.

the spectrum

Definition

$\mathcal{A}$ unital C*-algebra and $GL(\mathcal{A})$ the group of invertible elements of $\mathcal{A}$. The spectrum of an element $a \in \mathcal{A}$ is

$$\sigma(a) = \sigma_{\mathcal{A}}(a) = \{\lambda \in \mathbb{C} : \lambda \mathbf{1} - a \notin GL(\mathcal{A})\}.$$

If $\mathcal{A}$ is non-unital, the spectrum of $a \in \mathcal{A}$ is defined by

$$\sigma(a) = \sigma_{\mathcal{A}^{\sim}}(a).$$

In this case, necessarily $0 \in \sigma(a)$.

the spectrum

Examples

- $\mathcal{A} = M_n(\mathbb{C})$ and $a \in \mathcal{A}$, then $\sigma(A)$ is the set of eigenvalues of A.
- $\mathcal{A} = C([0,1])$ and $f \in \mathcal{A}$, then:

$$f - \lambda \mathbf{1} \ \text{invertible} \Leftrightarrow f(x) - \lambda \mathbf{1}(x) \neq 0, \forall x$$

$$\Leftrightarrow f(x) - \lambda 1 \neq 0, \forall x \Leftrightarrow \lambda \neq f(x), \forall x.$$

$$\Rightarrow \sigma(f) = \{f(x) : x \in [0,1]\}$$

the spectrum

Proposition

The spectrum $\sigma(a)$ is a compact nonempty subset of $\mathbb{C}$.

(i) $\sigma(a)$ is bounded: In a unital C*-algebra, if $\|x\| < 1$ then since $\sum \|x^n\| \leq \sum \|x\|^n$, the series $\sum x^n$ converges absolutely, and so converges to an element y such that $(\mathbf{1} - x)y = y(\mathbf{1} - x) = \mathbf{1}$ and $(\mathbf{1} - x) \in GL(\mathcal{A})$.

If $a \in \mathcal{A}$ and $\lambda \in \mathbb{C}$ satisfies $|\lambda| > \|a\|$ then:

$$\|\frac{a}{\lambda}\| < 1 \Rightarrow \mathbf{1} - \frac{a}{\lambda} \text{ is invertible}$$

$$\Rightarrow \lambda\mathbf{1} - a \text{ is invertible} \Rightarrow \lambda \notin \sigma(a)$$

and the spectrum is bounded by $\|a\|$.

the spectrum

The spectral radius of $a \in \mathcal{A}$ is defined to be

$$\rho(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\}.$$

It satisfies $\rho(a) \leq \|a\|$, but equality may fail. In fact, it can be shown that

$$\rho(a) = \lim_n \|a^n\|^{1/n}$$

This is the Gelfand-Beurling formula.

the spectrum

Lemma

If $a = a^*$ *then* $\rho(a) = \sup\{|\lambda| : \lambda \in \sigma(a)\} = \|a\|$.

proof

$\|a\|^2 = \|a^2\|$ and inductively $\|a\|^{2^n} = \|a^{2^n}\|$ for all n. Thus, by the Gelfand - Beurling formula, $\rho(a) = \lim \left\|a^{2^n}\right\|^{2^{-n}} = \|a\|$. □

the spectrum

Theorem

A morphism $\pi : \mathcal{A} \to \mathcal{B}$ is contractive (i.e. $\|\pi(a)\| \leq \|a\|$ for all $a \in \mathcal{A}$).

proof WLOG we may assume that $\mathcal{A}$ and π are unital.
If $x, y \in \mathcal{A}$ and $xy = \mathbf{1}$ then $\pi(x)\pi(y) = \mathbf{1}$.

$a - \lambda\mathbf{1}$ invertible implies $\pi(a) - \lambda\mathbf{1}$ invertible and hence,
$\sigma(\pi(a) \subseteq \sigma(a) \Rightarrow \rho(\pi(a)) \leq \rho(a)$.

$$\|\pi(a)\|^2 = \|\pi(a)^*\pi(a)\|$$

$$= \|\pi(a^*a)\| = \rho(\pi(a^*a)) \leq \rho(a^*a) = \|a^*a\| = \|a\|^2$$

Gelfand theory for commutative C*-algebras

Theorem (Gelfand-Naimark 1)

Every commutative C-algebra $\mathcal{A}$ is isometrically *-isomorphic to $C_0(\hat{\mathcal{A}})$ where $\hat{\mathcal{A}}$ is the set of nonzero morphisms $\phi : \mathcal{A} \to \mathbb{C}$ which, equipped with the topology of pointwise convergence, is a locally compact Hausdorff space. For each $a \in \mathcal{A}$ the function $\hat{a} : \hat{\mathcal{A}} \to \mathbb{C} : \ \phi \to \phi(a)$ is in $C_0(\hat{\mathcal{A}})$. The Gelfand transform:*

$$\mathcal{A} \to C_0(\hat{\mathcal{A}}) : \ a \to \hat{a}$$

*is an isometric *-isomorphism. The space $\hat{\mathcal{A}}$ is compact if and only if $\mathcal{A}$ is unital.*

commutative C*-algebras

$\mathcal{A}$ unital.

- $\hat{\mathcal{A}}$ is the set of all nonzero multiplicative linear forms (characters) $\phi : \mathcal{A} \to \mathbb{C}$.
  $\phi(\mathbf{1})^2 = \phi(\mathbf{1}) \Rightarrow \phi(\mathbf{1}) = 1$ (for if $\phi(\mathbf{1}) = 0$ then $\phi(a) = \phi(a\mathbf{1}) = 0$ for all a, a contradiction).
  Each $\phi \in \hat{\mathcal{A}}$ satisfies $\|\phi\| \leq 1$ and $\|\phi\| = \phi(\mathbf{1}) = 1$. The topology on $\hat{\mathcal{A}}$ is pointwise convergence: $\phi_i \to \phi$ iff $\phi_i(a) \to \phi(a)$ for all $a \in \mathcal{A}$.

commutative C*-algebras

- The inequality $|\phi(a)| \leq ||a||$ shows that $\hat{\mathcal{A}}$ is contained in the space $\Pi_{a\in\mathcal{A}}\mathbb{D}_a$, the Cartesian product of the compact spaces $\mathbb{D}_a = \{z \in \mathbb{C} : |z| \leq ||a||\}$; and the product topology is the topology of pointwise convergence.

  $\hat{\mathcal{A}}$ is closed in this product: if $\phi_i \to \psi$ pointwise, then it is clear that ψ is linear and multiplicative, because each ϕ_i is linear and multiplicative, and $\psi \neq 0$ because $\psi(\mathbf{1}) = \lim_i \phi_i(\mathbf{1}) = 1$; thus $\psi \in \widehat{\mathcal{A}}$.

commutative C*-algebras

- The Gelfand map $\mathcal{G} : a \to \hat{a}$. For each $a \in \mathcal{A}$ the function

$$\hat{a} : \hat{\mathcal{A}} \to \mathbb{C} \quad \text{where} \quad \hat{a}(\phi) = \phi(a), \ (\phi \in \hat{\mathcal{A}})$$

is continuous by the definition of the topology on $\hat{\mathcal{A}}$. This gives a well defined map

$$\mathcal{G} : \mathcal{A} \to C(\hat{\mathcal{A}}) : a \to \hat{a} .$$

If $a, b \in \mathcal{A}$, since each $\phi \in \hat{\mathcal{A}}$ is linear, multiplicative and *-preserving, we have

$$\widehat{(a+b)}(\phi) = \phi(a+b) = \phi(a) + \phi(b) = \hat{a}(\phi) + \hat{b}(\phi)$$
$$\widehat{(ab)}(\phi) = \phi(ab) = \phi(a)\phi(b) = \hat{a}(\phi)\hat{b}(\phi)$$
$$\widehat{(a^*)}(\phi) = \phi(a^*) = \overline{\phi(a)} = \overline{\hat{a}(\phi)}$$

commutative C*-algebras

therefore

$$\mathcal{G}(a+b) = \mathcal{G}(a)+\mathcal{G}(b), \quad \mathcal{G}(ab) = \mathcal{G}(a)\mathcal{G}(b) \quad \text{and} \quad \mathcal{G}(a^*) = (\mathcal{G}(a))^*$$

$$\hat{a}(\phi) = \phi(a) \Rightarrow \|\hat{a}(\phi)\| \leq \|\phi\| \, \|a\| \Rightarrow \|\hat{a}\| \leq \|a\|$$

It can be seen that $\mathcal{G}$ is isometric.

commutative C*-algebras

- The Gelfand map is onto $C(\hat{\mathcal{A}})$. Consider the range $\mathcal{G}(\mathcal{A})$: it is a *-subalgebra of $C(\hat{\mathcal{A}})$, because $\mathcal{G}$ is a *-homomorphism. It contains the constants, because $\mathcal{G}(\mathbf{1}) = \mathbf{1}$. It separates the points of $\hat{\mathcal{A}}$, because if $\phi, \psi \in \hat{\mathcal{A}}$ are different, they must differ at some $a \in \mathcal{A}$, so

$$\mathcal{G}(a)(\phi) = \phi(a) \neq \psi(a) = \mathcal{G}(a)(\psi).$$

By the Stone -- Weierstrass Theorem, $\mathcal{G}(\mathcal{A})$ must be dense in $C(\hat{\mathcal{A}})$. But it is closed, since $\mathcal{A}$ is complete and $\mathcal{G}$ is isometric. Hence $\mathcal{G}(\mathcal{A}) = C(\hat{\mathcal{A}})$. □

commutative C*-algebras

When $\mathcal{A}$ is abelian but non-unital every $\phi \in \hat{\mathcal{A}}$ extends uniquely to a character $\phi^\sim \in \widehat{\mathcal{A}^\sim}$ by $\phi^\sim(\mathbf{1}) = 1$, and there is exactly one $\phi_\infty \in \widehat{\mathcal{A}^\sim}$ that vanishes on $\mathcal{A}$. Thus $\mathcal{A}$ is *-isomorphic the algebra of those continuous functions on the 'one-point compactification' $\hat{\mathcal{A}} \cup \{\phi_\infty\}$ of $\hat{\mathcal{A}}$ which vanish at ϕ_∞; this algebra is in fact isomorphic to $C_0(\hat{\mathcal{A}})$.

commutative C*-algebras

Example

c_0 the space of sequences converging to 0.

$\phi_n : c_0 \to \mathbb{C}$, $\phi_n((a_k)) = a_n$. Then $\hat{c_0} \simeq \mathbb{N}$.

(ϕ_n) converges pointwise to the zero character, since

$$\lim_n \phi_n((a_k)) = \lim_n a_n = 0.$$

Thus, $\hat{c_0}$ is not compact.

commutative C*-algebras

Example

Consider the unitization c of c_0 which is the space of convergent sequences.

Extend ϕ_n to c by the same formula $\phi_n^{\sim}((a_k)) = a_n$.

A new nonzero character appears: $\phi_\infty((a_k)) = \lim(a_k)$.
This is the pointwise limit of the $\phi_n^{\sim}$, since

$$\lim_n \phi_n^{\sim}((a_k)) = \lim_n(a_n) = \phi_\infty((a_n)).$$

$\hat{c}$ is the one point compactification of $\mathbb{N}$.

commutative C*-algebras

remark

When $\mathcal{A}$ is non-abelian there may be no characters. $M_2(\mathbb{C})$ has no ideals, hence the only character is the trivial one.

positivity

Definition

Let $\mathcal{A}$ be a C^*-algebra. An element $a \in \mathcal{A}$ is selfadjoint if $a = a^*$.

Definition

An element $a \in \mathcal{A}$ is positive (written $a \geq 0$) if $a = a^*$ and $\sigma(a) \subseteq \mathbb{R}_+$.
We write $\mathcal{A}_+ = \{a \in \mathcal{A} : a \geq 0\}$.

Definition

If a, b are selfadjoint, we define $a \leq b$ by $b - a \in \mathcal{A}_+$.

positivity

Theorem

Let $\mathcal{A}$ be a C^*-algebra and $a \in \mathcal{A}$. The following are equivalent:

- *a is positive*
- $a = b^2$ *for some positive* $b \in \mathcal{A}$
- $a = b^* b$ *for some* $b \in \mathcal{A}$
- *If* $\mathcal{A} \subseteq \mathcal{B}(\mathcal{H})$, $\langle ax, x \rangle \geq 0, \forall x \in \mathcal{H}$ *(complex* $\mathcal{H}$*)*

Examples

- In $C(X)$: $f \geq 0$ iff $f(t) \in \mathbb{R}_+$ for all $t \in X$ because $\sigma(f) = f(X)$.
- In $\mathcal{B}(\mathcal{H})$: $T \geq 0$ iff $\langle T\xi, \xi \rangle \geq 0$ for all $\xi \in \mathcal{H}$.
- In $M_2(\mathbb{C})$:

$$\begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix}^* \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 5 & -4 \\ -4 & 5 \end{bmatrix}$$

positivity

Proposition

Every positive element of a C-algebra has a unique positive square root. In fact*

$$a \in \mathcal{A}_+ \quad \text{if and only if there exists} \ \ b \in \mathcal{A}_+ \ \text{such that} \ \ a = b^2.$$

Proposition

For any C-algebra the set $\mathcal{A}_+$ is a cone:*

$$a, b \in \mathcal{A}_+, \ \lambda \geq 0 \quad \Longrightarrow \quad \lambda a \in \mathcal{A}_+, a + b \in \mathcal{A}_+.$$

positivity

remark

Any morphism $\pi : \mathcal{A} \to \mathcal{B}$ *between C*-algebras preserves order:*

$$a \geq 0 \quad \Rightarrow \quad \pi(a) \geq 0.$$

proof If $a = a^*$ and $\sigma(a) \subseteq [0, +\infty)$ then $\pi(a)^* = \pi(a^*)$ and

$$\sigma(\pi(a)) \subseteq \sigma(a) \cup \{0\} \subseteq [0, +\infty)$$

so $\pi(a) \geq 0$.

states

Definition

Let $\mathcal{A}$ be a C^*-algebra. A linear form on $\mathcal{A}$ is positive if $f(a^*a) \geq 0$ $\forall a \in \mathcal{A}$.

Lemma

Let f be a positive linear form on $\mathcal{A}$. Then

1. $f(b^*a) = \overline{f(a^*b)}$
2. $|f(b^*a)|^2 \leq f(a^*a)f(b^*b)$
3. $f(a^*) = \overline{f(a)}$
4. $|f(a)|^2 \leq f(e)f(a^*a)$

states

Lemma

Let f be a positive linear form on $\mathcal{A}$. Then f is bounded and $\|f\| = f(e)$.

Definition

Let $\mathcal{A}$ be a C^*-algebra. A state is a linear form on $\mathcal{A}$ which is positive and satisfies $f(e) = 1$.

states

The set of states $S(\mathcal{A})$ of a C^*-algebra $\mathcal{A}$ is a w^*-compact set of the dual of $\mathcal{A}$. It is convex, hence by the Krein-Milman theorem it has extreme points.

Definition

A state is pure if it is an extreme point of $S(\mathcal{A})$.

states

Examples

- $\mathcal{C}(X)$, for X compact. A state on $\mathcal{C}(X)$ is a probability measure. A pure state is a Dirac measure.
- $\mathcal{B}(\mathcal{H})$ for a Hilbert space $\mathcal{H}$. If $\xi \in \mathcal{H}$, $f(a) = \langle a\xi, \xi\rangle$ is a state. These are called vector states.

states

Examples

- Let $\mathcal{D}$ be the C^*-algebra of 2×2 diagonal complex matrices. A linear form on $\mathcal{D}$ is of the form

$$f\left(\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\right) = xa + yd$$

for some $x, y \in \mathbb{C}$.
f is a state if and only if
$x + y = 1$ and $xa + yd \geq 0$ when $a \geq 0$ and $d \geq 0$.
That is $x \geq 0$ and $y \geq 0$.
f is pure iff $x = 0$ or $y = 0$.

the GNS construction

Examples

• For a C*-algebra $\mathcal{A}$, if $\pi : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a representation and $\xi \in \mathcal{H}$ a unit vector, then $\phi(a) = \langle \pi(a)\xi, \xi \rangle$ is a state.

the GNS construction

Conversely,

Theorem (Gelfand, Naimark, Segal)

For every state f on a C-algebra $\mathcal{A}$ there is a triple $(\pi_f, \mathcal{H}_f, \xi_f)$ where π_f is a representation of $\mathcal{A}$ on $\mathcal{H}_f$ and $\xi_f \in \mathcal{H}_f$ a cyclic [a] unit vector such that*

$$f(a) = \langle \pi_f(a)\xi_f, \xi_f \rangle \quad \text{for all } a \in \mathcal{A}.$$

The GNS triple $(\pi_f, \mathcal{H}_f, \xi_f)$ is uniquely determined by this relation up to unitary equivalence.

[a] i.e. $\pi_f(\mathcal{A})\xi_f$ is dense in $\mathcal{H}_f$.

the universal representation

Theorem (Gelfand, Naimark)

For every C-algebra $\mathcal{A}$ there exists a representation $(\pi, \mathcal{H})$ which is one to one (called faithful).*

Idea of proof Enough to assume $\mathcal{A}$ unital. Let $\mathcal{S}(\mathcal{A})$ be the set of all states. For each $f \in \mathcal{S}(\mathcal{A})$ consider $(\pi_f, \mathcal{H}_f)$ and `add them up' to obtain $(\pi, \mathcal{H})$. Why is this faithful? Because

Lemma

For each nonzero $a \in \mathcal{A}$ there exists $f \in \mathcal{S}(\mathcal{A})$ such that $f(a^*a) > 0$.

... and then

$$\|\pi(a)\xi_f\|^2 = \langle \pi(a^*a)\xi_f, \xi_f \rangle = \langle \pi_f(a^*a)\xi_f, \xi_f \rangle = f(a^*a) > 0$$

so $\pi(a) \neq 0$.

von Neumann algebras

Definition

A von Neumann algebra is a unital wot closed $*$-subalgebra of $\mathcal{B}(\mathcal{H})$.

Theorem

$\mathcal{A}$ a unital $*$- subalgebra of $\mathcal{B}(\mathcal{H})$. The following are equivalent:

1. $\mathcal{A} = \mathcal{A}''$
2. *$\mathcal{A}$ is wot closed*

the center

$\mathcal{A}$ von Neumann algebra, p a central projection, $a \in \mathcal{A}$

$$a = (p + p^\perp)a(p + p^\perp) =$$

$$pap + pap^\perp + p^\perp ap + p^\perp ap^\perp = pap + p^\perp ap^\perp.$$

$$\mathcal{A} = p\mathcal{A}p \oplus p^\perp \mathcal{A} p^\perp,$$

$$\mathcal{A} = \left\{ \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} : x \in p\mathcal{A}p, y \in p^\perp \mathcal{A} p^\perp \right\}$$

Definition

A von Neumann algebra is a factor iff its center is $\mathbb{C}I$.

vN(G)

G locally compact topological group, λ the left regular representation of G

Definition

The von Neumann algebra of G is the wot closure of the linear span of the $\lambda(x), x \in G$.

Examples

- vN($\mathbb{R}$) is $L^\infty(\mathbb{R})$.
- vN($\mathbb{T}$) is $\ell^\infty(\mathbb{Z})$.
- vN($\mathbb{Z}$) is $L^\infty(\mathbb{T})$.

G discrete, λ the left regular representation of G,
$\{e_x : x \in G\}$ basis of $\ell^2(G)$.

$$\langle \lambda(x) e_y, e_z \rangle = \langle e_{xy}, e_z \rangle = \delta_{xy,z} 1$$

$$\langle \lambda(x) e_y, e_z \rangle$$

depends only on zy^{-1}.

Example

$G = \mathbb{Z}$

$$a = \begin{pmatrix} \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ \cdots & a_0 & a_1 & a_2 & a_3 & a_4 & \cdots \\ \cdots & a_{-1} & a_0 & a_1 & a_2 & a_3 & \cdots \\ \cdots & a_{-2} & a_{-1} & a_0 & a_1 & a_2 & \cdots \\ \cdots & a_{-3} & a_{-2} & a_{-1} & a_0 & a_1 & \cdots \\ \cdots & a_{-4} & a_{-3} & a_{-2} & a_{-1} & a_0 & \cdots \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{pmatrix}.$$

$a \in \mathrm{vN}(G)$.

$$ae_e = \sum_{x \in G} c_x e_x = \sum_{x \in G} c_x \lambda(x) e_e$$

$(c_x) \in \ell^2$.

the center

a in the center of vN(G),

$$\lambda(y)ae_e = \lambda(y)\left(\sum_{x\in G} c_x e_x\right) = \left(\sum_{x\in G} c_x e_{yx}\right) = \left(\sum_{x\in G} c_{y^{-1}xy} e_{xy}\right)$$

and

$$a\lambda(y)e_e = ae_y = a\rho(y^{-1})e_e = \rho(y^{-1})ae_e =$$
$$\rho(y^{-1})\left(\sum_{x\in G} c_x e_x\right) = \sum_{x\in G} c_x e_{xy}$$

Thus

$$c_x = c_{y^{-1}xy}$$

for all $y \in G$.

$x \mapsto c_x$ is constant on conjugacy classes and since $(c_x) \in \ell^2$ it is 0 on infinite conjugacy classes.

Theorem

If G is icc (that is, every conjugacy class except the class of e is infinite) then $\mathrm{vN}(G)$ factor.

Examples

- F_2
- $S_\infty = \{\phi : \mathbb{N} \to \mathbb{N},\ 1-1\ \} \ : \exists\, k_\phi,\ \phi(n) = n, \forall n \geq k_\phi\}$
- $\left\{ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} : a \in \mathbb{Q}^*, b \in \mathbb{Q} \right\}$

Theorem (Cuntz)

$K_0(C^*(F_n)) = \mathbb{Z}, \quad K_1(C^*(F_n)) = \mathbb{Z}^n.$
Hence $C^*(F_m)$ is not isomorphic to $C^*(F_n)$

Question

Is $\mathrm{vN}(F_2)$ *isomorphic to* $\mathrm{vN}(F_3)$*?*

$C^*(G)$

G locally compact topological group, $f, g \in L^1(G)$

$$f * g(x) = \int_{y \in G} f(xy^{-1}) g(y) d\mu(y)$$

$$f^*(x) = \overline{f(x^{-1})}$$

$L^1(G)$ with multiplication and involution as above is the group algebra of G.

$C^*(G)$

Proposition

$(\pi, \mathcal{H})$ unitary representation of G. Then
$f \mapsto \pi(f) = \int_G f(x)\pi(x)d\mu(x)$ satisfies

1. $\pi : L^1(G) \to \mathcal{B}(\mathcal{H})$ *is linear*
2. $\pi(f * g) = \pi(f)\pi(g)$
3. $\pi(f^*) = \pi(f)^*$
4. $\overline{\pi(L^1(G))\mathcal{H}} = \mathcal{H}$

Conversely, if
If $\phi : L^1(G) \to \mathcal{B}(\mathcal{H})$ satisfies the above statements, then there exists a unitary representation π of G st $\phi(f) = \pi(f)$

$C^*(G)$

Definition

Define a norm on $L^1(G)$:

$$\|f\| = sup_{\pi \in \hat{G}} \|\pi(f)\|.$$

$C^*(G)$ is the completion of $L^1(G)$ wrt to this norm and is a C^* algebra. It is called the C^*-algebra of G.

There is a 1-1 correspondence between unitary representation of G and non-degenerate $*$ representations of $C^*(G)$.

$C^*(G)$

Proposition

G abelian
$C^*(G)$ is $C_0(\hat{G})$.

Examples

- $C^*(\mathbb{R})$ is $C_0(\mathbb{R})$.
- $C^*(\mathbb{T})$ is $C_0(\mathbb{Z})$.
- $C^*(\mathbb{Z})$ is $C_0(\mathbb{T})$.
- G compact
  $C^*(G)$ is the c_0 sum $\sum_{\pi \in \hat{G}} \oplus \mathcal{B}(\mathcal{H}_\pi)$

primitive ideals

Definition

An ideal of a C^*-algebra $\mathcal{A}$ is primitive if it is the kernel of an irreducible representation of $\mathcal{A}$. $\mathrm{Prim}(\mathcal{A})$ is the set of the primitive ideals.

Consider the space $\mathrm{Prim}(\mathcal{A})$ with the hull-kernel topology:
If

$$U \subseteq \mathrm{Prim}(\mathcal{A})$$

then

$$\overline{U} = \{I \in \mathrm{Prim}(\mathcal{A}) : I \supseteq \cap_{J \in U} J\}.$$

The space $\mathrm{Prim}(\mathcal{A})$ is T_0. Denote $\mathrm{Prim}(G) = \mathrm{Prim}(C^*(G))$.
There is a map $\hat{G} \to \mathrm{Prim}(G)$, $\pi \mapsto \ker \pi$.

Definition

G is of type I if: whenever π is a representation of G such that $\pi(G)''$ is a factor, then $\pi(G)''$ is a factor of type I (that is π is a multiple of an irreducible representation).

Proposition

G second countable locally compact group. The following are equivalent

1. *G is type I.*
2. *The map $\hat{G} \to \mathrm{Prim}(G)$ is $1-1$.*
3. *If $(\pi, \mathcal{H}) \in \hat{G}$, then $\pi(C^*(G))$ contains the compact operators on $\mathcal{H}$.*

Proposition

1. *If G is abelian then it is of type I*
2. *If G is compact then it is of type I*
3. *If G is discrete then it is of type I iff it contains an abelian normal subgroup of finite index.*

Example

$G = \{< a, b >: a^2 = e, aba = b^{-1}\}$
G as a set is $\{b^n, b^n a : n \in \mathbb{Z}\}$

Examples

- The Heisenberg group H:
  $\lambda \in \mathbb{R}^*$, π_λ $L^2(\mathbb{R})$:

$$(\pi_\lambda(x, y, z)f)(t) = e^{i\lambda z} e^{-i\lambda yt} f(t - x)$$

  $(s, t) \in \mathbb{R}^2$, define $\pi_{s,t}$ acting on $\mathbb{C}$:

$$\pi_{s,t}(x, y, z) = e^{i(sx+ty)}$$

$$\hat{H} = \{\pi_\lambda : \lambda \in \mathbb{R}^*\} \cup \{\pi_{s,t} : (s, t) \in \mathbb{R}^2\}$$

Examples

- Discrete Heisenberg H_d:

$$u = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, v = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}, w = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

  π irreducible then $\pi(w) = e^{2\pi i\theta}$ for some θ.
  If θ irrational $\pi(u), \pi(v)$ generate the irrational rotation algebra A_θ.
  If ρ_1, ρ_2 are two inequivalent representations of A_θ, $\rho_1\pi$, $\rho_2\pi$ are inequivalent representations of $C^*(G)$ with the same kernel

$$\ker \rho_1\pi = \ker \rho_2\pi = \ker \pi.$$