Αρμονική Ανάλυση και Άλγεβρες Τελεστών: μια εισαγωγή

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Σεμινάριο: Συναρτησιακή Ανάλυση και Άλγεβρες Τελεστών
Απρίλιος 2024

The Wiener or Fourier algebra of the circle group $\mathbb{T}$

This is the set $A(\mathbb{T})$ of (autom. continuous) functions $f : \mathbb{T} \to \mathbb{C}$ whose Fourier series $\sum \hat{f}(k)e^{ikt}$ converges absolutely (to f, of course).

• It is an algebra under pointwise multiplication.
Qu: If $f \in A(\mathbb{T})$ never vanishes, is $1/f$ in $A(\mathbb{T})$?

• $A(\mathbb{T})$ is a Banach algebra with the norm

$$\|f\|_A = \sum |\hat{f}(k)| = \|\hat{f}\|_{\ell^1}$$

•... and its character space is (homeo to) the group $\mathbb{T}$. So, YES!

The measure algebra

Locally compact (Hausdorff) group: $G \times G \to G : (s,t) \mapsto st^{-1}$ continuous.

Recall $M(G) := C_0(G)^* : \langle \mu, \phi \rangle := \int_G \phi d\mu$

$G \ni t \mapsto \delta_t \in M^+(G)$ w*-continuous.

$\mathrm{span}\{\delta_t : t \in G\}$ w*-dense in $M(G)$.

Multiplication $(s,t) \mapsto st$ extends to $M(G)$:

$$\langle \mu * \nu, \phi \rangle := \iint \phi(st) d\mu(s) d\nu(t), \quad \phi \in C_0(G).$$

Inversion extends to involution: $\langle \mu^*, \phi \rangle := \int \phi(s^{-1}) d\bar{\mu}(s)$.

$M(G)$ is a Banach *-algebra (abelian, iff G is) and $G \hookrightarrow \mathrm{Inv} M(G)$ as a topological group.

Haar measure

$G \curvearrowright M(G)$ by left translation: $\mu \mapsto \delta_t * \mu$

Theorem (Haar measure)

There is $m \in M^+(G)$ which is a fixed point under this action: for all $t \in G$, $\delta_t * m = m$ (equivalently, $m(tE) = m(E)$ for all Borel $E \subseteq G$).
This m is unique up to a multiplicative constant.

Examples

- For $(\mathbb{R}, +)$: Lebesgue measure.
- For $(\mathbb{R} \backslash \{0\}, \cdot)$: $dm(x) = \frac{dx}{|x|}$.
- For $GL(n, \mathbb{R})$: $dm(T) = \frac{dT}{|\det T|^n}$ ($dT =$ Lebesgue measure on $M_n(\mathbb{R})$).
- For the "$ax + b$" group: $dm(a, b) = \frac{da db}{a^2}$ (but $dm_r(a, b) = \frac{da db}{a}$).

Here for $a > 0$ and $b \in R$, (a, b) is the map $\mathbb{R} \to \mathbb{R} : x \mapsto ax + b$ or the matrix $\left(\begin{smallmatrix} a & b \\ 0 & 1 \end{smallmatrix}\right)$.

$L^p(G)$

For $1 \leq p \leq \infty$, write $L^p(G) := L^p(G, m)$ and $\int_G f(s)ds := \int_G f dm$.

Identify $L^1(G)$ with $\{\nu \in M(G) : |\nu| \ll m\}$, so $d\nu(s) = f(s)dm(s)$ with $f \in L^1(G)$.

Then $L^1(G)$ is a self-adjoint, 2-sided ideal of $M(G)$, proper iff G is not discrete.

$L^1(G)$ always has an approx. identity: $\{\psi_i\}$ i.e. $\|\psi_i * f - f\|_1 \to 0$ and $\|f * \psi_i - f\|_1 \to 0$ for all $f \in L^1(G)$.

Representations

Unitary representation of G

$$\pi : G \to \mathcal{U}(H_\pi) \qquad \text{group homomorphism}$$

such that $\quad s \mapsto \pi(s)\xi : G \to H_\pi$ continuous $\forall \xi \in H_\pi$.

Example (Left regular rep.)

$\lambda : G \to \mathcal{U}(L^2(G))$ where

$$(\lambda_t f)(s) := f(t^{-1}s), \quad f \in L^2(G)\,.$$

Proposition

*Every unitary rep. π of G defines a *-representation $\tilde{\pi}$ of $M(G)$ by*

$$\langle \tilde{\pi}(\mu)\xi, \eta \rangle := \int_G \langle \pi(s)\xi, \eta \rangle \, d\mu(s)$$

$(\xi, \eta \in H_\pi)$ whose restriction to $L^1(G)$ is non-degenerate.
Every non-degenerate-representation of $L^1(G)$ extends uniquely to a *-representation of $M(G)$, which restricts to a (cts) rep. of G.*

The Universal representation, $C^*(G)$ and $C_r^*(G)$

For $f \in L^1(G)$, define

$$\|f\|_* := \sup\{\|\pi(f)\| : \pi \text{ a *-rep. of } L^1(G)\}$$

This is a norm(!) on $L^1(G)$ satisfying the C* condition $\|f * f^*\|_* = \|f\|_*^2$. The completion of $(L^1(G), \|\cdot\|_*)$ is a C* algebra, $C^*(G)$. Every unitary rep. of G extends uniquely to a *-rep. of $C^*(G)$.

The left regular rep. λ defines a faithful (i.e. 1-1) *-rep. of $L^1(G) \curvearrowright L^2(G)$. The closure (in the norm of $\mathcal{B}(L^2(G))$) is the reduced C* algebra $C_r^*(G)$. It is a quotient of $C^*(G)$, isomorphic to it iff G is amenable.

The Fourier algebra of the group $\hat{\mathbb{R}}$

$$A(\hat{\mathbb{R}}) := \{\hat{f} : f \in L^1(\mathbb{R})\} \subseteq C_0(\hat{\mathbb{R}}), \quad \|\hat{f}\|_A := \|f\|_{L^1(\mathbb{R})} .$$

To express $A(\hat{\mathbb{R}})$ in terms of $\hat{\mathbb{R}}$ only:
For $f \in L^1(\mathbb{R})$, can write $f = \xi\bar{\eta}$ with $\xi, \eta \in L^2(\mathbb{R})$ and

$$\hat{f}(s) = \int_{\mathbb{R}} (\xi\bar{\eta})(u) e^{-2\pi i s u} du = (\phi_s \xi, \eta)_{L^2(\mathbb{R})}$$

where $\phi_s(u) = e^{-2\pi i s u}$ $(u \in \mathbb{R})$, so after Fourier transform :

$$\hat{f}(s) = (\phi_s \xi, \eta)_{L^2(\mathbb{R})} = (\lambda_s \hat{\xi}, \hat{\eta})_{L^2(\hat{\mathbb{R}})}, \quad s \in \hat{\mathbb{R}} .$$

where λ_s is translation by s on $L^2(\hat{\mathbb{R}})$: $(\lambda_s g)(t) := g(t - s)$.

The Fourier algebra of a locally compact group G

The Fourier algebra $A(G)$ [Eym64] of a locally compact group G is the space of all functions $u : G \to \mathbb{C}$ of the form

$$u(s) = (\lambda_s \xi, \eta)$$

where λ is the left regular representation of G:

$$(\lambda_s \xi)(t) := \xi(s^{-1}t), \quad \xi \in L^2(G)$$

and ξ, η are in $L^2(G)$.

The Fourier-Stieltjes algebra of the group $\hat{\mathbb{R}}$

Let

$$B(\hat{\mathbb{R}}) := \{\hat{\mu} : \mu \in M(\mathbb{R})\} \subseteq C_b(\hat{\mathbb{R}}), \quad \|\hat{\mu}\|_B := \|\mu\|_{M(\mathbb{R})}$$

where $\hat{\mu}(s) = \int_{\mathbb{R}} e^{-2\pi i s u} d\mu(u)$.

To express $B(\hat{\mathbb{R}})$ in terms of $\hat{\mathbb{R}}$ only:

Recall Bochner's Theorem: If $u \in C(\hat{\mathbb{R}})$ is of positive type, i.e. the matrix $[u(s_i - s_j)] \succcurlyeq 0$ for all n and all $(s_i) \in \hat{\mathbb{R}}^n$, then (and only then) there exists $\mu \in M^+(\mathbb{R})$ such that $u = \hat{\mu}$.

Thus $B(\hat{\mathbb{R}}) = \text{span}\, P(\hat{\mathbb{R}})$ (=continuous functions of positive type).

The Fourier and Fourier-Stieltjes algebras

The Fourier-Stieltjes algebra $B(G)$ (Eymard, 1964 [Eym64]) of a locally compact group G is the set of all complex-linear combinations of continuous, functions $u : G \to \mathbb{C}$ *of positive type*: $B(G) = \text{span}\, P(G)$.
But note that each function $u \in P(G)$ defines, via GNS, a unitary cyclic representation (π, ξ, H) of G such that $u(s) = (\pi(s)\xi, \xi)$.
Hence equivalently
• $B(G)$ is the space of all functions $u : G \to \mathbb{C}$ of the form

$$u(x) = (\pi(x)\xi, \eta)$$

where π is a unitary representation of G and ξ, η are vectors in the space of the representation.
It is an algebra under pointwise multiplication and is a Banach algebra with norm

$$\|u\| = \inf\{\|\xi\|.\|\eta\| : u(\cdot) = (\pi(\cdot)\xi, \eta)\}\,.$$

The magic of the Fourier algebra

• The Fourier algebra $A(G)$ is a closed ideal of the Fourier-Stieltjes algebra $B(G)$. In fact $B(G) \cap C_c(G)$ is (densely) contained in $A(G)$.

• The Fourier algebra $A(G)$ is the predual of the von Neumann algebra $vN(G) := \text{w*span}\{\lambda_s : s \in G\} \subseteq \mathcal{B}(L^2(G))$ of G for the duality

$$\langle u, \lambda_s \rangle = u(s)$$

(Thus $u(s) = (\lambda_s \xi, \eta)$ uniquely defines the w*-cts linear form $T \mapsto (T\xi, \eta),\ T \in vN(G)$.)

• The spectrum (=max. ideal space) of the Banach algebra $A(G)$ is homeomorphic to G.

The Fourier and Fourier-Stieltjes algebras remember the group

Theorem (Walter, 1972 [Wal72])

Let G and H be locally compact groups. The following are equivalent:

(1) $B(G)$ *and* $B(H)$ *are isometrically isomorphic as Banach algebras,*

(2) $A(G)$ *and* $A(H)$ *are isometrically isomorphic as Banach algebras,*

(3) G *and* H *are isomorphic as topological groups.*

[Spr14]

References I

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