# Non-commutative notions of Entropy in Free Probability Theory and applications to Operator Algebras

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The field of *operator algebras* involves the study of subalgebras of $\mathcal{B}(H)$, which is the space of all bounded, linear operators on a (separable) Hilbert space $H$.

The main analytical objects of study are:

- **C*- algebras**, which are norm-closed $*$-subalgebras of $\mathcal{B}(H)$.
  In the *abelian* case : $C(X)$ with $X$ compact, Hausdorff.

- **von Neumann algebras**, which are $*$-subalgebras of $\mathcal{B}(H)$ closed in the topology of pointwise convergence.
  In the *abelian* case: $L_\infty(X, \mu)$ with $\mu$ a positive measure.

Let $G$ be a discrete group. If $(\delta_g)_{g \in G}$ is the canonical orthonormal basis of the Hilbert space $\ell_2(G)$ and $g \in G$, we define

$$\begin{aligned} \lambda(g) : \ell_2(G) &\rightarrow \ell_2(G) \\ \delta_h &\mapsto \delta_{gh} \end{aligned}$$

for all $h \in G$.
Then, the induced map

$$\begin{aligned} \lambda : G &\rightarrow \mathcal{B}(\ell_2(G)) \\ g &\mapsto \lambda(g) \end{aligned}$$

is an injective unitary representation of $G$ called the *left regular representation*. We similarly define the *right regular representation* $\rho : G \rightarrow \mathcal{B}(\ell_2(G))$.

The algebra $\mathrm{C}^*_{\text{red}}(G) = \overline{\text{span}\{\lambda(g) : g \in G\}}^{\|\cdot\|}$ is called the **reduced group $\mathrm{C}^*$-algebra** of $G$.

Similarly, the algebra $L(G) = \overline{\text{span}\{\lambda(g) : g \in G\}}^{\text{s.o.t.}}$ is called the **group von Neumann algebra** of $G$.

On $L(G)$ define the linear functional $\tau : L(G) \to \mathbb{C}$ by:

$$\tau(T) = \langle T(\delta_e), \delta_e \rangle.$$

Then, the functional $\tau$ is :

1. *positive*, i.e. $\tau(T^*T) \geq 0$, for all $T \in L(G)$,
2. *faithful*, i.e. if $\tau(T^*T) = 0$, then $T = 0$,
3. *tracial*, i.e. $\tau(TS) = \tau(ST)$, for all $T, S \in L(G)$.

A von Neumann algebra $M$ is called a *factor* if its center is trivial, i.e.

$$\{T \in M : TS = ST \text{ for all } S \in M\} = \mathbb{C}\,\mathrm{Id}\,.$$

The algebra $L(G)$ is a factor if and only if the group $G$ is **i.c.c.** , which means that for all $g \neq e \in G$ , the set

$$\{hgh^{-1} : h \in G\}$$

is infinite.

If $G$ is an amenable, i.c.c. group, then $L(G) \cong \mathcal{R}$, where $\mathcal{R}$ is the hyperfinite $II_1$ factor (Connes).

As a result, we are interested in non-amenable, i.c.c. groups.

Specifically, what can we say about $G = \mathbb{F}_n$?

Free probability theory was initiated by Voiculescu in the 1980's as an extension of classical probability to the non-commutative context.

**Motivation:**

- Is is true that $\mathrm{C}^*_{\text{red}}(\mathbb{F}_n) \cong \mathrm{C}^*_{\text{red}}(\mathbb{F}_m)$ for $n \neq m$?
  **No!** (Pimsner - Voiculescu)

- Is is true that $L(\mathbb{F}_n) \cong L(\mathbb{F}_m)$ for $n \neq m$?
  This is known as the **Free Group Factor Isomorphism Problem**.
  Remains an **open** problem, but...

### Theorem

*Let $A$ be a $\mathrm{C}^*$-algebra generated by generators $a_1, \ldots, a_n$ and let $B$ be another $\mathrm{C}^*$-algebra generated by generators $b_1, \ldots, b_n$. We consider $A$ and $B$ equipped with faithful states*

$$\varphi : A \to \mathbb{C} \text{ and } \psi : B \to \mathbb{C} \ .$$

*If*

$$\varphi\left(a_{i(1)}^{\varepsilon(1)} \cdot a_{i(2)}^{\varepsilon(2)} \cdot \ldots \cdot a_{i(k)}^{\varepsilon(k)}\right) = \psi\left(b_{i(1)}^{\varepsilon(1)} \cdot b_{i(2)}^{\varepsilon(2)} \cdot \ldots \cdot b_{i(k)}^{\varepsilon(k)}\right)$$

*for all $k \in \mathbb{N}$, $1 \leq i(1), \ldots, i(k) \leq n$ and $\varepsilon(1), \ldots, \varepsilon(k) \in \{1, *\}$, then the mapping*

$$a_i \mapsto b_i$$

*extends to an isometric $*$-isomorphism between $A$ and $B$.*

As a result, if $A$ is equipped with a faithful state $\varphi$ , then the isomorphism class of $A$ depends only on the **non-commutative distribution** of its generators $a_1, \ldots, a_n$, i.e. the family of complex numbers given by

$$\varphi\left(a_{i(1)}^{\epsilon(1)} \cdot a_{i(2)}^{\epsilon(2)} \cdots a_{i(k)}^{\epsilon(k)}\right)$$

where $1 \leq i(1), i(2), \ldots, i(k) \leq n$ and $\epsilon(1), \ldots, \epsilon(k) \in \{1, *\}$.

Definition

A **non-commutative probability space** consists of a pair $(A, \varphi)$, where $A$ is a unital $*$-algebra and $\varphi : A \to \mathbb{C}$ is a state, i.e. $\varphi$ is unital, linear and positive.

We are interested in the *non-commutative distributions* of elements of $A$. For $a_1, \ldots, a_n \in A$, these are given by

$$\{\varphi(c_1 \cdots c_k) : k \geq 1, c_i \in \{a_1, \ldots, a_n, a_1^*, \ldots, a_n^*\} \text{ for all } i = 1, \ldots, k\}.$$

## Examples

1. If $(X, \Sigma, \mu)$ is a (classical) probability space, then the pair $(A, \varphi)$ where
$$A = L_\infty(X, \Sigma, \mu)$$
and
$$\varphi : A \to \mathbb{C}, \ \ \varphi(f) = \int_X f d\mu$$
is a non-commutative probability space.
2. $(M_n(\mathbb{C}), tr)$, where $tr$ is the canonical normalized trace.
3. If $G$ is a group, then the algebras
$$\mathrm{C}^*_{\mathrm{red}}(G) \text{ and } L(G)$$
are non-commutative probability spaces when equipped with the canonical faithful tracial state $\tau$.

**Definition**

A family $(A_i)_{i\in I}$ of unital subalgebras of $(A, \varphi)$ is called **freely independent** if

$$\varphi(a_1 \cdots a_k) = 0$$

whenever

1. $a_j \in A_{i_j}$, for all $j = 1 \dots, k$,
2. $\varphi(a_j) = 0$, for all $j = 1, \dots, k$,
3. $i_1 \neq i_2, \dots, i_{k-1} \neq i_k$.

Operators $(a_i)_{i\in I}$ in $A$ are called freely independent if the unital algebras they generate are freely independent.

### Example

If $\{G_i\}_{i\in I}$ is a family of subgroups of a group $G$, then $\{G_i\}_{i\in I}$ is free in $G$ if and only if the family of algebras $\{L(G_i)\}_{i\in I}$ are freely independent in the non-commutative probability space $(L(G),\tau)$

### Theorem

*If the family of algebras $(A_i)_{i\in I}$ is freely independent and generates $A$, then the state $\varphi$ is uniquely determined by the restrictions $\varphi|_{A_i}$, $i\in I$.*

**Corollary**

*Let $M$ be a von Neumann algebra and $\varphi$ a faithful state on $M$. If $M$ is generated by self-adjoint operators $x_1, \dots, x_n$ that are freely independent and whose distributions are non-atomic, then $M \cong L(\mathbb{F}_n)$.*

Free probability theory replaces the tensor product of algebras

$$A \otimes B$$

as a model of independence by the free product of algebras

$$A * B.$$

The idea of Voiculescu is that the study of the free product of algebras should be done in analogy to classical independence and classical probability theory.

Free independence can be characterized by the **free cumulants**, which are given by a sequence of multilinear maps

$$\kappa_n : A^n \to \mathbb{C}$$

defined by

$$\kappa_n(a_1, \dots, a_n) = \sum_{\pi \in NC(n)} \varphi_\pi(a_1, \dots, a_n) \cdot \mu(\pi, 1_n),$$

by the following result:

**Theorem (Speicher)**

*The following are equivalent:*

1. *the algebras $(A_i)_{i \in I}$ are freely independent in $(A, \varphi)$,*
2. *mixed free cumulants vanish, i.e.*

$$\kappa_n(a_1, \dots, a_n) = 0,$$

*whenever there exist at least two entries coming from different algebras.*

There are certain operators that play a prominent role in free probability:

1. the **Haar unitary operators**, which are given by unitary operators $u$ whose distribution is given by the Haar measure on the circle.
   The canonical generators of $\mathrm{C}^*_{\text{red}}(\mathbb{F}_n)$ and $L(\mathbb{F}_n)$ are Haar unitary operators, which are freely independent.
2. the **semicircular operators**, i.e. self-adjoint operators whose distribution has density given by $\frac{1}{2\pi}\sqrt{4-t^2}$ on the interval $[-2,2]$.
   These are the analog of the Gaussian distribution in free probability.

### Theorem (Free Central Limit Theorem)

*Let $(A, \varphi)$ be a non-commutative probability space and let $(a_n)_{n\in\mathbb{N}} \subseteq A$ be a sequence of self-adjoint, freely independent, identically distributed random variables. Assume that $(a_n)_{n\in\mathbb{N}}$ is standardized, in the sense that $\varphi(a_n) = 0$ and $\varphi(a_n^2) = 1$ for all $n$. Then*

$$\frac{a_1 + a_2 + \ldots + a_n}{\sqrt{n}} \xrightarrow{distr} s,$$

*where $s$ is a semicircular random variable.*

In a von Neumann algebra we can "deform" Haar unitaries into semicirculars.
In particular $L(\mathbb{F}_n)$ is generated by $n$ freely independent semicircular random variables.

Voiculescu noted the following central connection between random matrix theory and non-commutative distibutions of operators:

For each $n \in \mathbb{N}$ let $(X_i^{(n)})_{i=1}^{k}$ be a family of $n \times n$ independent self-adjoint Gaussian random matrices. Then for all $1 \leq i_1, \ldots, i_m \leq k$ we have

$$\lim_{n\to\infty} \mathbb{E}\left[tr\left(X_{i_1}^{(n)} \cdots X_{i_m}^{(n)}\right)\right] = \varphi(x_{i_1} \cdots x_{i_m}),$$

where $\{x_1, \ldots, x_k\}$ is a family of freely independent semicirculars.
Using the matrix model, one obtains:

**Theorem (Dykema-Radulescu Dichotomy)**

*Exactly of the following holds:*

1. $L(\mathbb{F}_n) \cong L(\mathbb{F}_m)$ *for all* $n, m \geq 2$,
2. $L(\mathbb{F}_n) \ncong L(\mathbb{F}_m)$ *for all* $n \neq m$.

The asymptotic convergence actually holds in a much stronger sense:

**Theorem (Haagerup - Thorbjornsen)**

*Suppose that $x_1, \ldots, x_k$ are freely independent semicirculars in a non-commutative probability space $(A, \varphi)$. Then for any non-commutative polynomial $P$ we have*

$$\lim_{n\to\infty} \left\| P\left(X_1^{(n)}(\omega), \ldots, X_k^{(n)}(\omega)\right) \right\| = \|P(x_1, \ldots, x_k)\| \qquad a.e.$$

**Corollary**

$\mathrm{C}^*_{\text{red}}(\mathbb{F}_n)$ *is projectionless.*

**Corollary**

*There exist (many) embeddings of* $\mathrm{C}^*_{\text{red}}(\mathbb{F}_n)$ *into* $\prod M_n(\mathbb{C}) / \sum M_n(\mathbb{C})$

**Corollary**

$Ext\left(\mathrm{C}^*_{\text{red}}(\mathbb{F}_n)\right)$ *is not a group.*

If $H$ is a Hilbert space, define its **full Fock space**

$$\mathcal{F}(H) = \mathbb{C}\Omega \oplus \left[ \bigoplus_{n=1}^{\infty} H^{\otimes n} \right]$$

For every $\xi \in H$ we define the *left creation operator* $\ell(\xi) \in \mathcal{B}(\mathcal{F}(H))$ by

$$\ell(\xi)(\xi_1 \otimes \ldots \otimes \xi_n) = \xi \otimes \xi_1 \otimes \ldots \otimes \xi_n$$

**Theorem**

*Let $H$ be a Hilbert space and let $\xi_1, \ldots, \xi_n$ be pairwise orthogonal unit vectors in $H$. Then*

$$\ell(\xi_1) + \ell(\xi_1)^*, \ldots, \ell(\xi_n) + \ell(\xi_n)^*$$

*are freely independent semicirculars in $\mathcal{B}(\mathcal{F}(H))$.*

There are certain operators that play a prominent role in free probability:

1. the **Haar unitary operators**, which are given by unitary operators $u$ whose distribution is given by the Haar measure on the circle. The canonical generators of $\mathrm{C}^*_{\text{red}}(\mathbb{F}_n)$ and $L(\mathbb{F}_n)$ are Haar unitary operators, which are freely independent.
2. the **semicircular operators**, i.e. self-adjoint operators whose distribution has density given by $\frac{1}{2\pi}\sqrt{4-t^2}$ on the interval $[-2,2]$. These are the analog of the Gaussian distribution in free probability.
3. The **R-diagonal operators** are operators $a \in (A, \varphi)$ whose only non-vanishing free cumulants are of the form

$$\kappa_n(a, a^*, \ldots, a, a^*) \text{ or } \kappa_n(a^*, a, \ldots, a^*, a).$$

R-diagonal operators form a class of particularly well-behaved non-normal operators. In particular:

**Theorem (Nica, Speicher)**

*If $a, b \in (A, \varphi)$ are such that $a$ is R-diagonal and $a, b$ are free, then $a \cdot b$ is also R-diagonal.*

**Theorem (Larsen)**

*If $a$ is R-diagonal, then so is $a^n$ for all $n$.*

**Theorem (Nica, Shlyakhtenko, Speicher)**

*For $a \in (A, \varphi)$, the following are equivalent:*

1. *$a$ is R-diagonal,*
2. *the distribution of $a$ arises in the form $u \cdot p$, where $u$ is a Haar unitary and $u, p$ are free,*
3. *the matrix*
$$\begin{bmatrix} 0 & a \\ a^* & 0 \end{bmatrix} \in M_2(A)$$
*is free from $M_2(\mathbb{C})$ with amalgamation over the diagonal scalar matrices.*

In a series of groundbreaking papers, Voiculescu developed analogues of notions of entropy and Fisher information to the free probability setting, introducing the concepts of microstates and non-microstates free entropy.

Microstates free entropy measures the volumes of tuples of self-adjoint scalar matrices which approximate in moments the distribution of tuples of self-adjoint operators within tracial von Neumann algebras, motivated by the connection between free probability and random matrix theory.

**Theorem (Voiculescu)**

*$L(\mathbb{F}_n)$ does not have property $\Gamma$.*

**Theorem (Voiculescu)**

*$L(\mathbb{F}_n)$ does not contain Cartan subalgebras.*

**Theorem (Ge)**

*$L(\mathbb{F}_n)$ is prime.*

If $A$ is a $\mathrm{C}^*$-algebra or von Neumann algebra, then associated to a state $\varphi : A \to \mathbb{C}$ is the Hilbert space $L_2(A, \varphi)$ obtained by completing $A$ with respect to the inner product $< a, b >= \varphi(b^* a)$. $A$ acts on $L_2(A, \varphi)$ via left multiplication.

For the development of *non-microstates free entropy*, the notion of conjugate variables is central.

### Definition

Let $(M, \varphi)$ be a tracial von Neumann algebra and $x_1, \dots, x_n \in M$ for $i = 1, \dots, n$. A family of vectors $\xi_1, \dots, \xi_n$ in $L_2(M, \varphi)$ is a **conjugate system** for $x_1, \dots, x_n$ if

1. for all $i \leq n$ and $z_1, \dots, z_k \in \{x_1, \dots, x_n\}$ we have that
$$\varphi(z_1 \cdots z_k \cdot \xi_i) = \sum_{z_q = x_i} \varphi(z_1 \cdots z_{q-1}) \cdot \varphi(z_{q+1} \cdots z_k),$$
2. $\xi_1, \dots, \xi_n \in \overline{\mathrm{alg}(x_1, \dots, x_n)}^{||\cdot||_2} \subseteq L_2(M, \varphi)$.

The **free Fisher information** of $x_1, \ldots, x_n$ is defined as

$$\Phi^*(x_1, \ldots, x_n) = \begin{cases} \sum_{i=1}^n ||\xi_i||_2^2, & \text{if a conjugate system exists,} \\ +\infty, & \text{otherwise.} \end{cases}$$

One then defines the non-microstates free entropy as an integral of the Fisher information of the tuple $x_1, \ldots, x_n$.

If $x_1 = x_1^*, \dots, x_n = x_n^* \in (M, \varphi)$ are such that $\Phi^*(x_1, \dots, x_n) < +\infty$, then:

1. for $n \geq 2$, $W^*(x_1, \dots, x_n)$ does not have property $\Gamma$ and hence has trivial center (Dabrowski),
2. for every non-zero non-commutative polynomial $P$, there exists no non-zero self-adjoint element $w \in W^*(x_1, \dots, x_n)$ such that $P(x_1, \dots, x_n) \cdot w = 0$. In particular, the distribution of $P(x_1, \dots, x_n)$ is not atomic (Mai, Speicher, Weber).

**Theorem (Nica, Shlyakhtenko, Speicher)**

*The minimum*

$$\min\{\Phi^*(a, a^*) : a^*a \text{ has a prescribed distribution } \}$$

*is attained when $a$ is R-diagonal.*

*Bi-free probability theory* was introduced by Voiculescu in 2013 as an extension of the free setting and involves the simultaneous study of left and right actions of algebras on reduced free product spaces.
The corresponding notion of *bi-freeness* was combinatorially characterised by Charlesworth, Nelson and Skoufranis with the use of the *bi-free cumulants*

$$\kappa_\chi : A^n \to \mathbb{C}.$$

The underlying combinatorial objects are the lattices of *bi-non-crossing partitions*, which are obtained by applying permutations on non-crossing partitions that are naturally derived by keeping track of the positions where the left and right operators appear.

The theory of R-diagonal operators was recently developed in the bi-free setting, based on the $*$-alternating condition on bi-free cumulants.

### Definition

Let $(A, \varphi)$ be a non-commutative probability space and $x, y \in A$. We say that the pair $(x, y)$ is *bi-R-diagonal* if for every $n \in \mathbb{N}$, $\chi \in \{l, r\}^n$ and $a_1, \ldots, a_n \in A$ such that

$$a_i \in \begin{cases} \{x, x^*\}, & \text{if } \chi(i) = l \\ \{y, y^*\}, & \text{if } \chi(i) = r \end{cases} \qquad (i = 1, \ldots, n)$$

we have that

$$\kappa_\chi(a_1, \ldots, a_n) = 0$$

unless the sequence $(a_{s_\chi(1)}, \ldots, a_{s_\chi(n)})$ is of even length and alternating in $*$-terms and non-$*$-terms.

**Theorem (K.)**

1. *Let $(x, y)$ and $(z, w)$ be two bi-free pairs in $(A, \varphi)$ such that $(x, y)$ is bi-R-diagonal. Then, the pair $(xz, wy)$ is also bi-R-diagonal.*
2. *If $(x, y)$ is bi-R-diagonal, then so is $(x^n, y^n)$ for all $n$.*
3. *For $x, y \in (A, \varphi)$, the following are equivalent:*
   - A. *the pair $(x, y)$ is bi-R-diagonal,*
   - B. *the joint distribution of the pair $(x, y)$ arises in the form $(u_l z, wu_r)$, where $(u_l, u_r)$ is a bi-Haar unitary that is bi-free from $(z, w)$.*

## Definition (K., Skoufranis)

Given a unital $*$-algebra $B$, an *analytical $B$-$B$-non-commutative probability space* consists of a tuple $(A, L, R, E, \tau)$, where $A$ is a unital $*$-algebra, $\tau : A \to \mathbb{C}$ is a state and

1. the maps $L : B \to A$ and $R : B^{op} \to A$ are injective unital $*$-homomorphisms with commuting images,
2. $E : A \to B$ is a unital linear map satisfying
$$E(L_{b_1} R_{b_2} a) = b_1 E(a) b_2, \text{ for all } b_1, b_2, a \in A,$$
and is also $\tau$-preserving in the sense that
$$\tau(a) = \tau(L_{E(a)}) = \tau(R_{E(a)}) \text{ for all } a \in A,$$
3. the canonical state $\tau_B : B \to \mathbb{C}$ given by $\tau_B(b) = \tau(L_b)$ is tracial,
4. left multiplication of $A$ on $A/N_\tau$ yields bounded linear operators which extend to $L_2(A, \tau)$.

We have the identifications

$$L_2(B,\tau_B) \cong \overline{\{L_b : b \in B\}}^{||\cdot||_\tau} \cong \overline{\{R_b : b \in B\}}^{||\cdot||_\tau} \subseteq L_2(A,\tau).$$

If $\tilde{E} : L_2(A,\tau) \to L_2(B,\tau_B)$ denotes the orthogonal projection, then we have that

$$\tilde{E}(a) = E(a),$$

for all $a \in A$.

This allows us to develop notions of $L_2$-valued moment and cumulant functions by extending the projection map to the lattice of bi-non-crossing partitions.

## Example

Let $(M, \tau)$ be a tracial von Neumann algebra, let $B$ be a unital von Neumann subalgebra of $M$ and let $A$ be the algebra generated by the left and right actions of $M$ on $L_2(M, \tau)$. If

$$P : L_2(M, \tau) \to L_2(B, \tau)$$

denotes the orthogonal projection, $E : A \to B$ is defined as

$$E(T) = P(T1_M)$$

and $\tau : A \to \mathbb{C}$ is given by

$$\tau(T) =< T1_M, 1_M >_{L_2(M,\tau)},$$

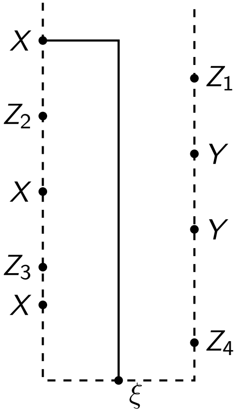
then the tuple $(A, L, R, E, \tau)$ is an analytical $B$-$B$-non-commutative probability space.

### Definition (K., Skoufranis)

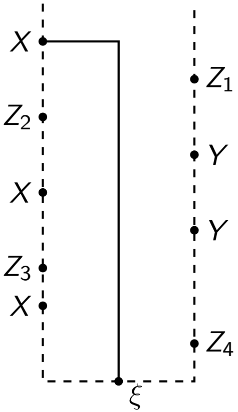
In an analytical $B$-$B$-non-commutative probability space $(A, L, R, E, \tau)$, let $C_l, C_r$ be subalgebras containing the left and right images of $B$ respectively. If $X \in A$ and $\eta : B \to B$ is a completely positive map, an element $\xi \in \overline{\mathrm{alg}(X, C_l, C_r)}^{||\cdot||_2} \subseteq L_2(A, \tau)$ is said to be the **left bi-free conjugate variable of $X$ with respect to $\eta$ and $\tau$ in the presence of** $(C_l, C_r)$ if for all $Z_1, \ldots, Z_k \in \{X\} \cup C_l \cup C_r$ we have

$$\tau(Z_1 \cdots Z_n \cdot \xi) = \sum_{Z_q = X_i} \tau((Z_1, \ldots, Z_k)|_{V_q^c} \cdot L_{\eta \circ E((Z_1, \ldots, Z_k)|_{V_q})}),$$

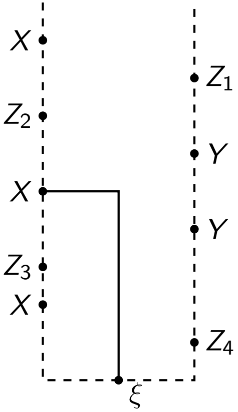
where $V_q = \{m > q : Z_m \text{ is a left operator}\}$.



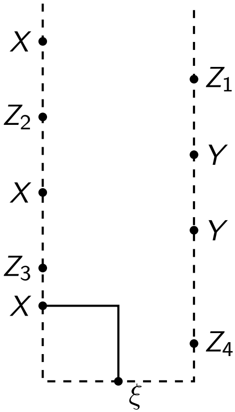

$$\tau(XZ_1Z_2YXYZ_3XZ_4 \cdot \xi) =$$




$$\tau(XZ_1Z_2YXYZ_3XZ_4 \cdot \xi) = \tau(Z_1YYZ_4 \cdot L_{\eta \circ E(Z_2XZ_3X)})$$




$$\tau(XZ_1Z_2YXYZ_3XZ_4 \cdot \xi) = \tau(Z_1YYZ_4 \cdot L_{\eta \circ E(Z_2XZ_3X)}) + \tau(XZ_1Z_2YYZ_4 \cdot L_{\eta \circ E(Z_3X)})$$




$$
\begin{aligned}
\tau(XZ_1Z_2YXYZ_3XZ_4 \cdot \xi) = & \tau(Z_1YYZ_4 \cdot L_{\eta\circ E(Z_2XZ_3X)}) \\
& + \tau(XZ_1Z_2YYZ_4 \cdot L_{\eta\circ E(Z_3X)}) \\
& + \tau(XZ_1Z_2YXYZ_3Z_4 \cdot L_{\eta\circ E(1)})
\end{aligned}
$$

**Theorem (K., Skoufranis)**

*The minimum*

$$\min\left\{\Phi^*(\{x,x^*\}\sqcup\{y,y^*\}) : \substack{\text{the joint distribution of} \\ (x,y)\text{is prescribed}}\right\}$$

*is attained whenever the pair $(x, y)$ is bi-R-diagonal and alternating adjoint flipping.*

**Thank you!**