Σεμινάριο Συναρτησιακής Ανάλυσης και Αλγεβρών Τελεστών: Matrix Convexity

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Matrix Convexity

Definition

A matrix convex set $\mathbf{K} = \{K_n\}_{n\in\mathbb{N}}$ in a vector space V is a collection of non-empty convex sets $K_n \subseteq M_n(V)$ such that:

- For $a \in M_{r,n}$ with $a^*a = 1$ we have $a^*K_ra \subseteq K_n$
- For $m, n \in \mathbb{N}$ we have

$$K_m \oplus K_n := \left\{ \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} : \text{ where } x \in K_n \text{ and } y \in K_m \right\} \subseteq K_{m+n}.$$

Proposition

A collection $\mathbf{K} = \{K_n\}_n$ *where* $K_n \subseteq M_n(V)$*, is a matrix convex set of* V *if and only if*

$$\sum_{i=1}^{k} \gamma_i^* u_i \gamma_i \in K_n,$$

for all $u_i \in K_{n_i}$ *and* $\gamma_i \in M_{n_i,n}$ *such that* $\sum_{i=1}^{k} \gamma_i^* \gamma_i = I_n$.
We call the element $\sum_{i=1}^{k} \gamma_i^* u_i \gamma_i$ *a matrix convex combination.*

Theorem

Let V, V' be in duality and let $\mathbf{K} = \{K_n\}_n$ be a closed matrix convex set of V with $0 \in K_1$. For any $u_0 \notin K_n$ there exists a weakly continuous $\phi : V \to M_n$ such that $\mathrm{Re}(\phi_r|K_r) \leq I_n \otimes I_r$ for all $r \in \mathbb{N}$ and $\mathrm{Re}(\phi(u_0)) \nleq I_n \otimes I_n$.

Definition

Let $B(\mathcal{H})$ denote the bounded operators of a Hilbert space $\mathcal{H}$.

- A closed linear subspace $V \subseteq B(\mathcal{H})$ will be called an operator space.
- An operator space $S \subseteq B(\mathcal{H})$ that is self-adjoint and contains the identity operator will be called a (unital) operator system.

- The space $M_n(V)$ inherits a norm from $M_n(B(\mathcal{H}))$ and $M_n(S)$ also inherits a positive cone of elements $C_n = \{x \in M_n(S) : x \in M_n(B(\mathcal{H}))^+\}$.

Definition

Let V, W be vector spaces and $\mathbf{K} = \{K_n\}_n$ be a matrix convex set. A matrix affine mapping on $\mathbf{K}$ is a sequence $\theta = \{\theta_n\}_n$ of mappings $\theta_n : K_n \to M_n(W)$ such that

$$\theta_n \left(\sum_{i=1}^{k} \gamma_i^* v_i \gamma_i \right) = \sum_{i=1}^{k} \gamma_i^* \theta_{n_i}(v_i) \gamma_i,$$

for all $v_i \in K_{n_i}$ and $\gamma_i \in M_{n_i,n}$ satisfying $\sum_{i=1}^{k} \gamma_i^* \gamma_i = I_n$.

Example

Let R be an operator system and consider the collection $\mathbf{CS}(R) = \{CS_n(R)\}_n$ where

$$CS_n(R) := \{\phi : R \to M_n : \phi \text{ is completely positive and unital}\}.$$

We can consider $CS_n(R)$ to be a subset of $M_n(R^d)$ via the identification

$$\phi = [\phi_{ij}],$$

where

$$\phi_{ij}(x) = e_i^* \phi(x) e_j, \text{ for } x \in R,$$

and e_j is the column matrix with 1 on the j-th coordinate and 0 elsewhere. Then $\mathbf{CS}(R)$ is a compact matrix convex set of R^d.

- Let $\mathbf{K} = \{K_n\}_n$ be a matrix convex set, we define $A(\mathbf{K}, M_r)$ to be the set of all matrix affine mappings

$$F = \{F_n\}_n \text{ where } F_n : K_n \to M_n(M_r),$$

such that F_1 is continuous.

- $A(\mathbf{K}, M_r)$ becomes a $*$-vector space if we define the $*$-operation $F^* = \{F_n^*\}_n$ where

$$F_n^*(v) = F_n(v)^*, \text{ for every } v \in K_n \text{ and } n \in \mathbb{N}.$$

- We say that $F \geq 0$ in $A(\mathbf{K}, M_r)$ if

$$F_n(v) \geq 0 \text{ for all } v \in K_n \text{ and } n \in \mathbb{N}.$$

- We define $E = \{E_n\}_n$ in $A(\mathbf{K}, \mathbb{C})$ where $E_n(v) = I_n \in M_n$ for every $v \in K_n$ and $n \in \mathbb{N}$.
- We identify $M_r(A(\mathbf{K}, \mathbb{C}))$ with $A(\mathbf{K}, M_r)$ where for $F = [F_{ij}] \in M_r(A(\mathbf{K}, \mathbb{C}))$ and $v \in K_n$ we have

$$F_n(v) = [(F_{ij})_n(v)].$$

- We use the ordering of $A(\mathbf{K}, M_r)$ to define a positive cone in $M_r(A(\mathbf{K}, \mathbb{C}))$.
- Then $A(\mathbf{K}, \mathbb{C})$ becomes an (abstract) operator system with E as an Archimedean matrix order unit, which we will simply denote by $A(\mathbf{K})$.

Compact Matrix Convex Sets

Theorem

Let R be an operator system, then there exists a unital complete order isomorphism $\psi : R \to A(\boldsymbol{CS}(R))$.

Theorem

Let $\mathbf{K}$ be a compact matrix convex set in a locally convex space V, then the spaces $\mathbf{K}$ and $\boldsymbol{CS}(A(\mathbf{K}))$ are matrix affinely homeomorphic.

Compact Matrix Convex Sets

Example

The matrix interval $[aI, bI] = \{[aI_n, bI_n]\}_n$ where for each n we have $[aI_n, bI_n] = \{x \in M_n : aI_n \leq x \leq bI_n\}$ is a compact matrix convex set of $\mathbb{C}$.

Proposition

Suppose that $\mathbf{K} = \{K_n\}_n$ *is a matrix convex set of* $\mathbb{C}$ *and* K_1 *is a compact subset of* $\mathbb{R}$. *Then*

$$\mathbf{K} = [aI, bI],$$

for some $a, b \in \mathbb{R}$.

Proof.

- Since K_1 is a non-empty, convex and compact subset of $\mathbb{R}$ it must be a closed interval of the form $[a, b]$ for some $a < b$ in $\mathbb{R}$.
- Suppose that $\gamma \in K_n$ and let ξ be a unit vector in $\mathbb{C}^n$ and consider it as a column matrix $\xi = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix}$.

- Then $\xi^*\xi = 1$ and

$$\langle \gamma\xi, \xi \rangle = \xi^*\gamma\xi \in K_1 = [a, b]$$

  and thus $aI_n \leq \gamma \leq bI_n$.
- Conversely, if $aI_n \leq \gamma \leq bI_n$ for some matrix γ in M_n, we may pick a unitary U and scalars $\lambda_i \in [a, b]$ such that

$$\gamma = U^*(\lambda_1 \oplus \cdots \oplus \lambda_n)U$$

  and therefore $\gamma \in K_n$.

Definition

Let K be a convex set in some vector space V. We say that a point $v \in K$ is extreme if whenever we write

$$v = \sum_{i=1}^{k} \lambda_i v_i,$$

where $v_i \in K$ and $0 < \lambda_i < 1$ for $i = 1, \ldots, n$ then $v_i = v$ for all $i = 1, \ldots, n$. We denote the set of extreme points of K by $\text{ext}(K)$.

- For a set $S \subseteq V$ we denote by $\text{co}(S)$ the smallest convex subset of V that contains S.

Theorem

Krein-Milman in Matrix Convexity Let K be a compact convex set in some locally convex space V. Then

$$\overline{\mathrm{co}}(\mathrm{ext}(K)) = K.$$

- In particular, $\mathrm{ext}(K)$ is non-empty for a non-empty compact convex set K in a locally convex space.

Krein-Milman in Matrix Convexity

Definition

Let K be a convex subset of a vector space V. We say that a convex set $F \subseteq K$ is a face of K if for all $x, y \in K$ and $0 < \lambda < 1$ whenever $\lambda x + (1-\lambda)y \in F$ then $x, y \in F$.

- If $x \in \text{ext}(K)$ for some convex set K, then $F = \{x\}$ is a face of K.
- If F is a face of a convex set K, then $\text{ext}(F) \subseteq \text{ext}(K)$.
- Suppose that K, C are convex sets and that $f : K \to C$ is an affine map. If F is a face of C then $f^{-1}(F)$ is a face of K.

Krein-Milman in Matrix Convexity

Definition

- Let $\mathbf{K}$ be a matrix convex set. We say that a matrix convex combination
$$v = \sum_{i=1}^{k} \gamma_i^* v_i \gamma_i,$$
where $v_i \in K_{n_i}$ and $\gamma_i \in M_{n_i,n}$ such that $\sum_{i=1}^{k} \gamma_i^* \gamma_i = I_n$, is proper if each γ_i has a right inverse in M_{n,n_i}.
- We say that $v \in K_n$ is matrix extreme point if whenever v is a proper matrix convex combination as above then each $n_i = n$ and $v = u_i^* v_i u_i$ for unitaries $u_i \in M_n$,

- We denote by ∂K_n the (possibly empty) set of matricial extreme points in K_n and set $\boldsymbol{\partial}\mathbf{K} = \{\partial K_n\}_n$.

- We observe that for $n = 1$ the matrix extreme points of K_1 are exactly the extreme points of K_1.
- Indeed, let $v \in K_1$ be a matrix extreme point and suppose that

$$v = \sum_{i=1}^{k} \lambda_i v_i,$$

for some $v_i \in V$ and $0 < \lambda_i < 1$.
- Set $\gamma_i = \sqrt{\lambda_i}$ and thus

$$v = \sum_{i=1}^{k} \overline{\gamma_i} v_i \gamma_i,$$

where $\sum_{i=1}^{k} \overline{\gamma_i} \gamma_i = 1$.

- Since v is matrix extreme, for each i there exists a $\mu_i \in \mathbb{C}$ such that $|\mu_i|^2 = 1$ and
$$\overline{\mu_i} v_i \mu_i = v \iff v_i = v.$$
- The converse is similar.
- If **K** is compact matrix convex set, we obtain by the Krein-Milman theorem that ∂K_1 is non-empty.
- This is not always the case for ∂K_n for $n > 1$.

Example

Let a, b be in $\mathbb{R}$ where $a < b$ and $[aI, bI] = \{[aI_n, bI_n]_n\}_n$. Then

$$\partial[aI_n, bI_n] = \begin{cases} \{a, b\} & \text{if } n = 1, \\ \emptyset & \text{if } n > 1. \end{cases}$$

Indeed,

$$\partial[a, b] = \text{ext}([a, b]) = \{a, b\}.$$

For $n > 1$ let v be in $[aI_n, bI_n]$ then

$$v = U^*(\lambda_1 \oplus \cdots \oplus \lambda_n)U$$

for some unitary $U \in M_n$ and $\lambda_i \in [a, b]$.

Example

We may write

$$U = \begin{bmatrix} \gamma_1 \\ \vdots \\ \gamma_n \end{bmatrix}, \text{ for some } \gamma_i \in M_{1,n}.$$

Since U is a unitary we obtain that

$$\sum_{i=1}^{n} \gamma_i^* \lambda_i \gamma_i = v$$

is a proper matrix convex combination and therefore v is not a matrix extreme point.

Krein-Milman in Matrix Convexity

Proposition

Let $\mathbf{K}$ *be a compact matrix convex set of a locally convex space* V. *If* v *is a matrix extreme point in* K_n, *then* v *is also an extreme point of* K_n.

Krein-Milman in Matrix Convexity

Definition

Let $S_n \subseteq M_n(V)$ for each $n \geq 1$ for some locally convex space V and $\mathbf{S} = \{S_n\}_n$. We define the closed matrix convex hull $\overline{\text{co}}(\mathbf{S})$ to be the smallest closed matrix convex set containing $\mathbf{S}$.

- If $\overline{\text{co}}(\mathbf{S}) = \{K_n\}_n$ then each K_n is the closure of the set of all elements $v \in M_n(V)$ of the form

$$v = \sum_{i=1}^{k} \gamma_i^* v_i \gamma_i,$$

where $v_i \in K_{n_i}$ and $\gamma_i \in M_{n_i,n}$ such that $\sum_{i=1}^{k} \gamma_i^* \gamma_i = I_n$.

Example

We already saw that that the matrix extreme points of the matrix convex set $[aI, bI] = \{[aI_n, bI_n]\}_n$ are

$$\partial[aI_n, bI_n] = \begin{cases} \{a, b\} & \text{if } n = 1, \\ \emptyset & \text{if } n > 1. \end{cases}$$

Since $\overline{\text{co}}(\partial[aI, bI])_1 = [a, b]$ and $\overline{\text{co}}(\partial[aI, bI])$ is a matrix convex set we obtain that $\overline{\text{co}}(\partial[aI, bI]) = [aI, bI]$.

- The above is just an example of the following theorem.

Theorem

Let $\mathbf{K}$ *be a compact matrix convex set of a locally convex space* V *and let* $\boldsymbol{\partial}\mathbf{K} = \{\partial K_n\}_n$. *Then* $\boldsymbol{\partial}\mathbf{K}$ *is non-empty and*

$$\overline{\mathrm{co}}(\boldsymbol{\partial}\mathbf{K}) = \mathbf{K}.$$

- The essential idea of the proof is to perform a reduction to the classical Krein-Milman theorem. In order to do so we have to introduce some convex sets related to $\mathbf{K}$.

Definition

Let $\mathbf{K}$ be compact matrix convex set of a locally convex space V, we define $\Delta_n(\mathbf{K})$ to be the subset of $M_n(V)$ such that

$$\Delta_n(\mathbf{K}) = \{\xi^* v \xi : v \in K_r, \xi \in M_{r,n}, \|\xi\|_2 = 1, r \in \mathbb{N}\},$$

where $\|.\|_2$ is the Hilbert-Schmidt norm.

- We may pick ξ to be right-invertible and also $r \leq n$. Indeed, let s be the dimension of the range of $\xi \in M_{r,n}$ and let $\alpha \in M_{r,s}$ be an isometry of $\mathbb{C}^s$ onto the range of ξ.
- Then, for $v \in K_r$ we have that

$$\xi^* v \xi = (\alpha^* \xi)^* (\alpha^* v \alpha)(\alpha^* \xi)$$

and $\alpha^* \xi \in M_{s,n}$ and also $\alpha^* \xi$ is right-invertible.

- Therefore,

$$\Delta_n(\mathbf{K}) = \{\xi^* v \xi : v \in K_r, \xi \in M_{r,n}, \|\xi\|_2 = 1, r \leq n\}$$

  and hence it follows that $\Delta_n(\mathbf{K})$ is compact as a finite union of compact sets.
- We prove now that $\Delta_n(\mathbf{K})$ is a convex set.

- Let $\xi^* v \xi$ and $\eta^* w \eta$ be in $\Delta_n(\mathbf{K})$ where $v \in K_r$ and $w \in K_s$ and $\xi \in M_{r,n}$ and $\eta \in M_{s,n}$ satisfying $\|\xi\|_2 = \|\eta\|_2 = 1$ and $0 \leq t \leq 1$.
- We have that

$$\begin{aligned} & t\xi^* v \xi + (1-t)\eta^* w \eta \\ & = \begin{bmatrix} t^{1/2}\xi^* & (1-t)^{1/2}\eta^* \end{bmatrix} \begin{bmatrix} v & 0 \\ 0 & w \end{bmatrix} \begin{bmatrix} t^{1/2}\xi \\ (1-t)^{1/2}\eta \end{bmatrix} \end{aligned}$$

where

$$\left\| \begin{bmatrix} t^{1/2}\xi \\ (1-t)^{1/2}\eta \end{bmatrix} \right\|_2^2 = t\|\xi\|_2^2 + (1-t)\|\eta\|_2^2 = 1.$$

Krein-Milman in Matrix Convexity

Lemma

Let R be an operator system and let $\bar{\varphi}$ be an extreme point of $\Delta_n(\mathbf{CS}(R))$, then there exists a matrix extreme point $\varphi \in CS_r(R)$ for some $r \in \mathbb{N}$ and a right-invertible element $\xi \in M_{r,n}$ with $\|\xi\|_2 = 1$ such that

$$\bar{\varphi} = \xi^* \varphi \xi.$$

Proof.

- Let $\bar{\varphi}$ be an extreme point of $\Delta_n(\mathbf{CS}(R))$, then there exist a right-invertible $\xi \in M_{r,n}$ with $\|\xi\|_2 = 1$ and $\varphi \in CS_r(R)$ for some $r \in \mathbb{N}$ such that

$$\bar{\varphi} = \xi^* \varphi \xi$$

- We will prove that φ is a matrix extreme point.

- Assume that φ is written as a proper matrix convex combination

$$\varphi = \sum_{i=1}^{k} \gamma_i^* \varphi_i \gamma_i,$$

where $\gamma_i \in M_{r_i,r}$ and $\varphi_i \in CS_{r_i}(R)$ for $i = 1, \dots, k$.

- Set $t_i = \|\gamma_i \xi\|_2^2$, then $t_i \neq 0$, since both γ_i and ξ have right-inverses and we have that

$$\bar{\varphi} = \xi^* \varphi \xi = \sum_{i=1}^{k} \xi^* \gamma_i^* \varphi_i \gamma_i \xi = \sum_{i=1}^{k} t_i \frac{(\gamma_i \xi)^*}{\|\gamma_i \xi\|_2} \varphi_i \frac{(\gamma_i \xi)}{\|\gamma_i \xi\|_2},$$

and also

$$\sum_{i=1}^{k} t_i = \sum_{i=1}^{k} \|\gamma_i \xi\|_2^2 = \sum_{i=1}^{k} \mathrm{Tr}(\xi^* \gamma_i^* \gamma_i \xi) = \mathrm{Tr}(\xi^* \xi) = \|\xi\|_2^2 = 1.$$

- Since $\bar{\varphi}$ is an extreme point we obtain that

$$\xi^*\varphi\xi = \|\gamma_i\xi\|_2^{-2}(\gamma_i\xi)^*\varphi_i(\gamma_i\xi)$$

and using the fact that ξ has a right-inverse we have

$$\varphi\|\gamma_i\xi\|_2^2 = \gamma_i^*\varphi_i\gamma_i.$$

- Note that φ and φ_i are unital and therefore

$$I_r\|\gamma_i\xi\|_2^2 = \gamma_i^*\gamma_i$$

and hence $\|\gamma_i\xi\|_2^{-1}\gamma_i$ is an isometry.

- Thus, γ_i is both injective and surjective for $i = 1, \dots, k$ which implies that $r = r_1 = \cdots = r_k$.
- We also have that

$$\varphi = \frac{\gamma_i^*}{\|\gamma_i \xi\|_2} \varphi_i \frac{\gamma_i}{\|\gamma_i \xi\|_2},$$

where $\|\gamma_i \xi\|_2^{-1} \gamma_i$ is a surjective isometry i.e. a unitary for $i = 1, \dots, k$.

- Hence φ is a matrix extreme point and the proof is complete.

Lemma

Let $\mathbf{K}$ *be compact matrix convex set of a locally convex space* V*. If* $\bar{v}$ *is an extreme point of* $\Delta_n(\mathbf{K})$*, then there exists a matrix extreme point* $v \in K_r$ *for some* $r \in \mathbb{N}$ *and a right-invertible element* $\xi \in M_{r,n}$ *with* $\|\xi\|_2 = 1$ *such that*

$$\bar{v} = \xi^* v \xi.$$

Proof.

- There exists an operator system R and a matrix affine homeomorphism $\theta = \{\theta_n\}_n$ of $\mathbf{CS}(R)$ onto $\mathbf{K}$.

- It suffices to prove that $\Gamma : \Delta_n(\mathbf{CS}(R)) \to \Delta_n(\mathbf{K})$ is well-defined and continuous affine surjection where

$$\Gamma(\xi^* \varphi \xi) = \xi^* \theta_r(\phi) \xi,$$

for $\varphi \in CS_r(R)$ and $\xi \in M_{r,n}$ satisfying $\|\xi\|_2 = 1$.

- Indeed, in that case if $\bar{v} \in \Delta_n(\mathbf{K})$ is an extreme point then $\Gamma^{-1}(\bar{v})$ is a compact face of $\Delta_n(\mathbf{CS}(R))$. By the Krein-Milman theorem, this set has an extreme point which is also an extreme point of $\Delta_n(\mathbf{CS}(R))$ and since θ preserves matrix extreme points, from the preceding lemma we are done.

- Note that if $\varphi \in CS_r(R)$ and $\xi \in M_{r,n}$ as before we may pick an isometry $\alpha \in M_{r,s}$ such that $s \leq n$ and $\alpha\alpha^*\xi = \xi$ and in that case we have that

$$\xi^*\theta_r(\varphi)\xi = \xi^*\alpha\alpha^*\theta_r(\varphi)\alpha\alpha^*\xi = \xi^*\alpha\theta_s(\alpha^*\varphi\alpha)\alpha^*\xi.$$

- Thus, in order to prove that Γ is well-defined we may pick

$$\xi^*\varphi\xi = \eta^*\psi\eta,$$

where $\varphi \in CS_r(R)$ and $\psi \in CS_t(R)$ and right-invertible elements $\xi \in M_{r,n}$ and $\eta \in M_{t,n}$ where $\|\xi\|_2 = \|\eta\|_2 = 1$ and show that

$$\Gamma(\xi^*\varphi\xi) = \Gamma(\eta^*\psi\eta).$$

- Thus,
$$\varphi = (\eta\xi^{-1})^*\psi(\eta\xi^{-1})$$
and since φ and ψ are unital $\eta\xi^{-1}$ is an isometry.
- We have that
$$\theta_r(\varphi) = \theta_r((\eta\xi^{-1})^*\psi(\eta\xi^{-1})) = (\eta\xi^{-1})^*\theta_t(\psi)(\eta\xi^{-1}).$$
- Since $\xi^*\xi = \eta^*\eta$ and η^* is left-invertible we have that
$$\xi^*\xi\xi^{-1}\xi = \xi^*\xi \iff \eta^*\eta\xi^{-1}\xi = \eta^*\eta$$
$$\iff \eta\xi^{-1}\xi = \eta$$
- Therefore,
$$\xi^*\theta_r(\varphi)\xi = \eta^*\theta_t(\psi)\eta,$$
as desired.

- The fact that Γ is affine and surjective is immediate.
- To see that Γ is continuous consider a convergent net

$$\xi_\lambda^* \varphi_\lambda \xi_\lambda \to \xi^* \varphi \xi$$

  in $\Delta_n(\mathbf{CS}(R))$, where $\varphi_\lambda \in CS_{r_\lambda}(R)$ and $\varphi \in CS_r(R)$ and $\xi_\lambda \in M_{r_\lambda,n}$ and $\xi \in M_{r.n}$ are right-invertible.
- Set $\eta_\lambda := \xi_\lambda \xi^{-1}$, since φ_λ and φ are unital we obtain that

$$\eta_\lambda^* \eta_\lambda \to I_r$$

  and therefore there exists a λ_0 such that for each $\lambda \geq \lambda_0$ we have that $\eta_\lambda^* \eta_\lambda$ is invertible.

- If $\eta_\lambda = \nu_\lambda |\eta_\lambda|$ is the polar decomposition of η_λ, then $|\eta_\lambda|$ is surjective for $\lambda \geq \lambda_0$ and hence ν_λ becomes an isometry.
- Since $\nu_\lambda - \eta_\lambda \to 0$ we obtain that

$$\nu_\lambda^* \varphi_\lambda \nu_\lambda = \eta_\lambda^* \varphi_\lambda \eta_\lambda + (\nu_\lambda - \eta_\lambda)^* \varphi_\lambda \nu_\lambda + \eta_\lambda^* \varphi_\lambda (\nu_\lambda - \eta_\lambda) \to \varphi.$$

- Then continuity of θ_r yields

$$\eta_\lambda^* \theta_{r_\lambda}(\varphi_\lambda) \eta_\lambda = |\eta_\lambda| \theta_r(\nu_\lambda^* \varphi_\lambda \nu_\lambda) |\eta_\lambda| \to I_r \theta_r(\varphi) I_r = \theta_r(\varphi),$$

equivalently

$$\Gamma(\xi_\lambda^* \varphi_\lambda \xi_\lambda) = \xi_\lambda^* \theta_{r_\lambda}(\varphi_\lambda) \xi_\lambda \to \xi^* \theta_r(\varphi) \xi = \Gamma(\xi^* \varphi \xi).$$

Krein-Milman in Matrix Convexity

Theorem

Let $\mathbf{K}$ *be a compact matrix convex set of a locally convex space* V *and let* $\boldsymbol{\partial}\mathbf{K} = \{\partial K_n\}_n$*. Then* $\boldsymbol{\partial}\mathbf{K}$ *is non-empty and*

$$\overline{\mathrm{co}}(\boldsymbol{\partial}\mathbf{K}) = \mathbf{K}.$$

Proof.

- Since ∂K_1 coincides with the usual extreme points of the compact convex set K_1, Krein-Milman theorem implies that ∂K_1 is non-empty and hence $\boldsymbol{\partial}\mathbf{K}$ is non-empty.
- Clearly $\boldsymbol{\partial}\mathbf{K} \subseteq \mathbf{K}$.
- Without loss of generality by translating ∂K_n by $v \otimes I_n$ where $v \in \partial K_1$ we may assume that $0 \in \overline{\mathrm{co}}(\boldsymbol{\partial}\mathbf{K})$.

Krein-Milman in Matrix Convexity

- For the converse inclusion suppose that there exists $v_0 \in K_n \backslash \overline{\text{co}}(\boldsymbol{\partial}\mathbf{K})_n$
- By the bipolar theorem there exists a weakly continuous linear mapping $\Phi : V \to M_n$ such that

$$\text{Re}(\Phi_r(v)) \leq I_n \otimes I_r, \text{ for all } v \in \overline{\text{co}}(\boldsymbol{\partial}\mathbf{K})_r \text{ and } r \in \mathbb{N},$$

and

$$\text{Re}(\Phi_n(v_0)) \not\leq I_n \otimes I_n.$$

- Then Φ induces a continuous linear functional $F : M_n(V) \to \mathbb{C}$ such that

$$F(\eta^* v \xi) = \langle \Phi_r(v)\xi, \eta \rangle,$$

for $v \in M_r(V)$ and $\xi, \eta \in M_{r,n}$ considered as vectors of $\mathbb{C}^{rn}$ on the right-hand side of the equality.

- Let $\bar{v}$ be an extreme point of $\Delta_n(\mathbf{K})$ then from the previous lemma we may write

$$\bar{v} = \xi^* v \xi,$$

where $v \in \partial K_r$ and $\xi \in M_{r,n}$ with $\|\xi\|_2 = 1$ and $r \leq n$.

- Therefore, we have that

$$\begin{aligned} \operatorname{Re} F(\bar{v}) &= \operatorname{Re} F(\xi^* v \xi) = \operatorname{Re} \langle \Phi_r(v)\xi, \xi \rangle \\ &= \langle \operatorname{Re}(\Phi_r(v))\xi, \xi \rangle \leq \langle \xi, \xi \rangle = \|\xi\|_2^2 = 1, \end{aligned}$$

for all extreme points $\bar{v}$ of $\Delta_n(\mathbf{K})$.

- Since $\Delta_n(\mathbf{K})$ is compact by the Krein-Milman theorem we obtain that
$$\operatorname{Re} F(\Delta_n(\mathbf{K})) \leq 1.$$
- Thus, for each unit vector $\xi = (\xi_1, \ldots, \xi_n) \in (\mathbb{C}^n)^n$ where $\xi_i \in \mathbb{C}^n$ we have that
$$\begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix}^* v_0 \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} \in \Delta_n(\mathbf{K}),$$
and hence
$$\operatorname{Re} \langle \Phi_n(v_0)\xi, \xi \rangle = \operatorname{Re} F \left(\begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix}^* v_0 \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} \right) \leq 1$$
which implies that $\operatorname{Re}(\Phi_n(v_0)) \leq I_n \otimes I_n$, a contradiction.

Theorem

Let $\mathbf{K}$ *be a compact matrix convex set in a locally convex space* V *and let* $\mathbf{S} = \{S_n\}_n$ *be a collection of closed subsets* $S_n \subseteq K_n$ *such that* $\nu^* S_m \nu \subseteq S_n$ *for all isometries* $\nu \in M_{m,n}$*. If* $\overline{\mathrm{co}}(\mathbf{S}) = \mathbf{K}$ *then*

$$\partial \mathbf{K} \subseteq \mathbf{S}.$$

Σας ευχαριστώ πολύ!