Σεμινάριο Συναρτησιακής Ανάλυσης και Άλγεβρων Τελεστών: Matrix Convexity

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Table of contents

Function Systems

There is a correspondence between compact convex sets and function systems, described by the following theorems:

Theorem

Let K be a compact convex set of a locally convex space and let $\mathcal{S}(A(K))$ be the state space of the continuous affine function on K. Then the map

$$\psi : K \to (\mathcal{S}(A(K)), w^*) : \quad x \mapsto \hat{x}$$

is an affine homeomorphism.

Function Systems

Let R be a function system, then every $f \in R$ defines a map

$$\tilde{f} : \mathcal{S}(R) \to \mathbb{C} : \quad \tilde{f}(s) = s(f),$$

which is w^*-continuous and affine.

Theorem

The map

$$\phi : R \to A(\mathcal{S}(R)) : \quad f \mapsto \tilde{f}$$

is an order preserving isomorphism (i.e. ϕ^{-1} is also positive).

Operator systems are considered to be the ”quantization” of function systems, so a natural question is whether there is a similar correspondence between operator systems and a suitable notion of ”quantized” convexity.

There is an affirmative answer to this question and the suitable notion turns out to be matrix convexity.

Definition

Let $B(\mathcal{H})$ denote the bounded operators of a Hilbert space $\mathcal{H}$.

- A closed linear subspace $V \subseteq B(\mathcal{H})$ will be called an operator space.
- An operator space $S \subseteq B(\mathcal{H})$ that is self-adjoint and contains the identity operator will be called a (unital) operator system.

- The space $M_n(V)$ inherits a norm from $M_n(B(\mathcal{H}))$ and $M_n(S)$ also inherits a positive cone of elements $C_n = \{x \in M_n(S) : x \in M_n(B(\mathcal{H}))^+\}$.

Matrix Convexity

Definition

A matrix convex set $\mathbf{K} = \{K_n\}_{n\in\mathbb{N}}$ in a vector space V is a collection of non-empty convex sets $K_n \subseteq M_n(V)$ such that:

- For $a \in M_{r,n}$ with $a^*a = 1$ we have $a^*K_ra \subseteq K_n$
- For $m, n \in \mathbb{N}$ we have

$$K_m \oplus K_n := \left\{ \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} : \text{ where } x \in K_n \text{ and } y \in K_m \right\} \subseteq K_{m+n}.$$

Matrix Convexity

Proposition

A collection $\mathbf{K} = \{K_n\}_n$ *where* $K_n \subseteq M_n(V)$*, is a matrix convex set of* V *if and only if*

$$\sum_{i=1}^{k} \gamma_i^* u_i \gamma_i \in K_n,$$

for all $u_i \in K_{n_i}$ *and* $\gamma_i \in M_{n_i,n}$ *such that* $\sum_{i=1}^{k} \gamma_i^* \gamma_i = I_n$.
We call the element $\sum_{i=1}^{k} \gamma_i^* u_i \gamma_i$ *a matrix convex combination.*

The Bipolar Theorem

Theorem

Let V, V' *be in duality and let* $\mathbf{K} = \{K_n\}_n$ *be a closed matrix convex set of* V *with* $0 \in K_1$. *For any* $u_0 \notin K_n$ *there exists a weakly continuous* $\phi : V \to M_n$ *such that* $\mathrm{Re}(\phi_r | K_r) \leq I_n \otimes I_r$ *for all* $r \in \mathbb{N}$ *and* $\mathrm{Re}(\phi(u_0)) \nleq I_n \otimes I_n$.

The Bipolar Theorem

Corollary

Let V, V' be in duality and let $\mathbf{K} = \{K_n\}_n$ be a closed matrix convex set of V. For any $u_0 \notin K_n$ there exists a weakly continuous $\phi : V \to M_n$ and a self-adjoint $\alpha \in M_n$ such that $\mathrm{Re}\,\phi_r|K_r \leq \alpha \otimes I_r$ for all $r \in \mathbb{N}$ and $\phi(u_0) \nleq \alpha \otimes I_n$.

Proof.

- Let x be in K_1 and set $x_1 = x$ and for $n \geq 2$ set $x_n = x \otimes I_n \in K_n$.
- Then each $L_n = K_n - x_n$ is a weakly closed convex set containing 0 and for $\gamma \in M_{r,n}$ and $v = v' - x_r \in L_r$ we have that

$$\gamma^* v \gamma = \gamma^* v' \gamma - \left[\sum_l \overline{\gamma}_{li} x \gamma_{lj}\right] = \gamma^* v' \gamma - \left[\sum_l \overline{\gamma}_{li} \gamma_{lj}\right] \otimes x$$

$$\gamma^* v' \gamma - (\gamma^* \gamma) \otimes x = \gamma^* v' \gamma - x_n \in L_n.$$

- If $v = v' - x_n \in L_n$ and $w = w' - x_m \in L_m$ we have that

$$v \oplus w = \begin{bmatrix} v' - x_n & 0 \\ 0 & w' - x_m \end{bmatrix} = v' \oplus w' - x_{n+m} \in L_{n+m}.$$

- Thus, $\mathbf{L} = \{L_n\}_n$ is a closed matrix convex set and $u_0 - x_n \notin L_n$. From the bipolar theorem there exists $\Phi : V \to M_n$ such that for every $v' \in K_r$

$$\mathrm{Re}(\Phi_r(v' - x_r)) = \mathrm{Re}(\Phi_r(v')) - \mathrm{Re}(\Phi_r(x \otimes I_r))$$
$$= \mathrm{Re}(\Phi_r(v')) - \mathrm{Re}(\Phi(x)) \otimes I_r \leq I_n \otimes I_r$$

and

$$\mathrm{Re}(\Phi(u_0)) - \mathrm{Re}(\Phi(x)) \otimes I_n \not\leq I_n \otimes I_n$$

Setting $\alpha = \mathrm{Re}(\Phi(x)) + I_n$ completes the proof.

The Bipolar Theorem

Definition

A $*$-vector space V is a complex vector space together with a map $* : V \to V$ that satisfies

1. $(v^*)^* = v$ for each $v \in V$,
2. $(v + \lambda w)^* = v^* + \bar{\lambda} w^*$ for each $v, w \in V$ and $\lambda \in \mathbb{C}$.

- We denote the set of self-adjoint elements of V by V_{sa}.
- $M_n(V)$ becomes a $*$-vector space for each n if we set

$$[v_{ij}]^* := [v_{ji}^*].$$

Preliminaries

Definition

If V is a $*$-vector space, we say that (V, V^+) is an ordered $*$-vector space if $V^+ \subseteq V_{sa}$ and the following two conditions hold

1. $$V^+ + V^+ \subseteq V^+ \text{ and } aV^+ \subseteq V^+ \text{ for all } a \in \mathbb{R}^+,$$
2. $$V^+ \cap (-V^+) = \{0\}.$$

- For $u, v \in V_{sa}$ we say that $u \leq v$ if $v - u \in V^+$.

Preliminaries

Definition

Let (V, V^+) be an ordered $*$-vector space, we say that an element $e \in V_{sa}$ is an Archimedean order unit if

1. For each $v \in V_{sa}$ there exists $r \in \mathbb{R}^+$ such that $re \geq v$.
2. If $v \in V_{sa}$ and $re + v \geq 0$ for all $r \in \mathbb{R}^+$ then $v \geq 0$.

- Each function system R is an ordered $*$-vector space with Archimedean order unit $e = 1_R$.

Preliminaries

Definition

We say that a linear map $\phi : (V, V^+) \to (W, W^+)$ is order preserving if $\phi(V^+) \subseteq W^+$ and we call ϕ an order isomorphism if in addition ϕ is bijective and $\phi(V^+) = W^+$.

Theorem

Let (V, V^+) be an ordered $*$-vector space with an Archimedean order unit e, then there exist a compact set K, a function system $R \subseteq C(K)$ and an order isomorphism $\Phi : V \to R$ such that $\Phi(e) = 1_K$.

- Therefore each V as above can be considered as an (abstract) function system.

Definition

We say that $*$-vector space V is matrix ordered if

1. For each n we are given a cone $M_n(V)^+ \subseteq M_n(V)_{sa}$ such that $(M_n(V), M_n(V)^+)$ is an ordered $*$-vector space,
2. for every n and m and $\alpha \in M_{n,m}$, we have that $\alpha^* M_n(V)^+ \alpha \subseteq M_m(V)^+$.

- We call the collection $\{M_n(V)^+\}_n$ a matrix order on V.

Definition

Let V a matrix ordered $*$-vector space V, we say that $e \in V_{sa}$ is an Archimedean matrix order unit if $e_n = e \otimes I_n$ is an Archimedean order unit for the ordered $*$-vector space $(M_n(V), M_n(V)^+)$ for each n.

- Let S be an operator system. Then S is a matrix ordered $*$-vector space with $e = 1_S$ as an Archimedean matrix order unit.

Definition

We say that a linear map $\phi : (V, \{M_n(V)^+\}_n) \to (W, \{M_n(W)^+\}_n)$ between two matrix ordered $*$-vector spaces is a completely order preserving if each $\phi_n : (M_n(V), M_n(V)^+) \to (M_n(W), M_n(W)^+)$ is order preserving. If in addition each ϕ_n is an order isomorphism we call ϕ a complete order isomorphism.

Theorem

If V is a matrix ordered $*$-vector space with an Archimedean matrix order unit e, then there exists a Hilbert space $\mathcal{H}$, an operator system $S \subseteq B(\mathcal{H})$ and a complete order isomorphism $\Phi : V \to S$ such that $\Phi(e) = 1_{B(\mathcal{H})}$.

- Therefore each V as above can be considered as an (abstract) operator system.

Definition

Let V and V' be in duality. We say that a matrix convex set **K** of V is compact if each K_n is a weakly compact subset of $M_n(V)$.

- Note that if V is a locally convex space and V^d is the topological dual of V, we have that

$$\langle .,. \rangle : V \times V^d \to \mathbb{C} : (v, f) \mapsto \langle v, f \rangle = f(v)$$

is a pairing of V and V^d and the weak topology on V^d defined by the pairing is exactly the w^*-topology.

Example

Let R be an operator system and consider the collection $\mathbf{CS}(R) = \{CS_n(R)\}_n$ where

$$CS_n(R) := \{\phi : R \to M_n : \phi \text{ is completely positive and unital}\}.$$

We can consider $CS_n(R)$ to be a subset of $M_n(R^d)$ via the identification

$$\phi = [\phi_{ij}],$$

where

$$\phi_{ij}(x) = e_i^* \phi(x) e_j, \text{ for } x \in R,$$

and e_j is the column matrix with 1 on the j-th coordinate and 0 elsewhere. Then $\mathbf{CS}(R)$ is a compact matrix convex set of R^d.

Compact Matrix Convex Sets

Proposition

Let R be an operator system and $x \in M_r(R)$. Then

$$x \geq 0 \iff \phi_r(x) \geq 0 \text{ for every } \phi \in CS_n(R) \text{ and } n \in \mathbb{N}.$$

Theorem

Let R be an operator system and $\phi : R \to M_n$ be a linear map. Then ϕ is completely positive if and only if ϕ is n-positive i.e. $\phi_n : M_n(R) \to M_n(M_n))$ is positive.

Compact Matrix Convex Sets

Definition

Let V, W be vector spaces and $\mathbf{K} = \{K_n\}_n$ be a matrix convex set. A matrix affine mapping on $\mathbf{K}$ is a sequence $\theta = \{\theta_n\}_n$ of mappings $\theta_n : K_n \to M_n(W)$ such that

$$\theta_n\left(\sum_{i=1}^{k} \gamma_i^* v_i \gamma_i\right) = \sum_{i=1}^{k} \gamma_i^* \theta_{n_i}(v_i)\gamma_i,$$

for all $v_i \in K_{n_i}$ and $\gamma_i \in M_{n_i,n}$ satisfying $\sum_{i=1}^{k} \gamma_i^* \gamma_i = I_n$.

- If V and W are locally convex then we say that θ as above is a matrix affine homeomorphism if each θ_n is a homeomorphism.
- In this case $\theta^{-1} = \{\theta_n^{-1}\}_n$ is automatically matrix affine.
- Note that if **K** is a compact matrix convex set, it suffices to prove that each θ_n is a continuous and bijective affine map in order to have that θ is a matrix affine homeomorphism.

Compact Matrix Convex Sets

- Let $\mathbf{K} = \{K_n\}_n$ be a matrix convex set, we define $A(\mathbf{K}, M_r)$ to be the set of all matrix affine mappings

$$F = \{F_n\}_n \text{ where } F_n : K_n \to M_n(M_r),$$

such that F_1 is continuous.

- $A(\mathbf{K}, M_r)$ becomes a $*$-vector space if we define the $*$-operation $F^* = \{F_n^*\}_n$ where

$$F_n^*(v) = F_n(v)^*, \text{ for every } v \in K_n \text{ and } n \in \mathbb{N}.$$

- We say that $F \geq 0$ in $A(\mathbf{K}, M_r)$ if

$$F_n(v) \geq 0 \text{ for all } v \in K_n \text{ and } n \in \mathbb{N}.$$

- We define $E = \{E_n\}_n$ in $A(\mathbf{K}, \mathbb{C})$ where $E_n(v) = I_n \in M_n$ for every $v \in K_n$ and $n \in \mathbb{N}$.
- Then $A(\mathbf{K}, \mathbb{C})$ becomes an (abstract) function system with E as an Archimedean order unit and the ordering defined above.

Proposition

The map $\phi : A(\mathbf{K}, \mathbb{C}) \to A(K_1)$ *where* $\phi(\{F_n\}_n) = F_1$ *is a unital order isomorphism.*

Proof.

- ϕ is clearly order preserving and unital.
- If $F = \{F_n\}_n$ is self adjoint then for every n and $v \in K_n$ and unit vector $\xi \in \mathbb{C}^n$ considered as row matrix we have that

$$\langle F_n(v)\xi, \xi\rangle = \xi^* F_n(v)\xi = F_1(\xi^* v \xi).$$

- Therefore, ϕ is injective.

Compact Matrix Convex Sets

- To see that ϕ is surjective suppose that $F' \in A(K_1)$ and $n \geq 2$. Then for every $u \in K_n$ we have that

$$\gamma_u(\xi) = \|\xi\|^2 F' \left(\frac{\xi^* u \xi}{\|\xi\|^2} \right), \xi \in \mathbb{C}^n$$

  is a bounded quadratic form and therefore there exists a unique $F_n(u) \in M_n$ such that for all $\xi \in \mathbb{C}^n$

$$\langle F_n(u)\xi, \xi \rangle = \gamma_u(\xi).$$

- Then $F = \{F_n\}_n \in A(\mathbf{K}, \mathbb{C})$ where $F_1 = F'$ and $\phi(F) = F_1$.
- We also have $F = \{F_n\}_n \geq 0$ if and only if $F_1 \geq 0$, which completes the proof.

- We should also note that if $F = \{F_n\}_n \in A(\mathbf{K}, \mathbb{C})$ then the formula above implies that each F_n is also continuous.
- We identify $M_r(A(\mathbf{K}, \mathbb{C}))$ with $A(\mathbf{K}, M_r)$ where for $F = [F_{ij}] \in M_r(A(\mathbf{K}, \mathbb{C}))$ and $v \in K_n$ we have

$$F_n(v) = [(F_{ij})_n(v)].$$

- We use the ordering of $A(\mathbf{K}, M_r)$ to define a positive cone in $M_r(A(\mathbf{K}, \mathbb{C}))$.
- Then $A(\mathbf{K}, \mathbb{C})$ becomes an (abstract) operator system with E as an Archimedean matrix order unit, which we will simply denote by $A(\mathbf{K})$.

Compact Matrix Convex Sets

Theorem

Let R be an operator system, then there exists a unital complete order isomorphism $\psi : R \to A(\boldsymbol{CS}(R))$.

Proof. For brevity set $\mathbf{K} := \mathbf{CS}(R)$. We have that $K_1 = CS_1(R) = \mathcal{S}(R)$ and therefore R, $A(K_1)$ and $A(\mathbf{K})$ are isomorphic as function systems. Let $\psi : R \to A(\mathbf{K})$ be the order preserving isomorphism where

$$x \in R \mapsto F = \{F_n\}_n \text{ such that } F_1 = \tilde{x}.$$

- It suffices to check that the matrix orderings are also preserved.

- For $x \in R$ and $[\phi_{ij}] \in K_n$ and a unit vector $\xi = \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} \in \mathbb{C}^n$ we have that

$$\begin{aligned} \left\langle F_n([\phi_{ij}])\xi, \xi \right\rangle &= F_1(\xi^*[\phi_{ij}]\xi) \\ &= \tilde{x}\left(\sum_{i,j=1}^{n} \xi_j \overline{\xi_i} \phi_{ij} \right) = \langle \tilde{x}_n([\phi_{ij}])\xi, \xi \rangle \end{aligned}$$

and therefore $F_n = \tilde{x}_n$.

Compact Matrix Convex Sets

- Now let $x = [x_{ij}]$ be in $M_r(R)$ and $[\phi_{kl}] \in K_n$ then

$$\psi_r([x_{ij}]_{i,j})_n([\phi_{kl}]_{k,l}) = ([\psi(x_{ij})]_{i,j})_n([\phi_{kl}]_{k,l})$$
$$= [\psi(x_{ij})_n([\phi_{kl}]_{k,l})]_{i,j} = [(\tilde{x}_{ij})_n([\phi_{kl}]_{k,l})]_{i,j}$$
$$= [\phi_{kl}(x_{ij})]_{i,j,k,l} = [(\phi_{kl})_r(x)]_{k,l} = \phi_r(x).$$

- Therefore $\psi_r([x_{ij}]) = F \in M_r(A(\mathbf{K}))$ where $F_n(\phi) = \phi_r(x)$ for each $\phi \in K_n$ and $n \in \mathbb{N}$.
- Finally,

$$x = [x_{ij}] \geq 0 \iff \phi_r(x) \geq 0, \text{ for each } \phi \in K_n, n \in \mathbb{N}$$
$$\iff F_n(\phi) \geq 0, \text{ for each } \phi \in K_n, n \in \mathbb{N}$$
$$\iff F \geq 0 \iff \psi_r(x) \geq 0,$$

which proves that ψ is a complete order isomorphism.

Theorem

Let $\mathbf{K}$ be a compact matrix convex set in a locally convex space V, then the spaces $\mathbf{K}$ and ***CS***$(A(\mathbf{K}))$ *are matrix affinely homeomorphic.*

Proof.

- For each $n \in \mathbb{N}$ set $\theta_n : K_n \to CS_n(A(\mathbf{K}))$ where for $v \in K_n$ and $F \in A(\mathbf{K})$
$$\theta_n(v)(F) = F_n(v).$$
- Note that
$$\theta_n(v)(E) = E_n(v) = I_n,$$
thus $\theta_n(v)$ is unital.

- In order to prove that $\theta_n(v)$ is completely positive it suffices to prove that $\theta_n(v)$ is n-positive, so let $[F_{ij}] \geq 0$ be in $M_n(A(\mathbf{K}))$. Then we have that

$$(\theta_n(v))_n([F_{ij}]) = [\theta_n(v)F_{ij}]$$
$$= [(F_{ij})_n(v)] = [F_{ij}]_n(v) \geq 0.$$

- We prove now that $\theta = \{\theta_n\}_n$ is matrix affine and that each θ_n is continuous.

- Suppose that $v_i \in K_{n_i}$ and $\gamma_i \in M_{n_i,n}$ for $i = 1, \dots, k$ and $\sum_{i=1}^{k} \gamma_i^* \gamma_i = I_n$ and $F \in A(\mathbf{K})$. Then

$$\theta_n \left(\sum_{i=1}^{k} \gamma_i^* v_i \gamma_i \right) (F) = F_n \left(\sum_{i=1}^{k} \gamma_i^* v_i \gamma_i \right)$$
$$= \sum_{i=1}^{k} \gamma_i^* F_{n_i}(v_i) \gamma_i = \sum_{i=1}^{k} \gamma_i^* \theta_{n_i}(v_i)(F) \gamma_i .$$

- Note that

$$\sum_{i=1}^{k} \gamma_i^* \theta_{n_i}(v_i)(F) \gamma_i = \sum_{i=1}^{k} \gamma_i^* \theta_{n_i}(v_i) \gamma_i (F).$$

- Indeed, for $v \in K_r$ we view $\theta_r(v)$ as element of $M_r(A(\mathbf{K})^d)$ and thus $\theta_r(v) = [\phi_{ij}]$ for some $\phi_{ij} \in A(\mathbf{K})^d$. For $\gamma = [\gamma_{kl}] \in M_{r,n}$ and $F \in A(\mathbf{K})$ we have that

$$\gamma^* \theta_r(v)\gamma(F) = \left[\sum_{j,k} \overline{\gamma_{ji}} \phi_{jk} \gamma_{kl}\right](F)$$

$$= \left[\sum_{j,k} \overline{\gamma_{ji}} \phi_{jk}(F) \gamma_{kl}\right] = \gamma^*[\phi_{ij}(F)]\gamma = \gamma^* \theta_r(v)(F)\gamma.$$

- Suppose that $v_\lambda \to v$ in K_r and $F \in A(\mathbf{K})$ then

$$\theta_r(v_\lambda)(F) = F_r(v_\lambda) \to F_r(v) = \theta_r(v)(F),$$

since each F_r is continuous.

We now prove injectivity of each θ_r.

- Let V^d denote the topological dual of V and note that for each $f \in V^d$ the mapping $F = \{f_n|K_n\}_n$ is in $A(\mathbf{K})$.
- Suppose that $\theta_r(v) = \theta_r(w)$ for some $v, w \in K_n$. Then

$$\theta_r(v)(\{f_n|K_n\}_n) = \theta_r(w)(\{f_n|K_n\}_n) \Rightarrow f_r(v) = f_r(w).$$

- Since V^d separates the points of V we obtain that $v = w$ and hence injectivity of θ_r.
- It remains to prove surjectivity for each θ_n due to the compactness of each K_n.

- Let ϕ_0 be in $CS_n(A(\mathbf{K}))$ such that $\phi_0 \notin \theta_n(K_n)$. We apply the bipolar theorem to $(A(\mathbf{K})^d, w^*)$ and the w^*-closed matrix convex set $\theta(\mathbf{K}) = \{\theta_n(K_n)\}_n$.
- Hence there exists a w^*-continuous and linear map $\Phi : A(\mathbf{K})^d \to M_n$ and a self-adjoint $\alpha \in M_n$ such that

$$\mathrm{Re}(\Phi_r(\theta_r(v)) \leq \alpha \otimes I_r, \text{ for all } v \in K_r \text{ and } r \in \mathbb{N}$$

and also

$$\mathrm{Re}(\Phi_n(\phi_0)) \nleq \alpha \otimes I_n.$$

- For $\psi \in A(\mathbf{K})^d$ set

$$\Phi_{ij}(\psi) = e_i^* \Phi(\psi) e_j,$$

  where e_j is the column matrix with 1 on the j-th coordinate and 0 elsewhere.
- Then each $\Phi_{ij} : A(\mathbf{K})^d \to \mathbb{C}$ is w^*-continuous and hence there exists $F_{ij} \in A(\mathbf{K})$ such that $\hat{F}_{ij} = \Phi_{ij}$.
- Therefore we may identify Φ with $[\hat{F}_{ij}] \in M_n((A(\mathbf{K})^d)^d)$.

- Set $F = [F_{ij}] \in M_n(A(\mathbf{K})) \cong A(\mathbf{K}, M_n)$ then for $v \in K_r$ we have that

$$\begin{aligned}\mathrm{Re}(F_r(v)) &= \mathrm{Re}([(F_{ij})_r(v)]) = \mathrm{Re}([\theta_r(v)(F_{ij})]) \\ &= \mathrm{Re}([(\hat{F}_{ij})_r(\theta_r(v))]) = \mathrm{Re}(\Phi_r(\theta_r(v))) \leq \alpha \otimes I_r.\end{aligned}$$

- We also have that

$$\begin{aligned}\mathrm{Re}((\phi_0)_n(F)) &= \mathrm{Re}([\phi_0(F_{ij})]) = \mathrm{Re}([(\hat{F}_{ij})_n(\phi_0)]) \\ &= \mathrm{Re}([\hat{F}_{ij}]_n(\phi_0)) = \mathrm{Re}(\Phi_n(\phi_0)) \not\leq \alpha \otimes I_n.\end{aligned}$$

- Note that for $v \in K_r$

$$(\mathrm{Re}\, F)_r(v) = (\mathrm{Re}([F_{ij}]))_r(v) = \left(\frac{[F_{ij}] + [F_{ij}]^*}{2}\right)_r (v)$$
$$= \frac{([F_{ij}])_r(v) + ([F_{ji}^*])_r(v)}{2} = \frac{[(F_{ij})_r(v)] + ([(F_{ji})_r(v)^*])}{2}$$
$$= \frac{F_r(v) + F_r(v)^*}{2} = \mathrm{Re}(F_r(v)) \leq \alpha \otimes I_r = (\alpha \otimes E)_r(v).$$

- Thus,

$$\mathrm{Re}\, F \leq \alpha \otimes E$$

in $M_n(A(\mathbf{K}))$.

- Since ϕ_0 is completely positive and unital we obtain that

$$\begin{aligned}
&\mathrm{Re}((\phi_0)_n(F)) = \frac{(\phi_0)_n([F_{ij}]) + ((\phi_0)_n([F_{ij}]))^*}{2} \\
&= (\phi_0)_n\left(\frac{[F_{ij}] + [F_{ij}]^*}{2}\right) = (\phi_0)_n(\mathrm{Re}\, F) \\
&\leq (\phi_0)_n(\alpha \otimes E) = \alpha \otimes \phi_0(E) = \alpha \otimes I_n,
\end{aligned}$$

a contradiction.

Σας ευχαριστώ πολύ!