Σεμινάριο Συναρτησιακής Ανάλυσης και Άλγεβρων Τελεστών: Matrix Convexity

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Function Systems

Definition

Let X be a compact Hausdorff space. A function system R is a closed subspace of $C(X)$ that contains the constant functions and for every $f \in R$ we have that $\bar{f} \in R$.

- We denote the positive elements of a function system R by R^+.
- We say that a linear map between two function systems $\phi : R \to S$ is positive (or order preserving) and unital if

$$\phi(R^+) \subseteq S^+$$

and

$$\phi(1_R) = 1_S.$$

Function Systems

Definition

The state space $\mathcal{S}(R)$ of a function system $R \subseteq C(X)$ is the set of positive linear functionals (which are automatically bounded) that preserve the unit i.e.

$$\mathcal{S}(R) := \{\phi \in R^* : \phi(f) \geq 0 \text{ for every } f \geq 0 \text{ and } \phi(1_R) = 1\}.$$

- $\mathcal{S}(R)$ is a w^*-compact convex set of (R^*, w^*).

Function Systems

Definition

We say that a topological vector space $(X,\mathcal{T})$ is locally convex if $\mathcal{T}$ is the weak topology of a family of seminorms $\mathcal{P}$ that separates the points of X. In particular, a set U is open if for every $x_0 \in U$ there exist $p_1,\dots,p_n \in \mathcal{P}$ and $\epsilon_1,\dots\epsilon_n > 0$ such that

$$x_0 \in \bigcap_{i=1}^{n} \{x \in X : p(x-x_0) < \epsilon\} \subseteq U.$$

Definition

Let $F : K \to C$ be map between convex sets K,C. We say that F is affine if

$$F\left(\sum_{i=1}^{n} \lambda_i x_i\right) = \sum_{i=1}^{n} \lambda_i F(x_i),$$

for every $\lambda_1,\dots,\lambda_n \geq 0$ such that $\sum_{i=1}^{n} \lambda_i = 1$.

Function Systems

Let X be a locally convex space and $K \subseteq X$ be a compact convex set.

- We set $A(K) := \{F : K \to \mathbb{C} : F \text{ is continuous and affine}\}$.
- $A(K) \subseteq C(K)$ is a function system.
- Each $x \in K$ naturally defines a state $\hat{x} \in \mathcal{S}(A(K))$ where

$$\hat{x}(F) = F(x), \text{ for } F \in A(K).$$

There is a correspondence between compact convex sets and function systems, described by the following theorems:

Theorem

Let K be a compact convex set of a locally convex space and let $\mathcal{S}(A(K))$ be the state space of the continuous affine function on K. Then the evaluation map

$$\psi : K \to (\mathcal{S}(A(K)), w^*) : \quad x \mapsto \hat{x}$$

is an affine homeomorphism.

Function Systems

Let R be a function system, then every $f \in R$ defines a map

$$\tilde{f} : \mathcal{S}(R) \to \mathbb{C} : \quad \tilde{f}(s) = s(f),$$

which is w^*-continuous and affine.

Theorem

The map

$$\phi : R \to A(\mathcal{S}(R)) : \quad f \mapsto \tilde{f}$$

is an order preserving isomorphism (i.e. ϕ^{-1} is also positive).

Operator systems are considered to be the ”quantization” of function systems, so a natural question is whether there is a similar correspondence between operator systems and a suitable notion of ”quantized” convexity.
There is an affirmative answer to this question and the suitable notion turns out to be matrix convexity.

Let E, F be vector spaces.

Definition

A tensor product of E and F is a couple (M, ϕ) where M is a vector space and $\phi : E \times F \to M$ is a bi-linear map that satisfy

1. $M = \text{span}\{\phi(x, y) : x \in E, y \in F\}$.
2. If $\{x_i : i \in I\} \subseteq E$ and $\{y_j : j \in J\} \subseteq F$ are linear independent sets, then $\{\phi(x_i, y_j) : i \in I, j \in J\} \subseteq M$ is linear independent.

Proposition

A tensor product of E and F exists, is unique up to isomorphism and satisfies the following universal property: if $\psi : E \times F \to G$ is a bi-linear map then there exists a linear map $\Psi : M \to G$ that makes the following diagram commutative

$$
\begin{array}{ccc}
E \times F & \xrightarrow{\psi} & G \\
\downarrow{\scriptstyle \phi} & \nearrow{\scriptstyle \Psi} & \\
M & &
\end{array}
$$

- From now on we will denote M by $E \otimes F$ and $\phi(x, y)$ by $x \otimes y$.
- $E \otimes F \simeq F \otimes E$ and $E \otimes (F \otimes G) \simeq (E \otimes F) \otimes G$.

Example

Let V be a vector space. Then $V \otimes M_n \simeq M_n(V)$ via the isomorphism

$$v \otimes [\gamma_{ij}] \to [\gamma_{ij} v].$$

In particular,

$$M_n(M_m) \simeq M_m(M_n) \simeq M_n \otimes M_m.$$

Preliminaries

Definition

A C^*-algebra A is a Banach space equiped with a multiplication and a map $* : A \to A$ where for every $a, b \in A$ and every $\lambda \in \mathbb{C}$ the following properties are satisfied:

- $\|ab\| \leq \|a\|\|b\|$,
- $(a + \lambda b)^* = a^* + \overline{\lambda} b^*$,
- $a^{**} = a$,
- $(ab)^* = b^* a^*$,
- $\|a^*\| = \|a\|$,
- $\|a^* a\| = \|a\|^2$.

Preliminaries

Definition

A $*$-homomorphism between two C^*-algebras A and B is a map $\phi : A \to B$ that satisfies the following properties for every $a, b \in A$ and $\lambda \in \mathbb{C}$:

- $\phi(a + \lambda b) = \phi(a) + \lambda\phi(b)$,
- $\phi(ab) = \phi(a)\phi(b)$,
- $\phi(a^*) = \phi(a)^*$.

If ϕ is also bijective we say that ϕ is a $*$-isomorphism.

Preliminaries

Definition

We say that an element a of a C^*-algebra A is positive if $a = c^*c$ for some $c \in A$. We denote by A^+ the set of all postive elements in A.

Definition

Let A be a unital C^*-algebra i.e. there exists a unique element $1_A \in A$ such that $1_A a = a 1_A = a$ for all $a \in A$. We say that $\phi : A \to \mathbb{C}$ is a state of A if $\phi(A^+) \subseteq \mathbb{R}^+$ and $\phi(1_A) = 1$. We denote the set of states of A by $\mathcal{S}(A)$.

Proposition

For each self-adjoint element a (i.e. $a^* = a$) in a unital C^* algebra A there exists a state $\phi \in \mathcal{S}(A)$ such that

$$|\phi(a)| = \|a\|.$$

Definition

We say that a state $\phi \in \mathcal{S}(A)$ is faithful if

$$\phi(a^*a) > 0$$

for every $0 \neq a \in A$.

Let us denote by $B(\mathcal{H})$ the bounded operators on some Hilbert space $\mathcal{H}$.

Definition

Let $\pi : A \to B(\mathcal{H})$ be an injective $*$-homomorphism. Let ξ be in $\mathcal{H}$.

1. We say that ξ is cyclic for $\pi(A)$ if
$$\overline{\pi(A)\xi} = \mathcal{H}.$$
2. We say that ξ is separating for $\pi(A)$ if
$$\pi(a)\xi = 0 \Rightarrow a = 0.$$

Theorem

Let ϕ be a faithful state of a C^*-algebra A. Then there exists an injective $*$-homomorphism $\pi : A \to B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$ and a $\xi \in \mathcal{H}$ such that ξ is separating and cyclic for $\pi(A)$ and

$$\phi(a) = \langle \pi(a)\xi, \xi \rangle ,$$

for all $a \in A$.

Preliminaries

Let $M_n(B(\mathcal{H}))$ be the algebra of all $n \times n$-matrices with entries from $B(\mathcal{H})$, with the usual matrix multiplication. For $[T_{ij}] \in M_n(B(\mathcal{H}))$ we set

$$[T_{ij}]^* := [T_{ji}^*].$$

Consider the Hilbert space direct sum $\mathcal{H}^{(n)} = \mathcal{H} \oplus \ldots \oplus \mathcal{H}$. Let $T = [T_{ij}]$ be an element of $M_n(B(\mathcal{H}))$, then we can regard it as an element of $B(\mathcal{H}^{(n)})$ via the matrix multiplication rule :

$$[T_{ij}] \begin{bmatrix} \xi_1 \\ \vdots \\ \xi_n \end{bmatrix} = \begin{bmatrix} \sum_{j=1}^{n} T_{1j}\xi_j \\ \vdots \\ \sum_{j=1}^{n} T_{nj}\xi_j \end{bmatrix} \in \mathcal{H}^{(n)}.$$

One can prove that this defines a bounded operator of $\mathcal{H}^{(n)}$ and that this correspondence yields a $*$-isomorphism between $M_n(B(\mathcal{H}))$ and $B(\mathcal{H}^{(n)})$. Therefore we obtain a norm on $M_n(B(\mathcal{H}))$ which makes it a C^*-algebra.

Preliminaries

Definition

Let $B(\mathcal{H})$ denote the bounded operators of a Hilbert space $\mathcal{H}$.

- A closed linear subspace $V \subseteq B(\mathcal{H})$ will be called an operator space.
- An operator space $S \subseteq B(\mathcal{H})$ that is self-adjoint and contains the identity operator will be called a (unital) operator system.

- The space $M_n(V)$ inherits a norm from $M_n(B(\mathcal{H}))$ and $M_n(S)$ also inherits a positive cone of elements $C_n = \{x \in M_n(S) : x \in M_n(B(\mathcal{H}))^+\}$.

Preliminaries

Let V, W be operator spaces and $\phi : V \to W$ a linear map.

- For each n we define the n-th amplification $\phi_n : M_n(V) \to M_n(W)$ of ϕ to be the map such that

$$\phi_n([x_{ij}]) = [\phi(x_{ij})].$$

- If $\sup_n \|\phi_n\| < \infty$, we say that ϕ is completely bounded and we set

$$\|\phi\|_{cb} = \sup_n \|\phi_n\|.$$

- If $\|\phi\|_{cb} \leq 1$ we say that ϕ is completely contractive.

Preliminaries

Let S, T be operator systems and $\phi : S \to T$ be a linear map.

- Let $C_n \subseteq M_n(S)$ and $D_n \subseteq M_n(T)$ be the positive cones of $M_n(S)$ and $M_n(T)$, respectively.
- We say that ϕ is completely positive if

$$\phi_n(C_n) \subseteq D_n,$$

for all n.

Definition

A matrix convex set $K = \{K_n\}_{n\in\mathbb{N}}$ in a vector space V is a collection of non-empty convex sets $K_n \subseteq M_n(V)$ such that:

- For $a \in M_{r,n}$ with $a^*a = I_n$ we have $a^*K_ra \subseteq K_n$
- For $m, n \in \mathbb{N}$ we have

$$K_m \oplus K_n := \left\{ \begin{bmatrix} x & 0 \\ 0 & y \end{bmatrix} : \text{ where } x \in K_n \text{ and } y \in K_m \right\} \subseteq K_{m+n}.$$

Proposition

A collection $K = \{K_n\}_n$ *where* $K_n \subseteq M_n(V)$*, is a matrix convex set of* V *if and only if*

$$\sum_{i=1}^{k} \gamma_i^* u_i \gamma_i \in K_n,$$

for all $u_i \in K_{n_i}$ *and* $\gamma_i \in M_{n_i,n}$ *such that* $\sum_{i=1}^{k} \gamma_i^* \gamma_i = I_n$.
We call the element $\sum_{i=1}^{k} \gamma_i^* u_i \gamma_i$ *a matrix convex combination.*

Proof. ($\Rightarrow$)
Suppose that $K = \{K_n\}_n$ is a matrix convex set of V and let $u_i \in K_{n_i}$ and $\gamma_i \in M_{n_i,n}$ for $i = 1, \ldots, k$ such that $\sum_{i=1}^{k} \gamma_i^* \gamma_i = I_n$. Then

$$\sum_{i=1}^{k} \gamma_i^* u_i \gamma_i = \begin{bmatrix} \gamma_1 & 0 & \cdots & 0 \\ 0 & \gamma_2 & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & \gamma_k \end{bmatrix}^* [u_1 \oplus u_2 \cdots \oplus u_k] \begin{bmatrix} \gamma_1 & 0 & \cdots & 0 \\ 0 & \gamma_2 & \cdots & 0 \\ \vdots & & \ddots & \\ 0 & 0 & \cdots & \gamma_k \end{bmatrix}.$$

$(\Leftarrow)$
The first condition of the definition is obviously satisfied. For the second let u be in K_n and w be in K_m then

$$u \oplus w = \begin{bmatrix} I_n \\ 0 \end{bmatrix} u \begin{bmatrix} I_n & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ I_m \end{bmatrix} w \begin{bmatrix} 0 & I_m \end{bmatrix} \in K_{n+m}$$

and

$$\begin{bmatrix} I_n \\ 0 \end{bmatrix} \begin{bmatrix} I_n & 0 \end{bmatrix} + \begin{bmatrix} 0 \\ I_m \end{bmatrix} \begin{bmatrix} 0 & I_m \end{bmatrix} = I_{n+m}.$$

Finally, to see that every K_n is convex note that if $u_1, \ldots, u_k \in K_n$ and $0 \leq \lambda_1, \ldots, \lambda_k \leq 1$ s.t. $\sum_{i=1}^{k} \lambda_i = 1$ then

$$\sum_{i=1}^{k} \lambda_i u_i = \begin{bmatrix} \sqrt{\lambda_1} & \cdots & \sqrt{\lambda_k} \end{bmatrix} \begin{bmatrix} u_1 \oplus \cdots \oplus u_k \end{bmatrix} \begin{bmatrix} \sqrt{\lambda_1} \\ \vdots \\ \sqrt{\lambda_k} \end{bmatrix} \in K_n.$$

Matrix Convexity

Example

The matrix interval $[aI, bI] = \{[aI_n, bI_n]\}_n$ where for each n we have $[aI_n, bI_n] = \{x \in M_n : aI_n \leq x \leq bI_n\}$ is a matrix convex set of $\mathbb{C}$.

Example

Let $V \subseteq \mathrm{B}(\mathcal{H})$ be an operator space and for each $n \in \mathbb{N}$ set $K_n(V) = \{u \in M_n(V) : \|u\| \leq 1\}$. Then $K(V) = \{K_n(V)\}_n$ is a matrix convex set of V.

Example

Let $S \subseteq \mathrm{B}(\mathcal{H})$ be an operator system and for each $n \in \mathbb{N}$ set $C_n(S) = \{u \in M_n(S) : u \geq 0\}$. Then $C(S) = \{C_n(S)\}_n$ is a matrix convex set of S.

Let V, W be operator spaces and $CB(V,W)$ denote the space of completely bounded maps from V to W equipped with the norm $\|.\|_{cb}$. For each n we identify $M_n(CB(V,W))$ with $CB(V,M_n(W))$. Consider an element $[\phi_{ij}] \in M_n(CB(V,W))$ and $v \in V$. Then we can identify $[\phi_{ij}]$ with an element of $CB(V,M_n(W))$ via

$$[\phi_{ij}]v = [\phi_{ij}v].$$

Matrix Convexity

Example

Set

$$\mathcal{CC}_n(V,W) = \{\phi \in CB(V, M_n(W)) \text{ such that } \|\phi\|_{cb} \leq 1\},$$

then $\mathcal{CC}(V,W) = \{\mathcal{CC}_n(V,W)\}_n$ is a matrix convex set of $CB(V,W)$.

Example

Let S, T be operator systems. For each n set

$$\mathcal{CP}_n(S,T) = \{\phi \in CB(S,T) \text{ such that } \phi \text{ is completely positive}\},$$

then $\mathcal{CP}(S,T) = \{\mathcal{CP}_n(S,T)\}_n$ is a matrix convex set.

Proposition

If $K = \{K_n\}_n$ *is a matrix convex set in* V *and* $0 \in K_1$*, then* $a^* K_r a \subseteq K_n$ *for any* $a \in M_{r,n}$ *with* $\|a\| \leq 1$.

Proof. Since $0 \in K_1$ we have $0 = 0 \oplus 0 \oplus \ldots \oplus 0 \in K_n$. Set $b = [1 - a^* a]^{1/2}$, then for any $u \in K_r$ we have

$$a^* u a = \begin{bmatrix} a^* & b^* \end{bmatrix} (u \oplus 0) \begin{bmatrix} a \\ b \end{bmatrix},$$

where

$$\begin{bmatrix} a^* & b^* \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = I.$$

The Bipolar Theorem

Let V, W be vector spaces.

Definition

A pairing of V, W is a bilinear function

$$F = \langle .,. \rangle : V \times W \to \mathbb{C}$$

such that

- $\langle v, w \rangle = 0$ for all $w \in W$ implies $v = 0$,
- $\langle v, w \rangle = 0$ for all $v \in V$ implies $w = 0$.

- When such a pairing exists we say that V and W are in duality and that V is the dual of W and W is the dual of V.

Having a pairing of V, W determines for each n, a pairing of $M_n(V)$ and $M_n(W)$ by

$$M_n(V) \times M_n(W) \to \mathbb{C} : ([v_{ij}], [w_{ij}]) \mapsto \sum_{i,j} \langle v_{i,j}, w_{ij} \rangle .$$

Thus, $M_n(V)$ and $M_n(W)$ are in duality for every n.

The Bipolar Theorem

If two vector spaces are in duality, then each one determines a weak topology on its dual and the weakly continuous functionals on the space can be identified with the elements of the dual space.

- Therefore, if V and V' are in duality we can define a weak topology on $M_n(V)$.
- A net $v^\lambda = [v^\lambda_{ij}]$ converges to an element v with respect to this topology if and only if

$$f(v^\lambda_{ij}) = \langle v^\lambda_{ij}, E_{ij} \otimes f \rangle \xrightarrow{\lambda} \langle v_{ij}, E_{ij} \otimes f \rangle = f(v_{ij}),$$

  for every i, j and $f \in V'$.

The Bipolar Theorem

Let V and V' be in duality and K a convex subset of V.

- The polar of K is the set

$$K^\circ = \{f \in V' : \mathrm{Re}\,\langle v, f\rangle \leq 1 \text{ for all } v \in K\}.$$

- K° is a weakly closed subset of V' that contains 0.

Theorem

If K is a convex and weakly closed subset of V that contains 0, then

$$K^{\circ\circ} = K.$$

The Bipolar Theorem

The proof of the preceding theorem is essentially the same with the case $V = \mathbb{R}^n$ but one has to use the following separation theorem:

Theorem

Let V be a complex locally convex space and let K and C be two disjoint closed convex subsets of V. If C is compact, then there is a continuous linear functional $f : V \to \mathbb{C}$, an $a \in \mathbb{R}$ and an $\epsilon > 0$ such that for all $x \in K$ and $y \in C$,

$$\operatorname{Re} f(x) \le a < a + \epsilon \le \operatorname{Re} f(y).$$

Let V, V' be in duality and $K = \{K_n\}$ be a matrix convex set of V.

- We say that K is closed if each K_n is a weakly closed subset of $M_n(V)$.

The Bipolar Theorem

Theorem

Let V, V' be in duality and let $K = \{K_n\}_n$ be a closed matrix convex set of V with $0 \in K_1$. For any $u_0 \notin K_n$ there exists a weakly continuous $\phi : V \to M_n$ such that $\mathrm{Re}\,(\phi_r|K_r) \leq I_n \otimes I_r$ for all $r \in \mathbb{N}$ and $\mathrm{Re}\,(\phi(u_0)) \nleq I_n \otimes I_n$.

In order to prove this theorem we are going to need a few lemmas.

The Bipolar Theorem

Lemma

Let $\mathcal{E}$ be a cone of real continuous affine functions on a compact convex subset K of a vector space V and that for each $e \in \mathcal{E}$, there is a corresponding point $p_e \in K$ with $e(p_e) \geq 0$. Then there is a point $p_0 \in K$ such that $e(p_0) \geq 0$ for every $e \in \mathcal{E}$.

Proof.

- The sets $\{e \geq 0\} := \{p \in K : e(p) \geq 0\}$ are non-empty and compact.
- It suffices to prove that they have the finite intersection property.
- Suppose that

$$\{e_1 \geq 0\} \cap \cdots \cap \{e_n \geq 0\} = \emptyset,$$

for some $e_1, ..., e_n \in \mathcal{E}$.

The Bipolar Theorem

- Define $\theta : K \to \mathbb{R}^n$ where

$$\theta(p) = (e_1(p), \ldots, e_n(p)),$$

  then θ is continuous and affine and thus $\theta(K)$ is a compact convex set.
- Then

$$\theta(K) \cap (\mathbb{R}^n)^+ = \emptyset,$$

  where $(\mathbb{R}^n)^+ = \{(x_1, \ldots, x_n) : x_1, \ldots, x_n \geq 0\}$.
- There exists a linear functional $f : \mathbb{R}^n \to \mathbb{R}$ such that $f((\mathbb{R}^n)^+ \geq 0$ and $f(\theta(K)) < 0$.

The Bipolar Theorem

- $f(x_1, \dots, x_n) = c_1 x_1 + \dots c_n x_n$, for some $c_1, \dots, c_n \geq 0$.
- Then $\{e \geq 0\} = \emptyset$ for

$$e = f \circ \theta = c_1 e_1 + \cdots + c_n e_n \in \mathcal{E},$$

which is a contradiction.

The Bipolar Theorem

Lemma

Let V, V' be in duality and $K = \{K_n\}_n$ be a matrix convex set of V such that $0 \in K_1$. If $F : M_n(V) \to \mathbb{C}$ is a weakly continuous linear function that satisfies $\operatorname{Re} F|K_n \leq 1$ then there exists a $p \in \mathcal{S}(M_n)$ such that for all $v \in K_r$ and $\gamma \in M_{r,n}$ we have

$$\operatorname{Re} F(\gamma^* v \gamma) \leq p(\gamma^* \gamma).$$

Proof.

- Set $\mathcal{E}$ to be the set of continuous affine real functions on $\mathcal{S}(M_n)$ of the form
$$e_{v,\gamma}(p) = p(\gamma^*\gamma) - \operatorname{Re} F(\gamma^* v\gamma),$$
where $v \in K_r$ and $\gamma \in M_{r,n}$ and $r \in \mathbb{N}$.
- For $c \in \mathbb{R}^+$ we have that
$$ce_{v,\gamma} = e_{v,c^{1/2}\gamma}$$
- For $v, w \in K_r$ and $\beta, \gamma \in M_{r,n}$ we have that
$$e_{v,\gamma} + e_{w,\beta} = e_{u,\alpha},$$
where $u = v \oplus w$ and $\alpha = \begin{bmatrix} \gamma \\ \beta \end{bmatrix}$.

The Bipolar Theorem

We proved that $\mathcal{E}$ is a cone. We prove now that for $e := e_{v,\gamma} \in \mathcal{E}$ there exists a state $p_e \in \mathcal{S}(M_n)$ such that $e(p_e) \geq 0$. Wlog we may assume that $\gamma \neq 0$.

- Let p_e be a state of M_n such that
$$p_e(\gamma^*\gamma) = \|\gamma\|^2,$$
- Set $\beta = \gamma/\|\gamma\|$, then
$$\beta^* v \beta \in K_n,$$
since $0 \in K_1$ and thus
$$\operatorname{Re} F(\gamma^* v \gamma) = \|\gamma\|^2 \operatorname{Re} F(\beta^* v \beta) \leq p_e(\gamma^*\gamma).$$
- From the previous lemma we are done.

Proof of the theorem.

- From the classical bipolar theorem since K_n is a weakly closed convex set that contains the origin and $u_0 \notin K_n$ there exists a weakly continuous linear functional F on $M_n(V)$ such that

$$\operatorname{Re} F|K_n \leq 1 < \operatorname{Re} F(u_0).$$

- From the previous lemma there exists a state p of M_n such that

$$\operatorname{Re} F(\gamma^* v \gamma) \leq p(\gamma^* \gamma),$$

for all $\gamma \in M_{r,n}$ and $v \in K_r$ and $r \in \mathbb{N}$.

The Bipolar Theorem

- We may pick $1 > \epsilon > 0$ such that $G := (1-\epsilon)F$ and $q := (1-\epsilon)p + \epsilon\tau$, where $\tau : M_n \to \mathbb{C}$ is the normalized trace, satisfy
$$\operatorname{Re} G(\gamma^* v\gamma) \leq q(\gamma^*\gamma)$$
and
$$\operatorname{Re} G|K_n \leq 1 < \operatorname{Re} G(u_0).$$
- The state q is faithful and therefore it induces a representation $\pi : M_n \to B(\mathcal{H})$ on a finite dimensional Hilbert space $\mathcal{H}$, with a separating and cyclic vector ξ where
$$q(\alpha) = \langle \pi(\alpha)\xi, \xi \rangle .$$

The Bipolar Theorem

- For a row matrix $\gamma = [\gamma_1 ... \gamma_n]$ we define $\tilde{\gamma} \in M_n$ where

$$\tilde{\gamma} = \begin{bmatrix} \gamma_1 & \gamma_2 & \cdots \\ 0 & 0 & \cdots \\ \cdots & \cdots & \cdots \end{bmatrix}$$

  and we set $\tilde{M}_{1,n}$ be the linear space of all such matrices.

- Since ξ is separating

$$\mathcal{H}_0 = \pi(\tilde{M}_{1,n})\xi$$

  is n-dimensional and also

$$B_v(\pi(\tilde{\alpha})\xi, \pi(\tilde{\beta})\xi) = G(\beta^* v \alpha),$$

  is a well-defined sesquilinear form for every $v \in V$.

The Bipolar Theorem

- Since B_v is defined on a finite dimensional space it is bounded and therefore there exists a $\phi(v) \in B(\mathcal{H}_0)$ such that

$$G(\beta^* v \alpha) = \left\langle \phi(v)\pi(\tilde{\alpha})\xi, \pi(\tilde{\beta})\xi \right\rangle.$$

- The map

$$\phi : V \to B(\mathcal{H}_0),$$

is linear and weakly continuous.

- For $v \in M_n(V)$ we have that

$$v = [v_{ij}] = \sum_{i,j} e_i v_{ij} e_j^*,$$

where $e_i = \begin{bmatrix} 0 \\ \vdots \\ 1 \\ 0 \\ \vdots \end{bmatrix} \in M_{n,1}.$

- We may identify $\mathcal{H}_0$ with $\mathbb{C}^n$ and $B(\mathcal{H}_0)$ with M_n by fixing a basis of $\mathcal{H}_0$.
- If we denote by f_j the row matrix e_j^* then

$$G(v) = \sum_{i,j} \left\langle \phi(v_{ij})\pi(\tilde{f}_i)\xi, \pi(\tilde{f}_j)\xi \right\rangle = \langle \phi_n(v)\eta_0, \eta_0 \rangle ,$$

where $\eta_0 = \begin{bmatrix} \pi(\tilde{f}_1)\xi \\ \vdots \\ \pi(\tilde{f}_n)\xi \end{bmatrix} \in (\mathcal{H}_0)^n$ and

$$\|\eta_0\|^2 = \sum_{j=1}^{n} \|\pi(\tilde{f}_j)\xi\|^2 = \sum_{j=1}^{n} q(f_j^* f_j) = q(I) = 1.$$

The Bipolar Theorem

- Let $v = [v_{ij}] \in K_r$. We wish to show that

$$\mathrm{Re}(\phi_r(v)) \leq I_r \otimes I_n.$$

After the identification of $M_r \otimes M_n$ with $M_r(M_n)$ we obtain that this is equivalent to

$$\mathrm{Re}\,\langle \phi_r(v)\eta, \eta\rangle = \langle \mathrm{Re}(\phi_r(v))\eta, \eta\rangle \leq \langle \eta, \eta\rangle\,,$$

for every $\eta \in (\mathbb{C}^n)^r$.

The Bipolar Theorem

- Let $\eta = \begin{bmatrix} \pi(\tilde{\alpha}_1)\xi \\ \vdots \\ \pi(\tilde{\alpha}_r)\xi \end{bmatrix}$ be in $(\mathcal{H}_0)^r$, where $\alpha_i \in M_{1,n}$.
- Then

$$\|\eta\|^2 = \sum_{i=1}^{n} \|\pi(\tilde{\alpha}_i)\xi\|^2 = \sum_{i=1}^{n} q(\alpha_i^* \alpha_i) = q(\alpha^* \alpha),$$

where $\alpha = \begin{bmatrix} \alpha_1 \\ \vdots \\ \alpha_r \end{bmatrix} \in M_{r,n}$.

The Bipolar Theorem

- Thus,

$$\begin{aligned}\langle \mathrm{Re}(\phi_r(v))\eta, \eta\rangle &= \mathrm{Re}\left\langle \sum_{i,j} \phi(v_{ij})\pi(\tilde{\alpha}_j)\xi, \pi(\tilde{\alpha}_i)\xi \right\rangle \\ &= \mathrm{Re} \sum_{i,j} G(\alpha_i^* v_{ij} \alpha_j) \\ &= \mathrm{Re}\, G(\alpha^* v \alpha) \\ &\leq q(\alpha^* \alpha) = \|\eta\|^2\end{aligned}$$

- On the other hand,

$$\mathrm{Re}\,\langle \phi_n(u_0)\eta_0, \eta_0\rangle = G(u_0) > 1,$$

and thus

$$\mathrm{Re}(\phi_n(u_0)) \nleq I_n \otimes I_n.$$

Σας ευχαριστώ πολύ!