

Quantum Hypergraph Homomorphisms and Applications to Non-Local Games

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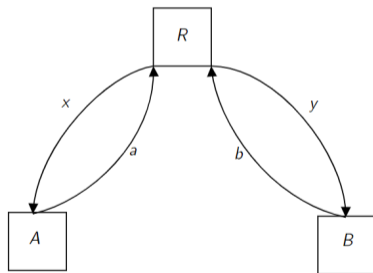
Motivation

Functional analytic methods \rightsquigarrow study combinatorial structures

- **Non-local game**: discrete object we study using **operator algebraic techniques**
- Similarity between non-local games? Strategy transport? Allow comparison for chance of “winning”?
- **Quantum homomorphisms of discrete structures** \rightsquigarrow studied for graphs, but few others

Non-local games

- The (classical) definition of a **non-local game** is a tuple (X, Y, A, B, λ) where X, Y, A, B are finite sets and $\lambda : X \times Y \times A \times B \rightarrow \{0, 1\}$ is a “verifier” function which encodes the rules of the game.



Hypergraphs

A **hypergraph** is a subset $E \subseteq V \times W$, where V and W are finite sets.

- For $w \in W$, $E(w) = \{v \in V : (v, w) \in E\}$ is an edge, and V are the vertices of a hypergraph.
- The **dual** E^* is

$$E^* := \{(w, v) : (v, w) \in E\}.$$

- Reformulate non-local games: a non-local game on (V_2, W_1, V_1, W_2) is a hypergraph $\Lambda \subseteq V_2 W_1 \times V_1 W_2$.
 - So Λ corresponds to the **support of λ** in classical definition.

Channels

When V, W are finite sets, a classical **information channel** from V to W is a positive trace preserving linear map $\mathcal{E} : \mathcal{D}_V \rightarrow \mathcal{D}_W$.

- A channel $\mathcal{E} : \mathcal{D}_V \rightarrow \mathcal{D}_W$ induces a hypergraph

$$E_{\mathcal{E}}(w) = \{(v, w) \in V \times W : \mathcal{E}(w|v) > 0\}.$$

(Here $\mathcal{E}(w|v) = \langle \mathcal{E}(\epsilon_{v,v}), \epsilon_{w,w} \rangle = \text{Tr}(\mathcal{E}(\epsilon_{v,v})\epsilon_{w,w})$ viewing M_W as a **self-dual** space, and $\epsilon_{v,v}$ are the basis elements for \mathcal{D}_V .)

- For a given hypergraph $E \subseteq V \times W$, we form the collection

$$\mathcal{C}(E) = \{\mathcal{E} : \mathcal{D}_V \rightarrow \mathcal{D}_W, \text{ a channel with } E_{\mathcal{E}} \subseteq E\}.$$

- \mathcal{E} is unital if $\mathcal{E}(I_V) = I_W$; in this case, \mathcal{E}^* is also a channel.

- Channel \mathcal{E} , hypergraph $E_{\mathcal{E}} \iff$ channel \mathcal{E}^* , hypergraph $E_{\mathcal{E}^*} = (E_{\mathcal{E}})^*$

Correlations

Let V_i, W_i be finite sets with $i = 1, 2$. A **no-signalling (NS) correlation** on the quadruple (V_2, W_1, V_1, W_2) is an information channel $\Gamma : \mathcal{D}_{V_2 W_1} \rightarrow \mathcal{D}_{V_1 W_2}$ for which **marginal channels**

$$\Gamma_{V_2 \rightarrow V_1} : \mathcal{D}_{V_2} \rightarrow \mathcal{D}_{V_1}, \quad \Gamma_{V_2 \rightarrow V_1}(v_1 | v_2) := \sum_{w_2 \in W_2} \Gamma(v_1, w_2 | v_2, w_1),$$

$$\Gamma^{W_1 \rightarrow W_2} : \mathcal{D}_{W_1} \rightarrow \mathcal{D}_{W_2}, \quad \Gamma^{W_1 \rightarrow W_2}(w_2 | w_1) := \sum_{v_1 \in V_1} \Gamma(v_1, w_2 | v_1, w_1)$$

are well-defined.

The collection of no-signalling correlations is denoted by \mathcal{C}_{ns} ; other classes of correlations $(\mathcal{C}_{\text{loc}}, \mathcal{C}_{\text{q}}, \mathcal{C}_{\text{qa}}, \mathcal{C}_{\text{qc}})$ are defined by additional restrictions we place on $\Gamma \in \mathcal{C}_{\text{ns}}$.

Local correlations: $\Gamma \in \mathcal{C}_{\text{loc}}$ is a convex combination of correlations of the form

$$\Gamma(v_1 w_2 | v_2 w_1) = \Gamma_1(v_1 | v_2) \Gamma_2(w_2 | w_1),$$

for probability distributions $\Gamma_1(\cdot | v_2), \Gamma_2(\cdot | w_1)$.

Quantum commuting: $\Gamma \in \mathcal{C}_{\text{qc}}$ if

$$\Gamma(v_1 w_2 | v_2 w_1) = \langle E_{v_2 v_1} F_{w_1 w_2} \xi, \xi \rangle$$

for mutually commuting **POVM's** $(E_{v_2 v_1})_{v_1 \in V_1}, (F_{w_1 w_2})_{w_2 \in W_2}$ acting on Hilbert space H and $\xi \in H$ is a unit vector.

- **POVM:** (finite) family of positive operators $(E_i)_i$ with $\sum_{i \in I} E_i = I$.

Quantum correlations: $\Gamma \in \mathcal{C}_{\text{q}}$ if Γ is quantum commuting, but we replace operator product $E_{v_2 v_1} F_{w_1 w_2}$ with tensor product $E_{v_2 v_1} \otimes F_{w_1 w_2}$, where our operators act on $H = H_V \otimes H_W$ with H_V, H_W finite-dimensional.

Approximately quantum: $\Gamma \in \mathcal{C}_{\text{qa}}$ if it is a limit of quantum strategies.

$$\mathcal{C}_{\text{loc}} \subset \mathcal{C}_{\text{q}} \subset \mathcal{C}_{\text{qa}} \subset \mathcal{C}_{\text{qc}} \subset \mathcal{C}_{\text{ns}}$$

Simulation paradigm

For an NS correlation Γ on (V_2, W_1, V_1, W_2) and a channel $\mathcal{E} : \mathcal{D}_{V_1} \rightarrow \mathcal{D}_{W_1}$, the map $\Gamma[\mathcal{E}] : \mathcal{D}_{V_2} \rightarrow \mathcal{D}_{W_2}$ (see [1]) defined by

$$\Gamma[\mathcal{E}](w_2|v_2) = \sum_{v_1 \in V_1} \sum_{w_1 \in W_1} \Gamma(v_1, w_2|v_2, w_1) \mathcal{E}(w_1|v_1)$$

is another channel.

- We “wire” the output for the marginal channel $\Gamma_{V_2 \rightarrow V_1}$ to the input for \mathcal{E} , and the output of \mathcal{E} back into Γ .

$$\begin{array}{ccc} V_1 & \xrightarrow{\mathcal{E}} & W_1 \\ \uparrow & & \downarrow \\ V_2 & \xrightarrow{\Gamma[\mathcal{E}]} & W_2 \end{array}$$

- When $\Gamma[\mathcal{E}] \in \mathcal{C}(V_2 \times W_2)$ we call Γ the **simulator**.

Hypergraph homomorphisms

Fix finite sets V_i, W_i and hypergraphs $E_i \subseteq V_i \times W_i$, for $i = 1, 2$.

- Let

$$E_1 \leftrightarrow E_2 = \{(v_2, w_1, v_1, w_2) : (v_1, w_1) \in E_1 \iff (v_2, w_2) \in E_2\}.$$

- If $V_1 = V_2 = V, W_1 = W_2 = W$, the class of **no-signalling bicorrelations** (see [2]) is the collection of channels

$$\mathcal{C}_{\text{ns}}^{\text{bi}} = \{\Gamma \in \mathcal{C}_{\text{ns}}(VW \times VW) : \Gamma \text{ is unital and } \Gamma^* \in \mathcal{C}_{\text{ns}}\}.$$

- For $t \neq \text{ns}$, $\Gamma \in \mathcal{C}_t^{\text{bi}}$ now has slight additional restriction: POVM's $(E_{v_2, v_1})_{v_1, v_2} \in V$ and $(F_{w_1, w_2})_{w_1, w_2} \in W$ are **magic squares**.
- **Note:** $\mathcal{C}_t(\Lambda) = \mathcal{C}(\Lambda) \cap \mathcal{C}_t, \mathcal{C}_t^{\text{bi}}(\Lambda) = \mathcal{C}(\Lambda) \cap \mathcal{C}_t^{\text{bi}}$.

Definition

We say that

- E_1 is t -homomorphic to E_2 (denoted $E_1 \rightarrow_t E_2$) if $\mathcal{C}_t(E_1 \leftrightarrow E_2) \neq \emptyset$.
- E_1 is t -isomorphic to E_2 (denoted $E_1 \simeq_t E_2$) if $V_1 = V_2, W_1 = W_2$ with $\mathcal{C}_t^{\text{bi}}(E_1 \leftrightarrow E_2) \neq \emptyset$.

Local homomorphisms

A map $f: V_2 \rightarrow V_1$ is a (classical) **homomorphism between hypergraphs** E_1 and E_2 if pre-images under f preserve edge relations; that is, f is a homomorphism if there exists a map $g: W_1 \rightarrow W_2$ so that

$$f^{-1}(E_1(w_1)) = E_2(g(w_1)), \text{ for every } w_1 \in W_1.$$

- If $V_1 = V_2, W_1 = W_2$ then f is an isomorphism when it is a **bijective homomorphism**, with g **bijective** as well.
- **Perfect local strategies** for hypergraph homomorphism (resp. isomorphism) $E_1 \rightarrow E_2$ (resp. $E_1 \simeq E_2$) **correspond precisely** with **classical homo/isomorphisms** f between E_1 and E_2 .
 - If Γ is perfect for $E_1 \rightarrow E_2$, assume Γ is an extreme point in $\mathcal{C}_{\text{loc}} + \text{no-signalling} \rightsquigarrow$ homomorphism (f, g) .
 - If (f, g) a homomorphism between hypergraphs, let $\Phi: \mathcal{D}_{V_2} \rightarrow \mathcal{D}_{V_1}, \Psi: \mathcal{D}_{W_1} \rightarrow \mathcal{D}_{W_2}$ where $\Phi(v_1|v_2) = \delta_{v_1, f(v_2)}$ and $\Psi(w_2|w_1) = \delta_{w_2, g(w_1)}$. Then $\Gamma = \Phi \otimes \Psi \in \mathcal{C}_{\text{loc}}$, and

$$(\Phi \otimes \Psi)(v_1 w_2 | v_2 w_1) = \begin{cases} 1 & \text{if } v_1 = f(v_2) \text{ and } w_2 = g(w_1), \\ 0 & \text{otherwise.} \end{cases}$$

implies $\Gamma \in \mathcal{C}_{\text{loc}}(E_1 \leftrightarrow E_2)$.

An operator algebraic approach

Start with a finite set V , and a block operator matrix $U = (u_{v,v'})_{v,v' \in V}$ such that U and U^t are **isometries**. Let \mathcal{V}_V be the (universal) ternary ring of operators generated by $u_{v,v'}$ for $v, v' \in V$ and the relations

$$\sum_{a \in V} [u_{a'',x''}, u_{a,x}, u_{a,x'}] = \delta_{x,x'} u_{a'',x''}, \quad \sum_{x \in V} [u_{a'',x''}, u_{a,x}, u_{a',x}] = \delta_{a,a'} u_{a'',x''}.$$

- A **(concrete) ternary ring of operators (TRO)** is a subspace $\mathcal{U} \subseteq \mathcal{B}(H, K)$ (where H, K are Hilbert spaces) so that $S, T, R \in \mathcal{U} \Rightarrow ST^*R \in \mathcal{U}$.
- For a **faithful ternary representation** $\theta : \mathcal{V}_V \rightarrow \mathcal{B}(H, K)$ (where H, K are Hilbert spaces), the **right C^* -algebra** $\mathcal{C}_V \simeq \overline{\text{span}}(\theta(\mathcal{V}_V)^* \theta(\mathcal{V}_V))$.
- Write $e_{v_1, v'_1, v_2, v'_2} := u_{v_2, v_1}^* u_{v'_2, v'_1}$, $v_1, v_2, v'_1, v'_2 \in V$.
 - The C^* -algebra \mathcal{C}_V is generated by elements e_{v_1, v'_1, v_2, v'_2} for $v_i, v'_i \in V, i = 1, 2$.
- Set $e_{v_2, v_1} := e_{v_1, v_1, v_2, v_2}$ for $v_1, v_2 \in V$ and generate **operator system** $\mathcal{S}_V = \text{span}\{e_{v_1, v_2} : v_1, v_2 \in V\}$.
 - **(Concrete) operator system:** $S \subset \mathcal{B}(H)$ is a $*$ -closed subspace with $I \in S$.
- Consider $\mathcal{J} = \text{span}\{e_{v_2, v_1} \otimes f_{w_1, w_2} : (v_2, w_1, v_1, w_2) \notin E_1 \leftrightarrow E_2\}$ as a subspace in $\mathcal{S}_V \otimes \mathcal{S}_W$.

Theorem (H.-Todorov, 2022)

The map $s \mapsto \Gamma_s$ is an affine surjective correspondence between

- the states of $\mathcal{S}_V \otimes_{\max} \mathcal{S}_W$ which annihilate \mathcal{J} and the perfect ns-strategies of $E_1 \leftrightarrow E_2$.
- the states of $\mathcal{S}_V \otimes_c \mathcal{S}_W$ which annihilate \mathcal{J} and the perfect qc-strategies of $E_1 \leftrightarrow E_2$.
- the states of $\mathcal{S}_V \otimes_{\min} \mathcal{S}_W$ which annihilate \mathcal{J} and the perfect qa-strategies of $E_1 \leftrightarrow E_2$.

Note: Proof of previous theorem is a consequence of (Theorem 5.10, [2]) using the theory of representations of bicorrelations. For $\tau \in \{\min, c, \max\}$ and state s on $\mathcal{S}_V \otimes_{\tau} \mathcal{S}_W$ which annihilates \mathcal{J} , map

$$\Gamma_s(v_1, w_2 | v_2, w_1) = s(e_{v_2, v_1} \otimes f_{w_1, w_2})$$

gives us the correspondence with perfect t-strategies on $E_1 \leftrightarrow E_2$ (for $t \in \{\text{qa}, \text{qc}, \text{ns}\}$).

Remark: When H is a Hilbert space, a **quantum magic square** over V on H is a block operator matrix $(E_{v_2, v_1})_{v_1, v_2 \in V}$ with positive entries, and

$$\sum_{v'_2 \in V} E_{v_1, v'_2} = \sum_{v'_1 \in V} E_{v'_1, v_2} = I, \quad v_1, v_2 \in V.$$

Operator system \mathcal{S}_V is **universal for quantum magic squares**:

ucp maps $\phi : \mathcal{S}_V \rightarrow \mathcal{B}(H) \leftrightarrow$ quantum magic square $(E_{v_1, v_2})_{v_1, v_2 \in V}$ via $E_{v_1, v_2} = \phi(e_{v_1, v_2})$

Local vs. quantum strategies

For a given finite graph G with vertex set X , we can form hypergraphs

$$E_G = \{(x, x') : x \sim x'\}, \quad F_G = \{((x, y), y) : x \sim_G y\}$$

in $X \times X$ and $XX \times X$, respectively.

- There exists graphs G_1, G_2 which are not **locally** isomorphic, but **quantum** isomorphic ([8] Atserias et. al, 2019).
- **Local**: **classical graph isomorphism** between G_1 and G_2 .
- **Quantum**: we can **interwine the adjacency matrices** A_{G_1}, A_{G_2} by some **unitary block permutation matrix** P whose entries act on finite-dimensional space H .

Theorem (H.-Todorov, 2022)

Let G_1, G_2 be graphs with vertex set X such that $G_1 \cong_q G_2$ (quantum) but $G_1 \not\cong G_2$. Then:

- $E_{G_1} \cong_q E_{G_2}$, but $E_{G_1} \not\cong_{\text{loc}} E_{G_2}$;
- $F_{G_1} \cong_q F_{G_2}$, but $F_{G_1} \not\cong_{\text{loc}} F_{G_2}$.

Local vs. quantum strategies

Proof: (Sketch)

(i) As $G_1 \cong_q G_2$, find permutation $P \in M_X \otimes M_d$ intertwining $A_{G_1} \otimes I_d$ and $A_{G_2} \otimes I_d$; this implies $E_{G_1} \cong_q E_{G_2}$. To show local separation, assume towards contradiction we have an isomorphism (f, g) on X (f, g bijections preserving edge relations). These induce an isomorphism from $L(G_1)$ to $L(G_2)$ (by considering the confusability graphs of E_{G_i}). Use Whitney's Isomorphism Theorem to show $G_1 \cong G_2$ - a contradiction.

(ii) Using permutation $P = (P_{x,y})_{x,y}$ as before, we know

$$P_{x,x'} P_{y,y'} = 0 \text{ if } \text{rel}(x, y) \neq \text{rel}(x', y').$$

For pairs $(x, y), (a, b) \in X \times X$, let $Q_{xy,ab} = P_{y,b} P_{x,a} P_{y,b}$. We can show:

- $(Q_{xy,ab})_{ab \in XX}$ is a POVM for every $xy \in XX$.
- $Q_{xy,ab} P_{y,c} = 0$ for $(xy, y) \in E_{G_1}, (ab, c) \notin E_{G_2}$.

If $\xi \in H \otimes H$ is maximally entangled, set

$$p(ab, c|xy, z) = \langle (Q_{xy,ab} \otimes P_{y,c}^t) \xi, \xi \rangle, \quad x, y, z, a, b, c \in X.$$

Then p gives us a perfect quantum strategy for $F_{G_1} \cong F_{G_2}$.

Strong no-signalling correlations

We restrict ourselves to considering hypergraphs which are themselves non-local games. We assume:

- X_i, Y_i, A_i, B_i are finite sets, $E_i \subseteq X_i Y_i \times A_i B_i, i = 1, 2$ are non-local games.
- Ordered pairs $(x, y) \in X \times Y$ are abbreviated as xy .

A channel $\Gamma : \mathcal{D}_{X_2 Y_2 \times A_1 B_1} \rightarrow \mathcal{D}_{X_1 Y_1 \times A_2 B_2}$ is **strongly no-signalling (SNS)** if

$$\sum_{b_2 \in B_2} \Gamma(x_1 y_1, a_2 b_2 | x_2 y_2, a_1 b_1) = \sum_{b_2 \in B_2} \Gamma(x_1 y_1, a_2 b_2 | x_2 y_2, a_1 b'_1), \quad b_1, b'_1 \in B_1,$$

$$\sum_{a_2 \in A_2} \Gamma(x_1 y_1, a_2 b_2 | x_2 y_2, a_1 b_1) = \sum_{a_2 \in A_2} \Gamma(x_1 y_1, a_2 b_2 | x_2 y_2, a'_1 b_1), \quad a_1, a'_1 \in A_1,$$

$$\sum_{y_1 \in Y_1} \Gamma(x_1 y_1, a_2 b_2 | x_2 y_2, a_1 b_1) = \sum_{y_1 \in Y_1} \Gamma(x_1 y_1, a_2 b_2 | x_2 y'_2, a_1 b_1), \quad y_2, y'_2 \in Y_2,$$

$$\sum_{x_1 \in X_1} \Gamma(x_1 y_1, a_2 b_2 | x_2 y_2, a_1 b_1) = \sum_{x_1 \in X_1} \Gamma(x_1 y_1, a_2 b_2 | x'_2 y_2, a_1 b_1), \quad x_2, x'_2 \in X_2.$$

- Positive operator matrix $P = (P_{xy,ab})$ is **NS** if marginal operators $P_{xa} = \sum_b P_{xy,ab}$ and $P^{yb} = \sum_a P_{xy,ab}$ are well-defined with $(P_{xa})_{a \in A}, (P^{yb})_{b \in B}$ POVM's.

SNS correlation classes

A NS operator matrix $P = (P_{xy,ab})_{xy,ab}$ is **dilatable** if there is an isometry $V: H \rightarrow K$ between Hilbert spaces and mutually commuting POVM's $(E_{xa})_{a \in A}, (F_{yb})_{b \in B}$ on K with

$$P_{xy,ab} := V^* E_{xa} F_{yb} V, \quad x \in X, y \in Y, a \in A, b \in B. \quad (1)$$

We have corresponding classes for SNS correlations: SNS correlation $\Gamma \in \mathcal{C}_{\text{SNS}}$ is

- **quantum commuting** if $\Gamma(x_1 y_1, a_2 b_2 | x_2 y_2, a_1 b_1) = \langle P_{x_2 y_2, x_1 y_1} Q_{a_1 b_1, a_2 b_2} \xi, \xi \rangle$ for mutually commuting dilatable operator matrices P, Q and unit vector $\xi \in H$.
- **quantum** if we replace operator product by tensor product in quantum commuting case, with quantum dilatable matrices M, N (replace product in (1) with tensor product as well) acting on finite dimensional $H \otimes K$.
- **approximately quantum** if Γ is a limit of quantum SNS correlations.
- **local** if Γ is quantum and individual entries in operator matrices P, Q commute with themselves as well.

Homomorphisms of non-local games

Theorem (H.-Todorov, 2022)

Let Γ be an SNS correlation over the quadruple $(X_2 Y_2, A_1 B_1, X_1 Y_1, A_2 B_2)$ and \mathcal{E} be an NS correlation over (X_1, Y_1, A_1, B_1) . Then

- $\Gamma[\mathcal{E}] \in \mathcal{C}_{\text{ns}}$;
- if $\Gamma \in \mathcal{C}_{\text{sqc}}, \mathcal{E} \in \mathcal{C}_{\text{qc}}$ then $\Gamma[\mathcal{E}] \in \mathcal{C}_{\text{qc}}$;
- if $\Gamma \in \mathcal{C}_{\text{sqqa}}, \mathcal{E} \in \mathcal{C}_{\text{qqa}}$ then $\Gamma[\mathcal{E}] \in \mathcal{C}_{\text{qqa}}$;
- if $\Gamma \in \mathcal{C}_{\text{sq}}, \mathcal{E} \in \mathcal{C}_{\text{q}}$ then $\Gamma[\mathcal{E}] \in \mathcal{C}_{\text{q}}$;
- if $\Gamma \in \mathcal{C}_{\text{sloc}}, \mathcal{E} \in \mathcal{C}_{\text{loc}}$ then $\Gamma[\mathcal{E}] \in \mathcal{C}_{\text{loc}}$.

Note: For $\Gamma \in \mathcal{C}_{\text{sqc}}, \mathcal{E} \in \mathcal{C}_{\text{qc}}$ case, say $\Gamma(x_1 y_1, a_2 b_2 | x_2 y_2, a_1 b_1) = \langle P_{x_2, x_1} P^{y_2, y_1} Q_{a_1, a_2} Q^{b_1, b_2} \xi, \xi \rangle$, and $\mathcal{E}(a_1, b_1 | x_1, y_1) = \langle E_{x_1, a_1} F_{y_1, b_1} \eta, \eta \rangle$ where $\xi \in H, \eta \in K$ are unit vectors, and families of operators are mutually commuting POVM's on resp. Hilbert spaces. Set

$$\tilde{E}_{x_2, a_2} = \sum_{x_1 \in X_1} \sum_{a_1 \in A_1} P_{x_2, x_1} Q_{a_1, a_2} \otimes E_{x_1, a_1}, \quad \tilde{F}_{y_2, b_2} = \sum_{y_1 \in Y_1} \sum_{b_1 \in B_1} P^{y_2, y_1} Q^{b_1, b_2} \otimes F_{y_1, b_1}.$$

We then have **qc-decomposition** $\Gamma[\mathcal{E}](a_2, b_2 | x_2, y_2) = \langle \tilde{E}_{x_2, a_2} \tilde{F}_{y_2, b_2} (\xi \otimes \eta), \xi \otimes \eta \rangle$. (Others follow similarly).

Homomorphisms of non-local games

Theorem (H.-Todorov, 2022)

Let Γ be an SNS correlation over the quadruple $(X_2 Y_2, A_1 B_1, X_1 Y_1, A_2 B_2)$ and \mathcal{E} be an NS correlation over (X_1, Y_1, A_1, B_1) . Then

- $\Gamma[\mathcal{E}] \in \mathcal{C}_{\text{ns}}$;
- if $\Gamma \in \mathcal{C}_{\text{sqc}}, \mathcal{E} \in \mathcal{C}_{\text{qc}}$ then $\Gamma[\mathcal{E}] \in \mathcal{C}_{\text{qc}}$;
- if $\Gamma \in \mathcal{C}_{\text{sqqa}}, \mathcal{E} \in \mathcal{C}_{\text{qa}}$ then $\Gamma[\mathcal{E}] \in \mathcal{C}_{\text{qa}}$;
- if $\Gamma \in \mathcal{C}_{\text{sq}}, \mathcal{E} \in \mathcal{C}_{\text{q}}$ then $\Gamma[\mathcal{E}] \in \mathcal{C}_{\text{q}}$;
- if $\Gamma \in \mathcal{C}_{\text{slqc}}, \mathcal{E} \in \mathcal{C}_{\text{loc}}$ then $\Gamma[\mathcal{E}] \in \mathcal{C}_{\text{loc}}$.

- Holds for SNS bicorrelations as well.
- Non-local games are now **t-isomorphic** if we can find perfect SNS strategies $\Gamma \in \mathcal{C}_t^{\text{bi}}(E_1 \leftrightarrow E_2)$.
- If $E_1 \rightarrow_{\text{st}} E_2$ or $E_1 \simeq_{\text{st}} E_2$, we simulate optimal **strategies for E_1** using SNS bicorrelation Γ and get **strategies for E_2** .

Game values

Let $E \subseteq V_2 W_1 \times V_1 W_2$ be a non-local game with probability distribution π on $V_2 W_1$.

The t -value of the game:

$$\omega_t(E, \pi) = \sup_{\Gamma \in \mathcal{C}_t} \sum_{(v_2 w_1, v_1 w_2) \in E} \Gamma(v_1 w_2 | v_2 w_1) \pi(v_2 w_1)$$

Product of games: $E \otimes E := \{(v, x, w, y) : (v, w) \in E, (x, y) \in E\} \Rightarrow E^{\otimes n} := \otimes_{i=1}^n E$, with $\pi^n := \otimes_{i=1}^n \pi$ probability distribution on $(V_2 W_1)^n$.

The optimal t -value of the game:

$$\bar{\omega}_t(E, \pi) = \limsup_{n \in \mathbb{N}} \omega_t(E^{\otimes n}, \pi^n)^{1/n}.$$

Theorem (H.-Todorov, 2022)

Let $E_i \subseteq X_i Y_i \times A_i B_i$, $i = 1, 2$ be a non-local game, π_2 be a probability distribution on $X_2 Y_2$, and $\pi_1 = \Gamma_{X_2 Y_2 \rightarrow X_1 Y_1}(\pi_2)$.

(i) If $E_1 \rightarrow_{\text{st}} E_2$ via $\Gamma \in \mathcal{C}_{\text{st}}$ then

$$\omega_t(E_1, \pi_1) \leq \omega_t(E_2, \pi_2) \text{ and } \bar{\omega}_t(E_1, \pi_1) \leq \bar{\omega}_t(E_2, \pi_2).$$

(ii) Suppose $X_1 = X_2 = X$, $Y_1 = Y_2 = Y$, and π is a Γ -stationary probability distribution on XY . If $E_1 \simeq_{\text{st}} E_2$ via $\Gamma \in \mathcal{C}_t^{\text{bi}}$ then

$$\omega_t(E_1, \pi) = \omega_t(E_2, \pi) \text{ and } \bar{\omega}_t(E_1, \pi) = \bar{\omega}_t(E_2, \pi).$$

Thank you for listening!

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